TOTALLY REAL IMMERSIONS OF SURFACES

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ABSTRACT. Totally real immersions \( f \) of a closed real surface \( \Sigma \) in an almost complex surface \( M \) are completely classified, up to homotopy through totally real immersions, by suitably defined homotopy classes \( \mathcal{M}(f) \) of mappings from \( \Sigma \) into a specific real 5-manifold \( E(M) \), and the class \( \mathcal{M}(f) \) is subject to a single cohomology constraint. This follows from Gromov's observation that totally real immersions satisfy the \( h \)-principle. For the receiving complex surfaces \( \mathbb{C}^2 \), \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), \( \mathbb{CP}^2 \) and \( \mathbb{CP}^2 \# k \mathbb{CP}^2 \), \( k = 1, 2, \ldots, 8 \), and all \( \Sigma \), we illustrate the above non-constructive result with explicit examples of immersions realizing all possible equivalence classes. We also determine which classes contain totally real embeddings, and provide examples of such embeddings for all classes that contain them.

0. Introduction

Given an almost complex surface, that is, an almost complex manifold \( M \) with \( \dim_{\mathbb{R}} M = 4 \), we ask which closed real surfaces \( \Sigma \) admit totally real immersions/embbeddings \( f : \Sigma \to M \), and how such \( f \) can be classified up to the equivalence relation \( \sim_{t\mathbb{R}} \) of being homotopic through totally real immersions.

We study these questions using a two-pronged approach. First, our Theorems 2.1 and 2.2 provide an answer for totally real immersions. Theorem 2.2 states that, when \( M \) is simply connected, the \( \sim_{t\mathbb{R}} \) equivalence class of a totally real immersion \( f : \Sigma \to M \) is completely determined by its Maslov index \( i = i(f) \) and degree \( d = d(f) \), which in turn form an arbitrary element \( (i, d) \) of a specific set depending on \( M \) and \( \Sigma \). Theorem 2.1 classifies such \( \sim_{t\mathbb{R}} \) equivalence classes for arbitrary \( M \), using the Maslov invariant \( \mathcal{M}(f) \), valued in a certain set of homotopy classes of mappings. We define \( \mathcal{M}(f) \) in \( \S 2 \) by modifying Arnold’s definition [1].

What makes Theorems 2.1 and 2.2 less than completely satisfactory is the reliance of their proofs on Gromov’s observation [8, p. 192] that totally real immersions satisfy the \( h \)-principle. Consequently, those proofs offer little information about how the immersions which are shown to exist might actually be constructed. In addition, the two theorems deal only with the case of totally real immersions, as opposed to embeddings.

To make up for such shortcomings, we devote most of this paper (beginning with \( \S 7 \)) to our second approach, which deals with totally real immersions and embeddings \( \Sigma \to M \) of arbitrary closed real surfaces \( \Sigma \) in one of the “model” simply connected complex surfaces

\[
\text{\( C^2 \), \( \mathbb{CP}^2 \), \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( \mathbb{CP}^2 \# k \mathbb{CP}^2 \), \( k \geq 1 \),}
\]

\( \mathbb{CP}^2 \# k \mathbb{CP}^2 \) being obtained by blowing up \( k \) points in \( \mathbb{CP}^2 \). For these \( M \), we provide explicit answers to the questions stated in the first paragraph. We begin by settling the existence question for totally real immersions/embbeddings \( \Sigma \to M \).
An almost complex manifold is a real manifold $M$ with an almost complex structure, that is, a $C^\infty$ bundle morphism $J: TM \to TM$ such that $J^2 = -\text{Id}$. The tangent bundle $TM$ then becomes a complex vector bundle, in which $J$ is the multiplication by $i$. We write $iv$ rather than $Jv$ for $v \in T_x M$ and $x \in M$.

A real vector subspace $W$ of a complex vector space $V$ is said to be totally real if $W \cap iW = \{0\}$. A totally real immersion/embedding of a real manifold $\Sigma$ (with or without boundary) in an almost complex manifold $M$ is an immersion/embedding $f: \Sigma \to M$ such that the image of the differential $df_x$ at any $x \in \Sigma$ is a totally real subspace of $T_{f(x)} M$. If $f$ is a totally real embedding, the image $f(\Sigma)$ is called a totally real submanifold of $M$. (See [4], [8], and Remark 1.4 below.)

Given an almost complex manifold $M$ with $\dim_C M = n$, we define $E^+(M)$ and $E(M)$ to be the unit circle bundles of the determinant bundle $\det_C TM = [TM]^\otimes n$ and, respectively, of its square $[\det_C TM]^\otimes 2$. Thus, $E(M)$ is the $\mathbb{RP}^1$ bundle over $M$ associated with $\det_C TM$. Both $E = E(M)$ and $E = E^+(M)$ are the total spaces of principal $U(1)$-bundles over $M$, leading to the homotopy exact sequences

$$
\pi_2 E \xrightarrow{\text{injective}} \pi_2 M \xrightarrow{\text{connecting}} \pi_1 [U(1)] = \mathbb{Z} \xrightarrow{\text{onto}} \pi_1 E \xrightarrow{\text{onto}} \pi_1 M.
$$

One also has an obvious twofold covering projection

$$
E^+(M) \to E(M) = E^+(M)/\mathbb{Z}_2,
$$

$U(1)$-equivariant relative to the homomorphism $U(1) \ni z \mapsto z^2 \in U(1)$. Thus,

$$
\pi_1 [E^+(M)] \subset \pi_1 [E(M)], \quad \pi_1 [E(M)]/\pi_1 [E^+(M)] = \mathbb{Z}_2,
$$

If, in addition, $M$ is simply connected, (1.3) and exactness of (1.1) imply that

$$
\pi_1 [E(M)] = \mathbb{Z}_q \text{ for some } q \in \{2, 4, \ldots, \infty\}, \quad \text{where we set } \mathbb{Z}_\infty = \mathbb{Z}.
$$

Given a manifold $\Sigma$ (always assumed connected) and an Abelian group $G$, we have natural isomorphic identifications

$$
H^1(\Sigma, G) = \text{Hom}(\pi_1 \Sigma, G) = \text{Hom}(H_1(\Sigma, \mathbb{Z}), G).
$$

For instance, according to (1.5) with $G = \mathbb{Z}_2$,

$$
w_1(\Sigma) \in H^1(\Sigma, \mathbb{Z}_2) \text{ is the orientation homomorphism } \pi_1 \Sigma \to \mathbb{Z}_2.
$$
We will also use Wu’s formula [10], valid whenever \( \Sigma \) is a closed real surface:

\[
(1.7) \quad \varpi_1(\Sigma) \sim \varpi_1(\Sigma) = [\chi(\Sigma) \mod 2] \in H^2(\Sigma, \mathbb{Z}_2) = \mathbb{Z}_2.
\]

**Remark 1.1.** In terms of (1.5), the homomorphism \( H^1(\Sigma, G) \rightarrow H^1(\Sigma, G') \) of coefficient reduction, corresponding to a given homomorphism \( h : G \rightarrow G' \) of Abelian groups, sends a homomorphism \( \varphi : H_1(\Sigma, \mathbb{Z}) \rightarrow G \) to the composite \( h \circ \varphi \).

**Remark 1.2.** Given a manifold \( \Sigma \) along with a continuous mapping \( g : \Sigma \rightarrow U(1) \), and \( q \in \{1, 2, 3, \ldots, \infty\} \), we define \([g \mod q] \in H^1(\Sigma, \mathbb{Z}_q) = \text{Hom}(\pi_1 \Sigma, \mathbb{Z}_q)\) to be the composite \( \pi_1 \Sigma \rightarrow Z \rightarrow \mathbb{Z}_q \) of the action of \( g \) on the fundamental groups and the projection \( Z \rightarrow \mathbb{Z}_q \), where \( Z_1 = \{0\} \) and \( \mathbb{Z}_\infty = \mathbb{Z} \).

(i) \([g \mod q] \), as a homomorphism \( \pi_1 \Sigma \rightarrow \mathbb{Z}_q \), sends the homotopy class of any loop \( S^1 \rightarrow \Sigma \) to the image under \( Z \rightarrow \mathbb{Z}_q \) of the degree of the composite \( S^1 \rightarrow \Sigma \rightarrow \).

(ii) \([g \mod q] = 0 \) if and only if either \( q = \infty \) and \( g \) has a lift \( \Sigma \rightarrow \mathbb{R} \) to the universal covering of \( U(1) \), or \( q < \infty \) and the \( q \)th root of \( g \) treated as a complex-valued function (with \( \mathbb{U}(1) = S^1 \subset \mathbb{C} \)) has a single-valued continuous branch \( \Sigma \rightarrow \mathbb{U}(1) \subset \mathbb{C} \).

In fact, (i) is obvious, and (ii) follows from (i).

**Remark 1.3.** Let \( E \) be any principal \( U(1) \)-bundle over a simply connected manifold \( M \). Exactness of (1.1) gives \( \pi_1 E = \mathbb{Z}_q \) for some \( q \in \{1, 2, 3, \ldots, \infty\} \), with \( Z_1 = \{0\} \) and \( \mathbb{Z}_\infty = \mathbb{Z} \). Defining \( j(\Theta) \in H^1(\Sigma, \mathbb{Z}_q) \), for a manifold \( \Sigma \) and a continuous mapping \( \Theta : \Sigma \rightarrow E \), to be the homomorphism of the fundamental groups induced by \( \Theta \) (cf. (1.5)), and letting \([g \mod q] \) as in Remark 1.2, we then have

\[
(1.8) \quad j(g\Theta) = j(\Theta) + [g \mod q]
\]

for any continuous mapping \( g : \Sigma \rightarrow U(1) \), where \( g\Theta \) is the valuewise product.

This is clear from Remark 1.2(i), since a principal \( U(1) \)-bundle over \( S^1 \) is trivial, and the degree is additive under valuewise multiplication of mappings \( S^1 \rightarrow U(1) \).

**Remark 1.4.** We define totally real subspaces \( W \subset V \) differently from Gromov [8], who requires \( \text{Span}_C W \) to have the maximum possible dimension \( \min(k, n) \), where \( k = \dim_R W \) and \( n = \dim_C V \). However, the two definitions agree when \( k \leq n \), and are both devoid of content when \( k \geq 2n - 1 \) (as one of them then makes every subspace totally real, and the other allows no such subspace); on the other hand, \( k \leq n \) or \( k \geq 2n - 1 \) whenever \( n = 2 \), which is the case of our main interest.

2. Statement of the results

Let \( E(M) \) and \( E^+(M) \) be defined as in §1 for an almost complex manifold \( M \) of complex dimension \( n \). In view of (1.3), there exists a unique homomorphism

\[
(2.1) \quad \varpi_1 : \pi_1[E(M)] \rightarrow \mathbb{Z}_2 \quad \text{with} \quad \text{Ker} \varpi_1 = \pi_1[E^+(M)].
\]

By (1.5), \( \varpi_1 \in H^1(E(M), \mathbb{Z}_2) \setminus \{0\} \). Clearly, \( \varpi_1 \) is the first Stiefel-Whitney class of the real line bundle over \( E(M) \) associated with the \( \mathbb{Z}_2 \) bundle (1.2).

If \( f : \Sigma \rightarrow M \) now is a totally real immersion of a real manifold \( \Sigma \) in an almost complex manifold \( M \) with \( \dim_R \Sigma = \dim_C M = n \), we define its Maslov invariant \( \mathcal{M}(f) \) to be the homotopy class of the mapping \( \Sigma \rightarrow E(M) \), for \( E(M) \) as in §1, that sends \( x \in \Sigma \) to the real line in \( [T_{f(x)} M]^{\wedge n} \) spanned by \( v_1 \wedge \ldots \wedge v_n \), where \( v_j = df_x e_j \) for any basis \( e_1, \ldots, e_n \) of \( T_x \Sigma \). (See also §3 below and [1].)
Obviously, $\mathcal{M}(f)$ depends only on the $\sim_{\text{tri}}$ equivalence class of $f : \Sigma \to M$, for $\sim_{\text{tri}}$ defined at the beginning of §8. In §4 we prove the following result.

**Theorem 2.1.** Given an almost complex surface $M$ and a closed real surface $\Sigma$, the assignment $f \mapsto \mathcal{M}(f)$ establishes a one-to-one correspondence between the set of all $\sim_{\text{tri}}$ equivalence classes of totally real immersions $f : \Sigma \to M$ and the set of those homotopy classes of mappings $\Theta : \Sigma \to E(M)$ for which

\begin{equation}
\Theta^* w_1 = w_1(\Sigma) \text{ in } H^1(\Sigma, \mathbb{Z}_2), \text{ with } w_1 \in H^1(E(M), \mathbb{Z}_2) \text{ as in (2.1).}
\end{equation}

If $M$ in Theorem 2.1 happens to be simply connected, $\mathcal{M}(f)$ may be replaced by a pair of more tangible invariants: the Maslov index $i(f)$ of a totally real immersion $f : \Sigma \to M$, and its degree $d(f)$, described below.

