



Kähler manifolds with geodesic holomorphic gradients

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Abstract. We prove a dichotomy theorem about compact Kähler manifolds admitting nontrivial real-holomorphic geodesic gradient vector fields, which has the following consequence: either such a manifold satisfies an additional integrability condition, or through every zero of the real-holomorphic geodesic gradient there passes an uncountable family of totally geodesic, holomorphically immersed complex projective spaces, each carrying a fixed multiple of the Fubini–Study metric. We also obtain a classification result for the case where the integrability condition holds, implying that the manifold must then be biholomorphically isometric to a bundle of complex projective spaces with a bundle-like metric.

1. Introduction

The present paper deals with *geodesic-gradient Kähler triples* (M, g, τ) consisting, by definition, of a Kähler manifold (M, g) and a nonconstant real-valued function τ on M such that the g -gradient of τ is real-holomorphic and its integral curves are reparametrized geodesics. We call (M, g, τ) *compact* if so is the manifold M .

Every compact Kähler manifold (M, g) of real cohomogeneity one which admits a nontrivial invariant Killing field with zeros provides an example of a geodesic-gradient Kähler triple (Lemma 4.4). Special cases of this construction yield *Grassmannian triples* and *CP triples*, described in Section 5. Further examples of compact geodesic-gradient Kähler triples arise from the above through what we call *nontrivial modifications* (Section 13) and *horizontal extensions of CP triples* (Section 17). Section 8 introduces a class of noncompact geodesic-gradient Kähler triples which serve as universal building blocks for all compact ones.

In a geodesic-gradient Kähler triple (M, g, τ) , of particular interest are

$$(1.1) \quad \text{the } \tau\text{-preimages } \Sigma^+ \text{ and } \Sigma^- \text{ of } \tau_+ = \max \tau \text{ and } \tau_- = \min \tau,$$

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whenever the extrema of τ exist. From now on, let us assume compactness of M , which has important known consequences –see Theorem 11.1: the maximum and minimum level sets Σ^\pm are connected totally geodesic compact complex submanifolds of M , their union $\Sigma^+ \cup \Sigma^-$ equals the set of critical points of τ (the zero set of the g -gradient $v = \nabla\tau$), $u = Jv$ is a Killing vector field with a periodic flow, and τ itself must be a Morse–Bott function.

Furthermore, according to Remark 11.3 and Theorem 11.6 (c), for either fixed sign \pm , every $x \in M \setminus \Sigma^\mp$ has a unique point y nearest x in Σ^\pm , and setting $y = \pi^\pm(x)$ one defines a *holomorphic disk-bundle projection* $\pi^\pm: M \setminus \Sigma^\mp \rightarrow \Sigma^\pm$. The word ‘disk’ is used here liberally, as the fibres of π^\pm are biholomorphic to complex vector spaces rather than disks in such spaces.

These two projections π^\pm or, more precisely, the punctured disk-bundle projections obtained by restricting them to the non-critical set $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$, are the main protagonists of our study.

Specifically, we consider any leaf Π^\pm of the (obviously integrable) vertical distribution $\text{Ker } d\pi^\pm$. While π^\pm sends Π^\pm to a point, the *other* projection π^\mp maps $\Pi^\pm \cap M'$, according to our Theorem 15.1, onto the image $F(\mathbb{C}\mathbb{P}^k)$ of some totally geodesic holomorphic immersion $F: \mathbb{C}\mathbb{P}^k \rightarrow \Sigma^\mp$ inducing on $\mathbb{C}\mathbb{P}^k$ a multiple of the Fubini–Study metric. The dimension $k = k_\pm \geq 0$ is given here by $k_\pm = m - 1 - d_\pm$, for $m = \dim_{\mathbb{C}} M$ and $d_\pm = \dim_{\mathbb{C}} \Sigma^\pm$.

There arises the question of how the restriction $\pi^\mp: M' \rightarrow \Sigma^\mp$ treats the individual leaves of $\text{Ker } d\pi^\pm$ passing through all points x of $\Pi^\mp \cap M'$, for a fixed leaf Π^\mp of $\text{Ker } d\pi^\mp$, that is, how their π^\mp -images vary with $x \in \Pi^\mp \cap M'$. Since these images are totally geodesic, the question may be reduced to its infinitesimal version, involving the complex Grassmannian $\text{Gr}_k(T_y \Sigma^\mp)$ and the assignment

$$(1.2) \quad M' \ni x \mapsto d\pi_x^\mp(\text{Ker } d\pi_x^\pm) \in \text{Gr}_k(T_y \Sigma^\mp), \quad \text{where } y = \pi^\mp(x),$$

with $k = k_\pm$ defined before. Our main result, Theorem 19.1, reveals a dichotomy about (1.2): one of the following two cases has to occur. First, (1.2) may be *constant* on each leaf of $\text{Ker } d\pi^\mp$ in M' , for both signs \pm . Otherwise, (1.2) must be “strongly nonconstant” on every such leaf Π^\mp , in the sense that the dimensions $l = k_\mp$ and $k = k_\pm$ are both positive, while (1.2) restricted to Π^\mp is a composite mapping $\Pi^\mp \rightarrow \mathbb{C}\mathbb{P}^l \rightarrow \text{Gr}_k(T_y \Sigma^\mp)$ formed by a holomorphic bundle projection $\Pi^\mp \rightarrow \mathbb{C}\mathbb{P}^l$, having the fibre $\mathbb{C} \setminus \{0\}$, and a *nonconstant holomorphic embedding* of $\mathbb{C}\mathbb{P}^l$ in the complex Grassmannian $\text{Gr}_k(T_y \Sigma^\mp)$.

Functions with geodesic gradients on arbitrary Riemannian manifolds, usually called *transnormal*, have been studied extensively as well [2], [11], [10]. One easily constructs examples showing that a dichotomy as above does not generally occur in the Riemannian case, without the Kähler and holomorphicity assumptions.

Our two vertical distributions $\text{Ker } d\pi^\pm$ always span a vector subbundle of TM' , cf. formulae (11.6)–(11.7). The *first case* of Theorem 19.1 occurs if and only if

$$(1.3) \quad \text{Ker } d\pi^+ \text{ and } \text{Ker } d\pi^- \text{ span an integrable distribution on } M',$$

and the immersions $\mathbb{C}\mathbb{P}^k \rightarrow \Sigma^\mp$, mentioned in the above discussion of Theorem 15.1, are then embeddings, for both signs \pm , while their images constitute

foliations of Σ^\mp , both with the same leaf space B appearing below in our summary of Theorem 17.3. In the aforementioned *second case* of Theorem 19.1, the images of these immersions, rather than being pairwise disjoint, are totally geodesic, holomorphically immersed complex projective spaces, an uncountable family of which passes through each point of Σ^\mp . See Remark 19.2.

Note that a compact geodesic-gradient Kähler triple *need not have the property (1.3)*. Examples are provided by those Grassmannian triples (Section 5), which do not constitute CP triples, as well as triples arising from them via non-trivial modifications close to the identity. See Remark 19.6. However, one easily sees that all CP triples satisfy condition (1.3).

Theorem 17.3 describes all compact geodesic-gradient Kähler triples (M, g, τ) with (1.3). In each of them, M is biholomorphic to a bundle of positive-dimensional complex projective spaces over some base manifold B having $\dim_{\mathbb{C}} B \geq 0$.

Three special classes of compact geodesic-gradient Kähler triples (M, g, τ) have been studied before. They all satisfy (1.3). In one, that of the gradient Kähler–Ricci solitons discovered by Koiso [9] and, independently, Cao [3], τ is the soliton function. The other two consist of the special Kähler–Ricci potentials τ on compact Kähler manifolds and, respectively, compact geodesic-gradient Kähler triples with $\dim_{\mathbb{C}} M = 2$. The papers [5], [4] provide complete explicit descriptions of the latter two classes, and our Theorem 17.3 generalizes their classification results: Theorem 16.3 of [5] and Theorem 6.1 in [4]. See the end of Section 17.

The text is organized as follows. After the preliminary Sections 2–3, 6 and 7, examples and basic properties of geodesic-gradient Kähler triples are presented in Sections 4–5, 8–10, and 13, while Sections 11–12 and 14–15 deal with general consequences of compactness. Sections 16–17 establish Theorem 17.3, that is, a classification result for the case (1.3). The two final sections are devoted to proving Theorem 19.1. The arXiv version [6] of this paper provides details of many arguments of secondary importance, addressed here only briefly.

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2. Preliminaries

Manifolds, mappings and tensor fields, including Riemannian metrics and functions, are by definition of class C^∞ . A (sub)manifold is always assumed connected.

Our sign convention for the curvature tensor $R = R^D$ of a connection D in a vector bundle N over a manifold Σ , any section ξ of N , and vector fields v, w tangent to Σ , is $R(v, w)\xi = D_w D_v \xi - D_v D_w \xi + D_{[v, w]}\xi$. The total space of N has the underlying set $N = \{(y, \xi) : y \in \Sigma \text{ and } \xi \in N_y\}$. We treat $R(v, w)$, the covariant derivative $D\xi$, and any function f on Σ as bundle morphisms $R(v, w)$, $f: N \rightarrow N$ and $D\xi: T\Sigma \rightarrow N$, sending ξ or v to $R(u, v)\xi$, $f\xi$ and $D_v \xi$. For a Riemannian manifold (M, g) , the symbol ∇ will always denote both the Levi-Civita connection of g and the g -gradient. Given a function τ , vector fields w, w', v, u on (M, g) , and

any bundle morphism $B: TM \rightarrow TM$, one has

$$(2.1) \quad \begin{aligned} & \text{a) } [\mathcal{L}_v g](w, w') = 2g(Sw, w'), \quad \text{for } v = \nabla\tau \text{ and } S = \nabla v: TM \rightarrow TM, \\ & \text{b) } \mathcal{L}_v B = \nabla_v B + [B, \nabla v], \\ & \text{c) } \nabla_v A = R(u, v), \quad \text{with } A = \nabla u, \text{ whenever } u \text{ is a Killing field,} \\ & \text{d) } \nabla Q = 2\nabla_v v \text{ and } d_v \tau = g(v, \nabla\tau) = Q \quad \text{if } v = \nabla\tau \text{ and } Q = g(v, v). \end{aligned}$$

Cf. Section 1 of [6]. Let Exp^\perp now denote the normal exponential mapping of a totally geodesic submanifold Σ in a Riemannian manifold (M, g) , so that Exp^\perp is defined on an open submanifold of the total space of the normal bundle $N\Sigma$. Given (y, ξ) in the domain of Exp^\perp and any $r \in [0, 1]$, the differential $d\text{Exp}^\perp_{(y, r\xi)}$ satisfies the following relation – see, for instance, formula (1.9) of [6]:

$$(2.2) \quad d\text{Exp}^\perp_{(y, r\xi)}(r\eta + w_r^{\text{hrz}}) = \hat{w}(r),$$

involving the normal connection D in $N\Sigma$ and any fixed $w \in T_y\Sigma$. Here $y \in \Sigma$ and $\xi \in N_y\Sigma$, while the vector $\eta \in N_y\Sigma = T_{(y, r\xi)}[N_y\Sigma]$ is vertical, w_r^{hrz} denotes the D-horizontal lift of w to $(y, r\xi)$, and $r \mapsto \hat{w}(r)$ stands for the Jacobi field along the geodesic $r \mapsto x(r) = \exp_y r\xi$ such that $\hat{w}(0) = w$ and $[\nabla_x \hat{w}](0) = \eta$.

As usual, given a mapping $\pi: M \rightarrow B$ between manifolds, a vector field w (or, a distribution \mathcal{E}) on M is said to be π -projectable if $d\pi_x w_x = u_{\pi(x)}$ or, respectively, $d\pi_x(\mathcal{E}_x) = \mathcal{H}_{\pi(x)}$ for some vector field u (or, some distribution \mathcal{H}) on B and all $x \in M$. We call such w , or \mathcal{E} , *projectable along an integrable distribution* \mathcal{V} on M , or \mathcal{V} -projectable, if it is π -projectable when restricted to any open submanifold of M on which \mathcal{V} forms the vertical distribution $\text{Ker } d\pi$ of a bundle projection π .

The following facts are well known. Details can be found in Section 2 of [6].

Remark 2.1. Let $\pi: M \rightarrow B$ be a bundle projection with the vertical distribution $\mathcal{V} = \text{Ker } d\pi$. One easily verifies that a vector field w on M is π -projectable if and only if, for every section v of \mathcal{V} , the Lie bracket $[v, w]$ is also a section of \mathcal{V} . On the other hand, the local flow of a vector field v on a manifold preserves a distribution \mathcal{E} if and only if, whenever w is a local section of \mathcal{E} , so is $[v, w]$.

Lemma 2.2. *For two integrable distributions \mathcal{E}^\pm on a manifold M such that the span \mathcal{E} of \mathcal{E}^+ and \mathcal{E}^- has constant dimension, the three conditions*

- (a) \mathcal{E} is integrable,
- (b) \mathcal{E}^+ is projectable along \mathcal{E}^- ,
- (c) \mathcal{E}^- is projectable along \mathcal{E}^+

are mutually equivalent. Each of (a)–(c) also implies integrability of the distributions that \mathcal{E}^\pm locally project onto.

Remark 2.3. For a connection D in a vector bundle N over a manifold Σ and vector fields u, v tangent to Σ , the vertical component, at any $x = (y, \xi) \in N$, of the Lie bracket of the horizontal lifts of u and v equals $R_y^D(u_y, v_y)\xi$.

3. Kähler manifolds

For Kähler manifolds we use symbols such as (M, g) , where M stands for the underlying complex manifold, and J usually denotes the complex-structure tensor.

Let u, v be vector fields on a Kähler manifold (M, g) . Since $\nabla J = 0$,

$$(3.1) \quad \begin{array}{l} \text{a) } A = JS \quad \text{if one sets } S = \nabla v \text{ and } A = \nabla u, \text{ for } u = Jv, \\ \text{b) } R(u, v) = R(Ju, Jv): TM \rightarrow TM \quad \text{commutes with } J: TM \rightarrow TM, \end{array}$$

R being the curvature tensor, with S, A viewed as bundle morphisms $TM \rightarrow TM$.

Real-holomorphic vector fields v on Kähler manifolds will always be briefly referred to as *holomorphic*. As they are characterized by $\mathcal{L}_v J = 0$, formula (2.1.b) for $B = J$ implies that, given a vector field v on a Kähler manifold (M, g) ,

$$(3.2) \quad v \text{ is holomorphic if and only if } \nabla v \text{ and } J \text{ commute,}$$

where $J, \nabla v: TM \rightarrow TM$, cf. Section 2. For any holomorphic vector field v ,

$$(3.3) \quad \begin{array}{l} Jv \text{ must be holomorphic as well, while } v \text{ is locally} \\ \text{a gradient if and only if } u = Jv \text{ is a Killing field.} \end{array}$$

In fact, for $S = \nabla v$ and $A = \nabla u$, (3.1)–(3.2) give $A = JS = SJ$, and so $A + A^* = J(S - S^*)$, while the local-gradient property of v amounts to $S - S^* = 0$, and the Killing condition for u reads $A + A^* = 0$.

Lemma 3.1. *Whenever a complex manifold M admits a Kähler metric g , with the Kähler form $\omega = g(J\cdot, \cdot)$, and $\varepsilon: \mathbb{C}P^k \rightarrow M$ is a nonconstant holomorphic mapping, $\varepsilon^*\omega$ must represent a nonzero de Rham cohomology class in $H^2(\mathbb{C}P^k, \mathbb{R})$.*

If a holomorphic mapping $\varepsilon: \mathbb{C}P^k \rightarrow M$ is constant/nonconstant, so are all holomorphic mappings $\mathbb{C}P^k \rightarrow M$ sufficiently close to ε in the C^0 topology.

Proof. See Lemma 3.2 in [6]. □

Lemma 3.2. *If $\Psi: \Pi \rightarrow M$ is a continuous mapping between complex manifolds, and a codimension-one complex submanifold Λ of Π , closed as a subset of Π , has the property that the restrictions of Ψ to Λ and to the complement $\Pi \setminus \Lambda$ are both holomorphic, then Ψ is holomorphic on Π .*

Proof. See Remark 3.5 of [6]. □

In any complex manifold, $d\omega = 0$ and $\omega(J\cdot, \cdot)$ symmetric whenever $\omega = i\partial\bar{\partial}f$ or, equivalently, $2\omega = -d[J^*df]$ for a real-valued function f , with the 1-form J^*df , also denoted by $(df)J$, which sends any tangent vector field v to $d_{Jv}f$. Clearly,

$$(3.4) \quad \begin{array}{l} \text{i) } 2i\partial\bar{\partial}f = 2if'\partial\bar{\partial}\chi - f''d\chi \wedge J^*d\chi, \quad \text{with } f' = df/d\chi, \\ \text{ii) } (d\kappa)(u, v) = d_u[\kappa(v)] - d_v[\kappa(u)] - \kappa([u, v]). \end{array}$$

In (3.4.i) we assume f to be a C^∞ function of a function χ on the manifold, while (3.4.ii) and $(\iota \wedge \kappa)(u, v) = \iota(u)\kappa(v) - \iota(v)\kappa(u)$ are our exterior-derivative and exterior-product conventions, for 1-forms ι, κ and vector fields u, v .

Remark 3.3. For the real part $\langle \cdot, \cdot \rangle$ of a Hermitian inner product in a complex vector space \mathcal{N} with $\dim \mathcal{N} < \infty$, let $\rho: \mathcal{N} \rightarrow [0, \infty)$ and \mathcal{V} be the *norm function*, $\rho(\xi) = \langle \xi, \xi \rangle^{1/2}$, and \mathbb{C} -*radial distribution* on $\mathcal{N} \setminus \{0\}$, so that $\mathcal{V}_\xi = \text{Span}_{\mathbb{C}}(\xi)$.

- (a) Obviously, $d\rho^2$ is given by $\xi \mapsto 2\langle \xi, \cdot \rangle$.
- (b) $i\partial\bar{\partial}\rho^2$ equals twice the Kähler form $\langle J\cdot, \cdot \rangle$ of the constant metric $\langle \cdot, \cdot \rangle$.
- (c) $d\rho^2 \wedge J^*d\rho^2$ restricted to \mathcal{V} , on $\mathcal{N} \setminus \{0\}$, coincides with $-4\rho^2\langle J\cdot, \cdot \rangle$.
- (d) $[d\rho^2 \wedge J^*d\rho^2](v, \cdot) = 0$ for any vector field v on $\mathcal{N} \setminus \{0\}$, orthogonal to \mathcal{V} .

In fact, (b)–(d) are immediate from (a) and the text following Lemma 3.2.

Remark 3.4. On any almost-complex manifold, the formula $2\omega = -d[J^*df]$ preceding (3.4) defines an operator associating with a real-valued function f the exact 2-form $\omega = i\partial\bar{\partial}f$. For instance, using the horizontal distribution \mathcal{H} of a connection D in a complex vector bundle N over an almost-complex manifold Σ , we define an almost-complex structure on the total space N by requiring \mathcal{H} to be a complex vector subbundle of TN , and the bundle projection $\pi: N \rightarrow \Sigma$, as well as the inclusion mappings of all fibres, to be holomorphic. (Holomorphicity means here \mathbb{C} -linearity of the differential at each point.) If a D -parallel Hermitian fibre metric on N has the norm function $\rho: N \rightarrow [0, \infty)$ and the real part $\langle \cdot, \cdot \rangle$, then, applying (3.4.ii) to $\kappa = J^*d\rho^2$ (which vanishes on \mathcal{H}) we see that, by Remarks 2.3 and 3.3 (a), for $\omega = i\partial\bar{\partial}\rho^2 = -d\kappa/2$ and $x = (y, \xi) \in N$, the restriction of ω_x to \mathcal{H}_x equals the $d\pi_x$ -pullback of the 2-form $-\langle R_y^D(\cdot, \cdot)\xi, i\xi \rangle$ at $y \in \Sigma$.

