TENSOR, N. S.
VOL. 34 (1980)

CONFORMALLY RECURRENT INDEFINITE METRICS ON TORI.

By Andrzej DERDZIŃSKI.

Let M be a manifold with a (possibly indefinite) Riemannian metric. A tensor field S of type (p, q) on M is said to be recurrent if at any $z \in M$ such that $S_z \neq 0$ there exists a covariant vector u (called the recurrence vector) which satisfies

$$(1) \qquad (\nabla S)_z = S_z \otimes u ,$$

∇ being the Riemannian connection. The coordinate expression for (1) is $S_{ij,k} = S_{ij}u_k$, where we have taken $(p, q) = (0, 2)$ for simplicity of notation. A Riemannian manifold is called recurrent [4][1] (Ricci-recurrent [2], conformally recurrent [1]) if its curvature tensor (resp. Ricci tensor, Weyl's conformal curvature tensor) is recurrent.

Suppose we are given an open subset M of the Euclidean n-space R^n, $n \geq 4$. Let G be a function of two variables, A and B functions of one variable and ε a non-zero constant. In the sequel we shall denote points of M by n-tuples $(x, y, \ldots)$, while partial differentiations will be marked by subscripts ($H_x = \partial H/\partial x$). Define the indefinite Riemannian metric g_{ij} on M by

$$(2) \qquad g_{ij} = \begin{cases} -2\varepsilon & \text{if } i = j = 1 \\ \exp F_i & \text{if } i + j = n + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where the functions $F_i = F_{n+1-i}$ are given by

$$(3) \qquad \begin{aligned} &F_1(x, y, \ldots) = G(x, y) + A(x) , \qquad F_2(x, y, \ldots) = G(x, y) + B(y) , \\ &F_\lambda(x, y, \ldots) = G(x, y) . \end{aligned}$$

We adopt here the convention that Greek indices $\lambda, \mu, \ldots$ range over the set $\{3, \ldots, n-2\}$ (empty for $n = 4$) and that repeated indices are not to be summed over.

The reciprocal g^{ij} of g_{ij} is clearly of the form

$$g^{ij} = \begin{cases} 2\varepsilon \exp(-2F_1) & \text{if } i = j = n \\ \exp(-F_i) & \text{if } i + j = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is now easy to verify that the only components of the Riemannian connection, curvature tensor, Ricci tensor and Weyl's conformal tensor, which may not vanish, are those related to

$$\begin{aligned} &\Gamma^1_{11} = G_x + A_x , \qquad \Gamma^1_{12} = \Gamma^\lambda_{2\lambda} = \Gamma^n_{2n} = \tfrac{1}{2}G_y , \qquad \Gamma^2_{12} = \Gamma^\lambda_{1\lambda} = \Gamma^{n-1}_{1,n-1} = \tfrac{1}{2}G_x , \\ &\Gamma^2_{22} = G_y + B_y , \qquad \Gamma^{n-1}_{1n} = -\tfrac{1}{2}G_y \exp(F_1 - F_2) , \qquad \Gamma^n_{12} = \varepsilon G_y \exp(-F_1) , \\ &\Gamma^{n-1}_{\lambda,n+1-\lambda} , \qquad \Gamma^n_{11} , \qquad \Gamma^n_{2,n-1} , \qquad \Gamma^n_{\lambda,n+1-\lambda} , \end{aligned}$$

Received November 17, 1977.