When dealing with mappings from a real $n$-manifold $\Sigma$, we will assume that an orientation of $\Sigma$ has been selected, as long as one exists; in other words,

\begin{equation}
\Sigma \text{ is either oriented, or nonorientable.}
\end{equation}

We then define a group $\mathbb{Z}_{[2]}$ associated with $\Sigma$ by

\begin{equation}
\mathbb{Z}_{[2]} = \mathbb{Z} \text{ if } \Sigma \text{ is oriented, } \mathbb{Z}_{[2]} = \mathbb{Z}_2 \text{ if } \Sigma \text{ is not orientable.}
\end{equation}

Let $[\Sigma] \in H_n(\Sigma, \mathbb{Z}_{[2]})$ now be the fundamental homology class of $\Sigma$. We set

\begin{equation}
d(f) = f_*[\Sigma] \in H_n(M, \mathbb{Z}_{[2]}) \text{ with } n = \dim H \Sigma \text{ and } \mathbb{Z}_{[2]} \text{ as in (2.4),}
\end{equation}

If, in addition, $\dim H \Sigma = \dim C M = n$, we define $i(f)$ to be the homomorphism $\pi_1 \Sigma \to \mathbb{Z}_q$ of the fundamental groups (see (1.4)) induced by $\mathcal{M}(f)$. Thus, by (1.5),

\begin{equation}
i(f) \in H^1(\Sigma, \mathbb{Z}_q).
\end{equation}

Rather than being arbitrary elements of the (co)homology groups in question, $i(f)$ and $d(f)$ are both confined to specific subsets. Namely, in Lemma 6.1 we verify that, given a totally real immersion $f : \Sigma \to M$ of a closed real surface $\Sigma$ in a simply connected almost complex surface $M$ we have, for $q$ as in (1.4),

\begin{equation}
i(f) \in \mathcal{I}_q(\Sigma) \subset H^1(\Sigma, \mathbb{Z}_q) \quad \text{and} \quad d(f) \in \mathcal{D}_\pm(M) \subset H_2(M, \mathbb{Z}_{[2]}).
\end{equation}

Here $\varepsilon$ and $\pm$ are $\mathbb{Z}_2$-valued parameters, determined by $\Sigma$ as follows:

\begin{equation}
\varepsilon = 1 \text{ if } \Sigma \text{ is orientable, } \varepsilon = 0 \text{ if it is not, } \text{ and } (-1)^{1(\Sigma)} = \pm 1,
\end{equation}

while the sets $\mathcal{D}_-^0(M) \subset H_2(M, \mathbb{Z}_{[2]})$ and $\mathcal{D}_+^1(M) \subset H_2(M, \mathbb{Z})$ are defined by

\begin{equation}
\mathcal{D}_-^0(M) = \text{Ker}[w_2(M)], \quad \mathcal{D}_+^0(M) = H_2(M, \mathbb{Z}_2) \setminus \text{Ker}[w_2(M)], \quad \mathcal{D}_+^1(M) = \mathcal{D}_+^1(M) = \emptyset,
\end{equation}

for $c_1(M), w_2(M)$ treated as homomorphisms $H_2(M, \mathbb{Z}) \to \mathbb{Z}$ or $H_2(M, \mathbb{Z}_2) \to \mathbb{Z}_2$.

Therefore, $\mathcal{D}_\varepsilon^0(M)$, with $\varepsilon \in \{0,1\}$, is either empty, or it is a coset of a subgroup of $H_2(M, \mathbb{Z}_2)$ or $H_2(M, \mathbb{Z})$.

Finally, $\mathcal{I}_q(\Sigma) \subset H^1(\Sigma, \mathbb{Z}_q)$ is given by

\begin{equation}
\mathcal{I}_q(\Sigma) = \{ \lambda \in H^1(\Sigma, \mathbb{Z}_q) : [\lambda \mod 2] = w_1(\Sigma) \},
\end{equation}

$H^1(\Sigma, \mathbb{Z}_q) \ni \lambda \mapsto [\lambda \mod 2] \in H^1(\Sigma, \mathbb{Z}_2)$ being the mod 2 reduction homomorphism corresponding to the unique nonzero homomorphism $\mathbb{Z}_q \to \mathbb{Z}_2$. (Recall that $q \in \{2, 4, 6, \ldots, \infty\}$ in (1.4).) In §6 we establish the following theorem.
Theorem 2.2. Given a simply connected almost complex surface $M$ and a closed real surface $\Sigma$, the assignment $f \mapsto (i(f), d(f))$ defines a bijective correspondence between the set of $\sim_{\text{tri}}$ equivalence classes of totally real immersions $f : \Sigma \to M$ and a specific subset $Z$ of the Cartesian product $\mathcal{I}_q(\Sigma) \times \mathcal{D}^\pm_q(M)$ of the sets defined by (2.9) - (2.10) for $q, \varepsilon, \pm$ as in (1.4) and (2.8).

The set $Z \subset \mathcal{I}_q(\Sigma) \times \mathcal{D}^\pm_q(M)$, defined in §6, coincides with $\mathcal{I}_q(\Sigma) \times \mathcal{D}^\pm_q(M)$ except in the case where, simultaneously, $\Sigma$ is nonorientable, $\chi(\Sigma)$ is even, while $q$ is finite and divisible by 4. In this latter case, $Z$ has half the (finite) number of elements of $\mathcal{I}_q(\Sigma) \times \mathcal{D}^\pm_q(M)$. More precisely, $Z$ then consists of all $(i, d)$ in $\mathcal{I}_q(\Sigma) \times \mathcal{D}^\pm_q(M)$ such that $i$ lies in a specific coset (depending on $d$) of the index 2 subgroup $\mathcal{G} \subset \text{Hom}(H_1(\Sigma, \mathbb{Z}), 2\mathbb{Z}_q)$ formed by those homomorphisms $H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}_q$ (cf. (1.5)) which, in addition to assuming even values only, send the unique nontrivial element of order 2 onto $0 \in \mathbb{Z}_q$.

Furthermore, we prove, in §17, the following six statements.

Theorem 2.3. Let $M$ be any almost complex surface. The class of closed real surfaces $\Sigma$ admitting a totally real embedding (or, immersion) in $M$ then includes the 2-torus $T^2$ and Klein bottle $K^2$ (and, for immersions, also the 2-sphere $S^2$), and is closed under the mapping $\Sigma \mapsto \Sigma \# T^2 \# K^2$ (and, for immersions, under the connected-sum operation $(\Sigma, \Sigma') \mapsto \Sigma \# \Sigma'$).

Corollary 2.4. Any closed real surface with an even Euler characteristic admits a totally real immersion in every almost complex surface.

Corollary 2.5. The 2-torus $T^2$ and all nonorientable closed real surfaces $\Sigma$ with $\chi(\Sigma) \equiv 0 \mod 4$ admit totally real embeddings in every almost complex surface.

Corollary 2.6. Let $M$ be an almost complex surface. If there exists a totally real immersion $\mathbb{R}P^2 \to M$, then every closed real surface $\Sigma$ admits a totally real immersion in $M$.

Corollary 2.7. Every closed real surface admits totally real immersions in $\mathbb{C}P^2$ and in $\mathbb{C}P^2 \# k\mathbb{C}P^2$ for all $k \geq 1$.

Corollary 2.8. The torus $T^2$, sphere $S^2$, and all nonorientable closed surfaces admit totally real embeddings in $\mathbb{C}P^2 \# k\mathbb{C}P^2$ for every integer $k \geq 2$.

Here and in the sequel, given a complex surface $M$, we identify the connected sum $M \# k\mathbb{C}P^2$ with the complex surface obtained by blowing up any fixed set of $k$ distinct points in $M$.

Corollaries 2.4, 2.8 and Theorem 2.3, combined with an obvious example of an “anti-diagonal” totally real 2-sphere in $S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$ (cf. (7) in §7), lead to the following three results. (Their detailed proofs are given in §17 and §18.)

Corollary 2.9. For any almost complex surface $M$ which is a spin manifold, the closed real surfaces $\Sigma$ that admit totally real immersions in $M$ are precisely those having even Euler characteristics. This is, for instance, the case for $M = \mathbb{C}P^2$ and $M = \mathbb{C}P^1 \times \mathbb{C}P^1$.

Corollary 2.10. The class of closed real surfaces admitting totally real embeddings in $\mathbb{C}P^1 \times \mathbb{C}P^1$ consists of the torus $T^2$, the sphere $S^2$, and all nonorientable closed surfaces with even Euler characteristics.
Corollary 2.11. For any fixed $k \in \{2, 3, \ldots, 9\}$, the class of closed real surfaces admitting a totally real embedding in $\mathbb{C}P^2 \# k \mathbb{C}P^2$ consists of the torus $T^2$, the sphere $S^2$, and all nonorientable closed surfaces.

In each of the last three corollaries an existence statement based on an explicit elementary construction is coupled with a nonexistence assertion that uses elementary topological obstructions: an intersection-number relation (5.4) or (5.5), and a condition involving either the first Chern class (for orientable surfaces $\Sigma$), or Stiefel-Whitney classes (in the nonorientable case); see also (18.1).

The Stiefel-Whitney class obstruction in (5.3,a) fails, however, to detect that some nonorientable closed surfaces $\Sigma$ do not admit totally real embeddings in complex surfaces such as $\mathbb{C}^2$, $\mathbb{C}P^2$ or $\mathbb{C}P^2 \# \mathbb{C}P^2$. Instead, we have to use Massey's formula (19.3) phrased in terms of mod 4 intersection numbers and Pontryagin squares. This leads to a proof, in §19, of the next three corollaries.

Corollary 2.12. A closed real surface $\Sigma$ admits a totally real embedding in $\mathbb{C}^2$ if and only if $\Sigma$ is either diffeomorphic to the torus $T^2$, or $\Sigma$ is nonorientable and $\chi(\Sigma) \equiv 0 \mod 4$.

Corollary 2.13. The class of closed real surfaces that admit a totally real embedding in $\mathbb{C}P^2$ consists of the torus $T^2$ and all nonorientable closed surfaces $\Sigma$ with $\chi(\Sigma) \equiv 0 \text{ or } \chi(\Sigma) \equiv 1 \mod 4$.

Corollary 2.14. For the complex surface $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, the closed real surfaces admitting totally real embeddings in $M$ are: the torus $T^2$, and all nonorientable closed surfaces whose first Euler characteristics are odd or divisible by 4.

Corollaries 2.7 and 2.9 - 2.14 are summarized below in Table 1. For a conclusion similar to but weaker than Corollaries 2.13 and 2.14, see Proposition 5.2.

Further such results can be derived from the following more general existence theorem, proved in §12 via another explicit argument (the blow-up construction).

Theorem 2.15. Let $M'$ be the complex surface obtained by blowing up $k$ distinct points, $k \geq 1$, in a given complex surface $M$. The class of closed real surfaces admitting a totally real embedding in $M'$ then includes

(a) the 2-sphere $S^2$, if $k \geq 2$,
(b) the connected sum $\Sigma \# s \text{ RP}^2$, whenever $s \in \{0, 1, \ldots, k\}$ and $\Sigma \subset M$ is any totally real embedded closed surface containing at least $s$ of the $k$ blow-up points.

Another method of obtaining totally real closed surfaces embedded in a given complex surface $M$ consists in deforming its exact "opposite", that is, a holomorphic curve $\Sigma \subset M$, in the direction of a suitable section $\psi$ of the normal bundle $\nu$ of $\Sigma$ in $M$. Since holomorphic deformations are characterized by $\nabla_\psi = 0$, it is not surprising that, if $\nabla_\psi \neq 0$ everywhere, all nearby deformed surfaces will be totally real. This technique, although less "constructive" than the direct geometric proofs of all the above existence assertions except Theorem 2.1, is still much more explicit than invoking the $h$-principle. Specifically, it requires finding sections $\psi$ of complex line bundles $\nu$ over closed Riemann surfaces $\Sigma$ such that $\nabla_\psi \neq 0$ everywhere, that is, $\nabla_\psi$ trivializes the line bundle $\text{Hom}_C(\mathcal{T}_\Sigma, \nu)$ (assumed trivial to begin with). In §21 we use this argument to prove the following result.
Theorem 2.16. Given a pseudoholomorphic immersion/embedding $f$ of a closed oriented real surface $\Sigma$ in an almost complex surface $M$, the following three conditions are equivalent:

(a) the tangent and normal bundles of $\Sigma$ and $f$ are anti-isomorphic as complex line bundles,
(b) $f^*\{\text{det}_C TM\}$ is trivial, that is, $f^*c_1(M) = 0$ in $H^2(\Sigma, \mathbb{Z})$,
(c) $f$ is homotopic through immersions/embbedings $\Sigma \to M$ to a totally real immersion/embedding $f' : \Sigma \to M$.

Moreover, $f'$ in (c) then can be chosen arbitrarily $C^1$-close to $f$.

Here we call an immersion $f : \Sigma \to M$ is a pseudoholomorphic [8] if $df_x(T_x \Sigma)$ is, for each $x \in \Sigma$, a complex subspace of $T_{f(x)}M$.

Blowing up 3d points on a nonsingular degree $d$ curve $Q \subset \mathbb{CP}^2$ and $k - 3d$ points in $\mathbb{CP}^2 \setminus Q$, and using Theorem 2.16, we obtain, in \S 21, the following

Corollary 2.17. For every pair of integers $d, k$ with $d \geq 1$ and $k \geq 3d$, the closed orientable surface of genus $g = (d - 1)(d - 2)/2$ admits a totally real embedding $f$ in any complex surface obtained from $\mathbb{CP}^2$ by successively blowing up $k$ points. Such $f$ may be chosen arbitrarily $C^1$-close to a holomorphic embedding.

Table 1. Totally-real immersibility/embeddability of closed real surfaces $\Sigma$ in the complex surfaces $(0,1)$ with $k \leq 9$. Here t.r. means ‘totally real’ and $\chi_4 \in \{0, 1, 2, 3\}$ stands for $\chi(\Sigma)$ mod 4.

