ON CONFORMALLY SYMMETRIC MANIFOLDS
WITH METRICS OF INDICES 0 AND 1.

By A. Derdziński and W. Roter.

§1. Introduction. An $n$-dimensional ($n \geq 4$) Riemannian manifold $M$ with a
metric $g_{ij}$ (which need not be positive definite) is said to be conformally symmetric
[1] if its Weyl's conformal curvature tensor

$$(1) \quad C^k_{\ ij\ k} = R^k_{\ ij\ k} - (g_{ij}R^k_k - g_{ik}R^j_j + \delta^k_j R_{ij} - \delta^k_j R_{ik}))/(n - 2)
+ R(\delta^k_ig_{ij} - \delta^k_ig_{ik})/(n - 1)(n - 2)$$

is parallel, i.e.

$$(2) \quad C^k_{\ ijk\ ,\ l} = 0 \ .$$

Clearly the class of conformally symmetric manifolds contains all conformally
flat as well as all locally symmetric manifolds of dimension $n \geq 4$. In this paper
we are interested in Riemannian manifolds which are essentially conformally sym-
metric, that is, lie beyond the two classes mentioned above. Their existence has
been proved by the second-named author in [5] (see also [6]). In Section 3 of this
paper we prove that essentially conformally symmetric $n$-manifolds ($n \geq 4$) cannot
have a positive definite metric (which has been proved in [6] by a similar argument
for $n \geq 5$). Section 4 is devoted to essentially conformally symmetric manifolds with
metrics of index one. Roughly speaking, we prove there that such a manifold always
admits a field of tangent isotropic lines. We recall the result of Tanno ([9], Theorem
6), which will be used below: Any non-conformally flat conformally symmetric
manifold has a constant scalar curvature.

Throughout this paper, by a manifold we shall mean a connected and paracompact
Hausdorff manifold of class $C^\infty$ or analytic.

§2. Some lemmas.

Lemma 1. Let an (algebraic) tensor $A_{lmhk\ldots sp}$ of type $(0, P + 3)$ be symmetric in
$(l, m)$ and skew-symmetric in $(m, h)$. Then $A_{lmhk\ldots sp} = 0$.

Proof. Fix $s_1, \ldots, s_p$ and set $B_{lmh} = A_{lmhk\ldots sp}$. We have $B_{lmh} = B_{mhl} = -B_{mlh} = -B_{hlm} = B_{hlm} = B_{ilm} = -B_{lhm}$, so $B_{lmh} = 0$, as desired.

Lemma 2. The Weyl's conformal curvature tensor satisfies the relations

$$(3) \quad C^h_{\ ijk\ h} = -C^h_{\ kih\ j} = -C^h_{\ jki\ h} = C^h_{\ jhi\ k} \ ,$$

$$(4) \quad C^h_{\ ijk} + C^h_{\ jki} + C^h_{\ kij} = 0 \ , \quad C^r_{\ ijr} = C^r_{\ rik} \ ,$$

$$(5) \quad C^r_{\ ijk\ ,\ r} = (n - 3)(R_{ij\ ,\ k} - R_{ik\ ,\ j} - R_{jik} - R_{jki} + g_{ij}g_{ik} - R_{k}g_{ij} - R_{i}g_{ik})/(n - 1)^2(n - 2) \ .$$

Lemma 3 ([4], Lemma 1). Every conformally symmetric Riemannian manifold

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1) Numbers in brackets refer to the references at the end of the paper.
satisfies the condition

\[(6) \quad R_{ij} R_{ijkl} + R_{ik} R_{ijl} + R_{il} R_{ijk} = 0.\]

**Lemma 4.** Every conformally symmetric Riemannian manifold satisfies the relations

\[(7) \quad R_{ij} C_{ijkl} + R_{ik} C_{ijl} + R_{il} C_{ijk} = 0,\]

\[(8) \quad R_{ijk} C_{ij} + R_{ikl} C_{ij} + R_{ilj} C_{ik} = 0.\]

The proof follows immediately from (6), (1) and (2).

**Remark.** Formulae (6) and (7) remain true under the assumption $C_{ijkl} = 0$ (see proof of Lemma 1 of [4]).

**Lemma 5.** Let $M$ be a conformally symmetric manifold of constant scalar curvature. For any positive integer $K$ we have

\[(9) \quad R_{ijkl} - C_{ijkl} = 0.\]

**Proof.** Differentiating (2) covariantly and making use of the Ricci identity, we obtain

\[(10) \quad C_{ijkl} R_{iklm} + C_{kijl} R_{iklm} + C_{kljm} R_{iklm} + C_{klij} R_{ik} = 0.\]

But the last relation, in virtue of

\[(11) \quad R_{ijkl} = (g_{ij} R_{k}^{l} - g_{ik} R_{jl} - g_{il} R_{jk} + g_{jk} R_{il})/(n - 2),\]

which is a consequence of (1) and $R = const$, leads immediately to

\[(12) \quad g_{kl} R_{ijkl} - g_{km} R_{ijkl} + R_{mk} C_{ijkl} - R_{km} C_{ijkl} - R_{in} C_{ijkl} - R_{ni} C_{ijkl} + R_{ijkl} - C_{ijkl} = 0.\]

