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ON CONFORMALLY SYMMETRIC MANIFOLDS WITH METRICS OF INDICES 0 AND 1.

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§1. Introduction. An n-dimensional ($n \geq 4$) Riemannian manifold M with a metric g_{ij} (which need not be positive definite) is said to be conformally symmetric [1][1)] if its Weyl's conformal curvature tensor

$$(1) \qquad C^h{}_{ijk} = R^h{}_{ijk} - (g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik})/(n-2) + R(\delta^h_k g_{ij} - \delta^h_j g_{ik})/(n-1)(n-2)$$

is parallel, i.e.

$$(2) \qquad C^h{}_{ijk,l} = 0 .$$

Clearly the class of conformally symmetric manifolds contains all conformally flat as well as all locally symmetric manifolds of dimension $n \geq 4$. In this paper we are interested in Riemannian manifolds which are essentially conformally symmetric, that is, lie beyond the two classes mentioned above. Their existence has been proved by the second-named author in [5] (see also [6]). In Section 3 of this paper we prove that essentially conformally symmetric n-manifolds ($n \geq 4$) cannot have a positive definite metric (which has been proved in [6] by a similar argument for $n \geq 5$). Section 4 is devoted to essentially conformally symmetric manifolds with metrics of index one. Roughly speaking, we prove there that such a manifold always admits a field of tangent isotropic lines. We recall the result of Tanno ([9], Theorem 6), which will be used below: Any non-conformally flat conformally symmetric manifold has a constant scalar curvature.

Throughout this paper, by a manifold we shall mean a connected and paracompact Hausdorff manifold of class C^∞ or analytic.

§2. Some lemmas.

Lemma 1. *Let an (algebraic) tensor $A_{lmhs_1\ldots s_P}$ of type $(0, P+3)$ be symmetric in (l, m) and skew-symmetric in (m, h). Then $A_{lmhs_1\ldots s_P} = 0$.*

Proof. Fix $s_1, \ldots, s_P$ and set $B_{lmh} = A_{lmhs_1\ldots s_P}$. We have $B_{lmh} = B_{mlh} = -B_{mhl} = -B_{hml} = B_{hlm} = B_{lhm} = -B_{lmh}$, so $B_{lmh} = 0$, as desired.

Lemma 2. *The Weyl's conformal curvature tensor satisfies the relations*

$$(3) \qquad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi} ,$$

$$(4) \qquad C^h{}_{ijk} + C^h{}_{jki} + C^h{}_{kij} = 0 , \qquad C^r{}_{ijr} = C^r{}_{irk} = C^r{}_{rjk} = 0 ,$$

$$(5) \qquad C^r{}_{ijk,r} = (n-3)[R_{ij,k} - R_{ik,j} - (R_{,k}g_{ij} - R_{,j}g_{ik})/2(n-1)]/(n-2) .$$

Lemma 3 ([4], Lemma 1). *Every conformally symmetric Riemannian manifold*

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1) Numbers in brackets refer to the references at the end of the paper.

satisfies the condition

$$R_{rj}R^{r}{}_{ikl} + R_{rk}R^{r}{}_{ilj} + R_{rl}R^{r}{}_{ijk} = 0 \,. \tag{6}$$

Lemma 4. *Every conformally symmetric Riemannian manifold satisfies the relations*

$$R_{rj}C^{r}{}_{ikl} + R_{rk}C^{r}{}_{ilj} + R_{rl}C^{r}{}_{ijk} = 0 \,, \tag{7}$$

$$R_{rj,p}C^{r}{}_{ikl} + R_{rk,p}C^{r}{}_{ilj} + R_{rl,p}C^{r}{}_{ijk} = 0 \,. \tag{8}$$

The proof follows immediately from (6), (1) and (2).

Remark. Formulae (6) and (7) remain true under the assumption $C^{r}{}_{ijk,r} = 0$ (see proof of Lemma 1 of [4]).