Remark 3.5. For a Kähler manifold (Π, h) with $\dim_{\mathbb{C}}\Pi = l$, any holomorphic mapping $F: \mathbb{C}P^l \rightarrow \Pi$ such that F^*h equals a positive constant times the Fubini–Study metric on $\mathbb{C}P^l$ is a biholomorphism. See Remark 3.9 in [6].

4. Geodesic-gradient Kähler triples

Given a manifold M endowed with a fixed connection ∇ , we refer to a vector field v on M as *geodesic* if the integral curves of v are reparametrized ∇ -geodesics. Equivalently, for some function ψ on the open set $M' \subseteq M$ on which $v \neq 0$,

$$(4.1) \quad \nabla_v v = \psi \quad \text{everywhere in } M'.$$

A function τ on a Riemannian manifold (M, g) is said to *have a geodesic gradient* if its gradient v is a geodesic vector field relative to the Levi-Civita connection ∇ .

Lemma 4.1. *For a function τ on a Riemannian manifold (M, g) , the gradient of τ is a geodesic vector field if and only if $Q = g(\nabla\tau, \nabla\tau)$ is, locally in M' , a C^∞ function of τ .*

Proof. By (2.1.d), condition (4.1) amounts to $dQ \wedge d\tau = 0$. □

Geodesic-gradient Kähler triples were defined in the Introduction. Speaking of their *dimension*, we always mean that of the underlying complex manifold, and we

call two such triples (M, g, τ) , $(\hat{M}, \hat{g}, \hat{\tau})$ *isomorphic* if $\tau = \hat{\tau} \circ \Phi$ and $g = \Phi^* \hat{g}$ for some biholomorphism $\Phi: M \rightarrow \hat{M}$.

Remark 4.2. Whenever the g -gradient $v = \nabla \tau$ of a function τ on a Riemannian manifold (M, g) is tangent to a submanifold Π with the submanifold metric g' , the restriction of v to Π obviously equals the g' -gradient of $\tau: \Pi \rightarrow \mathbb{R}$.

Remark 4.3. A geodesic-gradient Kähler triple (M, g, τ) can be *trivially modified* to yield $(M, g, p\tau + q)$, with any real constants $p \neq 0$ and q , and Σ^\pm in (1.1) then become switched if $p < 0$. Any such (M, g, τ) and any complex submanifold Π of M , tangent to $v = \nabla \tau$ (so that it is a union of integral curves of v), and not contained in a single level set of τ , give rise (cf. Remark 4.2) to the new geodesic-gradient Kähler triple (Π, g', τ') , where g', τ' are the restrictions of g, τ to Π .

According to the next lemma, geodesic-gradient Kähler triples naturally arise from suitable cohomogeneity-one isometry groups.

Lemma 4.4. *Let a connected Lie group G acting by holomorphic isometries on a Kähler manifold (M, g) , and having some orbits of real codimension 1, preserve a nontrivial holomorphic Killing field u with zeros. If $H^1(M, \mathbb{R}) = \{0\}$, then (M, g, τ) is a geodesic-gradient Kähler triple and $u = J(\nabla \tau)$ for some G -invariant function τ on M .*

Proof. Since $H^1(M, \mathbb{R}) = \{0\}$, (3.3) implies both the existence of a function τ with $u = J(\nabla \tau)$, and the fact that its gradient $v = \nabla \tau = -Ju$ is holomorphic. Thus, elements of G preserve τ up to additive constants. Let Σ now be a fixed connected component of the zero set of u , so that G , being connected, leaves Σ invariant, while τ is constant on Σ . The additive constants just mentioned are therefore equal to 0. Due to their G -invariance, the functions τ and $Q = g(\nabla \tau, \nabla \tau)$ are constant along codimension-one orbits of G and, consequently, functionally dependent. (Note that the union of such orbits is dense in M .) Consequently, by Lemma 4.1, the gradient $v = \nabla \tau$ is a geodesic vector field. \square

The assumptions about $H^1(M, \mathbb{R})$ and holomorphicity of u in Lemma 4.4 are well-known to be redundant when M is compact, by Corollary 4.5 on p. 95 of [8]; cf. formula (A.2c) and Theorem A.1 in [4]. For proving Theorem 10.1, we will also need the following fact.

Lemma 4.5. *If a vector field w on a Riemannian manifold (M, g) is orthogonal to a geodesic gradient v and commutes with v , then w is a Jacobi field along every integral curve of $v/|v|$ in the set M' where $v \neq 0$.*

Proof. See Lemma 4.6 of [6]. \square

A compact geodesic-gradient Kähler triple of complex dimension 1 is essentially, up to isomorphisms, nothing else than the sphere S^2 with a rotationally invariant metric. Cf. Remark 4.7 in [6].

5. Examples: Grassmannian and CP triples

In this section *vector spaces* are complex, except for the lines preceding (5.5), and finite-dimensional. By *k-planes* in a vector space V we mean k -dimensional vector subspaces of V . When $k = 1$, they will also be called *lines* in V .

Given a vector space V and $k \in \{0, 1, \dots, \dim_{\mathbb{C}}V\}$, the Grassmannian Gr_kV is the set of all k -planes in V . Each Gr_kV naturally forms a compact complex manifold, and $PV = \text{Gr}_1V$ is the projective space of V , provided that $\dim_{\mathbb{C}}V > 0$. We will use the standard identification

$$(5.1) \quad P(\mathbb{C} \times V) = V \cup PV,$$

of $P(\mathbb{C} \times V)$ with the disjoint union of an open subset biholomorphic to V and a complex submanifold biholomorphic to PV via the biholomorphism sending $v \in V$, or the line $\mathbb{C}v$ spanned by $v \in V \setminus \{0\}$, to the line $\mathbb{C}(1, v)$ or, respectively, $\mathbb{C}(0, v)$. The *projectivization* of a holomorphic vector bundle N over a complex manifold Σ is, as usual, the holomorphic bundle PN of complex projective spaces over Σ with the fibres $[PN]_y = PV$ for $V = N_y$, whenever $y \in \Sigma$.

For a subspace L of a vector space V such that $\dim_{\mathbb{C}}V \geq 2$, let G be the group of all complex-linear automorphisms of V preserving both L and a fixed Hermitian inner product in V . We now define a compact complex manifold M by

$$(5.2) \quad \begin{array}{l} \text{i) } M = \text{Gr}_kV, \text{ where } 0 < k < \dim_{\mathbb{C}}V \text{ and } \dim_{\mathbb{C}}L = 1, \text{ or} \\ \text{ii) } M = PV, \text{ allowing } \dim_{\mathbb{C}}L \in \{1, \dots, \dim_{\mathbb{C}}V - 1\} \text{ to be arbitrary.} \end{array}$$

The hypotheses (and conclusions) of Lemma 4.4 then are satisfied by these M, G , any G -invariant Kähler metric g on M , and some u . Specifically, u is a vector field arising from the central circle subgroup S^1 of G formed by all unimodular elements of G acting in both L, L^\perp as multiples of Id . See the discussion below.

The triples (M, g, τ) arising via Lemma 4.4 in cases (5.2.i) and (5.2.ii) will from now on be called *Grassmannian triples* and, respectively, *CP triples*.

Since G as above contains all unit complex multiples of Id , its action on M is not effective. Lemma 4.4 does not require effectiveness of the action.

For detailed justifications of the facts stated below, see Section 5 of [6].

The cohomogeneity-one assumption of Lemma 4.4 holds here as the orbits of G are easily seen to be the levels of the nonconstant real-analytic function f on M defined, in case (5.2.i) (or (5.2.ii)) by $f(W) = |\text{pr}(X, W)|^2$, where $\text{pr}(X, W)$ denotes the orthogonal projection of X onto W , and X is some/any unit vector spanning L (or, respectively, $f(W) = |\text{pr}(X_W, L)|^2$, with X_W standing for some, or any, unit vector that spans the line W).

For a Grassmannian or CP triple (M, g, τ) , critical points of τ (that is, zeros of $u = J\nabla\tau$ or, equivalently, fixed points of the central circle subgroup S^1 mentioned above) form the disjoint union of two (connected) compact complex submanifolds, coinciding – due to obvious constancy of τ on either of them – with Σ^\pm in (1.1). If $Y = L^\perp$ and \approx denotes biholomorphic equivalence, these Σ^\pm clearly equal

$$(5.3) \quad \begin{array}{ll} \text{a) } \{W \in M : L \subseteq W\} \approx \text{Gr}_{k-1}Y, & \text{b) } \{W \in M : W \subseteq Y\} \approx \text{Gr}_kY, \\ \text{c) } \{W \in M : W \subseteq L\} \approx PL, & \text{d) } \{W \in M : W \subseteq Y\} \approx PY, \end{array}$$

where (5.3.a)–(5.3.b) correspond to (5.2.i), and (5.3.c)–(5.3.d) to (5.2.ii).

Let \langle, \rangle be the real part of a Hermitian inner product in a vector space V . The *Fubini–Study metric* on PV associated with \langle, \rangle is, as usual, uniquely characterized by requiring that the restriction of the projection $\xi \mapsto \mathbb{C}\xi$ to the unit sphere of \langle, \rangle be a Riemannian submersion.

Given a vector space V and $k \in \{1, \dots, \dim_{\mathbb{C}}V\}$, we use the symbols St_kV and $\pi: \text{St}_kV \rightarrow \text{Gr}_kV$ for the Stiefel manifold of all linearly independent ordered k -tuples of vectors in V (forming an open submanifolds of the k th Cartesian power of V) and, respectively, the holomorphic submersion sending each $\mathbf{e} \in \text{St}_kV$ to $\pi(\mathbf{e}) = \text{Span}(\mathbf{e})$. One then also has the canonical isomorphic identification

$$(5.4) \quad T_W[\text{Gr}_kV] = \text{Hom}(W, V/W)$$

for the tangent space of the complex manifold Gr_kV at any k -plane W , where Hom means ‘the space of linear operators’: for any linear lift $\tilde{H}: W \rightarrow V$ of $H \in \text{Hom}(W, V/W)$ and any basis \mathbf{e} of W , (5.4) identifies $d\pi_{\mathbf{e}}(\tilde{H}\mathbf{e}) \in T_W[\text{Gr}_kV]$ with H , which does not depend on how \tilde{H} and \mathbf{e} were chosen.

For a real (or, complex) manifold U and real (or, complex) vector spaces \mathcal{T} and \mathcal{Y} , let $F: U \rightarrow \text{Hom}(\mathcal{T}, \mathcal{Y})$ be a C^∞ (or, holomorphic) mapping giving rise to a *constant* function $U \ni \xi \mapsto \text{rank } F(\xi)$ or, equivalently, leading to the same value of $k = \dim \text{Ker } F(\xi)$ over all $\xi \in U$. Then the mapping $U \ni \xi \mapsto \text{Ker } F(\xi) \in \text{Gr}_k\mathcal{T}$ is of class C^∞ (or, holomorphic) and its differential $T_\xi U \rightarrow \text{Hom}(W, \mathcal{T}/W)$ at any $\xi \in U$, where $W = \text{Ker } F(\xi)$, cf. (5.4), sends $\eta \in T_\xi U$ to the unique $H: W \rightarrow \mathcal{T}/W$ having a linear lift $\tilde{H}: W \rightarrow \mathcal{T}$ such that, as one easily sees,

$$(5.5) \quad F(\xi) \circ \tilde{H} \text{ equals the restriction of } -dF_\xi\eta \text{ to } W,$$

with $dF_\xi: T_\xi U \rightarrow \text{Hom}(\mathcal{T}, \mathcal{Y})$, both $F(\xi)$ and $dF_\xi\eta$ being linear operators $\mathcal{T} \rightarrow \mathcal{Y}$.

All compact geodesic-gradient Kähler triples of complex dimension 1 are isomorphic to those arising from the data (5.2.ii) with $(\dim_{\mathbb{C}}V, \dim_{\mathbb{C}}L) = (2, 1)$.

6. Some relevant types of data

We will repeatedly consider quadruples τ_-, τ_+, a, Q formed by

$$(6.1) \quad \begin{aligned} & \text{a nontrivial closed interval } [\tau_-, \tau_+], \text{ a constant } a \in (0, \infty), \\ & \text{and a } C^\infty \text{ function } Q \text{ of the variable } \tau \in [\tau_-, \tau_+], \text{ positive} \\ & \text{on } (\tau_-, \tau_+), \text{ such that } Q = 0 \text{ and } dQ/d\tau = \mp 2a \text{ at } \tau = \tau_\pm, \\ & \mp \text{ being the opposite of } \pm; \text{ then (see below), we choose a} \\ & \text{sign } \pm, \text{ a } C^\infty \text{ diffeomorphism } (\tau_-, \tau_+) \ni \tau \mapsto \rho \in (0, \infty) \\ & \text{having } d\rho/d\tau = \mp a\rho/Q, \text{ a function } (0, \infty) \ni \rho \mapsto f \in \mathbb{R} \\ & \text{with the derivative characterized by } a\rho df/d\rho = 2|\tau - \tau_\pm|, \end{aligned}$$

and the unique increasing diffeomorphism $(0, \infty) \ni \rho \mapsto \sigma \in (0, \delta)$ such that

$$(6.2) \quad \begin{aligned} & a\rho d\sigma/d\rho = Q^{1/2} \text{ and } \sigma \rightarrow 0 \text{ as } \rho \rightarrow 0, \text{ where} \\ & \delta \in (0, \infty) \text{ is the integral of } Q^{-1/2} d\tau \text{ over } (\tau_-, \tau_+). \end{aligned}$$

(We use the inverse of $\tau \mapsto \rho$ in (6.1) to treat τ, Q as functions of ρ .) Namely, one easily verifies that $\tau \mapsto \rho$ and $\rho \mapsto \sigma$ with the stated properties exist, while δ is

finite. Details are provided by Remark 5.1 in [5], Theorem 10.2 (iii) in [4] and p. 1661 of [4]. Note that it is the limit condition in (6.2) that makes σ unique; by contrast, ρ with (6.1) is only unique up to a positive constant factor.

Any f satisfying (6.1) has a C^∞ extension to $[0, \infty)$, which is also a C^∞ function of $\rho^2 \in [0, \infty)$. See Lemma 6.1 in [6]. The first and second derivatives of f with respect to ρ^2 obviously equal $|\tau - \tau_\pm|/(a\rho^2)$ and $(Q - 2a|\tau - \tau_\pm|)/(2a^2\rho^4)$. If ρ simultaneously denotes a positive function on a complex manifold M' , which turns f into a function $M' \rightarrow \mathbb{R}$, (3.4.i) for $\chi = \rho^2$ and these formulae give

$$(6.3) \quad 4ia^2\rho^4\partial\bar{\partial}f = 4ia\rho^2|\tau - \tau_\pm|\partial\bar{\partial}\rho^2 + (2a|\tau - \tau_\pm| - Q)d\rho^2 \wedge J^*d\rho^2.$$

Remark 6.1. The composite $(0, \delta) \ni \sigma \mapsto \rho \mapsto \tau \in (\tau_-, \tau_+)$ of the inverses of the diffeomorphisms in (6.1)–(6.2) is the unique solution of the autonomous equation $d\tau/d\sigma = \mp Q^{1/2}$, with the sign \pm fixed as in (6.1), such that $\tau \rightarrow \tau_\pm$ as $\sigma \rightarrow 0$. (We say ‘autonomous’ since (6.1) makes Q a function of τ .) In fact, any two solutions of the equivalent equation $d\sigma/d\tau = \mp Q^{-1/2}$ differ by a constant.

7. The Chern connection

Let \langle, \rangle be the real part of a Hermitian fibre metric in a holomorphic complex vector bundle N over a complex manifold Σ . The *Chern connection* of \langle, \rangle is the unique connection D in N which makes \langle, \rangle parallel and satisfies the condition $D^{0,1} = \bar{\partial}$, meaning that, for any section ξ of N , the complex-antilinear part of the real vector-bundle morphism $D\xi: T\Sigma \rightarrow N$ equals $\bar{\partial}\xi$, the image of ξ under the Cauchy–Riemann operator. Cf. Sect. 1.4 of [7]. The following properties of D are well known; see p. 32 in [1] and Propositions 1.3.5, 1.7.19, 1.4.18 of [7].

- (a) D depends on N and \langle, \rangle functorially with respect to all natural operations, including Hom, direct sums, and pullbacks under holomorphic mappings.
- (b) $R^D(Jw, Jw') = R^D(w, w')$, with the notation of Section 2, where w, w', R^D are any vector fields on Σ and, respectively, the curvature tensor of D .
- (c) D is the Levi-Civita connection of \langle, \rangle if \langle, \rangle is a Kähler metric in $N = T\Sigma$.
- (d) D coincides with the normal connection in the normal bundle $N\Sigma$ for any totally geodesic complex submanifold Σ a Kähler manifold (M, g) and the Riemannian fibre metric \langle, \rangle in N induced by g . (In addition, it follows then that N must be a holomorphic subbundle of TM .)

Lemma 7.1. *For $N, \Sigma, \langle, \rangle$ as above, the bundle projection $\pi: N \rightarrow \Sigma$, the norm function $\rho: N \rightarrow [0, \infty)$ of \langle, \rangle , the Chern connection D of \langle, \rangle , its curvature tensor R^D , and the 2-form $\omega = i\partial\bar{\partial}\rho^2$ have the following properties.*

- (i) *The horizontal distribution of D and the vertical distribution are mutually ω -orthogonal, in an obvious sense, complex vector subbundles of TN .*
- (ii) *Remark 3.3 (b) describes ω restricted to any fibre N_y of N , where $y \in \Sigma$.*
- (iii) *Whenever $x = (y, \xi) \in N$, the restriction of ω_x to the horizontal space of D at x equals the $d\pi_x$ -pullback of the 2-form $-\langle R_y^D(\cdot, \cdot)\xi, i\xi \rangle$ at $y \in \Sigma$.*

Proof. For (i), (ii) see Lemma 7.1 of [6]; (iii) follows from (i) and Remark 3.4. \square

8. Examples: Vector bundles

The geodesic-gradient Kähler triples constructed in this section are all noncompact. What makes them relevant is the fact that some of them serve as universal building blocks for compact geodesic-gradient Kähler triples (Theorem 14.2).

We begin with data $\Sigma, h, N, \langle, \rangle, \tau_-, \tau_+, a, Q, \pm, \tau \mapsto \rho$ and $\rho \mapsto f$ formed by

- (i) the real part \langle, \rangle of a Hermitian fibre metric in a holomorphic complex vector bundle N of positive fibre dimension over a Kähler manifold (Σ, h) ,
- (ii) objects $\tau_-, \tau_+, a, Q, \pm, \tau \mapsto \rho$ and $\rho \mapsto f$ satisfying (6.1).

Letting $\pi: N \rightarrow \Sigma$ stand for the bundle projection, D for the Chern connection of \langle, \rangle (see Section 7), and ρ both for the variable in (ii) and for the norm function $N \rightarrow [0, \infty)$, we use the inverse mapping of $\tau \mapsto \rho$, cf. (6.1), to

$$(8.1) \quad \text{treat } \tau, Q \text{ and } f \text{ as functions } N \rightarrow \mathbb{R}, \text{ denoted here by } \hat{\tau}, \hat{Q}, \text{ and } \hat{f}.$$

We define a Kähler metric \hat{g} on N by requiring the Kähler forms $\hat{\omega} = \hat{g}(\hat{J} \cdot, \cdot)$ and $\omega^h = h(J \cdot, \cdot)$ to be related by $\hat{\omega} = \pi^* \omega^h + i \partial \bar{\partial} \hat{f}$. (Here \hat{J} is the complex-structure tensor of N .) This amounts to

$$(8.2) \quad \begin{aligned} \text{a) } & \hat{g} = \pi^* h - (i \partial \bar{\partial} \hat{f})(\hat{J} \cdot, \cdot), \text{ where we are also assuming that} \\ \text{b) } & \pi^* h - (i \partial \bar{\partial} \hat{f})(\hat{J} \cdot, \cdot) \text{ is positive-definite at every point of } N. \end{aligned}$$

The above construction starts from the data (i)–(ii) with (8.2.b), and yields a geodesic-gradient Kähler triple $(N, \hat{g}, \hat{\tau})$. See Lemma 8.1 below.