Contracting now (12) with $g^{kl}$ and applying (4), we find

\[(13) \quad (n - 2)R_{ijkl} - R_{ijkl} + R_{ij} C_{ik} - R_{ij} C_{ik} - R_{ik} C_{ij} = 0.\]

On the other hand, it follows easily from (5) and (2) that $R_{ijkl} = R_{ijkl}$, which together with (13) yields

\[(14) \quad (n - 2)R_{ijkl} + R_{ij} C_{ik} + R_{ij} C_{ik} + R_{ik} C_{ij} - g_{lm} R_{ijkl} - g_{km} R_{ijkl} = 0.\]

Contracting (14) with $g^{ik}$ and taking into account the obvious formulæ $R_{ijkl} = 0$ and $R_{ijkl} = R_{ijkl}$, we obtain

\[(15) \quad (n - 3)R_{ijkl} + R_{ij} C_{ik} + R_{ij} C_{ik} = 0.\]

whence

\[(16) \quad (n - 3)R_{ijkl} + R_{ij} C_{ik} + R_{ij} C_{ik} = 0.\]

But in view of (8) and $R = const$, $R_{ijkl} R_{ijkl} = R_{ijkl} R_{ijkl}$, which reduces (16) to

\[(n - 2)R_{ijkl} + R_{ij} C_{ik} = 0,\]

that is, $R_{ijkl} = (2 - n)R_{ijkl} = (2 - n)R_{ijkl} = (2 - n)R_{ijkl}$. 

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so \( n > 3 \) yields

\[
R^r_{s,t}C_{rlij} = 0.
\]

Substituting (17) into (13) and using (8) we obtain (\( n - 1 \))\( R_{r,m}C_{rijkl} - R_{ri,p}C_{rjkm} = 0 \),

hence \( R_{ri,p}C_{rjkm} = (1 - n)R_{r,m}C_{rijkl} = (1 - n)^2 R_{ri,p}C_{rjkl} \), which implies

\[
R_{ri,p}C_{rjkm} = 0.
\]

This proves (9) in view of (2).

§ 3. Conformally symmetric manifolds with positive definite metrics.

**Theorem 1.** Let \( M \) be a conformally symmetric Riemannian manifold. If \( M \) is not conformally flat, then

\[
R_{r,s,t,...}^{i,j,...} = 0
\]

for arbitrary integers \( K, N \geq 1 \).

**Proof.** Since \( R = \text{const.} \), Lemma 5 works. Substituting (18) into (12) we obtain

\[
R_{j_l,p_l,...,p_N}C_{mijk} - R_{km,p_l,...,p_N}C_{mijk} + R_{i_l,p_l,...,p_N}C_{kmjk} - R_{im,p_l,...,p_N}C_{hiljk} + R_{j_l,p_l,...,p_N}C_{kimk} - R_{jm,p_l,...,p_N}C_{hilk} + R_{ki,p_l,...,p_N}C_{khjm} - R_{km,p_l,...,p_N}C_{khjl} = 0
\]

for \( N = 1 \), which extends to arbitrary \( N \) in view of (2). Define the tensor \( A \) of type \((0, K + N + 6)\) by \( A_{imkn,...,p_N} = R_{i_l,...,l_r, p_l,...,p_N}C_{mjk} \). Transvecting now (20) with \( R_{r,s,...}^{i,j,...} \) and using (9) and Lemma 2, we conclude that \( A \) satisfies the hypotheses of Lemma 1. Therefore \( A = 0 \), which implies our assertion, since \( M \) is not conformally flat.

**Theorem 2.** Let \( M \) be a conformally symmetric \( n \)-dimensional \((n \geq 4)\) Riemannian manifold with a positive definite metric. Then either \( M \) is conformally flat or \( M \) is locally symmetric.

**Proof.** Suppose \( M \) is not conformally flat. For \( K = N = 1 \) (19) yields \( R_{i,j,k} = 0 \), so

\[
R_{i,j,k} = 0
\]

and \( M \) is locally symmetric by (11).

**Theorem 3.** Suppose that a Riemannian manifold \( M \) satisfies the relation

\[
C_{kijk,...,r_p} = 0
\]

for some \( P \geq 1 \). If the metric of \( M \) is positive definite, then either \( M \) is conformally flat or \( M \) is locally symmetric.

**Proof.** By the result of Tanno ([9], Theorem 2), (22) leads to (2) and our assertion reduces to Theorem 2.

Generalizing a result of Simon ([7], Theorem 1) we obtain (cf. [6], Theorem 3)

**Corollary 1.** Let \( M \) be a complete simply connected conformally symmetric \( n \)-manifold \((n \geq 4)\) with a positive definite metric. If \( M \) is not conformally flat, then either \( M \) is a symmetric Einstein manifold or it is a product of such manifolds.
The proof can be obtained from Theorem 2 and (21) by standard de Rham decomposition techniques.

§ 4. Conformally symmetric manifolds with metrics of index one.