Lemma 5. *Let M be a conformally symmetric manifold of constant scalar curvature. For any positive integer K we have*

$$R_{rk,s_1\ldots s_K}C^{r}{}_{lmp} = 0 \,. \tag{9}$$

Proof. Differentiating (2) covariantly and making use of the Ricci identity, we obtain

$$C_{rijk}R^{r}{}_{hlm,p} + C_{hrjk}R^{r}{}_{ilm,p} + C_{hirk}R^{r}{}_{jlm,p} + C_{hijr}R^{r}{}_{klm,p} = 0 \,. \tag{10}$$

But the last relation, in virtue of

$$R^{h}{}_{ijk,p} = (g_{ij}R^{h}{}_{k,p} - g_{ik}R^{h}{}_{j,p} + \delta^{h}_{k}R_{ij,p} - \delta^{h}_{j}R_{ik,p})/(n-2) \,, \tag{11}$$

which is a consequence of (1) and $R = \text{const.}$, leads immediately to

$$\begin{aligned} & g_{hl}R_{rm,p}C^{r}{}_{ijk} - g_{hm}R_{rl,p}C^{r}{}_{ijk} + R_{hl,p}C_{mijk} - R_{hm,p}C_{lijk} - g_{il}R_{rm,p}C^{r}{}_{hjk} \\ & + g_{im}R_{rl,p}C^{r}{}_{hjk} + R_{il,p}C_{hmjk} - R_{im,p}C_{hljk} + g_{jl}R_{rm,p}C^{r}{}_{khi} \\ & - g_{jm}R_{rl,p}C^{r}{}_{khi} + R_{jl,p}C_{himk} - R_{jm,p}C_{hilk} + g_{kl}R_{rm,p}C^{r}{}_{jih} \\ & - g_{km}R_{rl,p}C^{r}{}_{jih} + R_{kl,p}C_{hijm} - R_{km,p}C_{hijl} = 0 \,. \end{aligned} \tag{12}$$

Contracting now (12) with g^{hl} and applying (4), we find

$$\begin{aligned} & (n-2)R_{rm,p}C^{r}{}_{ijk} + R_{ri,p}C^{r}{}_{mjk} + R_{rj,p}C^{r}{}_{imk} + R_{rk,p}C^{r}{}_{ijm} \\ & \qquad - g_{jm}R^{rs}{}_{,p}C_{rksi} - g_{km}R^{rs}{}_{,p}C_{rjis} = 0 \,. \end{aligned} \tag{13}$$

On the other hand, it follows easily from (5) and (2) that $R_{ij,p} = R_{ip,j}$, which together with (13) yields

$$\begin{aligned} & (n-2)R_{rp,m}C^{r}{}_{ijk} + R_{rp,i}C^{r}{}_{mjk} + R_{rp,j}C^{r}{}_{imk} + R_{rk,p}C^{r}{}_{ijm} \\ & \qquad - g_{jm}R_{rp,s}C^{r}{}_{k}{}^{s}{}_{i} - g_{km}R^{rs}{}_{,p}C_{rjis} = 0 \,. \end{aligned} \tag{14}$$

Contracting (14) with g^{pk} and taking into account the obvious formulae $R_{rp,s}C^{rps}{}_{i} = 0$ and $R^{rs}{}_{,m}C_{rjis} = R^{rs}{}_{,m}C_{rijs}$, we obtain

$$(n-3)R^{rs}{}_{,m}C_{rijs} + R^{rs}{}_{,i}C_{rmjs} + R^{rs}{}_{,j}C_{rims} = 0 \,, \tag{15}$$

whence

$$(n-3)R_{rm,s}C^{r}{}_{ij}{}^{s} + R_{rs,i}C^{r}{}_{mj}{}^{s} + R_{rj,s}C^{r}{}_{im}{}^{s} = 0 \,. \tag{16}$$

But in view of (8) and $R = \text{const.}$, $R_{rj,s}C^{r}{}_{im}{}^{s} = R^{rs}{}_{,m}C_{rijs}$, which reduces (16) to $(n-2)R^{rs}{}_{,m}C_{rijs} + R^{rs}{}_{,i}C_{rmjs} = 0$, that is, $R^{rs}{}_{,i}C_{rmjs} = (2-n)R^{rs}{}_{,m}C_{rijs} = (2-n)^2R^{rs}{}_{,i}C_{rmjs}$, so $n > 3$ yields

$$R^{rs}{}_{,m} C_{rijs} = 0 \,. \tag{17}$$

Substituting (17) into (13) and using (8) we obtain $(n-1)R_{rm,p}C^r{}_{ijk} + R_{ri,p}C^r{}_{mjk} = 0$, hence $R_{ri,p}C^r{}_{mjk} = (1-n)R_{rm,p}C^r{}_{ijk} = (1-n)^2 R_{ri,p}C^r{}_{mjk}$, which implies

$$R_{ri,p}C^r{}_{mjk} = 0 \,. \tag{18}$$

This proves (9) in view of (2).