<table>
<thead>
<tr>
<th>the complex surface</th>
<th>$M = \mathbb{CP}^2$</th>
<th>$M = \mathbb{CP}^2 # k \mathbb{CP}^2$</th>
<th>$M = \mathbb{CP}^1 \times \mathbb{CP}^1$</th>
<th>$M = \mathbb{C}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>which orientable $\Sigma$ are t.r. embeddable in $M$</td>
<td>$T^2$ only</td>
<td>$T^2$ only if $k = 1$; $S^2, T^2$ if $2 \leq k &lt; 9$</td>
<td>$S^2, T^2$ only</td>
<td>$T^2$ only</td>
</tr>
<tr>
<td>t.r. embeddability condition for nonorientable $\Sigma$</td>
<td>$\chi_4 \in {0, 1}$</td>
<td>$\chi_4 \neq 2$ if $k = 1$; all $\Sigma$ t.r. embeddable if $2 \leq k &lt; 9$</td>
<td>$\chi(\Sigma)$ even</td>
<td>$\chi(\Sigma)$ divisible by 4</td>
</tr>
<tr>
<td>when a t.r. immersion $\Sigma \to M$ exists</td>
<td>exists for all $\Sigma$ (since it does for $\Sigma = \mathbb{RP}^2$)</td>
<td>if and only if $\chi(\Sigma)$ is even</td>
<td>(just because $M$ is spin)</td>
<td></td>
</tr>
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</table>

3. A $\mathbb{Z}_2$ COHOMOLOGY CONSTRAINT

Given a complex vector space $V$ with $\dim \mathbb{C} V = n \geq 1$, let $\text{TR}(V)$ (or, $\text{TR}^+(V)$) denote the set of all totally real (or, respectively, oriented totally real) vector subspaces of real dimension $n$ in $V$. (See \S 1.) Also, let $\mathbb{RP}(W)$ (or, $S(W)$) be the real projective space (or, sphere) of all real lines (or, respectively, rays) emanating from 0 in any given real vector space $W$. We have natural mappings

$$(3.1) \quad \mathcal{L} : \text{TR}(V) \to \mathbb{RP}(V^{\wedge n}), \quad \mathcal{L}^+ : \text{TR}^+(V) \to S(V^{\wedge n})$$

sending each $W$ in $\text{TR}(V)$ or $\text{TR}^+(V)$ to the real line/ray containing $e_1 \wedge \ldots \wedge e_n$, where $e_1, \ldots, e_n$ is any basis (or, positive-oriented basis) of $W$, and $V^{\wedge n}$ is the $n$th complex exterior power of $V$. Thus, $V^{\wedge n}$ is a complex line, while $\mathbb{RP}(V^{\wedge n})$ and $S(V^{\wedge n})$ are circles. For a totally real immersion $f$ of a real manifold $\Sigma$ in an
almost complex manifold $M$ with $\dim_{\mathbb{R}} \Sigma = \dim_{\mathbb{C}} M = n$, the Maslov invariant $\mathcal{M}(f) \in [\Sigma, E(M)]$ was defined, in §2, to be the homotopy class of the mapping
\begin{equation}
\Theta(f) : \Sigma \rightarrow E(M)
\end{equation}
which sends each $x \in \Sigma$ to the real line $\mathcal{L}(df_x(T_x \Sigma))$ in $[T_{f(x)}M]^\wedge n$, with $\mathcal{L}$ as in (3.1) for $V = T_{f(x)}M$. If, in addition, $\Sigma$ is orientable, $\mathcal{M}(f)$ can be lifted to $E^+(M)$, that is, there is a naturally distinguished homotopy class
\begin{equation}
\mathcal{M}^+(f) \in [\Sigma, E^+(M)],
\end{equation}
whose composite with (1.2) is $\mathcal{M}(f)$. Specifically, a representative of (3.3) is
\begin{equation}
\Theta^+(f) : \Sigma \rightarrow E^+(M),
\end{equation}
obtained by fixing an orientation of $\Sigma$, which does not affect the resulting homotopy class, and then assigning to any $x \in \Sigma$ the ray $\Sigma^+(df_x(T_x \Sigma)) \subset (T_{f(x)}M)^\wedge n$ (notation of (3.1)), with $df_x(T_x \Sigma)$ oriented via $df_x$.

Being a homotopy class of mappings, $\mathcal{M}(f)$ induces a homomorphism
\begin{equation}
[\Theta(f)]_* : \pi_1 \Sigma \rightarrow \pi_1 [E(M)]
\end{equation}
of the fundamental groups (with fixed base points). Under the identifications (1.4) and (1.5), the homomorphism (3.5) coincides with $\theta(f)$ in (2.6). When $\Sigma$ is orientable, (3.2) admits a lift to $E^+(M)$ (such as (3.4)), and so the image of (3.5) lies in the subgroup $\pi_1[E^+(M)]$ appearing in (1.3).

Proposition 3.1. Let $f : \Sigma \rightarrow M$ be a totally real immersion of a real $n$-manifold $\Sigma$ in an almost complex manifold $M$ with $\dim_{\mathbb{C}} M = n$. Then (2.2) holds for the homomorphism $\Theta : H^1(E(M), \mathbb{Z}_2) \rightarrow H^1(\Sigma, \mathbb{Z}_2)$ given by $\Theta = [\mathcal{M}(f)]^* = [\Theta(f)]^*$, with $\Theta(f)$ as in (3.2). In other words, \( \mathfrak{w}_1 \circ [\Theta(f)]_* = \mathfrak{w}_1(\Sigma) \), or, equivalently, the kernel of the orientation homomorphism \( \mathfrak{w}_1(\Sigma) : \pi_1 \Sigma \rightarrow \mathbb{Z}_2 \) coincides with the preimage under (3.5) of the subgroup $\pi_1[E^+(M)]$ in (1.3).

Proof. Let $\Gamma = \text{Ker} \ [\mathfrak{w}_1(\Sigma)]$. Thus, $\Gamma$ is the subgroup of index 1 or 2 in $\pi_1 \Sigma$ formed by all homotopy classes of loops $\gamma : S^1 \rightarrow \Sigma$ for which $\gamma^*(T \Sigma)$ is orientable. Orientability of $\gamma^*(T \Sigma)$ means that the composite $\Theta(f) \circ \gamma$ (see (3.2)) can be lifted to $E^+(M)$ (cf. the discussion following (3.3)), that is, its homotopy class lies in $\pi_1[E^+(M)] = \text{Ker} \ \mathfrak{w}_1$ (see (2.1)). In other words, $\Gamma$ is the kernel of the composite $\mathfrak{w}_1 \circ [\Theta(f)]_*$. The final clause of our assertion now is obvious from (2.1). As $\mathbb{Z}_2$-valued homomorphisms are determined by their kernels, this also gives relation $\mathfrak{w}_1 \circ [\Theta(f)]_* = \mathfrak{w}_1(\Sigma)$ which, according to (1.5), is equivalent to (2.2) for $\Theta = [\mathcal{M}(f)]^*$. This completes the proof. \hfill \Box

Remark 3.2. Let $\Sigma, M$ and $E(M)$ be a closed real manifold, an almost complex manifold and, respectively, the principal $U(1)$-bundle over $M$ defined in §1. Furthermore, let $f : \Sigma \rightarrow M$ be a continuous mapping.

(i) Continuous lifts $\hat{\Theta} : \Sigma \rightarrow E(M)$ of $f$ satisfying (2.2) are nothing else than real-line subbundles of $f^*[\det_{\mathbb{C}} T M]$ isomorphic to $\det_{\mathbb{R}} T \Sigma$.

(ii) A lift (or subbundle) as in (i) exists if and only if the complex line bundle $f^*[\det_{\mathbb{C}} T M]$ over $\Sigma$ is isomorphic to the complexification $[\det_{\mathbb{R}} T \Sigma]^C$.

In fact, a lift $\hat{\Theta}$ of $f$ to $E(M)$ is a selection of a real line $\Theta_x$ in $[\det_{\mathbb{C}} T M]/_{(x)}$ for each $x \in \Sigma$, that is, a real-line subbundle of $f^*[\det_{\mathbb{C}} T M]$, while (2.2) states that this subbundle is isomorphic to $\det_{\mathbb{R}} T \Sigma$. We thus obtain (i), and hence (ii).
4. Proof of Theorem 2.1

Let $\mathcal{P}$ be the set of all homotopy classes of mappings $\Theta : \Sigma \to E(M)$ with (2.2). Assuming (2.3), we denote by $\text{pr} : \tilde{\Sigma} \to \Sigma$ the identity transformation of $\tilde{\Sigma} = \Sigma$ (if $\Sigma$ is oriented), or a twofold covering projection from an oriented surface $\tilde{\Sigma}$ (if $\Sigma$ itself is not orientable). Also, given $(x, \rho) \in \tilde{\Sigma} \times E^+(M)$, with $E^+(M)$ as in §1, let $D[x, \rho]$ be the set of all injective real-linear operators $A : T_x \tilde{\Sigma} \to T_y M$ such that $y \in M$ is the image of $\rho$ under the bundle projection $E^+(M) \to M$ and the image $A(T_x \tilde{\Sigma})$ is a totally real subspace of $T_y M$ satisfying the condition $\mathcal{L}^+(A(T_x \tilde{\Sigma})) = \rho$ for $\mathcal{L}^+$ as in (3.1) with $V = T_y M$, where $A(T_x \tilde{\Sigma})$ is oriented through $A$ and the orientation of $\tilde{\Sigma}$. The Lie group $G_{x, \rho}$ of all complex automorphisms $B$ of $T_y M$ (for $y$ as above) with $\det B \in (0, \infty)$ now acts on $D[x, \rho]$ simply transitively by the left multiplication, giving rise to a homotopy equivalence $D[x, \rho] \approx \text{SU}(2)$.

In view of Proposition 3.1, the assignment $f \mapsto \mathcal{M}(f)$ descends to a mapping $\mathcal{E} \to \mathcal{P}$, where $\mathcal{E}$ is the set of all $\sim_{\text{SU}}$ equivalence classes of totally real immersions $\Sigma \to M$. To see that $\mathcal{E} \to \mathcal{P}$ is surjective, we fix a continuous mapping $\Theta : \Sigma \to E(M)$ satisfying (2.2), that is, having a continuous lift $\Theta^+ : \tilde{\Sigma} \to E^+(M)$ (whose composite with the projection (1.2) equals $\Theta \circ \text{pr}$). Using a CW-decomposition of the surface $\Sigma$ and relation $D[x, \rho] \approx \text{SU}(2)$, we see that the locally trivial bundle $Z$ over $\tilde{\Sigma}$ with the fibres $D[x, \Theta^+(x)]$ for $x \in \tilde{\Sigma}$, admits a global continuous section. Surjectivity now follows from the $h$-principle for totally real immersions [8, p. 192].

Finally, to show that our mapping $\mathcal{E} \to \mathcal{P}$ is injective, let us consider two totally real immersions $f, f' : \Sigma \to M$ with $\mathcal{M}(f) = \mathcal{M}(f')$. We may thus choose a homotopy $[0, 1] \ni t \mapsto \Theta_t$ between $\Theta_0 = \Theta(f)$ and $\Theta_1 = \Theta(f')$, defined as in (3.2), which, by (2.2) for $\Theta = [\mathcal{M}(f)]^+$, can be lifted to a homotopy $\Theta^+_t$ between $\Theta^+(f)$ and $\Theta^+(f')$ (notation of (3.4)). If $Y$ now is the locally trivial bundle over $\tilde{\Sigma} \times [0, 1]$ with the fibres $D[x, \Theta^+_t(x)]$ for $(x, t) \in \tilde{\Sigma} \times [0, 1]$, a CW-decomposition argument and relation $D[x, \rho] \approx \text{SU}(2)$ show, as before, that $Y$ admits a global continuous section which coincides with $df$ and $df'$ on $\tilde{\Sigma} \times \{0\}$, and, respectively, $\tilde{\Sigma} \times \{1\}$. Thus, according to the $h$-principle [8, p. 192], $f$ and $f'$ are homotopic through totally real immersions $\Sigma \to M$, which completes the proof of Theorem 2.1.

Note that, as $\text{SU}(3)$ is 2-connected, the surjectivity part of the above argument is still valid when the real/complex dimension $n = 2$ is replaced by $n = 3$.

5. Topological obstructions

Let $f : \Sigma \to M$ be a totally real immersion into an almost complex manifold $M$ with $\dim_{\mathbb{R}} \Sigma = \dim_{\mathbb{C}} M = n$. The multiplication by $i$ obviously provides an isomorphic identification between the tangent bundle $\tau$ of $\Sigma$ (treated as a subbundle of $f^*TM$) and the normal bundle $\nu$ of $f$, so that

$$(5.1) \quad \tau = (-1)^{n(n-1)/2} \nu, \quad \text{where} \quad \tau = df(T\Sigma) \quad \text{and} \quad \nu = [f^*TM]/\tau.$$  

The factor $(-1)^{n(n-1)/2}$ represents the orientation if $\Sigma$ is oriented, and is to be ignored otherwise. Thus, the isomorphism $\tau \approx \nu$ in (5.1) is orientation-preserving if and only if $n \equiv 0$ or $n \equiv 1 \mod 4$. (Both $\tau$ and $\nu$ carry natural orientations if $\Sigma$ is oriented; see Remark 5.1 below.)
Remark 5.1. Every almost complex manifold carries a natural orientation. Specifically, an $n$-dimensional complex vector space $V$ ($1 \leq n < \infty$) becomes an oriented real vector space if one declares the real basis $e_1, ie_1, \ldots, e_n, ie_n$ to be positive oriented for some (or any) complex basis $e_1, \ldots, e_n$. With this convention, the effect on the orientation of the direct sum operation for complex spaces agrees with that for oriented real spaces. For the oriented totally real subspace $W = \text{Span}_R\{e_1, \ldots, e_n\}$ of $V$, the orientations of the “normal spaces” $W' = iW$ and $V/W$ obtained using the direct-sum requirement (that $V = W \oplus W'$ as oriented spaces) make the projection isomorphism $V' \to V/W$ orientation-preserving. The same applies, fibre by fibre, when $V$ and $W$ are replaced by $TM$ (or $f^*TM$) and $T\Sigma$ (or $\tau = df(T\Sigma)$) for any totally real immersion $f : \Sigma \to M$ into an almost complex manifold $M$ with $\dim_R \Sigma = \dim_C M = n$. The isomorphisms $W \to W^\perp$ and $\tau \to \nu$ of multiplication by $i$ thus affect the orientation via the sign factor of $(-1)^{n(n-1)/2}$, as stated in (5.1).