It is convenient, however, to provide the following equivalent, though less concise, description of \hat{g} and \hat{J} restricted to the complement $N' = N \setminus \Sigma$ of the zero section in N . It uses the *complex* – due to Lemma 7.1 (i) – direct-sum decomposition $TN' = \hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp \oplus \hat{\mathcal{H}}^\bullet$, in which $\hat{\mathcal{H}}^\bullet$ is the horizontal distribution of D and $\hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp = \text{Ker } d\pi$ equals the vertical distribution, with the summands $\hat{\mathcal{V}}$ and $\hat{\mathcal{H}}^\mp$ forming, on each punctured fibre $N_y \setminus \{0\}$, the \mathbb{C} -radial distribution (Remark 3.3) and, respectively, its \langle, \rangle -orthogonal complement in $N_y \setminus \{0\}$. To describe \hat{g} and \hat{J} , we declare that the three summands $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet$ of TN' are \hat{J} -invariant, that \hat{J} restricted to $\hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp$ agrees, along each punctured fibre $N_y \setminus \{0\}$, with its standard complex-structure tensor of the complex vector space N_y , that the differential of π at every $(y, \xi) \in N_y \setminus \{0\}$, maps $\hat{\mathcal{H}}_{(y, \xi)}^\bullet$ complex-linearly onto $T_y \Sigma$ and, with the constant $a \in (0, \infty)$ and function $\hat{\tau}$ appearing in (ii) and (8.1),

$$(8.3) \quad \begin{aligned} \text{a) } & \text{the summands } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet \text{ of } TN' \text{ are mutually } \hat{g}\text{-orthogonal,} \\ \text{b) } & a^2 \rho^2 \hat{g} = \hat{Q} \langle, \rangle \text{ on } \hat{\mathcal{V}}, \quad a \rho^2 \hat{g} = 2|\hat{\tau} - \tau_\pm| \langle, \rangle \text{ on } \hat{\mathcal{H}}^\pm, \\ \text{c) } & \hat{g}_x(w_x, w'_x) = h_y(w, w') - \frac{|\hat{\tau}(x) - \tau_\pm|}{a \rho^2} \langle R_y^D(w, J_y w') \xi, i \xi \rangle \text{ with } \rho = |\xi|, \end{aligned}$$

at any $x = (y, \xi) \in N_y \setminus \{0\}$, where w, w' are any two vectors in $T_y \Sigma$, and w_x, w'_x denote their D -horizontal lifts to x . Vertical vector fields \hat{v}, \hat{u} along with the

restrictions of \hat{g} and \hat{J} to $\hat{\mathcal{V}} = \text{Span}(\hat{v}, \hat{u})$ may now be characterized by

$$(8.4) \quad \hat{v}_{(y, \xi)} = \mp a\xi, \quad \hat{u}_{(y, \xi)} = \mp ai\xi, \quad \hat{g}(\hat{v}, \hat{v}) = \hat{g}(\hat{u}, \hat{u}) = \hat{Q}, \quad \hat{g}(\hat{v}, \hat{u}) = 0, \quad \hat{u} = \hat{J}\hat{v}.$$

Note that the symmetry of $\hat{g}_x(w_x, w'_x)$ in w_x, w'_x reflects (b) in Section 7.

Formula (6.3), combined with (b) and (d) in Remark 3.3 and Lemma 7.1 (i)–(iii), easily implies that the definition (8.3) of \hat{g} is actually equivalent to (8.2.a), while (8.2.b) amounts to positivity of the right-hand side in (8.3.c) when $w = w' \neq 0$.

Lemma 8.1. *For any data (i)–(ii) with (8.2.b), define $\hat{g}, \hat{\tau}$ by (8.1)–(8.2.a).*

- (a) *$(N, \hat{g}, \hat{\tau})$ is a geodesic-gradient Kähler triple, and the fibres $N_y = \pi^{-1}(y)$, $y \in \Sigma$, are totally geodesic complex submanifolds of (N, \hat{g}) .*
- (b) *The zero section $\Sigma \subseteq N$ coincides with Σ^\pm , the τ_\pm level set of τ .*
- (c) *The \hat{g} -gradient $\hat{v} = \hat{\nabla}\hat{\tau}$ and $\hat{S} = \hat{\nabla}\hat{v}$ satisfy (8.4) and the equality*

$$(8.5) \quad 2\hat{g}_x(\hat{S}_x w_x, w'_x) = \pm \frac{\hat{Q}(x)}{a\rho^2} \langle R_y^D(w, J_y w')\xi, i\xi \rangle, \quad \text{where } \rho = |\xi| > 0,$$

the assumptions being the same as in (8.3.c).

Proof. See Theorem 8.1 of [6]. □

A *special Kähler–Ricci potential* [5] on a Kähler manifold (M, g) is any non-constant function $\tau: M \rightarrow \mathbb{R}$ such that $v = \nabla\tau$ is real-holomorphic, while, at points where $v \neq 0$, all nonzero vectors orthogonal to v and Jv are eigenvectors of both ∇v and the Ricci tensor, with $\nabla v: TM \rightarrow TM$ as in Section 2. We then call (M, g, τ) an *SKRP triple*. All SKRP triples (M, g, τ) are geodesic-gradient Kähler triples, as they satisfy (4.1).

Compact SKRP triples (M, g, τ) have been classified by Theorem 16.3 of [5]. They are divided into Class 1, in which M is the total space of a holomorphic $\mathbb{C}\mathbb{P}^1$ bundle, and Class 2, with M biholomorphic to $\mathbb{C}\mathbb{P}^m$ for $m = \dim_{\mathbb{C}} M$.

Lemma 8.2. *Up to isomorphisms, compact SKRP triples of Class 2 are the same as CP triples constructed using (5.2.ii) with $\dim_{\mathbb{C}} \mathbf{L} = 1$.*

Proof. See Remark 6.2 in [5]. (Note that (5.2.ii) with $\dim_{\mathbb{C}} \mathbf{L} = m - 1$ obviously leads to the same isomorphism type.) □

In (i) above, $\dim_{\mathbb{C}} \Sigma \geq 0$, which allows the possibility of a one-point base manifold $\Sigma = \{y\}$, so that, as a complex manifold, N is a complex vector space, namely, the fibre N_y . According to pp. 85–86 of [5], under the standard identification (5.1) for $\mathbf{V} = N_y$, both \hat{g} and $\hat{\tau}$ then can be extended to the projective space $\mathbb{P}(\mathbb{C} \times N_y)$, giving rise to a Class 2 SKRP triple $(M, \hat{g}, \hat{\tau})$ with $M = \mathbb{P}(\mathbb{C} \times N_y)$.

Lemma 8.3. *The SKRP triples (M, g, τ) just mentioned, with $M = \mathbb{P}(\mathbb{C} \times N_y)$, represent all isomorphism types of compact SKRP triples of Class 2. Such types include all compact geodesic-gradient Kähler triples of complex dimension 1.*

Proof. For the first part, see Remark 6.2 in [5]. The final clause is in turn immediate from Lemma 8.2 and the final two-line paragraph of Section 5. □

Remark 8.4. As a consequence of the second part of Remark 4.3, for $(N, \hat{g}, \hat{\tau})$ as in Lemma 8.1, every fibre N_y is the underlying complex manifold of a geodesic-gradient Kähler triple, realizing a special case of Lemma 8.1: that of a one-point base manifold $\{y\}$. Its projective compactification $P(\mathbb{C} \times N_y)$ constitutes, for reasons mentioned above, the underlying complex manifold of an SKRP triple of Class 2. The resulting submanifold metric on the complement of N_y in $P(\mathbb{C} \times N_y)$ (that is, on the projective hyperplane at infinity, identified via (5.1) with PN_y) equals $2(\tau_+ - \tau_-)/a$ times the Fubini–Study metric associated – as in Section 5 – with $\langle \cdot, \cdot \rangle$.

This easily-verified claim is also justified in Remark 8.4 of [6].

9. Local properties

Throughout this section (M, g, τ) is a fixed geodesic-gradient Kähler triple (see the Introduction). We use the symbols

$$(9.1) \quad J, v, u, M', \psi, Q, \mathcal{V}, \mathcal{V}^\perp, S, A$$

for the complex-structure tensor $J: TM \rightarrow TM$ of the underlying complex manifold M , the gradient $v = \nabla\tau$, its J -image $u = Jv$, the open set M' where $v \neq 0$, the function ψ on M' with (4.1), the function $Q = g(v, v)$ on M , the distribution $\mathcal{V} = \text{Span}(v, u)$ on M' , its orthogonal complement, as well as the endomorphisms $S = \nabla v$ and $A = \nabla u$ of TM , cf. Section 2. Under the above hypotheses,

$$(9.2) \quad \begin{aligned} & \text{a) } v, u \text{ are both holomorphic, } |v| = |u| = Q^{1/2}, \text{ and } A = JS = SJ, \\ & \text{b) } u = Jv \text{ is a Killing field commuting with } v, \text{ and orthogonal to } v, \\ & \text{c) } \nabla_w A = R(u, w) \text{ and } \nabla_w S = -J[R(u, w)] \text{ for any vector field } w, \\ & \text{d) } S \text{ is self-adjoint and } J, A \text{ are skew-adjoint at every point of } M, \\ & \text{e) } g([w, w'], u) = -2g(Aw, w') \text{ for any local sections } w, w' \text{ of } \mathcal{V}^\perp, \\ & \text{f) } \nabla_v v = \psi v = -\nabla_u u \text{ and } \nabla_u v = \nabla_v u = \psi u \text{ everywhere in } M', \\ & \text{g) } Q \text{ is, locally in } M', \text{ a function of } \tau, \text{ and } 2\psi = dQ/d\tau, \\ & \text{h) } J, S, A \text{ and the local flows of } u \text{ and } v \text{ leave } \mathcal{V} \text{ and } \mathcal{V}^\perp \text{ invariant.} \end{aligned}$$

In (9.2.c), R denotes the curvature tensor of g , with the convention of Section 2.

Namely, holomorphicity of v combined with (3.1)–(3.2) gives (9.2.a), u being holomorphic due to (3.2), as $A = JS = SJ$ commutes with J . Next, (9.2.b) follows from (3.3) and the Lie-bracket equality $[u, v] = \nabla_u v - \nabla_v u = Su - Av = Su - SJv = 0$, obvious in view of (9.2.a), while (9.2.c) (or, (9.2.d)) is a direct consequence of (2.1.c) and (9.2.a) or, respectively, of of (9.2.b) combined with the fact that v is a gradient. We now obtain (9.2.e) from (9.2.d), noting that $g(\nabla_w w', u) = -g(w', \nabla_w u) = -g(w', Aw)$. On the other hand, (9.2.b), (9.2.a) and (4.1) yield $\nabla_u v = \nabla_v u = \nabla_v(Jv) = J\nabla_v v = \psi Jv = \psi u$ and so $\nabla_u u = \nabla_u(Jv) = J\nabla_u v = \psi Ju = -\psi v$, establishing (9.2.f). Lemma 4.1, (2.1.d) and (9.2.f) in turn imply (9.2.g). That J, S, A all leave $\mathcal{V} = \text{Span}(v, u)$ invariant is clear as $Jv = u$ and $Ju = -v$ while, by (9.2.f), Sv, Su, Av, Au are sections of \mathcal{V} . The same conclusion for \mathcal{V}^\perp is

now immediate from (9.2.d). By (9.2.b), the local flows of v and u preserve v, u and $\mathcal{V} = \text{Span}(v, u)$. The u -invariance of \mathcal{V}^\perp now follows from (9.2.b). Finally, let w be a section of \mathcal{V}^\perp . Writing $\langle \cdot, \cdot \rangle$ for g , we get $\langle [v, w], v \rangle = \langle \nabla_v w - \nabla_w v, v \rangle = -\langle w, \nabla_v v \rangle - \langle Sw, v \rangle = -\langle Sw, v \rangle = -\langle w, Sv \rangle = 0$, cf. (9.2.d) and (9.2.f). Similarly, $\langle [v, w], u \rangle = \langle \nabla_v w - \nabla_w v, u \rangle = -\langle w, \nabla_v u \rangle - \langle Sw, u \rangle = -\langle Sw, u \rangle = -\langle w, Su \rangle = 0$. Thus, $[v, w]$ is a section of \mathcal{V}^\perp as well. Due to the final clause of Remark 2.1, this completes the proof of (9.2.h). For easy reference, note that, by (9.2.a)–(9.2.b),

$$(9.3) \quad g(v, v) = g(u, u) = Q, \quad g(v, u) = 0, \quad u = Jv.$$

Lemma 9.1. *Under the assumptions preceding (9.2), on M' ,*

- (a) *the distribution $\mathcal{V} = \text{Span}(v, u)$ is integrable and has totally geodesic leaves,*
- (b) *a local section w of \mathcal{V}^\perp is projectable along \mathcal{V} if and only if $[u, w] = [v, w] = 0$,*
- (c) *if local sections w and w' of \mathcal{V}^\perp commute with u and v , then*

$$(9.4) \quad \begin{array}{ll} \text{i) } d_v[g(w, w')] = 2g(Sw, w'), & \text{ii) } d_v[g(Sw, w')] = 2\psi g(Sw, w'), \\ \text{iii) } d_v[Q^{-1}g(Sw, w')] = 0, & \text{iv) } d_v Q = 2\psi Q, \\ \text{v) } d_u[g(w, w')] = d_u[g(Sw, w')] = d_u Q = 0, & \end{array}$$

- (d) $[\nabla_v S]w = 2(\psi - S)Sw$ whenever w is a local section of \mathcal{V}^\perp .

Proof. Assertions (a)–(b) are obvious from (9.2.b) and, respectively, Remark 2.1 combined with (9.2.h). Next, let $\mathcal{L}_v w = \mathcal{L}_v w' = \mathcal{L}_u w = \mathcal{L}_u w' = 0$. Since \mathcal{L}_v and \mathcal{L}_u act on functions as d_v and d_u , (2.1.a) implies (9.4.i), and $d_u[g(w, w')] = 0$ as $\mathcal{L}_u g = 0$ by (9.2.b). For similar reasons, $d_u[g(Sw, w')] = \mathcal{L}_u[g(Sw, w')] = 0$. (Namely, (9.2.c) gives $\nabla_u S = 0$, so that (9.2.a) and (2.1.b), with u, S rather than v, B , yield $\mathcal{L}_u S = 0$.) On the other hand, by (9.3), $g(v, v) = Q$. Now (2.1.d), (9.2.f) and (9.2.b) imply that $d_u \tau = d_u Q = 0$ and $d_v Q = 2\psi Q$, establishing (d).

Using (9.2.a) we get $g(Sw, w') = g(JSw, Jw') = g(Aw, Jw')$ which, by (9.2.e), is nothing else than $-g([w, Jw'], u)/2$. Hence $2d_v[g(Sw, w')] = 2\mathcal{L}_v[g(Sw, w')] = -\mathcal{L}_v[g(u, [w, Jw'])] = -[\mathcal{L}_v g](u, [w, Jw'])$. (Our assumption that $\mathcal{L}_v w = \mathcal{L}_v w' = 0$ gives $\mathcal{L}_v(Jw') = 0$, as v is holomorphic, which in turn yields $\mathcal{L}_v[w, Jw'] = 0$, while $\mathcal{L}_v u = 0$, cf. (9.2.b).) From (2.1.a), (9.2.f) and (9.2.a) we now obtain $d_v[g(Sw, w')] = -[\mathcal{L}_v g](u, [w, Jw'])/2 = -g(Su, [w, Jw']) = -2g(\psi u, [w, Jw']) = 2\psi g(Aw, Jw') = -2\psi g(JAw, w') = 2\psi g(Sw, w')$, that is, (9.4.ii), which, since $d_v Q = 2\psi Q$ by (d), also proves (9.4.iii).

Finally, (9.2.h) and the equality $\nabla_v S = -J[R(u, v)]$, cf. (9.2.c), combined with (a), imply that $\nabla_v S - (2\psi - S)S$ leaves \mathcal{V}^\perp invariant. To obtain (e), it now suffices to show that $[\nabla_v S]w - (2\psi - S)Sw$ is orthogonal to w' for any local sections w, w' of \mathcal{V}^\perp . We are free to assume here that $w = w'$ (due to self-adjointness of $S = \nabla v$) and that w commutes with u and v (see (b)). Differentiation by parts gives, by (9.4.iii) and (9.2.d), $g([\nabla_v S]w, w) = d_v[g(Sw, w)] - g(S\nabla_v w, w) - g(Sw, \nabla_v w) = 2\psi g(Sw, w) - 2g(Sw, Sw)$, as required, with $\nabla_v w = Sw$ since $[v, w] = 0$. \square

10. Horizontal Jacobi fields

In addition to using the assumptions and notations of Section 9, we now let Γ stand for the underlying one-dimensional manifold of a fixed maximal integral curve of v in M' . We restrict the objects in (9.1) to Γ without changing the notation, and select a unit-speed parametrization $t \mapsto x(t)$ of the geodesic Γ such that

$$(10.1) \quad \begin{aligned} \text{a)} \quad \dot{x} &= v/|v| = Q^{-1/2}v \quad \text{along } \Gamma, \quad \text{where } v = \nabla\tau, \\ \text{b)} \quad \dot{\tau} &= Q^{1/2}, \quad \dot{Q} = 2\psi Q^{1/2}, \quad \text{with } (\cdot)' = d/dt = d_{\dot{x}}, \end{aligned}$$

(10.1.b) being an obvious consequence of (10.1.a), (2.1.d) and (9.4.iv).

Any constant $c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\}$, where $\tau(\Gamma)$ is the range of τ on Γ , gives rise to the function $\lambda_c: \Gamma \rightarrow \mathbb{R}$ defined by

$$(10.2) \quad \lambda_c = Q/[2(\tau - c)],$$

the convention being that λ_c is identically zero when $c = \infty$. We denote by \mathcal{W} the set of all \mathcal{V}^\perp -valued vector fields $t \mapsto w(t) \in \mathcal{V}_{x(t)}^\perp$ along Γ such that

$$(10.3) \quad \nabla_{\dot{x}} w = Q^{-1/2} S w.$$

Of particular interest to us are c satisfying the conditions

$$(10.4) \quad \begin{aligned} \text{a)} \quad c &\in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\} \quad \text{and } \mathcal{W}[c] \neq \{0\}, \quad \text{where} \\ \text{b)} \quad \mathcal{W}[c] &= \{w \in \mathcal{W} : S w = \lambda_c w\}. \end{aligned}$$

About projectability along \mathcal{V} in (i) below, see Lemma 9.1 (b).