1) Numbers in brackets refer to the references at the end of the paper.

and

$$R_{1212} = -\tfrac{1}{2}\varepsilon G_y^2 , \qquad R_{121,n-1} = \tfrac{1}{4}(2G_{xx} - G_x^2 - 2G_x A_x) \exp F_2 ,$$
$$R_{121n} = \tfrac{1}{4}(G_x G_y - 2G_{xy}) \exp F_1 , \qquad R_{122,n-1} = -R_{121n} \exp (F_2 - F_1) ,$$
$$R_{122n} = \tfrac{1}{4}(G_y^2 + 2G_y B_y - 2G_{yy}) \exp F_1 , \qquad R_{1\lambda 1,n+1-\lambda} = R_{121,n-1} \exp (F_\lambda - F_2) ,$$
$$R_{1\lambda 2,n+1-\lambda} = -R_{121n} \exp (F_\lambda - F_1) , \qquad R_{2\lambda 2,n+1-\lambda} = -R_{122n} \exp (F_\lambda - F_1) ,$$

and

$$R_{11} = (2 - n)R_{121,n-1} \exp (-F_2) , \qquad R_{12} = (n - 2)R_{121n} \exp (-F_1) ,$$
$$R_{22} = (n - 2)R_{122n} \exp (-F_1) ,$$

and, respectively, $C_{1212} = \varepsilon(G_{yy} - G_y B_y - G_y^2)$. It follows now easily that $C_{hijk,l} = C_{hijk}F_{,l}$ wherever $C_{hijk} \neq 0$, the function F being given by

$$F = \log |C_{1212}| - 3G - 2A - 2B .$$

Thus, we have proved

Theorem 1. *The n-dimensional Riemannian manifold M with the metric given by* (2) *and* (3) *is conformally recurrent.*

Remark 1. In the above notations, let $M = R^n$ and suppose that G, A and B are (doubly) periodic. Then it is clear that the group of translations K generated by a suitably chosen basis of R^n leaves g_{ij} invariant. Thus g_{ij} determines a conformally recurrent indefinite metric on the n-torus $T^n = R^n/K$.

A Riemannian manifold is said to be conformally symmetric if its Weyl's conformal tensor is parallel. A conformally recurrent manifold is called essentially conformally recurrent if it is neither recurrent, nor conformally symmetric.

First examples of essentially conformally recurrent manifolds were given by Roter in [3]. All his examples are Ricci-recurrent and satisfy the relations

$$R_{ij,k} = R_{ik,j} \tag{4}$$

and

$$\operatorname{rank} R_{ij} \leq 1 . \tag{5}$$

Essentially conformally recurrent metrics with the properties just stated can also be constructed on tori. However, as we shall show, these properties do not follow from essential conformal recurrency.

Theorem 2. *For each $n \geq 4$ the n-torus T^n admits an essentially conformally recurrent indefinite metric which is Ricci-recurrent and satisfies* (4) *and* (5).

Proof. Setting in (3) $G(x, y) = \sin y$, $A = B = 0$ and $\varepsilon = 1$, we can define a metric with desired properties in R^n. In view of Remark 1, we may project it onto T^n.

Theorem 3. *For each $n \geq 4$ the n-dimensional torus T^n admits an essentially conformally recurrent indefinite metric which is not Ricci-recurrent and satisfies neither* (4) *nor* (5).

Proof. In view of Remark 1 it is sufficient to define the metric g_{ij} in R^n by

(2) and (3) with $G(x, y) = \sin x + 2\sin y$, $A = B = 0$, $\varepsilon = 1$. It is now easy to verify that

$$R_{11,2}/R_{11} \neq R_{12,2}/R_{12}, \qquad R_{11}R_{22} - (R_{12})^2 \neq 0, \qquad R_{11,2} \neq R_{12,1}$$

at some points, which implies our assertion.

Wrocław University
Mathematical Institute
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

REFERENCES

[1] T. Adati and T. Miyazawa: On a Riemannian space with recurrent conformal curvature, *Tensor, N.S.*, **18** (1967), 348-354.

[2] E. M. Patterson: Some theorems on Ricci-recurrent spaces, *Journ. London Math. Soc.*, **27** (1952), 287-295.

[3] W. Roter: On the existence of conformally recurrent Ricci-recurrent spaces, *Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys.*, **24** (1976), 973-979.

[4] A. G. Walker: On Ruse's spaces of recurrent curvature, *Proc. London Math. Soc.*, **52** (1950), 36-64.