Let $f : \Sigma \to M$ be a totally real immersion of a $k$-dimensional real manifold $\Sigma$ into an almost complex manifold $M$ of complex dimension $n$, so that $k \leq n$. (See, however, Remark 1.4.) We then also have an obvious isomorphic identification $\text{Span}_C \tau = [T\Sigma]^C$ of complex vector bundles over $\Sigma$, with $\tau \subset f^*TM$ as in (5.1), and $T\Sigma^C$ denoting complexification. In fact, since $f$ is an immersion, $\tau$ is isomorphic to $T\Sigma$, while $\text{Span}_C \tau = \tau \oplus i\tau \approx \tau^C$ as $f$ is totally real.

If, in addition, $\Sigma$ is closed and $\dim_R \Sigma = \dim_C M = n$, relations $\text{Span}_C \tau = [T\Sigma]^C$ reads $\text{Span}_C [d(f(T\Sigma))] = f^*TM$ and, followed by the operation $\det_C$, gives natural isomorphic identifications

\begin{equation}
(5.2) \quad \text{i) } f^*TM = [T\Sigma]^C, \quad \text{ii) } f^* [\det_C TM] = [\det_R T\Sigma]^C.
\end{equation}

Here and in the sequel, given a real manifold $\Sigma$ with $\dim_R \Sigma = n$ (or, an almost complex manifold $M$ with $\dim_C M = n$), we will denote by $\det_R T\Sigma = [T\Sigma]^\wedge n$ or $\det_C TM = [TM]^\wedge n$ the determinant bundle of the tangent bundle $TM$, that is, its highest real/complex exterior power. Taking $w_2$ (or $c_1$) of (5.2ii) and noting that $\det_R T\Sigma$ is trivial if $\Sigma$ is orientable, we obtain

\begin{align}
(5.3) \quad & \text{a) } f^*[w_2(M)] = w_1(\Sigma) \cdot w_1(\Sigma) \quad \text{in } H^2(\Sigma, \mathbb{Z}_2), \\
& \text{b) } f^*[c_1(M)] = 0 \quad \text{in } H^2(\Sigma, \mathbb{Z}) \quad \text{whenever } \Sigma \text{ is orientable,}
\end{align}

If, in addition, $f : \Sigma \to M$ is a totally real embedding and the closed manifold $\Sigma$ with $\dim_R \Sigma = \dim_C M = n$ is orientable, we have

\begin{equation}
(5.4) \quad f_*[\Sigma] \cdot f_*[\Sigma] = (-1)^{n(n-1)/2} \chi(\Sigma),
\end{equation}

where $f_*[\Sigma] \in H_n(M, \mathbb{Z})$ corresponds to either fixed orientation of $\Sigma$ and the dot denotes the $\mathbb{Z}$-valued intersection form. In fact, the Euler class $e(\nu)$ of the normal bundle $\nu = \nu_f$ of any embedding $f : \Sigma \to M$, integrated over $\Sigma$, yields the self-intersection number of $f_*[\Sigma]$ in $H_n(M, \mathbb{Z})$, while for totally real embeddings $f$, (5.1) gives $\int_\Sigma e(\nu) = (-1)^{n(n-1)/2} \chi(\Sigma)$. When $\Sigma$ is not assumed orientable, instead of (5.4) this argument gives

\begin{equation}
(5.5) \quad f_*[\Sigma] \cdot f_*[\Sigma] = [\chi(\Sigma) \mod 2],
\end{equation}

where, this time, $f_*[\Sigma] \in H_n(M, \mathbb{Z}_2)$ and $\cdot$ takes values in $\mathbb{Z}_2$. 
Proposition 5.2. Let a closed, orientable manifold \( \Sigma \) of even real dimension \( n \geq 2 \) admit a totally real embedding in \( M = \mathbb{CP}^n \) or in the complex manifold \( M = \mathbb{CP}^n \# \overline{\mathbb{CP}^n} \) obtained from \( \mathbb{CP}^n \) by blowing up a point.

(i) If \( M = \mathbb{CP}^n \), then \((-1)^{n/2} \chi(\Sigma) \geq 0 \) and either \( \chi(\Sigma) \equiv 0 \mod 4 \) or \( \chi(\Sigma) \equiv 1 \mod 4 \).

(ii) If \( M = \mathbb{CP}^n \# \overline{\mathbb{CP}^n} \), then \( \chi(\Sigma) \) is either odd, or divisible by 4.

Thus, if \( n \) is even, \( S^n \) admits no totally real embedding in \( \mathbb{CP}^n \) or \( \mathbb{CP}^n \# \overline{\mathbb{CP}^n} \).

Proof. The (quadratic) intersection form in \( H^n(M, \mathbb{Z}) \) is algebraically equivalent to \( \mathbb{Z} \oplus \mathbb{Z} \) or \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) (Proposition 3.1) states that \( w_1(\Sigma) \) equals (3.5) followed by the reduction homomorphism \( \mathbb{Z}_q \to \mathbb{Z}_2 \), which coincides with the mod 2 reduction of \( i(f) \) (see Remark 1.1). Thus, \( i(f) \in J_q(\Sigma) \). Finally, when \( n = 2 \), (5.3) and (1.7) give \( d(f) \in D_\Sigma(M) \) with \( \varepsilon, \pm \) as in (2.8), completing the proof. \( \square \)

Let \( f : \Sigma \to M \) now be a continuous mapping of a closed real surface \( \Sigma \) into an almost complex surface \( M \). Conditions (5.3) are not only necessary for \( f \) to be homotopic to a totally real immersion \( \Sigma \to M \), but also sufficient.

This is clear from Theorem 2.1 and the previous comment, since complex line bundles over surfaces are classified by their \( c_1 \) and \( w_2 \).

6. Proof of Theorem 2.2

We begin by establishing relations (2.7).

Lemma 6.1. Let \( f : \Sigma \to M \) be a totally real immersion of a closed real surface \( \Sigma \) in a simply connected almost complex surface \( M \). Then (2.7) holds for \( q, i(f), d(f), J_q(\Sigma), D_\Sigma(M), \varepsilon \) and \( \pm \) as in (1.4), (2.5), (2.6) and (2.8) - (2.10).

Proof. As \( i(f) \) is the mapping (3.5), while \( \pi_1[E(\Sigma)^+] = 2\mathbb{Z}_q \subset \mathbb{Z}_q \) by (1.3) - (1.4), \( w_1 \) in (2.1) must be the unique nonzero homomorphism \( \mathbb{Z}_q \to \mathbb{Z}_2 \), and so relation \( w_1 \circ \theta(f) = w_1(\Sigma) \) (Proposition 3.1) states that \( w_1(\Sigma) \) equals (3.5) followed by the reduction homomorphism \( \mathbb{Z}_q \to \mathbb{Z}_2 \), which coincides with the mod 2 reduction of \( i(f) \) (see Remark 1.1). Thus, \( i(f) \in J_q(\Sigma) \). Finally, when \( n = 2 \), (5.3) and (1.7) give \( d(f) \in D_\Sigma(M) \) with \( \varepsilon, \pm \) as in (2.8), completing the proof. \( \square \)

Proof of Theorem 2.2. By Lemma 6.1, the mapping \( [f] \mapsto (i, d) \) sending the \( \sim_{\text{tri}} \) equivalence class \([f]\) of a totally real immersion \( f : \Sigma \to M \) to \((i(f), d(f))\) takes values in the set \( J_q(\Sigma) \times D_\Sigma(M) \).

To prove injectivity of \([f] \mapsto (i, d)\), consider two totally real immersions \( f, f' \) of \( \Sigma \) in \( M \) having the same Maslov index \( i \) and degree \( d \). As \( d(f) \) with (2.5) uniquely determines the homotopy class of \( f \) (see [13]), we may lift a fixed homotopy between \( f \) and \( f' \) to the principal \( U(\mathbb{1}) \)-bundle \( E(\Sigma) \) over \( M \), so as to obtain a homotopy between \( \Theta(\Sigma) : \Sigma \to E(\Sigma) \) defined as in (3.2) and some lift \( \Theta' \) of \( f \) to \( E(M) \). Since \( \Theta' = \Theta(f) \) is also a lift of \( f \), we have \( \Theta' = g\Theta \) for some continuous mapping \( g : \Sigma \to U(\mathbb{1}) \), and so (1.8) with \( j(\Theta(f)) = i(f) \) gives \( i(f') = i(f) + [g \mod q] \), that is, \( [g \mod q] = 0 \). We can thus build a homotopy between \( \Theta \) and \( \Theta' \) on successive skeleta of a fixed CW-decomposition of \( \Sigma \), with the 1-skeleton step possible in view of Remark 1.2(ii) (since, if \( q < \infty \), the \( q \)th power of any element of the fundamental group of \( E(M) \) is trivial, by (1.4)), and the 2-skeleton step due to (5.3). Injectivity of \([f] \mapsto (i, d)\) now follows.

Proving its surjectivity amounts in turn to showing that its image \( \mathcal{Z} \) has the properties listed in the lines following Theorem 2.2. To this end, let us fix a pair
$(i, d) \in \mathcal{J}_q(S) \times \mathbb{D}^1(M)$. First, since $M$ is simply connected, the universal coefficients and Hurewicz theorems allow us to realize $d$ (or, more generally, any class in $H_2(M, \mathbb{Z}[\beta])$) by a mapping $\tilde{\Sigma} \to M$ and, with the aid of a degree 1
map $\Sigma \to \tilde{\Sigma}$, also by a mapping $f : \Sigma \to M$. However, $d = d(f)$ lies in
$\mathcal{D}_q^1(M)$ rather than just in $H_2(M, \mathbb{Z}[\beta])$. Therefore, by (2.9), (2.8) and (1.7),
either $\tilde{\Sigma}$ is orientable and $f^*[\det_{TM}]$ is trivial, or, if $\tilde{\Sigma}$ is nonorientable, the
line bundles $f^*[\det_{TM}]$ and $[\det_{R \Sigma}]^C$ have the same $w_2$. Thus, in either
case, $f^*[\det_{TM}]$ and $[\det_{R \Sigma}]^C$ are isomorphic [12, p. 798]. Hence, according
to Remark 3.2(ii), $f$ admits a continuous lift $\Theta : \Sigma \to E(M)$ with (2.2). Once
$\Theta$ is fixed, every such lift of $f$ has the form $g\Theta$ for some continuous mapping
$g : \Sigma \to U(1)$ (notation as in (1.8)) which also has the property that the homomorphism
$[g \mod q] : \pi_1\Sigma \to \mathbb{Z}_q$ defined in Remark 1.2 takes values in the even
subgroup $2\mathbb{Z}_q$ of $\mathbb{Z}_q$. (Evenness follows from (2.2) for both $\Theta$ and $g\Theta$, as $\Theta^*$ in
(2.2) is dual to $j(\Theta)$ in (1.8).) It is now clear from Theorem 2.1, (2.2) and an
obvious homotopy-lifting argument that our fixed pair $(i, d)$ lies in the image $\mathcal{Z}$ of $[f]$ in
if and only if $i$ lies in a specific coset, in $\text{Hom}(\pi_1\Sigma, 2\mathbb{Z}_q)$, of the
subgroup $\mathcal{G}$ formed by those of its elements which are also equal to $[g \mod q]$ for
some continuous mapping $g : \Sigma \to U(1)$. (That coset is in turn contained in the
set $\mathcal{J}_q(S)$, which is here assumed nonempty, and so, by (2.10) and Remark 1.1, is
itself a coset of $\text{Hom}(\pi_1\Sigma, 2\mathbb{Z}_q)$ in $\text{Hom}(\pi_1\Sigma, \mathbb{Z}_q) = H^1(S, \mathbb{Z}_q)$.

We now have $\mathcal{G} = \text{Hom}(\pi_1\Sigma, 2\mathbb{Z}_q)$ (and hence $\mathcal{Z} = \mathcal{J}_q(S) \times \mathbb{D}^1(M)$) if $\Sigma$ is
orientable, or $\chi(\Sigma)$ is odd, or $q$ is infinite, or, finally, $q$ is finite but not divisible
by 4. In all four cases the reason is that, for every closed surface $\Sigma$, continuous mappings $g : \Sigma \to U(1)$ realize every homomorphism $\pi_1\Sigma \to \mathbb{Z}$ of the fundamental
groups, while, in each of the four cases, a homomorphism $H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}_q$
valued in $2\mathbb{Z}_q$ is necessarily the composite $H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z}_q$ of some homomorphism
$H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$ and the projection $\mathbb{Z} \to \mathbb{Z}_q$. In the first and third cases, this is obvious since $H_1(\Sigma, \mathbb{Z})$ is free, or, respectively, $\mathbb{Z}_\infty = \mathbb{Z}$. In the second and
fourth cases, we may assume that $\Sigma$ is nonorientable: thus, $H_1(\Sigma, \mathbb{Z})$ has just one
nontrivial element of finite order, and its order is 2. If $\chi(\Sigma)$ is odd, that unique
element is not in the kernel of $w_1(S)$ (as it is realized by an embedded circle with
a nontrivial normal bundle), and so its image under any homomorphism that lies in
$\mathcal{J}_q(S)$, being odd and or order 2 in $\mathbb{Z}_q$, necessarily equals $q/2$, which shows that its image under the difference of two elements of $\mathcal{J}_q(S)$ is $0 \in \mathbb{Z}_q$, as required. In
the fourth case, the image is 0 as well, since the only nontrivial element of order $2$ in $\mathbb{Z}_q$, namely, $q/2$, is odd.