Theorem 10.1. *Under the above hypotheses, the following conclusions hold.*

- (i) \mathcal{V}^\perp -valued solutions w to (10.3) are precisely the restrictions to Γ of the local sections of \mathcal{V}^\perp with domains containing Γ that are projectable along \mathcal{V} .
- (ii) All w as in (i), that is, all elements of \mathcal{W} , are Jacobi fields along Γ .
- (iii) Every vector in $\mathcal{V}_{x(t)}^\perp$ equals $w(t)$ for some unique $w \in \mathcal{W}$.
- (iv) \mathcal{W} is a complex vector space of dimension $\dim_{\mathbb{C}} M - 1$, and the direct sum of all $\mathcal{W}[c]$ for c in (10.4.a), $w \mapsto Jw$ being the multiplication by $i \in \mathbb{C}$.
- (v) A function $t \mapsto \lambda(t)$ on the parameter interval of $t \mapsto x(t)$ satisfies the equation $d\lambda/dt = 2(\psi - \lambda)\lambda Q^{-1/2}$, with ψ, Q evaluated at $x(t)$, if and only if $\lambda(t) = \lambda_c(x(t))$, cf. (10.2), for some $c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\}$ and all t .
- (vi) At any $x = x(t) \in \Gamma$, the eigenvalues of $S_x: \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$, cf. (9.2.h), are precisely the values $\lambda_c(x)$ for all c in (10.4.a). The $\lambda_c(x)$ -eigenspace of $S_x: \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$ is $\{w(t) : w \in \mathcal{W}[c]\}$.
- (vii) $R(w, u)u = R(w, v)v = (\psi - S)Sw = R(v, u)Jw/2$ on M' for sections w of \mathcal{V}^\perp .
- (viii) If $\tau(\Gamma) = (\tau_-, \tau_+)$ is bounded, then $Q/(\tau - \tau_+) \leq 2S \leq Q/(\tau - \tau_-)$ on \mathcal{V}^\perp .

Proof. Any w as in the second line of (i), restricted to Γ , becomes both a Jacobi field (by Lemmas 4.5 and 9.1(b)) and a \mathcal{V}^\perp -valued solution to (10.3) (since $S = \nabla v$, so that (10.1.a) and Lemma 9.1(b) give $\nabla_x w = Q^{-1/2} \nabla_v w = Q^{-1/2} \nabla_u v = Q^{-1/2} S w$). With Γ replaced by suitable shorter subgeodesics covering all points of Γ , the inclusion just established between the two vector spaces appearing in (i) is actually an equality: in either class, the vector field in question is uniquely determined by its initial value at any given point $x \in \Gamma$. This proves (i)–(ii) as well as (iii)–(iv), the latter in view of the fact that $JS = SJ$, cf. (9.2.a).

For a C^1 function λ defined on the parameter interval of $t \mapsto x(t)$, one has

$$(10.5) \quad \dot{\lambda} = 2(\psi - \lambda)\lambda Q^{-1/2} \quad \text{with } (\cdot)' = d/dt$$

if and only if either $\lambda = 0$ identically, or $\lambda \neq 0$ everywhere and the function c characterized by $2c = 2\tau - Q/\lambda$ is constant. (In fact, the either-or claim about vanishing of λ is due to uniqueness of solutions of initial-value problems, while (10.1.b) yields $2\dot{c} = Q\lambda^{-2}[\dot{\lambda} - 2(\psi - \lambda)\lambda Q^{-1/2}]$.) Now (v) easily follows, all nonzero initial conditions for (10.5) at fixed t being realized by suitably chosen constants $c \in \mathbb{R} \setminus \tau(\Gamma)$ (and $\lambda = 0$ satisfying (v) with $c = \infty$).

On the other hand, from (10.1.a) and Lemma 9.1(d),

$$(10.6) \quad [\nabla_x S]w = 2Q^{-1/2}(\psi - S)Sw, \quad \text{if } w \text{ is a } \mathcal{V}^\perp\text{-valued vector field along } \Gamma.$$

Next, we fix $x = x(t) \in \Gamma$ and express any prescribed eigenvalue-eigenvector pair for $S_x: \mathcal{V}_x^\perp \rightarrow \mathcal{V}_x^\perp$ as $\lambda_c(x)$ and $w(t)$, with some unique $c \in [\mathbb{R} \setminus \tau(\Gamma)] \cup \{\infty\}$ and $w \in \mathcal{W}$. By (v), $\lambda = \lambda_c$ satisfies (10.5), so that, in view of (10.3) and (10.6), the vector field $\hat{w} = Sw - \lambda w$ is a solution of the linear homogeneous differential equation $\nabla_x \hat{w} = Q^{-1/2}(2\psi - 2\lambda - S)\hat{w}$. Since \hat{w} vanishes at $x = x(t)$, it must vanish identically, which establishes (vi).

Now let $w \in \mathcal{W}$. As $\hat{Q} = 2\psi Q^{1/2}$ (see the lines following (10.5)), the Jacobi equation and (10.3) give, by (ii) and (10.6), $R(w, \dot{x})\dot{x} = \nabla_x \nabla_x w = \nabla_x [Q^{-1/2} Sw] = Q^{-1}(\psi - S)Sw$, that is, $R(w, v)v = (\psi - S)Sw$, the second equality in (vii). Also, Lemma 9.1(d), (9.2.c) and (3.1.b) yield $2(\psi - S)Sw = [\nabla_v S]w = -J[R(u, v)w] = -R(u, v)Jw = R(v, u)Jw = R(v, Jv)Jw$, the last equality in (vii). Combining the two relations, and repeatedly using (3.1.b), we get $2R(w, v)v = R(v, Jv)Jw$, that is, $R(w, v)v = R(v, w)v + R(v, Jv)Jw = R(Jv, Jw)v + R(v, Jv)Jw$. Thus, from the Bianchi identity, $R(w, v)v = R(v, Jw)Jv = R(Jv, JJw)Jv = R(w, u)u$, which proves (vii). Finally, (viii) is an easy consequence of (vi) and (9.2.d). \square

11. Consequences of compactness

The six parts of the next theorem are all well known: (i)–(iv) and (v)–(vi) appear in [4] as Lemmas 11.1–11.2, Example 8.1 and, respectively, Lemma 8.4 (iv). More general versions of both (vi) and (iv) are originally due to Wang; see Lemmas 1 and 3 of [11]. However, we follow the referee's suggestion that –to make the presentation more autonomous– the proofs of these six facts should be at least outlined here, along with an argument justifying Remark 11.2.

Theorem 11.1. *Given a compact geodesic-gradient Kähler triple (M, g, τ) , cf. the Introduction, with Σ^\pm as in (1.1), let $v = \nabla\tau$ be the g -gradient of τ .*

- (i) $Q = g(v, v)$ is a C^∞ function of τ , leading to data τ_-, τ_+, a, Q with (6.1).
- (ii) $u = Jv$ is a Killing vector field with a periodic flow.
- (iii) Σ^\pm are connected totally geodesic compact complex submanifolds of M .
- (iv) $\Sigma^+ \cup \Sigma^-$ equals the zero set of v , that is, the set of critical points of τ .
- (v) τ has the Morse–Bott property.
- (vi) v is tangent to every geodesic normal to Σ^\pm .

Remark 11.2. For a as in Theorem 11.1 (i), $\mp a$ is the unique nonzero eigenvalue of the Hessian of τ (that is, of $S = \nabla v$) at any critical point $y \in \Sigma^\pm$. The $\mp a$ -eigenspace of S_y is the normal space $N_y\Sigma^\pm$, and $\text{Ker } S_y = T_y\Sigma^\pm$ (which thus constitutes the 0-eigenspace of S_y unless $\Sigma^\pm = \{y\}$). See Remark 11.2 in [6].

Sketch of proof. The zeros of the holomorphic Killing field $u = Jv$, cf. (9.2.a)–(9.2.b), obviously form a disjoint union of submanifolds with the properties named in (iii), the normal exponential mappings of which easily lead to local coordinate systems making u appear as a skew-adjoint linear vector field. This – combined with the equalities $\nabla u = JS = SJ$ in (9.2.a) – implies the rank condition needed for (v), since $S = \nabla v$, at any zero y of $v = \nabla\tau$, corresponds via g to the Hessian of τ . Taking the limits, when they exist, of (4.1) divided by $|v|$ and evaluated at points of sequences converging to any such y along any geodesic normal at y to the critical manifold Σ of τ through y , we see that the normal space $N_y\Sigma$ consists of 0 and eigenvectors of ∇v_y with an eigenvalue $a \neq 0$ independent of the normal direction ($N_y\Sigma$ being a vector space). Up to a sign change, a is also independent of the critical point y , as $2\pi/|a|$ must obviously be the minimum positive period of the flow of the linear vector field $\nabla u_y = J_y S_y$ in T_yM and, consequently, also the minimum positive period of the flow of u in M . The last clause – immediate since an isometry is uniquely determined by its 1-jet at a single point – in addition yields (ii). On the other hand, the Hessian of τ is semidefinite at every critical point, a or $-a$ being its only nonzero eigenvalue, while the levels of τ (being complex submanifolds) are of real codimensions greater than 1. A standard argument – see Prop. 11.4 of [5] – shows that, for a Morse–Bott function τ with these properties on a compact manifold, all levels of τ must be connected, which proves (iii) and (iv). Connectedness of the levels also allows us to skip the word ‘locally’ when applying Lemma 4.1, and so the first part of (i) follows, as one easily verifies smoothness of the assignment $\tau \mapsto Q$ at the extremum values τ_\pm in (1.1). Replacing a with its absolute value we now obtain both Remark 11.2, and the remainder of (i), the latter – by noting that the limits of (4.1) divided by $|v|$, mentioned earlier in the proof, give $dQ/d\tau = \mp 2a$ at $\tau = \tau_\pm$, due to (2.1.d) and (4.1). Finally, (vi) is a consequence of a much more general fact, namely, Lemma 8.2 of [5]: if a vector field v and a connection ∇ on a manifold M satisfy (4.1), where $\psi : M' \rightarrow \mathbb{R}$, then every geodesic segment emanating from a zero y of v , tangent at y to an eigenvector of ∇v_y for a nonzero eigenvalue, and containing no zeros of v other than y , must be a reparametrized integral curve of v . \square

Restricting $\tau \mapsto Q$ in Theorem 11.1 (i) to the open interval (τ_-, τ_+) we have

$$(11.1) \quad dQ/d\tau = 2\psi, \quad \text{and so } \psi \rightarrow \mp a \text{ as } \tau \rightarrow \tau_{\pm},$$

ψ being the function with (4.1) on the open set M' where $v \neq 0$. This makes ψ a C^∞ function of τ , and it follows as $dQ = 2\psi d\tau$ on M' , by (2.1.d) and (4.1).

Still assuming compactness of a geodesic-gradient Kähler triple (M, g, τ) , let $N^\delta \Sigma^\pm$ be the bundle of radius δ normal open disks around the zero section in the normal bundle $N\Sigma^\pm$, with δ characterized by (6.2). According to Lemma 10.3 of [4], δ is then the distance between Σ^+ and Σ^- , while, with Exp^\perp as in Section 2,

$$(11.2) \quad \begin{aligned} &\text{the restriction to } N^\delta \Sigma^\pm \text{ of the normal exponential mapping} \\ &\text{Exp}^\perp: N\Sigma^\pm \rightarrow M \text{ is a diffeomorphism } N^\delta \Sigma^\pm \rightarrow M \setminus \Sigma^\mp. \end{aligned}$$

Cf. [2], Lemma 2 in [11], and Theorem 1.1 of [10]. Its inverse $M \setminus \Sigma^\mp \rightarrow N^\delta \Sigma^\pm$, composed with the projection $N^\delta \Sigma^\pm \rightarrow \Sigma^\pm$, yields a new disk-bundle projection

$$(11.3) \quad \pi^\pm: M \setminus \Sigma^\mp \rightarrow \Sigma^\pm.$$

Remark 11.3. Clearly, $\pi^\pm \circ \text{Exp}^\perp$ is the normal-bundle projection $N\Sigma^\pm \rightarrow \Sigma^\pm$. Also, according to the lines preceding (11.3),

$$(11.4) \quad \begin{aligned} &\text{the image } \pi^\pm(x) \text{ of any } x \in M' \text{ is the unique } y \in \Sigma^\pm \text{ that can} \\ &\text{be joined to } x \text{ by a (necessarily unique) geodesic segment } \Gamma_x^\pm \text{ of} \\ &\text{length less than } \delta \text{ emanating from } y \text{ in a direction normal to } \Sigma^\pm, \end{aligned}$$

which implies (see Remark 4.6, Example 8.1 and Theorem 10.2 (iii)–(vi) of [4]) that π^\pm sends every $x \in M \setminus \Sigma^\mp$ to the unique point nearest x in Σ^\pm .

In the next lemma, by a *leaf* we mean—as usual—a maximal integral manifold.

Lemma 11.4. *Given a compact geodesic-gradient Kähler triple (M, g, τ) , the integrable distribution $\mathcal{V} = \text{Span}(v, u)$ on $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$, cf. Lemma 9.1 (a) and Remark 11.1 (iv), has the property that $\mathcal{V} \subseteq \text{Ker} d\pi^\pm$ while, if ξ is a unit vector normal to Σ^\pm at a point y , then, with δ as in (11.2),*

- (a) *the punctured radius δ disk $\{z\xi : z \in \mathbb{C} \text{ and } 0 < |z| < \delta\}$ in $N_y \Sigma^\pm$ is mapped by exp_y diffeomorphically onto a leaf Λ of \mathcal{V} .*

Also, for any leaf $\Lambda \subseteq M'$ of \mathcal{V} , its closure $\hat{\Lambda}$ in M , and the normal exponential mapping $\text{Exp}^\perp: N\Sigma^\pm \rightarrow M$, there exist $y_\pm \in \Sigma^\pm$ such that $\{y_\pm\} = \pi^\pm(\Lambda)$ and Λ arises from (a) applied to some unit normal vector ξ at the point $y = y_\pm$, while $\hat{\Lambda} = \Lambda \cup \{y_+, y_-\}$ is a totally geodesic complex submanifold of (M, g) , biholomorphic to $\mathbb{C}\mathbb{P}^1$, and

$$(11.5) \quad \pi^\mp(\text{Exp}^\perp(y_\pm, z\xi)) = y_\mp \quad \text{whenever } z \in \mathbb{C} \text{ and } 0 < |z| \leq \delta.$$

Proof. Let us fix x, y and Γ_x^\pm as in (11.4). Due to Theorem 11.1 (iv), the Killing field $u = Jv$ vanishes along Σ^\pm , so that its infinitesimal flow at y preserves both $T_y \Sigma^\pm$

and $N_y \Sigma^\pm$. The images of Γ_x under the flow transformations of u thus are geodesic segments normal to Σ^\pm emanating from y and, as a consequence of (11.4), π^\pm maps them all onto $\{y\}$. In other words, the union of such segments, with the point y removed, is simultaneously a subset of the π^\pm -preimage of y as well as – according to (9.3) and parts (ii), (iv), (vi) of Theorem 11.1 – a surface embedded in M' . This surface is, due to its very definition and Theorem 11.1(vi), tangent to both u and v which, in view of (11.2), yields (a); note that, by (9.2.a) and Remark 11.2, the orbit of ξ under the flow of $A = \nabla u$ at y consists of all unit complex multiples of ξ .

What we just observed about the orbit of ξ clearly ensures smoothness of the closure of the leaf at y . By (11.2) and Theorem 11.1(vi), the union of Γ_x and its analog for the same point x and the *other* projection π^\mp is a length δ geodesic segment joining $y \in \Sigma^\pm$ to its other endpoint $y_\mp \in \Sigma^\mp$. The above discussion of the images of such a segment under the flow of u applies equally well to y_\mp , so that (b)–(c) follow from Lemma 9.1 (a) and the fact that $x \in M'$ was arbitrary. \square

Remark 11.5. Let (M, g, τ) be a Grassmannian or CP triple, constructed as in Section 5 from some data (5.2.i) or (5.2.ii). We use the notation of (9.1) and (11.3).

- (a) We already know that the critical manifolds Σ^\pm of τ are given by (5.3).
- (b) In the case of (5.3.c) (or, (5.3.b) and (5.3.d)), π^\pm acts on W as the orthogonal projection into L (or, respectively, into L^\perp).
- (c) When Σ^\pm has the form (5.3.a), π^\pm sends W to $L \oplus (W \cap L^\perp)$.
- (d) The leaf of \mathcal{V} through any $W \in M'$ consists
 - (d1) for (5.2.i) – of all $L' \oplus W'$, where $W' = W \cap L^\perp$ and L' is any line in the plane $L \oplus (W' \cap W^\perp)$ other than the lines L and $W' \cap W^\perp$ themselves,
 - (d2) for (5.2.ii) – of all lines other than W' and W'' in the plane $W' \oplus W''$, where W', W'' denote the orthogonal projections of W into L and L^\perp .

Justifications of these easily-verified facts are given in Remark 11.5 of [6].

As $\mathcal{V} \subseteq \text{Ker } d\pi^\pm$ (Lemma 11.4), we define vector subbundles \mathcal{H}^\pm of TM' by

$$(11.6) \quad \mathcal{H}^\pm = \mathcal{V}^\perp \cap \text{Ker } d\pi^\mp, \quad \text{so that} \quad \text{Ker } d\pi^\pm = \mathcal{V} \oplus \mathcal{H}^\mp.$$

Theorem 11.6. *For a compact geodesic-gradient Kähler triple (M, g, τ) and v , M' , Q , \mathcal{V} , S , Σ^\pm , τ_\pm , π^\pm , \mathcal{H}^\pm defined in (9.1), (1.1), (11.6), the bundle endomorphism $2(\tau - \tau_\pm)S - Q$ of TM , restricted to \mathcal{V}^\perp , has constant rank on M' , and*

$$(11.7) \quad \mathcal{H}^\mp = \mathcal{V}^\perp \cap \text{Ker}[2(\tau - \tau_\pm)S - Q], \quad TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathcal{H}$$

with some subbundle \mathcal{H} of TM' , the decomposition being complex, S -invariant and orthogonal. Furthermore, the closure in M of any $\Gamma \subseteq M'$ chosen as at the beginning of Section 10 admits a unit-speed C^∞ parametrization $[t_-, t_+] \ni t \mapsto x(t)$ which, restricted to (t_-, t_+) , is a parametrization of Γ satisfying (10.1.a) and the following conditions.

- (a) The endpoint $y_{\pm} = x(t_{\pm})$ lies in Σ^{\pm} , while $\dot{x}(t_{\pm})$ is normal to Σ^{\pm} at y_{\pm} .
- (b) Every solution $(t_-, t_+) \ni t \mapsto w(t) \in \mathcal{V}_{x(t)}^{\pm}$ of (10.3) along Γ has a C^{∞} extension to $[t_-, t_+]$ such that $d\pi_{x(t)}^{\pm}[w(t)] = w(t_{\pm})$ whenever $t \in (t_-, t_+)$.
- (c) The bundle projection $\pi^{\pm}: M \setminus \Sigma^{\mp} \rightarrow \Sigma^{\pm}$ is holomorphic.
- (d) If $w \in \mathcal{W}[\tau_{\pm}]$, cf. (10.4), then, in (b), $w(t_{\pm}) = 0$, and $[\nabla_{\dot{x}}w](t_{\pm})$ is normal to Σ^{\pm} at $y_{\pm} = x(t_{\pm})$ as well as orthogonal to $\dot{x}(t_{\pm})$ and $J\dot{x}(t_{\pm})$.
- (e) If w lies in the direct sum of spaces $\mathcal{W}[c] \neq \{0\}$ with $c \neq \tau_{\pm}$, for a fixed sign \pm , then $w(t_{\pm})$ is tangent to Σ^{\pm} at $y_{\pm} = x(t_{\pm})$, and $[\nabla_{\dot{x}}w](t_{\pm}) = 0$.
- (f) Whenever $t \in (t_-, t_+)$, the assignment $w(t) \mapsto (w(t_{\pm}), [\nabla_{\dot{x}}w](t_{\pm}))$, with w as in (b), is a \mathbb{C} -linear isomorphism $\mathcal{V}_x^{\pm} \rightarrow T_y \Sigma^{\pm} \times N'_y$, where N'_y denotes the orthogonal complement of $\text{Span}(\dot{x}(t_{\pm}), J\dot{x}(t_{\pm}))$ in $N_y \Sigma^{\pm}$ and $(x, y) = (x(t), y_{\pm})$. At the same time, $w(t)$ then equals the image, under the differential of the normal exponential mapping $\text{Exp}^{\perp}: N\Sigma^{\pm} \rightarrow M$ at $(y, \xi) \in N\Sigma^{\pm}$ given by $y = x(t_{\pm})$ and $\xi = (t - t_{\pm})\dot{x}(t_{\pm})$, of the vector tangent to $N\Sigma^{\pm}$ at (y, ξ) which equals the sum of the vertical vector $\eta = (t - t_{\pm})[\nabla_{\dot{x}}w](t_{\pm})$ and the D-horizontal lift of $w(t_{\pm})$ to (y, ξ) , for the normal connection D in $N\Sigma^{\pm}$. Similarly, $u_{x(t)}$, for $u = Jv$, is the image, under the differential of Exp^{\perp} at (y, ξ) , of the vertical vector $\eta = \mp ai\xi$.
- (g) For any $w, w' \in \mathcal{W}$, the function $Q^{-1}g(Sw, w')$ is constant on Γ and the restriction of $g(w, w')$ to Γ is an affine function of $\tau: \Gamma \rightarrow \mathbb{R}$ with the derivative $d[g(w, w')]/d\tau = 2Q^{-1}g(Sw, w')$.
- (h) In (g), with a as in Remark 11.1 (i), either sign \pm , and $y = y_{\pm} = x(t_{\pm})$,
- (h1) $g(w, w') = (\tau_+ - \tau_-)^{-1}|\tau - \tau_{\mp}|g_y(w_{\pm}, w'_{\pm})$ if $w \in \mathcal{W}[\tau_{\mp}]$ and $w' \in \mathcal{W}$,
- (h2) $g(w, w') = g_y(w_{\pm}, w'_{\pm}) - a^{-1}|\tau - \tau_{\pm}|g_y(R_y(w_{\pm}, J_y w'_{\pm})\dot{x}_{\pm}, J_y \dot{x}_{\pm})$ if w, w' both satisfy the assumption of (e),
- (h3) $g(w, w') = 2a^{-1}|\tau - \tau_{\pm}|g_y([\nabla_{\dot{x}}w]_{\pm}, [\nabla_{\dot{x}}w']_{\pm})$ if $w, w' \in \mathcal{W}[\tau_{\pm}]$,

the subscript \pm next to $w, w', \nabla_{\dot{x}}w, \nabla_{\dot{x}}w'$ or \dot{x} representing the evaluation at t_{\pm} .