§ 3. Conformally symmetric manifolds with positive definite metrics.

Theorem 1. *Let M be a conformally symmetric Riemannian manifold. If M is not conformally flat, then*

$$R^i{}_{r,s_1\ldots s_K} R_{il,p_1\ldots p_N} = 0 \tag{19}$$

for arbitrary integers $K,\ N \geq 1$.

Proof. Since $R = \text{const.}$, Lemma 5 works. Substituting (18) into (12) we obtain

$$\begin{aligned} R_{hl,p_1\ldots p_N}C_{mijk} - R_{hm,p_1\ldots p_N}C_{lijk} + R_{il,p_1\ldots p_N}C_{hmjk} - R_{im,p_1\ldots p_N}C_{hljk} \\ + R_{jl,p_1\ldots p_N}C_{himk} - R_{jm,p_1\ldots p_N}C_{hilk} + R_{kl,p_1\ldots p_N}C_{hijm} \\ - R_{km,p_1\ldots p_N}C_{hijl} = 0 \end{aligned} \tag{20}$$

for $N = 1$, which extends to arbitrary N in view of (2). Define the tensor A of type $(0, K+N+6)$ by $A_{lmhs_1\ldots s_Kp_1\ldots p_Nrjk} = R^i{}_{r,s_1\ldots s_K}R_{il,p_1\ldots p_N}C_{hmjk}$. Transvecting now (20) with $R^i{}_{r,s_1\ldots s_K}$ and using (9) and Lemma 2, we conclude that A satisfies the hypotheses of Lemma 1. Therefore $A = 0$, which implies our assertion, since M is not conformally flat.

Theorem 2. *Let M be a conformally symmetric n-dimensional $(n \geq 4)$ Riemannian manifold with a positive definite metric. Then either M is conformally flat or M is locally symmetric.*

Proof. Suppose M is not conformally flat. For $K=N=1$ (19) yields $R^{ij,k}R_{ij,k}=0$, so

$$R_{ij,k} = 0 \tag{21}$$

and M is locally symmetric by (11).

Theorem 3. *Suppose that a Riemannian manifold M satisfies the relation*

$$C_{hijk,r_1\ldots r_P} = 0 \tag{22}$$

for some $P \geq 1$. If the metric of M is positive definite, then either M is conformally flat or M is locally symmetric.

Proof. By the result of Tanno ([9], Theorem 2), (22) leads to (2) and our assertion reduces to Theorem 2.

Generalizing a result of Simon ([7], Theorem 1) we obtain (cf. [6], Theorem 3)

Corollary 1. *Let M be a complete simply connected conformally symmetric n-manifold $(n \geq 4)$ with a positive definite metric. If M is not conformally flat, then either M is a symmetric Einstein manifold or it is a product of such manifolds.*

The proof can be obtained from Theorem 2 and (21) by standard de Rham decomposition techniques.

§ 4. Conformally symmetric manifolds with metrics of index one.

Lemma 6. *Let H be an analytic tensor field of type (P, Q) on the analytic manifold M and let $x \in M$. If $H_x = 0$ and $(\nabla^N H)_x = 0$ for each $N \geq 1$, then H vanishes identically on M.*

In fact, for any local chart at x we have $H^{i_1 \ldots i_P}_{j_1 \ldots j_Q}(x) = 0$ and $\partial_{s_1} \ldots \partial_{s_N} H^{i_1 \ldots i_P}_{j_1 \ldots j_Q}(x) = 0$, $N \geq 1$. Since M is connected, our assertion follows from analyticity of H.