**Lemma 6.** Let \( H \) be an analytic tensor field of type \((P, Q)\) on the analytic manifold \( M \) and let \( x \in M \). If \( H_x = 0 \) and \( (P^N)H_x = 0 \) for each \( N \geq 1 \), then \( H \) vanishes identically on \( M \).

In fact, for any local chart at \( x \) we have \( H_{i_1 \ldots i_P} \equiv 0 \) and \( \partial_{i_1} \ldots \partial_{i_P} H_{j_1 \ldots j_Q}(x) = 0 \), \( N \geq 1 \). Since \( M \) is connected, our assertion follows from analyticity of \( H \).

**Lemma 7.** Let \( M \) be an analytic conformally symmetric Riemannian manifold. If \( x \in M \) and \( M \) is neither conformally flat nor locally symmetric, then (i) there exists \( N \geq 1 \) such that \( R_{i_1 \ldots i_N}(x) \neq 0 \), (ii) for \( K, N \geq 1 \) and any vectors \( u, u_1, \ldots, u_K \), \( v, v_1, \ldots, v_N \in T_xM \), the vectors \( a, b \) given by

\[
\begin{align*}
   a_i &= R_{i j_1 \ldots j_K} u^j_1 u^j_1 \ldots u^j_K \\
   b_i &= R_{i j_1 \ldots j_N} v^j_1 v^j_1 \ldots v^j_N,
\end{align*}
\]

are isotropic and mutually orthogonal.

**Proof.** (i): Conversely, we would have (21) in view of Lemma 6, so \( M \) would be locally symmetric by (11). (ii): By Theorem 1, \( a^t a = b^t b = 0 \), which completes the proof.

Using Theorem 1 of [6] it is easy to see that essentially conformally symmetric manifolds may have metrics of index one. In the sequel, the notion of distribution will be used in the sense explained in [3], p. 10.

**Theorem 4.** Let \( M \) be a conformally symmetric analytic Riemannian manifold with a metric of index one. Suppose \( M \) is neither conformally flat nor locally symmetric. Then the assignment to each \( x \in M \) of the set \( L_x \) of all vectors \( a \in T_xM \) of the form (23), where \( u, u_1, \ldots, u_K \in T_xM \) and \( K \) runs through positive integers, defines an analytic 1-dimensional isotropic distribution \( L \) on \( M \).

**Proof.** By our assumption on the index and (ii) of Lemma 7, any two vectors of \( L_x \) are collinear. Hence \( L_x \) is a vector space of dimension at most one. Furthermore, by (i) of Lemma 7, \( L_x \neq 0 \). Thus \( L \) is an isotropic 1-dimensional distribution. Now suppose that a vector \( a \neq 0 \) of \( L_x \) is defined by (23). Extending \( u, u_1, \ldots, u_K \) to analytic vector fields in a neighbourhood of \( x \) we define, again by (23), an analytic extension of \( a \) to a vector field which spans \( L \) at all points sufficiently near to \( x \). Therefore \( L \) is analytic, which completes the proof.

**Corollary 2.** Any essentially conformally symmetric analytic manifold \( M \) with a metric of index one admits two 1-dimensional \( C^\infty \) distributions which are distinct at each point of \( M \).

**Proof.** It is well-known that any Riemannian manifold with a metric of index 1 admits a 1-dimensional \( C^\infty \) distribution \( V \) which is nowhere isotropic (for a standard argument see e.g. [2], proof of Lemma 3). Our assertion is now satisfied by \( L \) and \( V \).

**Corollary 3.** Every essentially conformally symmetric analytic manifold \( M \) with a metric of index one admits a 2-dimensional \( C^\infty \) distribution.
Theorem 5. Let $S$ be the standard sphere of dimension $n = 2q$, $q \geq 2$, or $n = 4q + 1$, $q \geq 1$. Then $S$ admits no analytic, essentially conformally symmetric Riemannian metric.

Proof. Conversely, suppose that $S$ admits an analytic, essentially conformally symmetric Riemannian metric $g_{ij}$. By Theorem 2 $g_{ij}$ must be indefinite. Even-dimensional spheres do not admit indefinite metrics ([8], Theorem 40.11, p. 207 and Theorem 27.18, p. 144), so we may restrict ourselves to spheres of dimension $n = 4q + 1$. Since such spheres admit indefinite metrics of indices 1 and $n - 1$ only ([8], l. cit.), we may assume index one by changing the sign of $g_{ij}$ if necessary. Now Corollary 3 contradicts Theorem 27.18 of [8], p. 144, which states that the spheres considered above do not admit 2-dimensional distributions. This completes the proof.

Theorem 6. Suppose $M$ is an analytic, essentially conformally symmetric Riemannian manifold of dimension $n = 4$ or $n = 5$. Then $M$ admits a 2-dimensional $C^\infty$ distribution.

Proof. Changing the sign of the metric, if necessary, and using Theorem 2, we may restrict our consideration to metrics of indices 1 and 2. For index 1 our assertion follows from Corollary 3, for index 2 it is clear (see e.g. Theorem 40.11 of [8], p. 207, in the compact case).

REFERENCES