Remark 11.7. Since $|\tau - \tau_{\pm}| = \mp(\tau - \tau_{\pm})$ and $\pm(\tau_+ - \tau_-) = \tau_{\pm} - \tau_{\mp}$, applying $d/d\tau$ to the right-hand side in (h1), or (h2), or (h3), we get the three values

$$(\tau_{\pm} - \tau_{\mp})^{-1}g_y(w_{\pm}, w'_{\pm}), \pm a^{-1}g_y(R_y(w_{\pm}, J_y w'_{\pm})\dot{x}_{\pm}, J_y \dot{x}_{\pm}), \mp 2a^{-1}g_y([\nabla_{\dot{x}}w]_{\pm}, [\nabla_{\dot{x}}w']_{\pm}).$$

As a consequence of parts (g)–(h) of Theorem 11.6, this triple provides the three expressions for $2Q^{-1}g(Sw, w')$ in the cases (h1), (h2) and (h3), respectively.

Note that the three different formulae for $g(w, w')$ in (h1), (h2) and – with the reversed sign – in (h3), are all simultaneously valid when $w, w' \in \mathcal{W}[\tau_{\mp}]$.

Remark 11.8. Under the assumptions of Theorem 11.6,

- (i) the relation $\xi = (t - t_{\pm})\dot{x}(t_{\pm})$ in (f) clearly gives $\dot{x}_{\pm} = \mp \xi/|\xi|$ in (h2),
- (ii) by (d)–(f), the images under the differential of Exp^{\perp} of vertical (or, horizontal) vectors tangent to $N\Sigma^{\pm}$ at the point (y, ξ) appearing in (f) have the form $w(t)$ for w satisfying the hypothesis of (d) (or, respectively, of (e)),

- (iii) the differential of π^\pm at any $x \in M'$ maps the summands \mathcal{H}_x^\pm and \mathcal{H}_x in (11.7) isomorphically onto the images $d\pi_x^\pm(\mathcal{H}_x^\pm)$ and $d\pi_x^\pm(\mathcal{H}_x)$, orthogonal to each other in $T_y\Sigma^\pm$ for $y = \pi^\pm(x)$,
- (iv) one has $(\tau_+ - \tau_-)g_x(w, w') = |\tau(x) - \tau_\mp|g_y(d\pi_x^\pm w, d\pi_x^\pm w')$ whenever $w \in \mathcal{H}_x^\pm$ and $w' \in \mathcal{V}_x^\perp$ at any $x \in M'$, while $y = \pi^\pm(x)$,
- (v) (1.1) and (a) imply the inequality of Theorem 10.1 (viii) everywhere in M' .

Only (iii)–(iv) require explanations, which can be found in Remark 11.8 of [6].

Remark 11.9. As another direct consequence of Theorem 11.6, the assignment $x \mapsto d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm)$ defines a holomorphic section of the bundle over M' arising via the pullback under π^\pm from $\text{Gr}_k(T\Sigma^\pm)$, for a suitable integer $k = k_\pm$. Here $\text{Gr}_k(T\Sigma^\pm)$ is the Grassmannian bundle over Σ^\pm with the fibres $\text{Gr}_k(T_y\Sigma^\pm)$, $y \in \Sigma^\pm$ (cf. Section 5), holomorphicity and the equality $d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm)$ are clear from Theorem 11.6 (c) (which also implies, due to (11.6), that $\mathcal{V} \oplus \mathcal{H}^\pm$ is a holomorphic subbundle of TM') and (11.7) (which, combined with (11.6), ensures constancy of the dimension $k = k_\pm$ of the spaces $d\pi_x^\pm(\mathcal{H}_x^\pm)$).

12. Proof of Theorem 11.6

We begin by establishing (a)–(f) under the stated assumptions about Γ .

Let $(t_-, t_+) \mapsto x(t)$ be a parametrization of Γ with (10.1.a). As τ then is clearly an increasing function of t , it has some limits $\hat{\tau}_\pm$ as $t \rightarrow t_\pm$, finite due to boundedness of τ . The length of Γ obviously equals the integral of $Q^{-1/2}$ over $(\hat{t}_-, \hat{t}_+) \subseteq (\tau_-, \tau_+)$, and so it is finite in view of (6.2). This implies the existence of limits $x(t_\pm)$ of $x(t)$ as $t \rightarrow t_\pm$. Furthermore, each $x(t_\pm)$ lies in Σ^\pm since, if one $x(t_\pm)$ did not, Theorem 11.1 (iv) would yield $v \neq 0$ at $x(t_\pm)$, contradicting maximality of Γ . Thus, $[t_-, t_+] \mapsto x(t)$ parametrizes the closure of Γ . Next, $M \setminus (\Sigma^+ \cup \Sigma^-)$ is, by (11.2) and Theorem 11.1 (vi), a disjoint union of maximal integral curves of v , each of which has two limit points, one in Σ^- and one in Σ^+ , and the corresponding limit directions of the curve are normal to Σ^- and Σ^+ . As Γ is one of these curves, (a) follows.

In (b), a C^∞ extension to $[t_-, t_+]$ must exist as w is a Jacobi field; see Theorem 10.1 (ii). To obtain (d)–(e), we fix $w \in \mathcal{W}[c]$, so that, from (10.2)–(10.4),

$$(12.1) \quad \text{i) } Sw = Qw/[2(\tau - c)], \quad \text{ii) } \nabla_x w = Q^{1/2}w/[2(\tau - c)].$$

Let $y = x(t_\pm)$ and $w_y = w(t_\pm)$. By Theorem 11.1 (iv), $Q = \tau - \tau_\pm = 0$ on Σ^\pm while, in view of (a) and (11.1), $Q/[2(\tau - \tau_\pm)]$ evaluated at $x(t)$ tends to $\mp a \neq 0$ as $t \rightarrow t_\pm$. If $c = \tau_\pm$, (12.1.ii) multiplied by $Q^{1/2}$ thus yields $w_y = 0$, and the relation $Sw' = Qw'/[2(\tau - c)]$ for $w' = \nabla_x w$, obvious from (12.1), implies that $[\nabla_x w](t_\pm)$ lies in the $\mp a$ -eigenspace of S_y . When $c \neq \tau_\pm$, (12.1.i) and (12.1.ii) give $S_y w_y = 0$ and $[\nabla_x w](t_\pm) = 0$. Due to Remark 11.2, this proves (d) and (e): orthogonality in (d) follows since w and $\nabla_x w$ take values in \mathcal{V}^\perp , for $\mathcal{V} = \text{Span}(v, u)$ (so that $g(w, v) = g(w, u) = 0$), while $\dot{x} = v/|v|$ by (10.1.a), and $u = Jv$.

Furthermore, the assignment in (f) is well-defined, injective, complex-linear and $(T_y\Sigma^\pm \times N'_y)$ -valued due to parts (iii), (ii), (iv) of Theorem 10.1 and, respectively, (d)–(e). The first claim of (f) thus follows since both spaces have the same dimension. The second (or, third) one is in turn immediate from (2.2) applied, at $r = 1$, to any $w \in \mathcal{W}$, cf. Theorem 10.1 (ii) (or, to $w = u$), with y, ξ, η as in (f), and \hat{w} defined by $\hat{w}(r) = w(rt + (1-r)t_\pm)$. (That $r \mapsto \hat{w}(r)$ then is a Jacobi field along the geodesic $r \mapsto x(rt + (1-r)t_\pm)$ follows from Theorem 10.1 (ii) or, respectively, (9.2.b) while, in the latter case, due to (9.2.a) along with Remarks 11.1 (iv) and 11.2, $w = u$ realizes the initial conditions $(u, \nabla_{dx/dx}u) = (0, \mp ai\xi)$ at $r = 0$.)

The remaining equality $d\pi_{x(t)}^\pm[w(t)] = w(t_\pm)$ in (b) now becomes an obvious consequence of the second part of (f) combined with the first line of Remark 11.3. This proves (b) and, combined with Theorem 10.1 (iv), implies (c).

Next, for $t \mapsto x(t)$ as in (a)–(f), any $t \in (t_-, t_+)$, a fixed sign \pm , and $x = x(t)$, Theorem 10.1 (iii), (11.6) and (b) give $\mathcal{H}_x^\mp = \{w(t) : w \in \mathcal{W} \text{ and } w(t_\pm) = 0\}$. Writing any $w \in \mathcal{W}$ as $w = w' + w''$, where $w' \in \mathcal{W}[\tau_\pm]$ and w'' lies in the direct sum of the spaces $\mathcal{W}[c] \neq \{0\}$ with $c \neq \tau_\pm$, cf. Theorem 10.1 (iv), we see that, by (d)–(e), the isomorphism in (f) sends $w'(t)$ and $w''(t)$, respectively, to pairs of the form $(0, \cdot)$ and $(\cdot, 0)$. Thus, $w(t) \in \mathcal{H}_x^\mp$ if and only if $w'' = 0$, that is, $w \in \mathcal{W}[\tau_\pm]$. From Theorem 10.1 (vi), (10.2) and (10.4.b), one now obtains the first equality in (11.7), so that (11.6) implies the constant-rank assertion preceding (11.7). On the other hand, \mathcal{H}_x^+ and \mathcal{H}_x^- are mutually orthogonal at every $x \in M'$, being, by (11.7), contained in eigenspaces corresponding to different eigenvalues of the self-adjoint operator S_x , cf. (9.2.d), so that the second part of (11.7) follows.

Let $w, w' \in \mathcal{W}$. Constancy of $Q^{-1}g(Sw, w')$ along Γ is immediate from (9.4.iii) and Theorem 11.1 (vi), cf. Lemma 9.1 (b) and parts (i)–(ii) of Theorem 10.1. The operators $d/d\tau$ and d_v acting on functions $\Gamma \rightarrow \mathbb{R}$ are in turn related by $d_v = Q d/d\tau$, since (10.1.a) gives $d_v = Q^{1/2}d_x = Q^{1/2}d/dt$, while $d/dt = Q^{1/2}d/d\tau$ due to (10.1.b). Now (g) trivially follows from (9.4.ii).

In (h), all three right-hand sides are affine functions of τ with the correct values at $t = t_\pm$ (that is, limits at the endpoint $y_\pm = x(t_\pm)$). Proving (h) is thus reduced by (g) to showing that, in each case, $\chi = 2Q^{-1}g(Sw, w')$ coincides with the derivative of the right-hand side provided by Remark 11.7, which – even though χ is constant on Γ , cf. (g) – will be achieved via evaluating the limit of χ at $y_\pm \in \Gamma$ or, equivalently, at $t_\pm \in [t_-, t_+]$. When $w \in \mathcal{W}[\tau_\mp]$, (10.4.b) and (10.2) easily imply that $2Q^{-1}Sw = (\tau - \tau_\mp)^{-1}w$ and, consequently, $\chi = (\tau - \tau_\mp)^{-1}g(w, w')$ has the value (and limit) $\pm(\tau_\pm - \tau_\mp)^{-1}g_y(w_\pm, w'_\pm)$ at $y = y_\pm$, as required in (h1).

Let w, w' now satisfy the hypotheses of (e). By (a), $Q(y) = 0$, where $y = y_\pm$. Also, $Q^{-1}Sw$ is bounded near the endpoint y of $\Gamma \setminus \{y\}$ (and similarly for w'); to see this, we may assume that $w \in \mathcal{W}[c]$ with $c \neq \tau_\pm$, cf. (e), and then (10.4.b) and (10.2) give $2Q^{-1}Sw = (\tau - c)^{-1}w$, which is bounded as $\tau \rightarrow \tau_\pm$ since, due to (b), w has a limit at $t = t_\pm$. Therefore, Q, Sw' and $Q^{-1}g(Sw, Sw')$, restricted to $\Gamma \setminus \{y_\pm\}$, all tend to 0 at y_\pm . Using this and (b) we now evaluate the limit of $\chi = 2Q^{-1}g(Sw, w')$ as $t \rightarrow t_\pm$ via l'Hôpital's rule: it coincides with the limit of $2d_x[g(Sw, w')]/\dot{Q}$. By (9.2.c) for \dot{x} rather than w , (10.3), (9.2.d) and (10.1.b), the last expression is the sum of two terms, $\psi^{-1}Q^{-1/2}g(R(u, \dot{x})w, Jw')$ and $2\psi^{-1}Q^{-1}g(Sw, Sw')$. According

to (11.1) and the above three limit relations at y_{\pm} , only the first term contributes to the limit and, as it equals $\psi^{-1}g(R(J\dot{x}, \dot{x})w, Jw')$, cf. (9.3) and Theorem 11.1 (vi), relation (11.1) yields (h2). Finally, suppose that $w, w' \in \mathcal{W}[\tau_{\pm}]$, and let $y = y_{\pm}$. As Q and w vanish at y (see (a), (d)), while $(Q^{1/2})' = \psi$ by (10.1.b), and so $[\nabla_{\dot{x}}w]/(Q^{1/2})' = \psi^{-1}\nabla_{\dot{x}}w$, l'Hôpital's rule and (11.1) now imply that $Q^{-1/2}w \rightarrow \mp a^{-1}[\nabla_{\dot{x}}w]_{\pm}$ at y , and analogously for w' . Since $S_y[\nabla_{\dot{x}}w]_{\pm} = \mp a[\nabla_{\dot{x}}w]_{\pm}$ from (d) and Remark 11.2, this yields (h3), completing the proof of Theorem 11.6.

Remark 12.1. With the same notations and assumptions as in Theorem 11.6,

$$(12.2) \quad d_+ + d_- \geq m - 1 \geq d_{\pm} \geq 0, \quad \text{where } m = \dim_{\mathbb{C}}M \text{ and } d_{\pm} = \dim_{\mathbb{C}}\Sigma^{\pm}.$$

Cf. the end of Section 13. In fact, denoting by k_{\pm} and q the complex fibre dimensions of the subbundles \mathcal{H}^{\mp} and \mathcal{H} of TM' , we have $d_+ + d_- = m - 1 + q$, as one sees adding the equalities $d_{\pm} = m - 1 - k_{\pm}$ and $m = 1 + k_+ + k_- + q$ (the former due to (11.3) and (11.6), the latter to (11.7)). Now (12.2) follows: $d_+ + d_- \geq m - 1$, with equality if and only if the distribution \mathcal{H} in (11.7) is 0-dimensional, that is, if $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$. The explicit descriptions of Σ^{\pm} in (5.3.c)–(5.3.d) clearly give $d_+ + d_- = m - 1$ for every CP triple (M, g, τ) .

13. Examples: Nontrivial modifications

Whenever two functions $\tau \mapsto Q$ and $\hat{\tau} \mapsto \hat{Q}$ have the properties listed in (6.1), with the same values of τ_{\pm} and a , there must exist an increasing C^{∞} diffeomorphism $[\tau_-, \tau_+] \ni \tau \mapsto \hat{\tau} \in [\tau_-, \tau_+]$ which realizes the equality $\hat{Q} d/d\hat{\tau} = Q d/d\tau$ of vector fields on $[\tau_-, \tau_+]$ expressed in terms of the two diffeomorphically-related coordinates $\hat{\tau}$ and τ . Such a diffeomorphism is unique up to compositions from the left (or, right) with transformations forming the flow of $\hat{Q} d/d\hat{\tau}$ (or, $Q d/d\tau$). Both the existence and uniqueness are immediate, cf. Remark 13.1 in [6].

Lemma 13.1. *For τ_{\pm} and $\tau \mapsto Q$ related via Remark 11.1 (i) to a given compact geodesic-gradient Kähler triple (M, g, τ) , and any increasing C^{∞} diffeomorphism $[\tau_-, \tau_+] \ni \tau \mapsto \hat{\tau} \in [\tau_-, \tau_+]$, there exists a C^{∞} function $[\tau_-, \tau_+] \ni \tau \mapsto \phi \in \mathbb{R}$, unique up to additive constants, such that $\hat{\tau} = \tau + Q d\phi/d\tau$.*

With $\hat{\tau}, \phi$ treated, due to their dependence on τ , as functions on M , the formula $\hat{g} = g - 2(i\partial\bar{\partial}\phi)(J \cdot, \cdot)$ then defines another Kähler metric on M , and

(a) $(M, \hat{g}, \hat{\tau})$ is a new geodesic-gradient Kähler triple.

Also, for the analog $\hat{\tau} \mapsto \hat{Q}$ of $\tau \mapsto Q$ in Remark 11.1 (i) and the \hat{g} -gradient $\hat{\nabla}\hat{\tau}$ of $\hat{\tau}$, one has $\hat{Q} d/d\hat{\tau} = Q d/d\tau$ and $\hat{\nabla}\hat{\tau} = \nabla\tau$.

Proof. See Theorem 13.2 of [6]. □

Let G be the group of all automorphisms of a given compact geodesic-gradient Kähler triple (M, g, τ) , in the sense of Section 4. Then every quadruple $\tau_-, \tau_+, a, \hat{\tau} \mapsto \hat{Q}$ satisfying the analog of (6.1) arises from an application of Theorem 11.1 (i)

to a suitably chosen G -invariant geodesic-gradient Kähler triple $(M, \hat{g}, \hat{\tau})$ with the same underlying complex manifold M . Cf. Remark 13.3 in [6].

As a special case of the last paragraph, for the first triple using the Fubini–Study metric g and G as in the lines preceding (5.2), all quadruples $\tau_-, \tau_+, a, \tau \mapsto Q$ with (6.1) are realized, via Theorem 11.1 (i), by CP triples (M, g, τ) having arbitrarily fixed values of $m = \dim_{\mathbb{C}} M$ and $d_{\pm} = \dim_{\mathbb{C}} \Sigma^{\pm}$ that satisfy (12.2). Conversely, Lemma 13.1 and the lines preceding it allow us to canonically modify any given CP triple, obtaining one with the Fubini–Study metric and the same automorphism group G .