Let us now suppose that $\Sigma$ is nonorientable, $\chi(\Sigma)$ is even, and $q$ is a (finite)
multiple of 4. A homomorphism $H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}_q$ valued in $2\mathbb{Z}_q$ may send the
unique nontrivial element of finite order in $H_1(\Sigma, \mathbb{Z})$ to 0 or to $q/2$, and only those homomorphisms sending it to 0 have a factorization $H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z}_q$ as above, that is, lie in $\mathcal{G}$. Hence $\mathcal{G}$ is contained in $\text{Hom}(\pi_1\Sigma, 2\mathbb{Z}_q)$ as a subgroup of index 2. This completes the proof.

Remark 6.2. Two totally real immersions $f, f' : \Sigma \to M$ of a closed real surface
$\Sigma$ into an almost complex surface $M$ which are homotopic must also be $C^1$-
homotopic through $C^1$ immersions. In fact, let us denote by $\eta = f^* TM$ the
pullback of the complex bundle $TM$ under $f$ and identify it with the pullback
of $TM$ under $f'$. All we need to do is show that the totally real monomorphisms
represent a single homotopy class of monomorphisms $T\Sigma \to \eta$. The latter are
sections of a bundle over $\Sigma$ with fibre $S^3 \times S^2$, in which the former are those sections lying in a subbundle with fibre (homotopically) $S^3 \times S^1$. The canonical homotopic trivialization of $S^3$ within $S^2$ now makes a totally real monomorphism appear as a section of an $S^3$ bundle over $\Sigma$, that is, a unit section of a rank 2 complex vector bundle $\zeta$ with a Hermitian fibre metric. However, any two such sections $\psi, \phi$ are homotopic. Namely, let $\text{SU}(\zeta)$ be the bundle over $\Sigma$ whose fibre over each $x \in \Sigma$ is the group $\text{SU}(\zeta_x)$ of all unimodular $C$-linear isometries $\zeta_x \to \zeta_x$. Due to simple transitivity of the action of $\text{SU}(\zeta_x)$ on the unit sphere in $\zeta_x$, there exists a unique section $A$ of $\text{SU}(\zeta)$ with $\phi = A\psi$. Moreover, $A$ is homotopic to the identity section of $\text{SU}(\zeta)$, since the fibre of $\text{SU}(\zeta)$ (the sphere $S^3$) is 2-connected, and so the required homotopy can be chosen successively over the 0-, 1- and 2-skeleta of $\Sigma$.

7. THE SIMPLEST EXAMPLES

We will repeatedly use the following constructions of totally real immersions.

Example 7.1. A real subspace $W$ of a complex vector space $V$ with $\dim V < \infty$ is totally real if and only if $\text{Span}_C W$ in $V$ has the complex dimension $\dim_R W$. Equivalently, this means that some (or, every) $R$-basis of $W$ is also linearly independent over $C$ in $V$. Thus, given complex-valued $C^1$ functions $f_1, \ldots, f_n$ on an open set $U$ in $R^n$, the mapping $f = (f_1, \ldots, f_n)$ is a totally real immersion $U \to C^n$ if and only if $J(f_1, \ldots, f_n) \neq 0$ at every $x \in U$, where $J(f_1, \ldots, f_n) = \det \mathfrak{J}$ for the complex $n \times n$ Jacobian matrix $\mathfrak{J} = F(x)$ with the entries $\partial f_j/\partial x_k$.

Example 7.2. Obviously, an embedding $f$ of a real manifold $\Sigma$ in an almost complex manifold $M$ is totally real if and only if so is the image $f(\Sigma)$ as a submanifold of $M$.

Example 7.3. Let $\Sigma = \{(z, v) \in \tilde{U} \times V : v = \varphi(z)\}$ be the graph of a $C^\infty$ mapping $\varphi : \tilde{U} \to V$ from a connected open set $\tilde{U} \subset C$ into a complex vector space $V$ with $\dim V < \infty$. Then $\Sigma$ is a totally real submanifold of $\tilde{U} \times V$ if and only if $\varphi$ satisfies, at each point of $U$, the “Cauchy-Riemann inequality” $\varphi_z \neq 0$, where $\varphi_z = (\varphi_x + i\varphi_y)/2$ with $z = \text{Re} z, y = \text{Im} z$, and the subscripts stand for the partial derivatives.

In fact, $\Sigma$ is totally real in $\tilde{U} \times V$ if and only if the mapping $\tilde{U} \to \tilde{U} \times V$ given by $z \mapsto (z, \varphi(z))$ is totally real, that is (cf. Example 7.1), if the vectors $(1, \varphi_x)$ and $(i, \varphi_y)$ are linearly independent at every $z \in \tilde{U}$, which amounts to $\varphi_z \neq 0$ everywhere.

Example 7.4. Given a complex vector space $V$ with $\dim_C V = 2$, any real subspace $W \subset V$ with $\dim_R W = 2$ which is not totally real must, obviously, be a complex 1-dimensional subspace of $V$. Suppose now that $\Sigma$ is a submanifold of an almost complex manifold $M$ and $\dim_R \Sigma = 2$. Removing from $\Sigma$ all complex points, that is, those $x \in \Sigma$ for which $T_x \Sigma$ is a complex line in $T_x M$, we clearly obtain an open subset $U$ of $\Sigma$ and, if $U$ is nonempty, each of its connected components is a totally real submanifold of $M$.

Totally real submanifolds naturally arise in many other common situations. For instance, an embedded real submanifold $\Sigma \subset M$ of an almost complex manifold $M$ is totally real in each of the seven obvious cases, listed below.
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(i) \( \dim \Sigma = 1 \).

(ii) \( \Sigma \) is a connected component of the fixed-point set \( \{ x \in M : \tau = x \} \) of any \( C^\infty \) involution \( M \ni x \mapsto \tau \in M \) reversing the almost complex structure. One then has \( \dim_\mathbb{R} \Sigma = \dim_\mathbb{C} M \).

(iii) \( N \) is an almost complex manifold admitting an involution \( x \mapsto \tau \) that reverses the almost complex structure and \( M \) is the product almost complex manifold \( N \times N \), while \( \Sigma \subset M \) is the “anti-diagonal” \( \{(\tau, x) : x \in N \} \), diffeomorphic to \( N \). (This is clear from (ii) applied to the involution \( (x, y) \mapsto (\tau, \tau) \) of \( N \times N \).)

(iv) \( \Sigma = \Sigma' \times \Sigma'' \), where \( \Sigma' \subset M' \) and \( \Sigma'' \subset M'' \) are totally real embedded submanifolds, and \( M \) is the product almost complex manifold \( M' \times M'' \). (For a more general construction, see Lemma 26.2.)

(v) By (i) and (iv), embedded closed curves \( K, K' \subset C \) give rise to a Clifford-like totally real embedded 2-torus \( \Sigma = K \times K' \subset M = C^2 \). Iterating this produces totally real embedded \( n \)-tori in \( C^n \).

(vi) The standard “real form” \( \Sigma = \mathbb{R}P^n \subset M = \mathbb{C}P^n \) is a totally real embedded submanifold, which is a special case of (ii) with the involution given in projective coordinates, by \( \mathbb{C}P^n \ni [x_0, \ldots, x_n] \mapsto [\bar{x}_0, \ldots, \bar{x}_n] \in \mathbb{C}P^n \).

(vii) Denoting by \( x \mapsto \tau \) any antiholomorphic involution of \( S^2 = \mathbb{C}P^1 \) (e.g., the complex conjugation in \( C \) extended to the Riemann sphere), and applying (iii), we see that the “anti-diagonal” \( \Sigma = \{(\tau, x) : x \in S^2 \} \) is a totally real embedded 2-sphere in \( M = S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1 \).

8. ZOOMING

We now establish what might be called a zooming principle.

**Proposition 8.1.** If a compact manifold \( \Sigma \) admits a totally real immersion/embedding \( f \) in \( C^n \), then it admits a totally real immersion/embedding \( f' \) in every almost complex manifold \( M \) of complex dimension \( n \). More precisely, we may choose \( f' \) to be the composite of \( f \) with a suitable \( C^\infty \)-diffeomorphic embedding in \( M \) of an open ball in \( C^n \) containing \( f(\Sigma) \).

Proposition 8.1 trivially follows from Lemma 8.3 below (see Remark 8.4). First, we need a definition and another lemma.

Given an immersion \( \Phi : K \rightarrow V \) of a (real) \( n \)-dimensional manifold \( K \) into a real or complex vector space \( V \), the corresponding Gauss mapping

\[
G_\Phi : K \rightarrow \text{Gr}_n(V)
\]

assigns to each \( \xi \in K \) the image \( d\Phi_\xi(T_\xi K) \). Here \( \text{Gr}_n(V) \) is the Grassmann manifold of all \( n \)-dimensional real vector subspaces of \( V \).

**Lemma 8.2.** Let \( J \) be a complex structure in a real vector space \( V \), that is, a linear operator \( V \rightarrow V \) with \( J^2 = -\text{Id} \), and let \( Y \) be a set of its \( J \)-totally real subspaces of a fixed dimension \( n \geq 0 \). If \( \dim V < \infty \) and \( Y \) is a compact as a subset of \( \text{Gr}_n(V) \), then all \( W \in Y \) are totally real relative to every complex structure that lies in a suitable neighborhood of \( J \) in \( \text{Hom}_R(V, V) \).

In fact, otherwise there would exist a sequence \( J_k \in \text{Hom}_R(V, V) \) of complex structures and sequences \( W_k \in Y \) and \( u_k \in V \) such that \( J_k \rightarrow J \) as \( k \rightarrow \infty \), while \( u_k \in W_k \cap J_k W_k \) and \( |u_k| = 1 \) for some fixed Euclidean norm \( || \) in \( V \). Using compactness of \( Y \) and the unit sphere, we could pass to subsequences for which
$W_k \to W$ and $u_k \to u$ with some $W \in Y$ and $u \in V$, so that $u \in W \cap JW$ and $|u| = 1$, contradicting the assumption that $W \cap JW = \{0\}$ for all $W \in Y$.

**Lemma 8.3.** Let there be given an almost complex manifold $M$, a point $y$ in $M$, a real manifold $\Sigma$, a neighborhood $U'$ of 0 in $T_y M$, as well as $C^\infty$ mappings $\Phi : \Sigma \to T_y M$ and $F : U' \to M$ such that $\Phi$ is a totally real immersion/embedding of $\Sigma$ into the complex vector space $T_y M$, while $F(0) = y$ and $dF_0$ is the identity mapping of $T_y M$. If, in addition, the image $Y$ of the Gauss mapping (8.1), with $V = T_y M$, is compact, and $\varepsilon : T_y M \to T_y M$ denotes the multiplication by $\varepsilon \in \mathbb{R}$, then, for some neighborhood $U$ of 0 in $T_y M$ contained in $U'$ and all sufficiently small $\varepsilon > 0$, the composite

$$F \circ \varepsilon \circ \Phi : \Phi^{-1}(\varepsilon^{-1}U) \to M$$

is a totally real immersion/embedding in $M$ of the open subset $\Phi^{-1}(\varepsilon^{-1}U)$ of $\Sigma$.

**Proof.** Let $U' \ni x \mapsto J(x) \in \text{Hom}_\mathbb{R}(V, V)$, with $V = T_y M$, be the $F$-pullback to $U'$ of the original almost complex structure in $M$, and let $U \subset U'$ be a neighborhood of 0 in $V$ such that and $J(x) \in \Omega$ for all $x \in U$, with $\Omega$ obtained by applying Lemma 8.2 to our $Y$ and $J = J(0)$. Obviously, $d\varepsilon v(W) = W$ whenever $\varepsilon \neq 0$, $v \in V$ and $W$ is a real vector subspace of $V = T_y V = T_{x_0} V$. Hence $\varepsilon \circ \phi : \Phi^{-1}(\varepsilon^{-1}U) \to U$ is a totally real immersion/embedding relative to the almost complex structure pulled back from $M$ via $F$. This completes the proof. \qed

**Remark 8.4.** Lemma 8.3 becomes particularly simple when $\Sigma$ is compact (and hence so is $Y$). Then, clearly, $\varepsilon(\Phi(\Sigma)) \subset U$ for all sufficiently small $\varepsilon > 0$, so that (8.2) is a totally real immersion/embedding of $\Sigma$ in $M$.

9. Immersed spheres

The immersions of spheres, described below, go back to Whitney. See [14, 11].

**Lemma 9.1.** Let $||$ be a fixed Euclidean norm in a totally real subspace $W$ of a complex vector space $V$ with $\dim_W = \dim_C V$, and let $S$ denote the sphere of some radius $r > 0$ in $W$, centered at 0. Furthermore, let $K \subset C \setminus \{0\}$ be an embedded $C^\infty$ curve with $0 \notin K + K$. Then the mapping $f : K \times S \to V$, given by $f(\lambda, v) = \lambda v$, is a totally real embedding.