Lemma 7. *Let M be an analytic conformally symmetric Riemannian manifold. If $x \in M$ and M is neither conformally flat nor locally symmetric, then* (i) *there exists $N \geq 1$ such that $R_{ij,s_1 \ldots s_N}(x) \neq 0$,* (ii) *for $K, N \geq 1$ and any vectors $u, u_1, \ldots, u_K, v, v_1, \ldots, v_N \in T_x M$, the vectors a, b given by*

$$a_i = R_{ij,p_1 \ldots p_K} u^j u_1^{p_1} \ldots u_K^{p_K} \tag{23}$$

and $b_i = R_{ik,s_1 \ldots s_N} v^k v_1^{s_1} \ldots v_N^{s_N}$, are isotropic and mutually orthogonal.

Proof. (i): Conversely, we would have (21) in view of Lemma 6, so M would be locally symmetric by (11). (ii): By Theorem 1, $a^i a_i = a^i b_i = b^i b_i = 0$, which completes the proof.

Using Theorem 1 of [6] it is easy to see that essentially conformally symmetric manifolds may have metrics of index one. In the sequel, the notion of distribution will be used in the sense explained in [3], p. 10.

Theorem 4. *Let M be a conformally symmetric analytic Riemannian manifold with a metric of index one. Suppose M is neither conformally flat nor locally symmetric. Then the assignment to each $x \in M$ of the set L_x of all vectors $a \in T_x M$ of the form* (23), *where $u, u_1, \ldots, u_K \in T_x M$ and K runs through positive integers, defines an analytic 1-dimensional isotropic distribution L on M.*

Proof. By our assumption on the index and (ii) of Lemma 7, any two vectors of L_x are collinear. Hence L_x is a vector space of dimension at most one. Furthermore, by (i) of Lemma 7, $L_x \neq 0$. Thus L is an isotropic 1-dimensional distribution. Now suppose that a vector $a \neq 0$ of L_x is defined by (23). Extending $u, u_1, \ldots, u_K$ to analytic vector fields in a neighbourhood of x we define, again by (23), an analytic extension of a to a vector field which spans L at all points sufficiently near to x. Therefore L is analytic, which completes the proof.

Corollary 2. *Any essentially conformally symmetric analytic manifold M with a metric of index one admits two 1-dimensional C^∞ distributions which are distinct at each point of M.*

Proof. It is well-known that any Riemannian manifold with a metric of index 1 admits a 1-dimensional C^∞ distribution V which is nowhere isotropic (for a standard argument see e.g. [2], proof of Lemma 3). Our assertion is now satisfied by L and V.

Corollary 3. *Every essentially conformally symmetric analytic manifold M with a metric of index one admits a 2-dimensional C^∞ distribution.*

Theorem 5. *Let S be the standard sphere of dimension $n = 2q$, $q \geq 2$, or $n = 4q + 1$, $q \geq 1$. Then S admits no analytic, essentially conformally symmetric Riemannian metric.*

Proof. Conversely, suppose that S admits an analytic, essentially conformally symmetric Riemannian metric g_{ij}. By Theorem 2 g_{ij} must be indefinite. Even-dimensional spheres do not admit indefinite metrics ([8], Theorem 40.11, p. 207 and Theorem 27.18, p. 144), so we may restrict ourselves to spheres of dimension $n = 4q + 1$. Since such spheres admit indefinite metrics of indices 1 and $n - 1$ only ([8], l. cit.), we may assume index one by changing the sign of g_{ij} if necessary. Now Corollary 3 contradicts Theorem 27.18 of [8], p. 144, which states that the spheres considered above do not admit 2-dimensional distributions. This completes the proof.

Theorem 6. *Suppose M is an analytic, essentially conformally symmetric Riemannian manifold of dimension $n = 4$ or $n = 5$. Then M admits a 2-dimensional C^∞ distribution.*

Proof. Changing the sign of the metric, if necessary, and using Theorem 2, we may restrict our consideration to metrics of indices 1 and 2. For index 1 our assertion follows from Corollary 3, for index 2 it is clear (see e.g. Theorem 40.11 of [8], p. 207, in the compact case).

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