14. The normal-geodesic biholomorphisms

In this section (M, g, τ) is a fixed compact geodesic-gradient Kähler triple. We use the notation of (9.1), denote by τ_-, τ_+, a, Q the data (6.1) associated with (M, g, τ) (see Theorem 11.1 (i)), and choose for them the further data (6.1)–(6.2), so that a sign \pm is fixed as well. We also let $\Sigma, N, h, \langle, \rangle$ and D stand for Σ^{\pm} , the normal bundle $N\Sigma^{\pm}$, the submanifold metric of Σ , the Riemannian fibre metric in N induced by g , and the Chern connection of \langle, \rangle in N , cf. (d) in Section 7. We write $(y, \xi) \in N$ when $y \in \Sigma$ and $\xi \in N_y$.

Using the normal exponential diffeomorphism $\text{Exp}^{\perp}: N^{\delta}\Sigma^{\pm} \rightarrow M \setminus \Sigma^{\mp}$ in (11.2), we define $\Phi = \Phi^{\pm}: N \rightarrow M \setminus \Sigma^{\mp}$, depending on the sign \pm , to be

$$(14.1) \quad \begin{aligned} \Phi &= \text{Exp}^{\perp} \circ \Delta \text{ for } \Delta: N \rightarrow N^{\delta}\Sigma^{\pm} \text{ given by } \Delta(y, \xi) = y \text{ if } \xi = 0 \\ &\text{and } \Delta(y, \xi) = (y, t\xi) \text{ otherwise, where } t = \sigma/\rho \text{ for } \rho = |\xi|, \text{ the} \\ &\text{function } \sigma \text{ of the variable } \rho \in [0, \infty) \text{ being chosen as in (6.2).} \end{aligned}$$

Note that Δ is a homeomorphism and, restricted to the complement $N' = N \setminus \Sigma$ of the zero section, it becomes a diffeomorphism $N' \rightarrow N^{\delta}\Sigma^{\pm} \setminus \Sigma^{\pm}$. In fact, $t\xi$ with $t = \sigma/\rho$ determines ξ (smoothly if $\xi \neq 0$), since $|t\xi| = \sigma$ and σ determines ρ according to the line preceding (6.2). Consequently, $\Phi: N \rightarrow M \setminus \Sigma^{\mp}$ is a homeomorphism, and the restriction $\Phi: N' \rightarrow M'$ a diffeomorphism. In addition,

$$(14.2) \quad \pi^{\pm} \circ \Phi^{\pm} \text{ equals the normal-bundle projection } N\Sigma^{\pm} \rightarrow \Sigma^{\pm}$$

due to (14.1), the fibre-preserving property of Δ , and the first line of Remark 11.3.

Remark 14.1. Suppose that a vector field w on N' is

- (a) the D -horizontal lift of a vector field on Σ , or
- (b) vertical, and has the form $(y, \xi) \mapsto \Theta\xi$ for some complex-linear vector-bundle morphism $\Theta: N \rightarrow N$, skew-adjoint relative to \langle, \rangle at every point.

Then Δ , restricted to N' , sends w onto its restriction to $N' \cap N^{\delta}\Sigma^{\pm}$.

In fact, let $r \mapsto (y(r), \xi(r))$ be an integral curve of w . Then the function $r \mapsto |\xi(r)|$ is constant, and so, by (14.1), $\Delta(y(r), \xi(r)) = (y(r), c\xi(r))$ with some $c \in \mathbb{R}$. This proves our claim since, in case (b), w restricted to every fibre N_y , being a linear vector field on N_y , is invariant under multiplications by scalars.

Theorem 14.2. *For either critical manifold Σ^\mp of τ in any compact geodesic-gradient Kähler triple (M, g, τ) , the triple $(M \setminus \Sigma^\mp, g, \tau)$ is isomorphic to one constructed in Section 8 from some data (6.1) and $\Sigma, h, N, \langle, \rangle$.*

The data consist of (6.1) associated with (M, g, τ) as in Remark 11.1 (i), any choice of $\tau \mapsto \rho$ with (6.1) for our fixed sign \pm , the submanifold metric h and normal bundle $N = N\Sigma^\pm$ of $\Sigma = \Sigma^\pm$, and the fibre metric \langle, \rangle in N induced by g . Furthermore,

- (i) *the required isomorphism $N \rightarrow M \setminus \Sigma^\mp$ is provided by the mapping $\Phi = \Phi^\pm$ with (14.1), which, in particular, must be biholomorphic,*
- (ii) *Φ sends the horizontal distribution of the Chern connection D of \langle, \rangle in N , cf. (d) of Section 7, onto the summand $\mathcal{V} \oplus \mathcal{H}^\pm$ in (11.7),*
- (iii) *the leaves of \mathcal{V} are precisely the same as the Φ -images of all punctured complex lines through 0 in the normal spaces of Σ .*

In the special case where $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$, that is, the summand distribution \mathcal{H} of (11.7) is 0-dimensional, formula (8.3.c) in the construction of Section 8 may be replaced by the following equality, with the simplified notation of (8.3.c):

$$(14.3) \quad \hat{g}(w, w') = \frac{|\tau - \tau_\mp|}{\tau_+ - \tau_-} h(w, w').$$

Proof. It suffices to prove that the restriction of Φ to $N' = N \setminus \Sigma$ is an isomorphism between the geodesic-gradient Kähler triples $(N', \hat{g}, \hat{\tau})$ and (M', g, τ) , since the analogous conclusion about Φ itself then follows from Lemma 16.1 in [5].

We start by establishing the equality $\tau \circ \Phi = \hat{\tau}$. Namely, $|\rho\xi| = \rho$ for any $\rho \in (0, \infty)$ and any $(y, \xi) \in N$ with $|\xi| = 1$, so that $\Phi(y, \rho\xi) = x_\sigma$, where $x_\sigma = \exp_y \sigma\xi$ and σ depends on ρ as in (6.2). As $\sigma \mapsto x_\sigma$ is a unit-speed geodesic, Theorem 11.1 (vi) and (10.1.b) give $d[\tau(x_\sigma)]/d\sigma = \mp Q^{1/2}$, the sign factor being due to the relation $d(x_\sigma)/d\sigma = \mp v/|v|$ (immediate from (1.1) with $v = \nabla\tau$). Here $Q = g(v, v)$ depends on $\tau(x_\sigma)$ as in Theorem 11.1 (i). Remark 6.1 and the text preceding (8.3.a) yield the same equation $d[\hat{\tau}(y, \rho\xi)]/d\sigma = \mp Q^{1/2}$ when $\tau(x_\sigma)$ is replaced by $\hat{\tau}(y, \rho\xi)$, with the same dependence of Q on the unknown function. The uniqueness in Remark 6.1 thus gives $\tau(\Phi(y, \rho\xi)) = \tau(x_\sigma) = \hat{\tau}(y, \rho\xi)$.

Next, one has two complex direct-sum decompositions, $TM' = \mathcal{V} \oplus \mathcal{H}^\mp \oplus \mathcal{H}^\bullet$ and $TN' = \hat{\mathcal{V}} \oplus \hat{\mathcal{H}}^\mp \oplus \hat{\mathcal{H}}^\bullet$, orthogonal relative to g and, respectively, \hat{g} . The former arises from (11.7) if one sets $\mathcal{H}^\bullet = \mathcal{H}^\pm \oplus \mathcal{H}$. In the latter $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ are the distributions introduced in the lines preceding (8.3). First, for \hat{u} as in (8.4) and our $u = Jv$, where $v = \nabla\tau$, we show that

$$(14.4) \quad \begin{aligned} & \text{i) } \Delta \text{ preserves } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet \text{ and } \hat{u}, \\ & \text{ii) } \text{Exp}^\perp \text{ sends } \hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet, \hat{u} \text{ to } \mathcal{V}, \mathcal{H}^\mp, \mathcal{H}^\bullet, u, \\ & \text{iii) } \text{both } \Delta \text{ and } \text{Exp}^\perp \text{ act complex-linearly on } \hat{\mathcal{H}}^\mp \text{ and } \hat{\mathcal{H}}^\bullet. \end{aligned}$$

More precisely, Δ (or, Exp^\perp) appearing in (14.1) (or, (11.2)), restricted to N' (or, $N' \cap N^{\delta\Sigma^\pm}$), sends $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp, \hat{\mathcal{H}}^\bullet, \hat{u}$ onto their restrictions to $N' \cap N^{\delta\Sigma^\pm}$ (or, respectively, onto $\mathcal{V}, \mathcal{H}^\mp, \mathcal{H}^\bullet, u$). The claims about $\hat{\mathcal{V}}$ in (14.4.i)–(14.4.ii) follow as Δ

clearly preserves each leaf of $\hat{\mathcal{V}}$, that is, each punctured complex line through 0 in the normal space $N_y\Sigma$ at any point $y \in \Sigma$, while, by Lemma 11.4 (a), Exp^\pm maps the leaves of $\hat{\mathcal{V}}$ intersected with $N' \cap N^\delta\Sigma^\pm$ onto leaves of \mathcal{V} . This also proves (ii). Next, the class of vertical vector fields of Remark 14.1 (b) obviously includes \hat{u} and, locally, some of them span $\hat{\mathcal{H}}^\mp$. Remark 14.1 thus yields the remainder of (14.4.i), while (14.4.iii) for Δ follows from complex-linearity of the D-horizontal lift operation (due to Lemma 7.1 (i)), and the fact that Δ acts on the vertical vector fields in Remark 14.1 (b) as the identity operator. On the other hand, (14.4.ii) in the case of $\hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ (or, of \hat{u}) is an immediate consequence of the second (or, third) claim in Theorem 11.6 (f). (To be specific, for $\hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ this is clear from Remark 11.8 (ii) combined with (11.6)–(11.7).) Finally, the complex-linearity assertion of Theorem 11.6 (f) implies (14.4.iii).

By (14.4), the diffeomorphism $\Phi = \text{Exp}^\pm \circ \Delta: N' \rightarrow M'$ maps $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$ onto $\mathcal{V}, \mathcal{H}^\mp$ and \mathcal{H}^\bullet . Proving the theorem is thus reduced to showing that \hat{J} and \hat{g} , on each of the three summands $\hat{\mathcal{V}}, \hat{\mathcal{H}}^\mp$ and $\hat{\mathcal{H}}^\bullet$, correspond under the differential $d\Phi$ to J and g on $\mathcal{V}, \mathcal{H}^\mp$ and \mathcal{H}^\bullet , respectively. This last claim is easily verified; for details, see formula (14.7) in [6]. \square

Corollary 14.3. *Suppose that (M, g, τ) is a compact geodesic-gradient Kähler triple. Then, for \mathcal{V} and \mathcal{H}^\pm appearing in (11.7), with either sign \pm , the distribution $\mathcal{V} \oplus \mathcal{H}^\pm$ is integrable and its leaves are totally geodesic in (M', g) .*

Proof. Use Theorem 14.2 and Lemma 8.1 (a) (or – for integrability – (11.6)). \square

15. Immersions of complex projective spaces

In the next result the inclusions $N_y \subseteq P(\mathbb{C} \times N_y)$ and $PN_y \subseteq P(\mathbb{C} \times N_y)$ come from the standard identification (5.1) for $V = N_y$, where $y \in \Sigma^\pm$. Note that, by (11.6) and Corollary 14.3, the biholomorphism $\Phi: N\Sigma^\pm \rightarrow M \setminus \Sigma^\mp$ (see Theorem 14.2), when restricted to the normal space $N_y = N_y\Sigma^\pm \subseteq N\Sigma^\pm$, constitutes

$$(15.1) \quad \text{a totally geodesic holomorphic embedding } \Phi: N_y \rightarrow M \setminus \Sigma^\mp.$$

Theorem 15.1. *Given a compact geodesic-gradient Kähler triple (M, g, τ) , a fixed sign \pm , and a point y of the critical manifold Σ^\pm , the following conclusions hold.*

- (a) *The embedding $\Phi: N_y \rightarrow M \setminus \Sigma^\mp$ with (15.1) has an extension to a totally geodesic holomorphic immersion $\Psi: P(\mathbb{C} \times N_y) \rightarrow M$.*
- (b) *The restriction of Ψ in (a) to the hyperplane $PN_y \subseteq P(\mathbb{C} \times N_y)$ at infinity is a totally geodesic holomorphic immersion $F: PN_y \rightarrow \Sigma^\mp$ and, for a, τ_\pm of Theorem 11.1 (i), the metric induced by F on PN_y equals $2(\tau_+ - \tau_-)/a$ times the Fubini–Study metric arising from the inner product g_y in N_y .*
- (c) *The images of the immersion $F: PN_y \rightarrow \Sigma^\mp$ in (b) and of its differential at any point $\mathbb{C}\xi$, where $(y, \xi) \in N\Sigma^\pm$ and $\xi \neq 0$, coincide with the π^\mp -image of the leaf of $\text{Ker} d\pi^\pm$ in M' passing through $x = \Phi(y, \xi)$ and, respectively, with the subspace $d\pi_x^\mp(\mathcal{H}_x^\mp) = d\pi_x^\mp(\mathcal{V}_x \oplus \mathcal{H}_x^\mp)$ of $T_y\Sigma^\mp$.*

Proof. By Theorems 14.2 and 11.6 (c), $\pi^\mp \circ \Phi$ maps the complement $N\Sigma^\pm \setminus \Sigma^\pm$ of the zero section in $N\Sigma^\pm$ holomorphically into Σ^\mp . Since the restriction of $\pi^\mp \circ \Phi$ to $N_y \setminus \{0\} \subseteq N\Sigma^\pm \setminus \Sigma^\pm$ is constant on each punctured complex line through 0, cf. (11.5) and (14.1), it descends to a holomorphic immersion $F: PN_y \rightarrow \Sigma^\mp$, the immersion property of F being an immediate consequence of the fact, established below, that both $\pi^\mp: \Phi(N_y \setminus \{0\}) \rightarrow \Sigma^\mp$ and $\pi^\mp \circ \Phi: N_y \setminus \{0\} \rightarrow \Sigma^\mp$ have constant (complex) rank, equal to $\dim_{\mathbb{C}} N_y - 1$. As Φ is a biholomorphism, it suffices to verify this last claim for the former mapping; we do it noting that $\Pi = \Phi(N_y \setminus \{0\})$ coincides with the π^\pm -preimage of y (due to (14.2) and Remark 11.3), and hence forms a leaf of $\text{Ker } d\pi^\pm = \mathcal{V} \oplus \mathcal{H}^\mp$ restricted to M' , cf. (11.6). That $\pi^\mp: \Pi \rightarrow \Sigma^\mp$ satisfies the required rank condition is now clear: its differential at any point x has, by (11.6)–(11.7), the kernel \mathcal{V}_x , while $\mathcal{V} = \text{Span}(v, u)$.

The mapping $\Psi: P(\mathbb{C} \times N_y) \rightarrow M$, equal to Φ on N_y and to F on PN_y , is continuous. Namely, if it were not, we could pick a sequence $\xi_j \in N_y$, $j = 1, 2, \dots$, such that $|\xi_j| \rightarrow \infty$ and $\xi_j/|\xi_j| \rightarrow \xi$ as $j \rightarrow \infty$ for some unit vector $\xi \in N_y$, while no subsequence of the image sequence $\Psi(\xi_j)$ tends to $F(\mathbb{C}\xi)$. The resulting limit relation $\sigma_j \rightarrow \delta$, where σ_j corresponds to $\rho_j = |\xi_j|$ as in the line preceding (6.2), combined with (14.1), now gives $\Psi(\xi_j) = \Phi(\xi_j) = \text{Exp}^\perp(y, \sigma_j \xi_j / \rho_j)$ which – due to continuity of Exp^\perp and (11.5) – converges to $\text{Exp}^\perp(y, \delta\xi) = y_\mp$, for a specific point y_\mp . However, Lemma 11.4 and the definition of F also give $y_\mp = F(\mathbb{C}\xi)$, which contradicts our choice of ξ_j , proving continuity of Ψ .

Holomorphicity of Ψ is now obvious from Lemma 3.2 applied to $\Pi = P(\mathbb{C} \times N_y)$ and its codimension-one complex submanifold $\Lambda = PN_y$. Furthermore, Ψ is an immersion. To see this, first note that Ψ has two restrictions, F to PN_y and Φ to the dense open submanifold N_y , already known to be immersions, cf. (15.1), the former into Σ^\mp . Next, for any unit vector $\xi \in N_y$, if A' denotes the projective line in $P(\mathbb{C} \times N_y)$ joining $\mathbb{C}(1, 0)$ to the point $\mathbb{C}\xi \in PN_y$ (identified via (5.1) with $\mathbb{C}(0, \xi)$), then the restriction of Ψ to A' is an embedding with the image $\Lambda = \Psi(A')$ forming a complex submanifold of M , biholomorphic to $\mathbb{C}P^1$, and intersecting each of Σ^+ and Σ^- orthogonally at a single point. Namely, Lemma 11.4 yields all the claims just made except the ‘embedding’ property. To obtain the latter, we invoke the well-known fact that the only injective holomorphic mappings $\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ are biholomorphisms, applied here to the resulting holomorphic mapping $\Psi: A' \rightarrow \Lambda$, the injectivity of which follows from that of Φ . Thus, Ψ is in fact an immersion.

Due to obvious reasons of continuity, (15.1) implies that the holomorphic immersion $\Psi: P(\mathbb{C} \times N_y) \rightarrow M$ is totally geodesic, which establishes (a). Finally, as the intersection of two totally geodesic submanifolds must itself be totally geodesic, Remarks 8.4, 11.1 (iii) and Theorem 14.2 imply (b). \square

Remark 15.2. For m, d_\pm, k_\pm, q as in Remark 12.1, the codimension $\dim_{\mathbb{C}} \Sigma^\mp - \dim_{\mathbb{C}} N_y$ of the immersion F in Theorem 15.1(b) equals q . Namely, $\dim_{\mathbb{C}} N_y = m - d_\pm - 1$, and so, from Remark 12.1, $\dim_{\mathbb{C}} \Sigma^\mp - \dim_{\mathbb{C}} N_y = (m - d_\pm - 1) - d_\mp = q$.

Remark 15.3. Suppose that the distribution \mathcal{H} in (11.7) is 0-dimensional or, in other words, $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$. Then, for either sign \pm , the critical manifold Σ^\pm , with its submanifold metric, must be biholomorphically isometric to a complex projective space carrying the Fubini–Study metric multiplied by $2(\tau_+ - \tau_-)/a$.

In fact, the isometric immersion F of Theorem 15.1(b), having codimension zero (cf. Remark 15.2), is necessarily a biholomorphism (Remark 3.5).

16. Consequences of condition (1.3)

The results stated and proved below use the notations of (9.1), (11.3), (11.6), and the notion of projectability introduced in Section 2.