**Proof.** The differential of $f$ at $(\lambda, v)$ sends $(\lambda, \dot{v}) \in T_\lambda K \times T_v S \subset C \times W$ onto $\dot{\lambda} \dot{v} + \lambda \dot{v}$, and so it transforms an $R$-basis of $T_\lambda K \times T_v S$, each of whose vectors $(\lambda, \dot{v})$ has either $\lambda = 0$ or $\dot{v} = 0$, onto a $C$-basis of $V$ (cf. Example 7.1). Thus, $f$ is a totally real immersion. On the other hand, injectivity of $f$ is clear since relation $\lambda \dot{v} = \mu w$ with $\lambda, \mu \in K$ and $v, w \in S$ implies $v = \lambda^{-1} \mu w$ and so, as $W$ is totally real and $|v| = |w| = r$, we have $\mu = \pm \lambda$ (see Example 7.1 again), while $\mu \neq -\lambda$ as $0 \notin K + K$. This completes the proof. \qed

Any fixed Euclidean norm in a totally real subspace $W$ of a complex vector space $V$ with $\dim_W = \dim_C V = n$ gives rise to the group $\text{SO}(W) \approx \text{SO}(n)$ of all orientation-preserving linear isometries of $W$, with the inclusion $\text{SO}(W) \subset \text{GL}(V)$ obtained by extending operators $W \to W$ complex-linearly to $V$.

**Proposition 9.2.** Let $W$ and $L$ be a totally real subspace and a complex subspace of a complex vector space $V$ with $\dim_W = \dim_C V = n$ and $\dim_C L = 1$. Next, let $\Gamma \subset L$ be a compact set such that $0 \in \Gamma$ and $\Gamma \setminus \{0\}$ is a 1-dimensional $C^\infty$
submanifold of $L$ with $v+v' \neq 0$ whenever $v, v' \in \Gamma \setminus \{0\}$, while the intersection of
$\Gamma$ with some neighborhood of $0$ in $L$ consists of two non-parallel real-line segments
eating from $0$. Finally, let $Q = \text{SO}(W) \Gamma = \{Ax : A \in \text{SO}(W), x \in \Gamma\}$,
with $\text{SO}(W) \subset \text{GL}(V)$ as above for any fixed Euclidean norm $\|\cdot\|$ in $W$.
Then $Q$ is a totally real $n$-sphere immersed in $V$. It has just one self-
intersection in the form of a double point at $0$, and its two tangent spaces $T, T'$ at $0$
related by $T' = zT$ for some $z \in C \setminus R$.

Proof. Let us fix $u \in W \setminus \{0\}$. Now $\Gamma \setminus \{0\} = K u$ for some embedded $C^\infty$ curve
$K \subset C \setminus \{0\}$, which approaches $0$ along two real-line segments
$(0,1)a$ and $(0,1)b$, where $aC \setminus \{0\}$, $b \in C$ and $b/a \notin R$. For the sphere $S$ of
radius $|u|$ in $W$, centered at $0$, we have $Q \setminus \{0\} = KS$, as $Q \setminus \{0\} = \text{SO}(W)Ku =
K \text{SO}(W)u$. Thus, in view of Lemma 9.1, $Q \setminus \{0\}$ is a totally real submanifold
of $V$, diffeomorphic to $K \times S$, that is, to $S^n$ minus two points. On the other
hand, because of how $K$ approaches $0$ in $C$, a neighborhood of $0$ in $Q$ is the
union of two open $n$-balls centered at $0$ in the totally real subspaces $T = aW$ and
$T' = bW$ spanned by $\text{SO}(W)au$ and $\text{SO}(W)bu$. This completes the proof. \qed

As an obvious consequence of Remark 8.4 and Propositions 8.1 and 9.2, we obtain

**Corollary 9.3.** Given an almost complex manifold $M$ with $\dim_{C} M = n \geq 2$,
a point $y \in M$, and a neighborhood $U$ of $y$ in $M$, there exists a totally real
immersion of the $n$-sphere in $U$ which has a double point with a transverse self-
intersection at $y$, and no other multiple points.

10. **Totally real blow-ups**

Let $M'$ be the complex manifold, diffeomorphic to $M \# \overline{CP^n}$, obtained by blowing
up a point $y$ in a given complex manifold $M$ with $n = \dim_{C} M$.

Totally real immersions/submanifolds in $M'$ may be constructed using various
given totally real immersions/submanifolds in $M$. We discuss three different
versions of such a procedure, two of which are summarized in the following lemma;
the third one will be presented, for $n = 2$ only, in §11.

**Lemma 10.1.** Let there be given a complex manifold $M$ with $\dim_{C} M = n$, a point
$y \in M$, and a totally real submanifold $\Sigma \subset M$ with $\dim_{R} \Sigma = k$ which is closed
as a subset of $M$ and carries the subset topology. Denoting by $M' = M \# \overline{CP^n}$
the complex manifold obtained from $M$ by blowing up the point $y$, we have

(a) if $y \notin \Sigma$, then $\Sigma$ is also totally real as a submanifold of $M' = M \# \overline{CP^n}$,
(b) if $y \in \Sigma$, the closure $\Sigma'$ of $\Sigma \setminus \{y\}$ in $M'$ is a totally real submanifold
of $M'$, obtained through the real blow-up of $y$ in $\Sigma$, and hence diffeomorphic
to $\Sigma \# \overline{RP^k}$.

Proof. (a) is obvious. As for (b), let us fix holomorphic coordinates $z_a$ in $M$,
a = 1, \ldots, n$ and $C^\infty$ coordinates $x_j$ in $\Sigma$, $j = 1, \ldots, k$, both defined near $y$
and satisfying, at $y$, the equalities $z_a = x_j = 0$ and $\partial z_a/\partial x_j = \delta_{aj}$ for all $a, j$. (This
amounts to choosing the vectors $\partial/\partial z_a$ at $a, j = 1, \ldots, k$, equal to the $\partial/\partial x_j$,
and can be achieved by applying linear changes to any given coordinate systems at
$y$, cf. Example 7.1.) We then obtain local coordinates $r, \xi_2, \ldots, \xi_k$ and $\lambda \xi_2, \ldots, \xi_k$,
for the blown-up manifolds $\Sigma' = \Sigma \# \overline{RP^k}$ and $M' = M \# \overline{CP^n}$ related to the
$z_a$ and $x_j$ by $(x_1, \ldots, x_k) = (r, r \xi_2, \ldots, r \xi_k)$ and $(z_1, \ldots, z_n) = (\lambda, \lambda \xi_2, \ldots, \lambda \xi_n)$,
with either of $r \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ varying in a neighborhood of 0. Note that the distinguished rôles of the first coordinates $x_1$ and $z_1$ lead to no restriction of generality, as they can be eliminated by a simultaneous permutation of the $x_j$ and $z_j$ with $j \leq k$; our argument thus applies to a whole neighborhood in $\Sigma'$ of the $\mathbb{RP}^{k-1}$ added to $\Sigma \setminus \{y\}$ during the blow-up. The integral form of the first-order Taylor formula now gives $z_a = \sum_{j=1}^{k} x_j h_{a_1}(x_1, \ldots, x_k)$ for some $C^\infty$ functions $h_{a_j}$ with $h_{a_j} = \partial z_a / \partial x_j = \delta_{a_1 j}$ at $x_1 = \ldots = x_k = 0$. In terms of $r, \xi_1, \ldots, \xi_k$ this becomes $\lambda = z_1 = r \mu(r_1, \xi_2, \ldots, \xi_k)$ for a $C^\infty$ function $\mu$ with $\mu(0, \xi_2, \ldots, \xi_k) = 1$. Similarly, for each $a = 2, \ldots, n$, $z_a / r$ is a $C^\infty$ function of $r, \xi_2, \ldots, \xi_k$ (where $r$ varies around 0 in $\mathbb{R}$), and hence so is $\zeta_a = z_a / \lambda = z_a / r \mu$ while, as $r \to 0$, $(\lambda, \xi_2, \ldots, \xi_n) \to (0, \xi_2, \ldots, \xi_k, 0, \ldots, 0)$ (so that $\partial \zeta_a / \partial \xi_j = \delta_{a_1 j}$ whenever $r = 0$).

The inclusion $\Sigma \setminus \{o\} \to M \setminus \{y\}$ thus can be extended to a $C^\infty$ mapping $f : \Sigma' \to M'$ represented by our assignment $(r, \xi_1, \ldots, \xi_k) \mapsto (\lambda, \xi_2, \ldots, \xi_n)$, which is again injective, since on the added $\mathbb{RP}^{k-1}$ (represented by $r = 0$) it acts by $f(0, \xi_2, \ldots, \xi_k) \mapsto (0, \xi_2, \ldots, \xi_k, 0, \ldots, 0)$.

At $r = 0$ we have $\mu = 1$, so that $\partial \mu / \partial \xi_j = 0$, while as we have just seen, $\partial \zeta_a / \partial \xi_j = \delta_{a_1 j}$ if $r = 0$. On the other hand, relation $\lambda = r \mu$ gives $\partial \lambda / \partial r = \mu = 1$ and $\partial \lambda / \partial \xi_j = 0$ when $r = 0$. Consequently, at points with $r = 0$, the matrix $\mathcal{A} = [\partial \zeta_a / \partial \xi_j]_1$ with $1 \leq a \leq n$ and $1 \leq j \leq k$ (where $\xi_1 = r$, $\xi_1 = \lambda$) has a nonzero $k \times k$ subdeterminant obtained by restricting $a$ to $\{1, \ldots, k\}$. Therefore, rank $\mathcal{A} = k$ and so $f$ is a totally real immersion (see Example 7.1). This completes the proof.

**Remark 10.2.** Let $M, \Sigma, n, k, o, M'$ and $\Sigma'$ be as in Lemma 10.1(b) with $n = k = 2$, and let $P \subseteq M'$ be the divisor created by the blow-up. Then the mod 2 intersection number of $\Sigma'$ and $P$ is nonzero. More precisely, some small perturbation of $\Sigma'$ near its intersection with $P$ produces a surface $\Sigma'' \subset M'$ having a single, transverse intersection with $P$. In fact, the coordinate description of $\Sigma'$ near $P$ in $M'$ obtained in the above proof shows that a neighborhood of $P$ in $M'$ may be identified with the total space $N$ of a bundle of 2-disk over $P$ such that, for every $L \in P$ (that is, every complex line through 0 in $T_y M$), the fibre $N_L$ of $N$ over $L$ is the $\varepsilon$-disk centered at 0 in $L$, for some fixed Hermitian norm $||$ in $T_y M$ and a fixed real $\varepsilon > 0$. Denoting by $\Gamma \subset P$ the circle $\Gamma \cong \mathbb{RP}^1$ formed by all complex spans $C \ell$ of the real lines $\ell$ through 0 in $T_y \Sigma \subset T_y M$, we see that $K = N \cap \Sigma'$ is nothing else than the total space of a bundle over $\Gamma$ whose fibre $K_\ell$ over each real line $\ell \in \Gamma$ is a real curve in the disk $N_L$, for $L = C \ell$, passing through 0 in $N_L$ with the tangent line $T_0[K_\ell] = \ell$. Let us now choose a $C^\infty$ section $\psi$ of the real tautological line bundle over $\Gamma \cong \mathbb{RP}^1$ which has exactly one, transverse zero; for instance, we may use the Euclidean inner product corresponding to the norm $||$ and a fixed nonzero vector $u \in T_y \Sigma$, and define $\psi(\ell)$ to be the orthogonal projection into $\ell$ of $u$. Finally, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ cut-off function, equal to 1 near 0 and vanishing outside the interval $(-\varepsilon / 2, \varepsilon / 2)$. Making $\varepsilon$ smaller, if necessary, we see that the required deformations $f_r$, with $r$ near 0 in $\mathbb{R}$, of the inclusion mapping $f : \Sigma' \to M'$ can be defined by setting, for $v \in K_\ell \subset N_L$, $f_r(v) = v + ri \varphi(|v|) \psi(\ell)$ (if $v \neq 0$) and $f_r(v) = ri \psi(\ell)$ (if $v$ is the zero vector of $\ell$). Outside a suitable neighborhood of $\Gamma$ in $\Sigma'$, we set $f_r = f$. Thus, for $r$ close to 0, $f_r$ intersects $P$ at just one point, which is the unique $\ell \in \Gamma$ with $\psi(\ell) = 0$, and the intersection is obviously transverse.
By the nonorientable closed surface of genus \( g \geq 1 \) we mean, as usual, the connected sum \( g \mathbb{RP}^2 \) of \( g \) copies of \( \mathbb{RP}^2 \). Lemma 10.1 now yields the following corollary. Note that a modified argument (see Corollary ..., below) leads to a more general conclusion.

**Corollary 10.3.** The surface \( 3 \mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \) admits totally real embeddings in \( \mathbb{CP}^2 \# k \mathbb{CP}^2 \) for all \( k \geq 1 \). More generally, for any integers \( k, g \) with \( k \geq g - 2 \geq 0 \), the nonorientable closed surface \( g \mathbb{RP}^2 \) of genus \( g \) admits a totally real embedding in the complex surface \( \mathbb{CP}^2 \# k \mathbb{CP}^2 \) obtained by blowing up \( k \) suitably chosen, distinct points in \( \mathbb{CP}^2 \).