Lemma 16.1. *For a compact geodesic-gradient Kähler triple (M, g, τ) , the following three conditions, involving distributions on M' , are mutually equivalent.*

- (i) *The span $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ of $\text{Ker}d\pi^+$ and $\text{Ker}d\pi^-$ is integrable.*
- (ii) *$\text{Ker}d\pi^- = \mathcal{V} \oplus \mathcal{H}^+$ is π^+ -projectable.*
- (iii) *$\text{Ker}d\pi^+ = \mathcal{V} \oplus \mathcal{H}^-$ is π^- -projectable.*

In (ii)–(iii) one may also replace $\mathcal{V} \oplus \mathcal{H}^\pm$ by \mathcal{H}^\pm or \mathcal{Z} . If (i)–(iii) hold, then:

- (iv) *The immersions of Theorem 15.1(c) are all embeddings.*
- (v) *The π^\pm -images \mathcal{Z}^\pm of the integrable distribution \mathcal{Z} on M' are integrable holomorphic distributions on Σ^\pm and have totally geodesic leaves biholomorphically isometric to complex projective spaces carrying $2(\tau_+ - \tau_-)/a$ times the Fubini–Study metric, cf. Theorem 15.1(b). These leaves coincide with the images of the embeddings in (iv), and form the fibres of holomorphic bundle projections $\text{pr}^\pm: \Sigma^\pm \rightarrow B^\pm$ for some compact complex base manifolds B^\pm .*
- (vi) *The summand \mathcal{H} in (11.7) is π^\pm -projectable and its π^\pm -image coincides with the orthogonal complement of \mathcal{Z}^\pm in $T\Sigma^\pm$.*
- (vii) *The leaf space $B = M'/\mathcal{Z}$ has a unique structure of a compact complex manifold making the quotient projection $M' \rightarrow M'/\mathcal{Z}$ a holomorphic fibration. For either sign \pm and $\text{pr}^\pm: \Sigma^\pm \rightarrow B^\pm$ as in (v), the mapping $B \rightarrow B^\pm$ sending each leaf of \mathcal{Z} to its $\text{pr}^\pm \circ \pi^\pm$ -image is a biholomorphism.*
- (viii) *There exists a unique holomorphic bundle projection $\pi: M \rightarrow B$ with $\text{Ker}d\pi = \mathcal{Z}$ on M' such that, for both signs \pm , the restriction of π to M' equals $\beta^\pm \circ \text{pr}^\pm \circ \pi^\pm$, where β^\pm is the inverse of $B \rightarrow B^\pm$ in (vii).*
- (ix) *$R^D(w, w') = -ia(\tau_+ - \tau_-)^{-1}h(Jw, w') : N \rightarrow N$, with the notation of Section 2, for the submanifold metric h of Σ^\pm , the normal connection D in its normal bundle $N = N\Sigma^\pm$, and any sections w of \mathcal{Z}^\pm , and w' of $T\Sigma^\pm$.*

Proof. Since $\mathcal{V} \oplus \mathcal{H}^\pm$ are both integrable by (11.6), the mutual equivalence of (i), (ii), (iii) and the integrability claim in (v) are all immediate from Lemma 2.2 applied to $\mathcal{E}^\pm = \mathcal{V} \oplus \mathcal{H}^\pm$, along with (11.6)–(11.7). The immersions mentioned

in Theorem 15.1 (c) thus have nonsingular images, namely, the leaves Π of the distribution \mathcal{Z}^\pm in (v), so that (iv) follows from Remark 3.5 applied to $\mathbb{P}N_y$ standing for $\mathbb{C}\mathbb{P}^l$, with $l = k_{\mp}$ defined in Remark 12.1, and such a leaf Π . The remaining part of (v) is a direct consequence of Theorem 15.1 (b), since an integrable distribution with compact simply connected leaves constitutes the vertical distribution of a bundle projection, cf. Remark 3.3 of [6]. At any $y \in \Sigma^\pm$, the image $d\pi_x^\pm(\mathcal{H}_x^\pm)$ is now independent of the choice of $x \in M'$ with $\pi^\pm(x) = y$, and hence so is its orthogonal complement $d\pi_x^\pm(\mathcal{H}_x)$ in $T_y\Sigma^\pm$ (see Remark 11.8 (iii)), proving (vi).

The mappings $B \rightarrow B^\pm$ in (vii) are obviously bijective, and lead to an identification $B^+ = B^-$ which is a biholomorphism, as one sees restricting π^\pm to “local” complex submanifolds of M' , sent biholomorphically, by the composite bundle projections $M' \rightarrow \Sigma^\pm \rightarrow B^\pm$ (having the leaves of \mathcal{Z} as fibres), onto open submanifolds of B^\pm . This yields (vii). For (viii), it suffices to note that the two composite bundle projections $\text{pr}^\pm \circ \pi^\pm : M \setminus \Sigma^\mp \rightarrow B$ agree, by (vii), on the intersection M' of their domains, cf. Theorem 11.1 (iv), while the union of their domains is M .

For (ix), Theorem 14.2 allows us to identify $M \setminus \Sigma^\mp$ with N so that (8.3.c) and (8.5) hold under the assumptions following (8.3). As w lies in the π^\pm -image \mathcal{Z}^\pm of \mathcal{H}^\pm , cf. (ii), (iii), (v), formula (11.7) gives $2Sw = Qw/(\tau - \tau_\mp)$ for its D-horizontal lift, also denoted by w . Replacing $2Sw$ in (8.5) with $Qw/(\tau - \tau_\mp)$ and multiplying the result by $(\tau - \tau_\mp)Q^{-1}$, we get an expression for $g(w, w')$ which, equated to (8.3.c), yields $\langle R^D(w, Jw')\xi, i\xi \rangle = -a(\tau_+ - \tau_-)^{-1} \langle \xi, \xi \rangle h(w, w')$, since $\rho^2 = \langle \xi, \xi \rangle$ while, clearly, $|\tau - \tau_\pm| = \mp(\tau - \tau_\pm)$. Assertion (ix) now follows if we apply the last equality to Jw rather than w , and use (b) of Section 7 along with Hermitian symmetry of $\langle R^D(w, w')\xi, i\eta \rangle = -\langle iR^D(w, w')\xi, \eta \rangle$ in ξ, η . \square

Let us now fix a Kähler manifold $(\hat{\Sigma}, \hat{h})$, and consider pairs $N, \langle \cdot, \cdot \rangle$ formed by a holomorphic complex vector bundle N over $\hat{\Sigma}$ and the real part $\langle \cdot, \cdot \rangle$ of a Hermitian fibre metric in N , the Chern connection of which – see Section 7 – is assumed to satisfy the curvature condition $R^D(w, w') = 2i\hat{h}(Jw, w') : N \rightarrow N$ for any vector fields w, w' tangent to $\hat{\Sigma}$, the conventions about the sign of R^D and the operators $R^D(w, w')$ being the same as in Section 2.

Lemma 16.2. *If $\hat{\Sigma}$ is simply connected and such $N, \langle \cdot, \cdot \rangle$ exist, they are essentially unique, in the sense that, given another pair $N', \langle \cdot, \cdot \rangle'$ with the same property, some holomorphic vector-bundle isomorphism $N \rightarrow N'$ takes $\langle \cdot, \cdot \rangle$ to $\langle \cdot, \cdot \rangle'$.*

Proof. See Lemma 16.2 of [6]. \square

Theorem 16.3. *For a compact geodesic-gradient Kähler triple (M, g, τ) , the following two conditions are equivalent.*

- (i) (M, g, τ) is isomorphic to a CP triple, defined as in Section 5.
- (ii) $d_+ + d_- = m - 1$, where $(m, d_\pm) = (\dim_{\mathbb{C}} M, \dim_{\mathbb{C}} \Sigma^\pm)$. In other words, cf. Remark 12.1, $TM' = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$, that is, \mathcal{H} in (11.7) is 0-dimensional.

In this case, the assertion of Theorem 14.2, including equation (14.3), is satisfied by $(M \setminus \Sigma^\mp, g, \tau)$, with either fixed sign \pm and (Σ, h) biholomorphically isometric to a complex projective space carrying $2(\tau_+ - \tau_-)/a$ times the Fubini–Study metric, N

and \langle, \rangle being, up to a holomorphic vector-bundle isomorphism, the normal bundle of the latter treated as a linear variety in $\mathbb{C}P^m$ and its Hermitian fibre metric induced by the Fubini–Study metric of $\mathbb{C}P^m$.

Furthermore, the isomorphism types of CP triples (M, g, τ) having any given values of d_{\pm} and m in (ii) are in a natural bijective correspondence, obtained by applying Remark 11.1 (i), with quadruples $\tau_-, \tau_+, a, \tau \mapsto Q$ that satisfy (6.1).

Proof. First, (i) implies (ii) according to the last line of Section 12. Assuming now (ii), we use the end of Section 13 to select a CP triple $(\mathbb{C}P^m, g', \tau')$ realizing the same data d_{\pm}, τ_{\pm}, a and $\tau \mapsto Q$, in (ii) above and Theorem 11.1 (i), as our (M, g, τ) (which also establishes the surjectivity part of the final clause). With either fixed sign \pm , denoting $\Sigma^{\mp}, \Sigma^{\pm}$ by Σ, Π , and their analogs for $(\mathbb{C}P^m, g', \tau')$ by Σ', Π' , we choose the isomorphisms $N \rightarrow M \setminus \Pi$ and $N' \rightarrow \mathbb{C}P^m \setminus \Pi'$ by applying Theorem 14.2 (i) to both triples. As (i) has already been shown to yield (ii), we may now also apply Remark 15.3 to both of them, identifying the critical manifolds Σ, Σ' (and their submanifold metrics) with a complex projective space $\hat{\Sigma}$ (and, respectively, with the Fubini–Study metric \hat{h} multiplied by $2(\tau_+ - \tau_-)/a$). Next, (ix) in Lemma 16.1 holds for both triples, so that the pairs N, \langle, \rangle and N', \langle, \rangle' associated with them via Theorem 14.2 satisfy, along with $\hat{\Sigma} = \Sigma = \Sigma'$ and \hat{h} , the assumptions – as well as the conclusion – of Lemma 16.2. Thus, some holomorphic vector-bundle isomorphism $N \rightarrow N'$ takes \langle, \rangle to \langle, \rangle' and, since the metrics \hat{g}, \hat{g}' on N and N' constructed in Section 8 depend only on $\langle, \rangle, \langle, \rangle'$ (aside from the data fixed above and shared by both triples), this isomorphism is a holomorphic isometry of (N, \hat{g}) onto (N', \hat{g}') , sending τ to its analog on N' . As a consequence of Lemma 16.1 in [5], it can be extended to an isomorphism between the triples (M, g, τ) and $(\mathbb{C}P^m, g', \tau')$. We thus obtain injectivity in the final clause and the fact that (ii) yields (i). \square

17. Horizontal extensions of CP triples

Once again, we use the notation of (9.1), (1.1) and (11.2), assuming (M, g, τ) to be a compact geodesic-gradient Kähler triple.

Lemma 17.1. *Suppose that conditions (i)–(iii) along with the other assumptions of Lemma 16.1 hold for a triple (M, g, τ) , and π, B are as in Lemma 16.1 (viii).*

- (a) *Given a π -projectable local section w of the distribution \mathcal{H} in (11.7),*
 - (a1) *w commutes with the vector fields $v = \nabla\tau$ and $u = Jv$,*
 - (a2) *w is π^{\pm} -projectable for both signs \pm ,*
 - (a3) *the local flow of w in M' preserves the distributions $\mathcal{V}, \mathcal{H}^+$ and \mathcal{H}^- .*
- (b) *The leaves of the integrable distribution $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ on M' are totally geodesic complex submanifolds of M' and all the local flows mentioned in (a3) act between them via local isometries.*

Proof. See Lemma 17.1 of [6]. \square

We say that a (locally-trivial) holomorphic fibre bundle carries a specific local-type *fibre geometry* if such a geometric structure is selected in each of its fibres and suitable local C^∞ trivializations make the structures appear the same in all nearby fibres. For instance, holomorphic complex vector bundles endowed with Hermitian fibre metrics may be referred to as

- (i) holomorphic bundles of Hermitian vector spaces.

The fact that (i) leads to the presence of the distinguished Chern connection (Section 7) has obvious generalizations to two situations (ii)–(iii) discussed below.

By a *horizontal distribution* for a holomorphic bundle projection $\pi: M \rightarrow B$ between complex manifolds, also called a *connection in the holomorphic bundle* M over B , we mean any C^∞ real vector subbundle \mathcal{H} of TM such that TM is the direct sum of the vertical distribution $\text{Ker } d\pi$ and \mathcal{H} . *Horizontal lifts* of vectors tangent to B , and of piecewise C^1 curves in B , as well as *parallel transports* along such curves, are then defined in the usual fashion, although the maximal domain of a lift of a curve (or, of a parallel transport) may in general be a proper subinterval of the original domain interval. This last possibility does not, however, occur in bundles with compact fibres, or in vector bundles with linear connections, where horizontal lifts of curves and parallel transports are all *global*.

The *Chern connection* \mathcal{H} naturally arises in the cases of

- (ii) holomorphic bundles of Fubini–Study complex projective spaces, and
- (iii) holomorphic bundles of CP triples, over any complex manifold B .

See Section 17 of [6]. The fibre geometries consist here of Fubini–Study metrics and, respectively, the structures of a CP triple (Section 5). In case (ii), or (iii),

$$(17.1) \quad \begin{array}{l} \text{the } \mathcal{H}\text{-parallel transports are holomorphic isometries or,} \\ \text{respectively, CP-triple isomorphisms between the fibres,} \end{array}$$

which holds for (ii) since it does for (i), cf. Section 7, and thus extends to (iii) via the canonical modifications in the two final paragraphs of Section 13. Case (iii) leads to a special situation: the critical manifolds – analogs of (1.1) – in the fibres now constitute two holomorphic bundles Σ^\pm of Fubini–Study complex projective spaces over B (with nonnegative fibre dimensions; see Remark 15.3), contained as subbundles in the original bundle, and invariant under all \mathcal{H} -parallel transports.

The following assumptions and notations will now be used to construct compact geodesic-gradient Kähler triples, each of which we call a *horizontal extension* of the CP triple provided by any fibre $(\pi^{-1}(z), g^z, \tau^z)$.

- (a) $\pi: M \rightarrow B$ and \mathcal{H} are the bundle projection and the Chern connection of a holomorphic bundle of CP triples with a compact base B and the CP-triple fibres $(\pi^{-1}(z), g^z, \tau^z)$, $z \in B$, while Σ^\pm stand for the above subbundles of Fubini–Study complex projective spaces, invariant under \mathcal{H} -parallel transports.
- (b) We let τ_\pm, a be the data associated with some/any fibre $(\pi^{-1}(z), g^z, \tau^z)$ as in Theorem 11.1(i), and $\tau: M \rightarrow \mathbb{R}$ (or $\pi^\pm: M \setminus \Sigma^\mp \rightarrow \Sigma^\pm$) be the C^∞ function (or holomorphic bundle projection) which, restricted to each $\pi^{-1}(z)$, equals τ^z or, respectively, the version of (11.3) corresponding to $(\pi^{-1}(z), g^z, \tau^z)$. We also set $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$.

- (c) One is given two Kähler metrics h^\pm on the total spaces Σ^\pm of our holomorphic bundles of Fubini–Study complex projective spaces such that either h^\pm makes the fibres Σ_z^\pm , $z \in B$, orthogonal to \mathcal{H} along Σ^\pm and, restricted to each fibre, h^\pm equals $2(\tau_+ - \tau_-)/a$ times the Fubini–Study metric of Σ_z^\pm .
- (d) We define a Riemannian metric g on M' by requiring that \mathcal{H} be g -orthogonal to the vertical distribution $\text{Ker } d\pi$, that g agree on the fibres $\pi^{-1}(z)$ with the metrics g^z , and that $(\tau_+ - \tau_-)g = (\tau - \tau_-)h^+ + (\tau_+ - \tau)h^-$ on \mathcal{H} , the symbols h^\pm being also used for the π^\pm -pullbacks of h^\pm , cf. (b)–(c).
- (e) Our final assumption: the Riemannian metric g on the dense open submanifold M' has an extension to a Kähler metric on M , still denoted by g .

Remark 17.2. Let a horizontal extension (M, g, τ) arise as above, under the hypotheses (a)–(e). The following is easily verified, cf. Remarks 17.2, 17.3 in [6].

- (i) (M, g, τ) actually constitutes a geodesic-gradient Kähler triple.
- (ii) Compactness of M implies integrability of the distribution $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ coming from the decomposition in (11.7) for (M, g, τ) , and \mathcal{Z} coincides, on M' , with the vertical distribution $\text{Ker } d\pi$ of the bundle projection $\pi: M \rightarrow B$.

Theorem 17.3. *A geodesic-gradient Kähler triple (M, g, τ) , with compact M , satisfies one/all of the mutually-equivalent conditions (i)–(iii) of Lemma 16.1, if and only if it is isomorphic to a horizontal extension of a CP triple, defined as above using (a)–(e).*

Proof. Remark 17.2 (ii) clearly yields the ‘if’ part of our claim.

Conversely, let (M, g, τ) satisfy (i)–(iii) in Lemma 16.1. Lemma 16.1 (viii) states that $\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ coincides, on M' , with the vertical distribution $\text{Ker } d\pi$ of the holomorphic bundle projection $\pi: M \rightarrow B$. Also, in view of Remark 4.3, the leaves of \mathcal{Z} form geodesic-gradient Kähler triples, due to their being complex submanifolds of M tangent to $v = \nabla\tau$ (since $\mathcal{V} = \text{Span}(v, u)$) and, as they are also totally geodesic (see Lemma 17.1 (b)), (11.7) and the S -invariance in (11.7), with $S = \nabla v$, imply via Theorem 16.3 that they are all isomorphic to CP triples. The local isometries of Lemma 17.1 (b) can obviously be made global due to compactness (see the lines preceding (ii) above) which, consequently, turns M into a holomorphic bundle of CP triples over B , in the sense of (iii).

On the other hand, the g -orthogonal complement of $\mathcal{Z} = \text{Ker } d\pi$ is equal, on M' , to the summand \mathcal{H} in (11.7). Thus, \mathcal{H} constitutes a connection in the bundle M over B , as defined in the lines following (i), and –being the intersection of the horizontal distribution of the Chern connections $\mathcal{V} \oplus \mathcal{H}^\pm$ in the normal bundles $N = N\Sigma^\pm$, cf. Theorem 14.2 (ii)– \mathcal{H} itself is, according to (a) in Section 7, the Chern connection of the holomorphic bundle M of CP triples over B .

This provides parts (a)–(b) of the data (a)–(e) required above, with Σ^\pm and τ_\pm , a given by (1.1) and, respectively, Theorem 11.1 (i). The submanifold metrics h^\pm of Σ^\pm have, by (v)–(vi) in Lemma 16.1 and the final clause of Theorem 11.6 (b), all the properties needed for (c).

To show that g satisfies (d), consider two π -projectable nonzero local sections w, w' of the distribution $\mathcal{H} = \mathcal{Z}^\perp$, cf. (11.7). According to Lemma 17.1 (a) and

Remark 2.1, w and w' are projectable along \mathcal{V} and \mathcal{H}^\pm , as well as π^\pm -projectable, for either sign \pm . Their restrictions to any fixed normal geodesic segment Γ emanating from Σ^\pm thus lie in the space \mathcal{W} (cf. Theorem 11.1 (vi) and (i)–(ii) in Theorem 10.1) and, by Theorem 11.6 (g), $g(w, w')$ restricted to Γ is a (possibly nonhomogeneous) linear function of τ . The same linearity condition obviously holds for $g(w, w')$ when g is defined as in (d), rather than being the metric of our triple (M, g, τ) . The two definitions of $g(w, w')$ must now agree, as the two linear functions have – in view of Remark 11.8 (iii) and the final clause of Theorem 11.6 (b) – the same values $h^\pm(w, w')$ at the endpoints τ_\pm of the interval $[\tau_-, \tau_+]$. \square

All compact SKRP triples of Class 1 (cf. Section 8) must be isomorphic to horizontal extensions of CP triples of complex dimension 1, while those of Class 2 are themselves CP triples of a special type. The former claim is easily verified using Theorem 16.3 of [5]; for the latter, see Lemma 8.2. The classification result of Theorem 6.1 in [4] may be rephrased as the conclusion that all compact geodesic-gradient Kähler triples (M, g, τ) with $\dim_{\mathbb{C}} M = 2$, other than Class 2 SKRP triples, are isomorphic to horizontal extensions of CP triples of complex dimension 1. The same conclusion holds – by their very construction – for the Koiso–Cao gradient Kähler–Ricci solitons [9], [3], mentioned in the Introduction.

18. Constant-rank multiplications

In this section vector spaces are finite-dimensional and complex. Bilinear mappings of the type discussed here arise in any compact geodesic-gradient Kähler triple (Theorem 18.3), leading to the dichotomy conclusion of Theorem 19.1.