This is clear if one blows up \( k \) distinct points of \( \mathbb{CP}^2 \), \( g - 2 \) of which lie in a given totally real torus embedded in \( \mathbb{C}^2 \subset \mathbb{CP}^2 \) (cf. (v) in §7), and then uses Lemma 10.1.

## 11. Removability of complex points by blow-up

Given a point \( y \) of a real surface \( \Sigma \) embedded in a complex surface \( M \), we will say that \( \Sigma \) contains \( y \) as a complex point removable by blow-up if, for some neighborhood \( U \) of \( y \) in \( \Sigma \) and some totally real \( C^\infty \) submanifold \( U' \) of the complex surface \( M' = M \# \mathbb{CP}^n \) obtained from \( M \) by blowing up the point \( y \), the blow-down projection \( \pi : M' \to M \) maps \( U' \) diffeomorphically onto \( U \). This amounts to requiring that \( \pi^{-1}(x) \) have a limit \( y' \in M' \) as \( z \in U \setminus \{y\} \) approaches \( y \), and that \( U' = \{y'\} \cup \pi^{-1}(U \setminus \{y\}) \) is a totally real \( C^\infty \) submanifold of \( M' \) transverse, at \( y' \), to the divisor \( \pi^{-1}(y) \). Note that \( y \) then must actually be an (isolated) complex point of \( \Sigma \), since \( T_y \Sigma \) coincides with the image of the differential of \( \pi \) at \( y' \) (which is the complex line in \( T_y M \) corresponding to \( y' \)), while \( \pi : M' \setminus \pi^{-1}(y) \to M \setminus \{y\} \) is a biholomorphism (and so \( U \setminus \{y\} = \pi(U' \setminus \{y'\}) \) is totally real in \( M \)).

**Example 11.1.** Let \( \Sigma = \{(z,w) \in D \times \mathbb{C} : w = h(z)\} \) be the graph of a \( C^\infty \) function \( h : D \to \mathbb{C} \) defined on a neighborhood \( D \) of 0 in \( \mathbb{C} \). Then the following two conditions are equivalent:

1. \( \Sigma \) treated as a real surface embedded in \( M = \mathbb{C}^2 \) contains \((0,h(0))\) as a complex point removable by blow-up;
2. \( h(z) = h(0) + z\varphi(z) \) for all \( z \in D \) and some \( C^\infty \) function \( \varphi : D \to \mathbb{C} \) satisfying at each point the Cauchy-Riemann inequality \( \varphi_z \neq 0 \) of Example 7.3.

In fact, in the complex surface \( M' \) obtained from \( \mathbb{C}^2 \) by blowing up \( h(0) \) we have the local holomorphic coordinates \((\zeta, \eta)\) related to \((z,w)\) by \((z,w) = (\zeta, \zeta h + h(0))\). Since relation \( w = h(z) \) for \( z \neq 0 \) now reads \( \eta = [h(\zeta) - h(0)]/\zeta \) whenever \( \zeta \neq 0 \), our claim follows from Example 7.3. Note that if a limit \( L \) of \([h(\zeta) - h(0)]/\zeta \) as \( \zeta \to 0 \) exists, it must be finite: using \( \zeta \in \mathbb{R} \), we get \( L = h_\ast(0) \).

**Example 11.2.** Let us choose \( D = \mathbb{C} \) and \( h(z) = ze^z \) in Example 11.1. The resulting graph surface \( \Sigma \subset \mathbb{C}^2 \) is a paraboloid of revolution in the real subspace \( \mathbb{C} \times \mathbb{R} \) of \( \mathbb{C}^2 \), and \( y = (0,0) \) is its unique complex point, as well as a complex point removable by blow-up. (Namely, \( U \setminus \{y\} \) is totally real in \( \mathbb{C}^2 \) since \( \mathbb{C} \times \mathbb{R} \) contains only one complex-line direction: that of the \( z \) axis \( \mathbb{C} \times \{0\} \).)
Example 11.3. Equation $a\pi = b\tau$ in the homogeneous coordinates $[a, b, c]$ defines an embedded 2-sphere $S \subset \mathbb{C}P^2$ with just two complex points, $x = [0, 0, 1]$ and $y = [0, 1, 0]$, both removable by blow-up. This is clear from Example 11.2, since if one removes $x$ (or $y$) from $S$, by setting $v = 1$ (or $w = 1$), one gets the paraboloid $c = |a|^2$ (or $b = |a|^2$) in the ac-plane (or, $ab$-plane).

Example 11.4. The 2-sphere $S$ of radius $R > 0$ in $\mathbb{C}^2$ with the coordinates $(\lambda, \mu)$, given by $|\lambda|^2 + |\mu|^2 = R^2$ and $\text{Im} \mu = 0$, has just two complex points $x^\pm = (0, \pm R)$, both removable by blow-up. In fact, $S \setminus \{x^+, x^-\}$ is totally real due to uniqueness of a complex-line direction in $\mathbb{C} \times \mathbb{R}$ (cf. Example 11.2). As for $x^\pm$, we may use Example 11.1: $x^\pm$ appears in the new coordinates $(z, w) = (\lambda, R \mp \mu)$ as $(z, w) = (0, 0)$, and $S$ near $x^\pm$ is the graph of $w = h(z)$, where $h(z) = R - \sqrt{R^2 - |z|^2}$, while for $\varphi(z) = h(z)/z$ we have $\varphi(z) = \frac{1}{2}F(z\pi)$, with the function $F(s) = [R - (R^2 - s)^{1/2}]/s$ of $s \in (-\infty, R^2)$, obviously real-analytic at $s = 0$. Finally, $i(\varphi_y - i\varphi_x) = 2F(z) > 0$ at $z = 0$.

Remark 11.5. In §10 we described two blow-up procedures leading from a given totally real submanifold $\Sigma \subset M$ to a new totally real submanifold $\Sigma' \subset M'$, where $M$ is a complex manifold and $M'$ is obtained by blowing up a point $y$ in $M$. Our third blow-up procedure assumes that, $\dim_\mathbb{C} M = \dim_\mathbb{R} \Sigma = 2$ and $\Sigma$ contains $y$ as a complex point removable by blow-up, while $\Sigma \setminus \{y\}$ is totally real in $M$. Obviously, $\Sigma' = \{y'\} \cup \pi^{-1}(\Sigma \setminus \{y\})$, for a suitable $y' \in M'$, then is a totally real surface in $M'$, diffeomorphic to $\Sigma$ under the blow-down projection $\pi$. For instance, the complex surface obtained by blowing up two points in $\mathbb{C}P^2$ contains a totally real embedded 2-sphere $\Sigma$, namely

(i) $\Sigma$ obtained as above from $S \subset \mathbb{C}P^2$ described in Example 11.3; also,

(ii) $\Sigma$ obtained in this way from $S \subset \mathbb{C}^2 \subset \mathbb{C}P^2$ of Example 11.4.

12. Deformations involving complex points

The definition of removability by blow-up given in §11 has an immediate extension to the case of immersions. Namely, if $f$ is an immersion of a real surface $\Sigma$ in a complex surface $M$, by a complex point of $f$ removable by blow-up we mean any $x \in \Sigma$ such that $f(x)$ is a complex point, removable by blow-up, for the surface $f(\Sigma') \subset M$, where $\Sigma'$ is some connected neighborhood of $x$ in $\Sigma$ with the property that $f$ restricted to $\Sigma'$ is injective. As in §11, $x$ then is a complex point of $f$, that is, $df_x(T_x \Sigma)$ forms a complex line in $T_{f(x)} M$. Cf. Example 7.4.

We will need the following lemma. (See also Remark 13.4 below.)

Lemma 12.1. Given $k$ distinct points $x_1, \ldots, x_k$ in a real surface $\Sigma$, a complex surface $M$, points $y_1, \ldots, y_k \in M$, and an immersion $f : \Sigma \to M$ for which

(*) $x_1, \ldots, x_k$ are complex points, removable by blow-up,

there exists an immersion $f' : \Sigma \to M$, homotopic to $f$, with the same complex points as $f$, for which we have both (*), and $f'(x_j) = y_j$, $j = 1, \ldots, k$.

If, in addition, the points $y_j$ are all distinct and lie in $M \setminus f(\Sigma)$, the above assertion remains valid also when instead of an immersion one speaks, in both instances, of an embedding whose image is closed as a subset of $M$.

Proof. We may assume that $k = 1$, since our assertion will then follow via induction on $k$. Namely, $f$ can be deformed to $f'$ in two stages: first, using the inductive assumption that our claim holds for $x_j, y_j$, $j = 2, \ldots, k - 1$, with $M$ replaced by
$M \setminus \{y_i\}$ if $f$ is an embedding; then, applying the $k = 1$ case of our claim to the resulting new immersion and the points $x_1, y_1$.

Setting $k = 1$ and writing $x, y$ for $x_j, y_j$, we may also assume that

\[(***)\] some biholomorphism $(z, w)$ maps an open set in $M$ containing $p = f(x)$ and $y$ onto a product $D \times D'$ of disks around $0$ in $\mathbb{C}$ with $1 \in D'$, sending $p$ to $(0, 0)$ and $y$ to $(0, 1)$, while, if $f$ is an embedding and $f(\Sigma)$ is closed in $M$, the $(z, w)$-preimage of $\{0\} \times \{0, 1\}$ does not intersect $f(\Sigma)$.

In fact, as $M$ is connected, there exist $p_0, p_1, \ldots, p_m \in M$ with $p_0 = p, p_m = y$ and such that $(***)$ holds with $p_l$ replaced by $p_{l-1}, p_l, p_{l+1}$ for any $l = 1, \ldots, m$. If our assertion (with $k = 1$) holds under the assumption $(***)$, applying it $m$ times in a row we will obtain totally real immersions $f_0, f_1, \ldots, f_m$ of $\Sigma$ in $M$, with $f_i(z) = p_l$ for $l = 0, \ldots, m$, all homotopic to $f_0 = f$, and we may set $f' = f_m$.

To prove our assertion for $k = 1$, under the assumption $(***)$, we may also assume that $dz$ at $p_l$ restricted to $df_x(T_x\Sigma)$, is nonzero. (This is achieved by replacing $(z, w)$ with $(z + \lambda w(w - 1), w)$ for a suitable $\lambda \in \mathbb{R}$, close to 0, and making $D$ smaller if necessary.) Let $U$ now be a neighborhood of $x$ in $\Sigma$ such that $f : U \to M$ is a homeomorphic embedding. Thus, $(z, w)$ maps some neighborhood of $p_l$ in $f(U)$ onto a graph surface $w = h(z)$ as in Example 11.1, with $h(0) = 0$, and, if $D$ is made even smaller, we may in addition require that $h$ be defined on the whole disk $D$, and (after the coordinate $z$ has been replaced by $e^{i\theta}z$ for a suitable $\theta \in \mathbb{R}$) that also $\varphi_\ell(0) \notin \mathbb{R}$, for $\varphi$ appearing in Example 11.1. (Notation of Example 7.3.) Hence $\varphi_\ell(z) \notin \mathbb{R}$ whenever $z \in \mathbb{C}$ and $|z| < \varepsilon$ for some fixed $\varepsilon > 0$ that is less than the radius of $D$. Let us now fix $\delta \in (0, \varepsilon)$ and a $C^\infty$ function $\alpha : \mathbb{R} \to [0, 1]$ with $\alpha = 1$ on $(-\infty, \delta]$ and $\alpha = 0$ on $[\varepsilon, \infty)$. The required deformation of $f$ consists in replacing the graph of $h : D \to D'$ by that of the function $\tilde{h} : D \to D'$ with $\tilde{h}(z) = h(z) + \alpha(r)z$ for $z \in D$ and $r = |z|$ (thus, $\tilde{h}(0) = 1$). This is clear from Example 11.1, as $\tilde{h}(z) = \tilde{h}(0) + z\varphi(z)$ with $\varphi(z) = \varphi(z)[a(r) - 1]/z$, so that relations $2\varphi_\ell = 2\varphi + r^{-1}d\alpha/dr$ (immediate since $\varphi_\ell(0) = 0$ and $(r^2)\varphi_\ell(r) = 0$, that is, $2\varphi_\ell = r\varphi_\ell$, and $\varphi_\ell(z) \notin \mathbb{R}$ whenever $|z| < \varepsilon$ (see above) give $\varphi_\ell \neq 0$ everywhere in $D$, completing the proof.

Proof of Theorem 2.15. Assertion (b) is obvious from Lemma 10.1. To establish (a), let $k \geq 2$ and let $y_1, y_2 \in M$ be any two of the blown-up points $(y_1 \neq y_2)$. A biholomorphic identification of an open set in $M$ with a neighborhood of $(0, 0)$ in $\mathbb{C}^2$ allows us to treat a small round sphere $S$ of Example 11.4 as a real surface in $M$ having only two complex points $x, y$, both removable by blow-up. Now (a) follows from the final clause of Lemma 12.1 applied to $k = 1$, the inclusion mapping $f : \Sigma \to M$ of the sphere $\Sigma = S$, our $y_1, y_2$, and $x = x^+, x = x^-$. \hfill \Box

13. Another immersed two-sphere

With complex points of immersions and their removability by blow-up defined as in §12, we have

Lemma 13.1. For a complex surface $M$ obtained as the total space of a holomorphic line bundle $\mathcal{L}$ over a complex curve $\Sigma$, a fixed holomorphic section $\phi$ of $\mathcal{L}$, and a $C^\infty$ function $F : \Sigma \to \mathbb{R}$, let us treat the product section $F\phi$ as an embedding $f : \Sigma \to M$. Then

(i) the complex points of $f$ are those $x \in \Sigma$ at which $\phi = 0$ or $dF = 0$. \hfill \Box
Secondly, a complex point $x$ of $f$ is removable by blow-up if

(i) $x$ is a simple zero of $\phi$ and $dF \neq 0$ at $x$, or

(ii) $\phi(x) \neq 0$ and there exists a holomorphic coordinate $z$ on a neighborhood of $x$ in $\Sigma$ with $z = 0$ at $x$ and $F = F(x) + z \varphi$ for some $C^\infty$ function $\varphi$ such that $\varphi(x) = 0$.