A *constant-rank multiplication* is any bilinear mapping $\mu: \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$, where $\mathcal{N}, \mathcal{T}, \mathcal{Y}$ are vector spaces, such that the function $\mathcal{N} \setminus \{0\} \ni \xi \mapsto \text{rank } \mu(\xi, \cdot)$ is constant or, equivalently, $\dim \text{Ker } \mu(\xi, \cdot)$ is the same for all nonzero $\xi \in \mathcal{N}$. If $\dim \text{Ker } \mu(\xi, \cdot) = k$ whenever $\xi \in \mathcal{N} \setminus \{0\}$, we also say that $\mu: \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ has the constant rank $\dim \mathcal{T} - k$. Using the notations of Section 5, we see that μ then gives rise to a mapping

$$(18.1) \quad \varepsilon: \mathcal{P}\mathcal{N} \rightarrow \text{Gr}_k \mathcal{T} \quad \text{defined by} \quad \varepsilon(\mathbb{C}\xi) = \text{Ker } \mu(\xi, \cdot) \quad \text{for } \xi \in \mathcal{N} \setminus \{0\}.$$

Lemma 18.1. *For μ and ε as above, $\mathcal{N} \setminus \{0\} \ni \xi \mapsto \text{Ker } \mu(\xi, \cdot) \in \text{Gr}_k \mathcal{T}$ and ε are both holomorphic. The differential of $\xi \mapsto \text{Ker } \mu(\xi, \cdot)$ at any point $\xi \in \mathcal{N} \setminus \{0\}$ sends $\eta \in \mathcal{N}$, in terms of the identification (5.4), to the unique $H \in \text{Hom}(\mathbb{W}, \mathcal{T}/\mathbb{W})$ such that $\mu(\eta, w) = \mu(\xi, -\tilde{H}w)$ whenever $w \in \mathbb{W} = \varepsilon(\mathbb{C}\xi)$, where $\tilde{H}: \mathbb{W} \rightarrow \mathcal{T}$ is any linear lift of H .*

Proof. This is obvious from (5.5) with $F(\xi) = \mu(\xi, \cdot)$. \square

Lemma 18.2. *If $\mu: \mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ has the constant rank $\dim \mathcal{T} - k$ and ε given by (18.1) is nonconstant, then ε is a holomorphic embedding.*

Whether ε is constant, or not, the same is the case for all multiplications $\mathcal{N} \times \mathcal{T} \rightarrow \mathcal{Y}$ of the constant rank $\dim \mathcal{T} - k$, sufficiently close to μ .

Proof. Let $W \in \text{Gr}_k \mathcal{T}$. The subset of \mathcal{N} consisting of 0 and all $\xi \in \mathcal{N} \setminus \{0\}$ with $\varepsilon(\mathbb{C}\xi) = W$ is a vector subspace. In fact, if $\xi, \eta \in \mathcal{N} \setminus \{0\}$ and $W = \text{Ker } \mu(\xi, \cdot) = \text{Ker } \mu(\eta, \cdot)$, then $W \subseteq \text{Ker } \mu(\zeta, \cdot)$ for any $\zeta \in \text{Span}(\xi, \eta)$ and, unless $\zeta = 0$, the last inclusion is actually an equality due to the constant-rank property of μ .

Thus, ε -preimages of points of $\text{Gr}_k \mathcal{T}$ are linear subvarieties in $P\mathcal{N}$. If ε is non-constant, all these subvarieties are zero-dimensional, that is, ε has to be injective. Namely, a projective line \mathcal{L} in $P\mathcal{N}$ cannot lie in the ε -preimage of a point: $\varepsilon^* \omega$ has a nonzero integral over \mathcal{L} , for the Kähler form ω of any Kähler metric on $\text{Gr}_k \mathcal{T}$ (Lemma 3.1). Also, Lemma 18.1 gives holomorphicity of ε .

Let ε now be nonconstant. Then ε must be an embedding, that is, $d\varepsilon_{\mathbb{C}\xi}$ is injective at any $\mathbb{C}\xi \in P\mathcal{N}$ or, equivalently, the differential of $\xi \mapsto \text{Ker } \mu(\xi, \cdot)$ at any $\xi \in \mathcal{N} \setminus \{0\}$ has the kernel $\mathbb{C}\xi$. Namely, in Lemma 18.1 we may set $\tilde{H} = 0$ when $H = 0$, and so η lies in the kernel if and only if the inclusion $W \subseteq \text{Ker } \mu(\eta, \cdot)$ holds for $W = \varepsilon(\mathbb{C}\xi)$. Unless $\eta = 0$, this inclusion is, as before, an equality, and injectivity of ε then yields $\eta \in \mathbb{C}\xi$, which completes the proof, the final clause being an immediate consequence of that in Lemma 3.1. \square

Given a compact geodesic-gradient Kähler triple (M, g, τ) , we use the notation of (9.1) and (1.1) to set $Z_y^\pm(\xi, \eta)w = ag_y(\xi, \eta)w + (\tau_+ - \tau_-)R_y(\xi, J_y\eta)J_y w$ for ξ, η in $N_y\Sigma^\pm$ and $w \in T_y\Sigma^\pm$, with either fixed sign \pm . Thus, $Z_y^\pm(\xi, \eta)w \in T_y\Sigma^\pm$, as ξ, η are tangent, and w normal, to the totally geodesic leaf through y of the J -invariant integrable distribution $\text{Ker } d\pi^\pm = \mathcal{V} \oplus \mathcal{H}^\mp$, cf. (11.6), Theorem 11.6 (c), Corollary 14.3, and the first line of Remark 11.3. As a consequence of (3.1.b),

$$(18.2) \quad Z_y^\pm(\xi, \eta) = Z_y^\pm(\eta, \xi) = Z_y^\pm(J_y\xi, J_y\eta)w, \quad J_y[Z_y^\pm(\xi, \eta)] = [Z_y^\pm(\xi, \eta)]J_y,$$

where $Z_y^\pm(\xi, \eta)$ denotes the endomorphism $w \mapsto Z_y^\pm(\xi, \eta)w$ of $T_y\Sigma^\pm$. We now define a complex-bilinear mapping $\mu_y^\pm: N_y\Sigma^\pm \times T_y\Sigma^\pm \rightarrow \overline{\text{Hom}}_{\mathbb{C}}(N_y\Sigma^\pm, T_y\Sigma^\pm)$ by

$$(18.3) \quad \mu_y^\pm(\xi, w) = Z_y^\pm(J_y\xi, \cdot)w + Z_y^\pm(\xi, \cdot)J_y w.$$

Here $\overline{\text{Hom}}_{\mathbb{C}}$ means ‘the space of antilinear operators’ and $\overline{\text{Hom}}_{\mathbb{C}}(N_y\Sigma^\pm, T_y\Sigma^\pm)$ is treated as a complex vector space in which the multiplication by i acts via composition with J_y from the left. (The product thus equals the given operator $N_y\Sigma^\pm \rightarrow T_y\Sigma^\pm$ followed by J_y .) Antilinearity of $\mu_y^\pm(\xi, w)$ and complex-bilinearity of μ_y^\pm are both obvious from (18.2).

Theorem 18.3. *For a compact geodesic-gradient Kähler triple (M, g, τ) , a fixed sign \pm , and any point $y \in \Sigma^\pm$, the mapping μ_y^\pm with (18.3) is a constant-rank multiplication. Furthermore, if $\varepsilon = \varepsilon_y^\pm$ corresponds to $\mu = \mu_y^\pm$ via (18.1) and ξ is any nonzero vector normal to Σ^\pm at y , then $\varepsilon_y^\pm(\mathbb{C}\xi) = d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm)$, where $x = \Phi(y, \xi)$, and $\varepsilon_y^\pm(\mathbb{C}\xi) = \text{Ker } Z_y^\pm(\xi, \xi)$ for $Z_y^\pm(\xi, \xi)$ as in (18.2), so that*

$$(18.4) \quad \text{Ker } Z_y^\pm(\xi, \xi) = d\pi_x^\pm(\mathcal{H}_x^\pm) = d\pi_x^\pm(\mathcal{V}_x \oplus \mathcal{H}_x^\pm).$$

Proof. Let $x = x(t) \in \Gamma$ as in Theorem 11.6, with some fixed $t \in (t_-, t_+)$. According to (11.7) and parts (iii), (iv), (vi) of Theorem 10.1, the vectors forming \mathcal{H}_x^\pm are precisely the values $w(t)$ for all w as in Theorem 11.6 (e) which also have the property that $2(\tau - \tau_\mp)Q^{-1}g(Sw, w') = g(w, w')$ whenever w' satisfies the hypotheses

of Theorem 11.6 (e). Since the values w'_\pm in Theorem 11.6 (h2) fill $T_y\Sigma^\pm$ (cf. assertions (d)–(f) of Theorem 11.6), replacing $g(w, w')$ and $g(Sw, w')$ in the last equality with the expressions provided by Theorem 11.6 (h2) and Remark 11.7, we easily verify, using (3.1.b) and Remark 11.8 (i), that $w(t) \in \mathcal{H}_x^\pm$ if and only if $Z_y^\pm(\xi, \eta)w_\pm = 0$. Now the final clause of Theorem 11.6 (b) (or, Remark 11.9) yields the first (or, second) equality in (18.4).

To simplify notations, let us write g, Z, J rather than g_y, Z_y^\pm, J_y . Since $x = \Phi(y, \xi)$ in (18.4) and Φ is holomorphic (Theorem 14.2), (18.4) and Remark 11.9 clearly imply that, for a suitable integer $k = k_\pm$, the resulting mapping

$$(18.5) \quad N_y\Sigma^\pm \setminus \{0\} \ni \xi \mapsto \text{Ker } Z(\xi, \xi) \in \text{Gr}_k(T_y\Sigma^\pm) \quad \text{is holomorphic.}$$

The C^∞ version of the assumptions preceding (5.5) thus holds for $U, \mathcal{T}, \mathcal{Y}$ equal to $N_y\Sigma^\pm \setminus \{0\}, T_y\Sigma^\pm, T_y\Sigma^\pm$ and $F(\xi) = Z(\xi, \xi)$. By (5.5), the differential of (18.5) at any $\xi \in N_y\Sigma^\pm \setminus \{0\}$ sends $\eta \in N_y\Sigma^\pm$ to the unique $H : \mathbb{W} \rightarrow \mathcal{T}/\mathbb{W}$, where $\mathbb{W} = \text{Ker } Z(\xi, \xi)$, with a linear lift $\tilde{H} : \mathbb{W} \rightarrow \mathcal{T} = T_y\Sigma^\pm$ such that $Z(\xi, \xi) \circ \tilde{H}$ equals the restriction of $-2Z(\xi, \eta)$ to \mathbb{W} . (We have $dF_\xi = 2Z(\xi, \cdot)$ since $Z(\xi, \eta)$ is real-bilinear and symmetric in ξ, η , cf. (18.2).) Hence $2Z(\xi, \eta)w = -Z(\xi, \xi)\tilde{H}w$ for all $w \in \text{Ker } Z(\xi, \xi)$. Complex-linearity of the differential, due to (18.5), means that this will still hold if we replace η with $J\eta$ and \tilde{H} with $J\tilde{H}$. Then, from (18.2), $2Z(J\xi, \eta)w = -2Z(\xi, J\eta)w = Z(\xi, \xi)J\tilde{H}w = J[Z(\xi, \xi)\tilde{H}w] = -2J[Z(\xi, \eta)w] = -2Z(\xi, \eta)Jw$. In other words, $Z(J\xi, \eta)w + Z(\xi, \eta)Jw = 0$ whenever $w \in \text{Ker } Z(\xi, \xi)$ and $\eta \in N_y\Sigma^\pm$. Therefore, by (18.3), $\text{Ker } Z(\xi, \xi) \subseteq \varepsilon_y^\pm(\mathbb{C}\xi) = \text{Ker } \mu_y^\pm(\xi, \cdot)$, while the opposite inclusion is obvious since (18.2) gives $Z(\xi, J\xi) = 0$, and so the expression $Z(J\xi, \eta)w + Z(\xi, \eta)Jw$ for $\eta = J\xi$ equals $Z(\xi, \xi)w$.

The equality $\text{Ker } Z(\xi, \xi) = \varepsilon_y^\pm(\mathbb{C}\xi)$ and (18.4)–(18.5) complete the proof. \square

19. The dichotomy theorem

This section uses the notations listed at the beginning of Section 11 and the symbols k_\pm of Remark 12.1. With $\Phi = \Phi^\mp$ as in (14.1), any $y \in \Sigma^\mp$ leads to the assignment

$$(19.1) \quad N_y\Sigma^\mp \setminus \{0\} \ni \xi \mapsto d\pi_x^\mp(\mathcal{H}_x^\mp) \in \text{Gr}_k(T_y\Sigma^\mp), \quad \text{where } x = \Phi(y, \xi) \text{ and } k = k_\pm.$$

(Due to (11.6)–(11.7), $d\pi_x^\mp$ must be injective on \mathcal{H}_x^\mp .) Under the identification, via Φ , between $N_y\Sigma^\mp \setminus \{0\}$ and the π^\mp -preimage of y , which forms a leaf of $\text{Ker } d\pi^\mp$ in M' , the mapping (19.1) is obviously the restriction of (1.2) to the leaf.

Theorem 19.1. *Given any compact geodesic-gradient Kähler triple (M, g, τ) , one and only one of the following two cases occurs.*

- (a) *Either the mappings (19.1) are all constant, for all y and both signs \pm , or*
- (b) *each of (19.1), for both signs \pm , descends to a nonconstant holomorphic embedding $\text{PN}_y \rightarrow \text{Gr}_k(T_y\Sigma^\mp)$ of the projective space PN_y of $N_y = N_y\Sigma^\mp$.*

Condition (a) holds if and only if (M, g, τ) satisfies (i)–(iii) in Lemma 16.1.

Proof. In view of Theorem 18.3, we may use Lemma 18.2 for $\varepsilon = \varepsilon_y^\mp$ corresponding to $\mu = \mu_y^\mp$ as in (18.1), concluding (from an obvious continuity argument) that, with either fixed sign \pm , all the mappings (19.1) descend to holomorphic embeddings of $\mathbb{P}N_y$ unless they are all constant. Their constancy for one sign implies, however, the same for the other, since it amounts to (ii) or (iii) in Lemma 16.1, while (ii) and (iii) are equivalent. This completes the proof. \square

Remark 19.2. Case (a) of Theorem 19.1 is equivalent to (1.3), as one sees combining Lemma 16.1 (i) with (11.6). According to (iv)–(vi) in Lemma 16.1, the immersions of Theorem 15.1 (c) are then embeddings and their images form the leaves of foliations on Σ^\mp , both of which have the same leaf space B . On the other hand, when (b) holds in Theorem 19.1, images of the totally geodesic holomorphic immersions of Theorem 15.1 (c) pass through every point $y \in \Sigma^\pm$, realizing an uncountable family of tangent spaces: the image of the embedding (19.1).

Lemma 19.3. *The leaf space M'/\mathcal{V} of the integrable distribution $\mathcal{V} = \text{Span}(v, u)$ on $M' = M \setminus (\Sigma^+ \cup \Sigma^-)$, cf. Lemma 9.1 (a), carries a natural structure of a compact complex manifold of complex dimension $m - 1$, with $m = \dim_{\mathbb{C}} M$, such that the quotient-space projection $M' \rightarrow M'/\mathcal{V}$ forms a holomorphic fibration and, for either sign \pm , the projectivization $\mathbb{P}N$ of the normal bundle $N = N\Sigma^\pm$, defined as in Section 5, is biholomorphic to M'/\mathcal{V} via the biholomorphisms sending each complex line \mathcal{L} through 0 in the normal space of Σ^\pm at any point to the Exp^\perp -image of the punctured radius δ disk in \mathcal{L} , the latter image being a leaf of \mathcal{V} according to Lemma 11.4 (a).*

Restricted to M' , (11.3) descend to further holomorphic bundle projections $\pi^\pm: M'/\mathcal{V} \rightarrow \Sigma^\pm$ which, under the biholomorphic identifications $M'/\mathcal{V} = \mathbb{P}(N\Sigma^\pm)$ of the preceding paragraph, coincide with the bundle projections $\mathbb{P}(N\Sigma^\pm) \rightarrow \Sigma^\pm$.

Proof. The restrictions $\Phi^\pm = \Phi: N\Sigma^\pm \setminus \Sigma^\pm \rightarrow M'$ given by (14.1) with the two possible signs \pm are biholomorphisms (Theorem 14.2), and hence so is the composite of one of them followed by the inverse of the other. At the same time, by Theorem 14.2 (iii), either of them descends to a bijection $\mathbb{P}(N\Sigma^\pm) \rightarrow M'/\mathcal{V}$, and the composite just mentioned yields a biholomorphism between $\mathbb{P}(N\Sigma^\pm)$ and $\mathbb{P}(N\Sigma^\mp)$. This turns M'/\mathcal{V} into a compact complex manifold in a manner independent of the bijection used. Our assertion is now immediate from (14.2). \square

Remark 11.5 (d) trivially implies that, for a Grassmannian triple (M, g, τ) obtained in Section 5 from data (5.2.i), with Σ^\pm described by (5.3.a), and $Y = \mathbb{L}^\perp$, one has a natural biholomorphic identification, written as the equality

$$(19.2) \quad M'/\mathcal{V} = \{(W, W') \in \text{Gr}_k Y \times \text{Gr}_{k-1} Y : W' \subseteq W\}, \text{ under which } \pi^\pm \\ \text{ of Lemma 19.3 correspond to } (W, W') \mapsto W \text{ and } (W, W') \mapsto W',$$

If (M, g, τ) is in turn a CP triple, arising from (5.2.ii), Σ^\pm must satisfy (5.3.b), and (19.2) is replaced by $M'/\mathcal{V} = \Sigma^+ \times \Sigma^-$, with π^\pm in Lemma 19.3 becoming the factor projections. For (easy) proofs of the next two claims, see Section 20 of [6].

Lemma 19.4. *For a complex vector space Y , any k with $1 \leq k \leq \dim Y < \infty$, and M'/\mathcal{V} given by (19.2), let $(W_0, W'_0), (W, W') \in M'/\mathcal{V}$. Then there exist an integer $p \geq 1$ and $(W_j, W'_j) \in M'/\mathcal{V}$, $j = 0, 1, \dots, p$, with $(W_p, W'_p) = (W, W')$ and $(W_{j-1}, W'_{j-1}) \sim (W_j, W'_j)$ whenever $j = 1, \dots, p$, the notation $(\tilde{W}, \tilde{W}') \sim (W, W')$ meaning that $W = \tilde{W}$ or $W' = \tilde{W}'$.*

Corollary 19.5. *Let (M, g, τ) be any Grassmannian triple, cf. Section 5. Then the direct sum $\mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ of Lemma 16.1 (i) constitutes a strongly bracket-generating distribution on M' , in the sense that any two points of M' can be joined by a piecewise C^∞ curve tangent to $\mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$.*

Remark 19.6. A compact geodesic-gradient Kähler triple need not, in general, satisfy conditions (i)–(iii) of Lemma 16.1, that is, (1.3). Examples are provided by all Grassmannian triples (M, g, τ) arising via Lemma 4.4 from data (5.2.i) such that $2 \leq k \leq n - 2$, where $n = \dim_{\mathbb{C}} \mathcal{V}$. Namely, the equality $d_+ + d_- = m - 1 + q$ in Remark 12.1 gives $q = (k - 1)(n - 1 - k)$ as $m = (n - k)k$ and, similarly, $\{d_+, d_-\} = \{(n - k)(k - 1), (n - 1 - k)k\}$ from (5.3.a)–(5.3.b), where $\dim_{\mathbb{C}} \mathcal{L} = 1$ by (5.2.i). Thus, $q > 0$ and $\mathcal{V} \oplus \mathcal{H}^+ \oplus \mathcal{H}^-$ in (11.7) is a proper subbundle of TM' . Corollary 19.5 now implies that it cannot be integrable.

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