Proof. Given $x \in \Sigma$, let us choose a holomorphic coordinate $z$ on a neighborhood $U$ of $x$ in $\Sigma$, with $z = 0$ at $x$, and a local holomorphic section $\psi$ trivializing $\mathcal{L}$ on $U$. In the resulting holomorphic local coordinates $(z, w)$ for $M$, the image $f(\Sigma)$ becomes a graph surface $w = h(z)$, with $h = \sigma F$ for the holomorphic function $\sigma$ such that $\phi = \sigma \psi$ on $U$. Example 7.3 now yields (i), since $\sigma_z = 0$ and so $h_z = \sigma F_z$ vanishes only where $\sigma = 0$ (that is, $\phi = 0$) or $F_z = 0$ (which, as $F$ is real-valued, amounts to $dF = 0$). Similarly, Example 11.1 gives removability of $x$ by blow-up, both in case (ii) (as $\sigma/z$ then is holomorphic on $U$ and $F_z(x) \neq 0$, so that $(\sigma F/z)_z \neq 0$ at $x$), and in case (iii) (since we then may choose $\psi = \phi$, that is, $\sigma = 1$). This completes the proof. \hfill $\square$

**Proposition 13.2.** The 2-sphere $\Sigma$ admits an immersion $f$ in $\mathbb{CP}^2$ such that, for some subset $Y \subset \Sigma$,

(a) $f_*[\Sigma]$ is a generator of $H_2(\mathbb{CP}^2, \mathbb{Z})$,

(b) $Y$ consists of three complex points of $f$ removable by blow-up,

(c) the immersion $f : \Sigma \setminus Y \to \mathbb{CP}^2$ is totally real.

Furthermore, such $f$ and $Y$ can be chosen so that either

(i) $f$ is an embedding, or

(ii) $Y$ is the $f$-preimage $f^{-1}(y)$ of a point $y \in \mathbb{CP}^2$.

**Proof.** To realize (a) – (c) and (i), we define $M \subset \mathbb{CP}^2$ to be the complement of a single point in $\mathbb{CP}^2$, so that $M$ is biholomorphic to the total space of the dual tautological line bundle $\mathcal{L}$ of a projective line $\Sigma \subset M$. Our claim is now obvious from Lemma 13.1 if we choose a holomorphic section $\phi$ of $\mathcal{L}$ with a unique, simple zero and a $C^\infty$ function $F : N \to \mathbb{R}$ having just two critical points, such that either critical point $x$ satisfies condition (iii) in Lemma 13.1. Specifically, identifying $\Sigma$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$, we may set $F = z\overline{z}/(z\overline{z} + 1)$ for $z \in \mathbb{C} \subset N$.

The role of the coordinate $z$ in Lemma 13.1(iii) is now played by $z$ at $x = 0 \in \mathbb{C}$, and by $1/z$ at $x = \infty$, while $F$ becomes the standard height function if we use the stereographic projection to identify $\mathbb{C} \cup \{\infty\}$ with the radius $1/2$ sphere around $(0,1/2)$ in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$.

To obtain (a) – (c) and (ii), we now fix $f$ and $Y$ satisfying (a) – (c) and (i), and then replace $f$ by $f'$ chosen as in Lemma 12.1 for $M = \mathbb{CP}^2$, our $f$, $\Sigma$, and $k = 3$, with the points $x_j \in Y$ and $y_j = y$, $j = 1, 2, 3$. This completes the proof. \hfill $\square$

Blowing up the point $y \in \mathbb{CP}^2$ in case (ii) of Proposition 13.2, we get

**Corollary 13.3.** There exists a totally real immersion $f : S^2 \to M$ of an oriented 2-sphere in the complex surface $M$ obtained from $\mathbb{CP}^2$ by blowing up a point $y$, such that $f_*[S^2] \cdot [L] = 1$ for any projective line $L \subset \mathbb{CP}^2 \setminus \{y\}$ treated as a submanifold of $M$, where $\cdot$ is the intersection form in $H_2(M, \mathbb{Z})$.

**Remark 13.4.** Lemma 12.1 allows us to modify an immersion $f$ by moving the images of any number $k$ of its complex points $x_j$, removable by blow-up, to arbitrary prescribed locations $y_j$. This is achieved by replacing the $f$-images of small
neighbourhoods of the $x_j$ with thin protrusions or "tentacles" reaching all the way to the $y_j$. In case (ii) of Proposition 13.2, $y_1 = y_2 = y_3 = y$ and so the three tentacles all intersect at $y$. In addition to $y$, these tentacles must have further pairwise intersection points; otherwise, the totally real immersion $S^2 \to M$ obtained in Corollary 13.3 would be an embedding, contrary to Corollary 18.2 in §18.

14. Connected sums

**Lemma 14.1.** Let $||$ and $||'$ be Euclidean norms on totally real planes $W, W'$ with $W \cap W' = \{0\}$ in a complex plane $V$, and let $S = \{u \in W : |u| = 1\}$. Then there exists a totally real embedding $\Phi : R \times S \to V$ such that

(a) $\Phi((0, \infty, -1] \times S) = \{w \in W : |w| \geq 1\}$ and $\Phi([1, \infty) \times S) = \{v \in W' : |v'| \geq 1\}$,

(b) the image of the Gauss mapping $G_{\Phi} : R \times S \to Gr_2(V)$ of $\Phi$, cf. (8.1), is a compact subset of $Gr_2(V)$.

**Proof.** Let us choose a $C^\infty$ curve segment $[-1, 1] \ni t \mapsto W_t \in Gr_n(V)$, joining $W$ to $W'$. $Nh$, and consisting of totally real planes in $V$ with $W_t \cap W_s = \{0\}$ whenever $t \neq s$. It exists since, under the standard diffeomorphic identification between $S^2 \times S^2$ and the Grassmanian of oriented planes in an oriented Euclidean 4-space (provided by decomposing 2-forms into their self-dual and anti-self-dual components, or, for instance, by a fixed spin$^c(4)$-geometry), complex lines, with one or the other orientation, form the subset $\Xi \times S^2$ of $S^2 \times S^2$, where $\Xi$ is a specific pair of antipodal points, while two planes intersect trivially if and only if $\langle x, y \rangle \neq \langle x', y' \rangle$ for the corresponding elements $(x, y)$ and $(x', y')$ of $S^2 \times S^2$. Joining $W$ and $W'$ by a geodesic (relative to the product metric of $S^2 \times S^2$) will thus lead to triviality of the pairwise intersections, and a small perturbation, if needed, will guarantee that all the planes forming the curve segment are totally real. Since fibre bundle over an interval are trivial, we may also choose a $C^\infty$ curve $[-1, 1] \ni t \mapsto A_t$ in the automorphism group $GL(V)$ with $A_{-1} = Id$ and $A_t W = W_t$ for all $t$, and such that $A_1 : W \to W'$ is a Euclidean isometry.

Let us now use a composite in which this last curve is preceded by a $C^\infty$ function $h : R \to [-1, 1]$ with $h(-1) = -1$ and $h(1) = 1$, which is strictly increasing on $[-1, 1]$. The resulting new curve, still written as $t \mapsto A_t$, is defined on $R$. Our totally real embedding $\Phi : R \times S \to V$ is given by $\Phi(t, u) = e^{d\varphi(t)} A_t u$, for a $C^\infty$ function $\varphi : R \to C$ with $\varphi(t) = \log |t|$ whenever $|t| \geq 1$, chosen as described below.

Let $J$ stand for the unique complex-linear operator $V \to V$ whose restriction to $W$ is a rotation in $W$ by the angle $\pi/2$ relative to the Euclidean structure of $W$. Clearly, $\Phi$ corresponding to any fixed $C^\infty$ function $\varphi : R \to C$ as above is injective (due to the trivial-intersections property of the $W_t$), and it is totally real if and only if, for every $u \in S$, the $(A_t$-images of the) vectors $(\dot{\varphi} + A^{-1} \dot{A})u$ and $Ju$ are linearly independent over $C$. (We omit the dependence on $t$ in our notation, and write $(\cdot) = d/dt$.) Identifying $V \wedge 2$ with $C$, we can rewrite the requirement that $\Phi$ be totally real as $\dot{\varphi} \neq - A^{-1} Au \wedge Ju / (u \wedge Ju$ for all $t \in [-1, 1]$ and $u \in S$. (Note that $u \wedge Ju \neq 0$ for all $u \in S$, as $W$ is totally real, and the requirement holds if $|t| \geq 1$ due to our choice of $\varphi(t)$ when $|t| \geq 1$.) Since such a function $\varphi$ obviously exists, we obtain (a).

Finally, by (a), $G_\Phi((-1, -1] \times S) = \{W\}$ and $G_\Phi([1, \infty) \times S) = \{W'\}$. Consequently, $G_\Phi(R \times S) = G_\Phi([-1, 1] \times S)$, and (b) follows. This completes the proof.

□
Lemma 14.2. Suppose that we are given a point \( y \) in an almost complex surface \( M \), totally real immersions \( \rho, \sigma \) of real surfaces \( P, Q \) in \( M \), and points \( x \in P, z \in Q \) with
\[
\rho(x) = \sigma(z) = y, \quad d\sigma_z(T_zQ) \cap d\rho_x(T_xP) = \{0\} \subset T_yM.
\]
Then there exist arbitrarily small disk-like neighborhoods \( U_x, U_z, U_y \) of \( x, z, y \) in \( P, Q \) and \( M \), respectively, a real surface \( \Sigma \), and a totally real immersion \( f : \Sigma \to M \) such that

(i) \( \Sigma \) is obtained from a formal disjoint union of \( P \) and \( Q \) by replacing the union of the closures of \( U_x \) and \( U_z \) with a "handle" \( \Gamma \) diffeomorphic to \([-1,1] \times S^1\),

(ii) \( f(\Gamma) \subset U_y \) and \( f \) restricted to \( \Gamma \) is injective,

(iii) \( f \) coincides with \( \rho \) on \( P \setminus U_x \) and with \( \sigma \) on \( Q \setminus U_z \).

Proof. Let us set \( V = T_yM, W = d\rho_x(T_xP), W' = d\sigma_z(T_zQ) \). We can obviously find a \( C^\infty \) mapping \( F \) of a neighborhood of \( 0 \) in \( V = T_yM \) into \( M \) with \( F(0) = y \), \( dF_0 = Id : V \to V \) and having the property that, for some fixed \( r > 0 \), \( F \) is defined on the whole of \( B_r = \{v \in V : |v| < r\} \) (for a fixed Hermitian norm \( || \) in \( V \)) and maps \( B_r \) diffeomorphically onto an open set \( F(B_r) \subset M \), while \( F(B_r \cap W) = \rho(U'_y), F(B_r \cap W') = \sigma(U'_y) \) for some neighborhoods \( U'_y, U'_x \) of \( x \) and \( z \) in \( P \) and \( Q \), such that \( \rho \) and \( \sigma \) restricted to \( U'_y \) and \( U'_z \), respectively, are injective. For instance, let us choose \( F \) so that \( F^{-1} \) is the composite of the local coordinate system \((x_1, \ldots, x_{2n})\) with a real-linear isomorphism \( h : \mathbb{R}^{2n} \to T_yM \), requiring that \( x_j(y) = 0 \) for all \( j \) and \( h \) send the standard basis of \( \mathbb{R}^{2n} \) onto the basis of \( T_yM \) dual to the \( dx_j \), while \( x_j \circ \rho \) for \( j > n \) (or, \( x_j \circ \sigma \) for \( j \leq n \)) be constant near \( x \) in \( P \) (or, respectively, near \( z \) in \( Q \)).

Let us now choose a \( \Phi : \mathbb{R} \times S \to V \) as in Lemma 14.1. By Lemmas 14.1(b) and 8.3, the composite of \( \Phi \) followed by multiplication by any sufficiently small \( \varepsilon \in (0, r) \) forms a totally real embedding relative to the original almost complex structure of \( M \). Using such a composite to modify \( \rho \) and \( \sigma \) near \( x \) and \( z \), we obtain our assertion. This completes the proof.

Remark 14.3. Let the real surfaces \( P, Q \) in Lemma 14.2 be, in addition, closed, and let at least one of them be orientable. For the totally real immersion \( f : \Sigma \to M \) of \( \Sigma = P \# Q \), obtained as in Lemma 14.2, we then have \( f_*[\Sigma] = \rho_*[P] + \sigma_*[Q] \in H_2(M, \mathbb{Z}_p) \), where \( [N] \in H_n(N, \mathbb{Z}_p) \) is the fundamental homology class of any closed \( n \)-manifold \( N \) with coefficients in the group \( \mathbb{Z}_p \) (notation of (2.4)). In fact, the handle \( \Gamma = F(\varepsilon \Phi([-1,1] \times S)) \subset M \) constructed in the proof of Lemma 14.2 can be filled with the "solid handle" \( F(\varepsilon \Phi([-1,1] \times B)) \subset M \), \( B = \{u \in W : |u| \leq 1\} \) being the unit ball in \( W \) with the boundary \( S \). Thus, \( f_*[\Sigma] \) is homologous in \( H_n(M, G) \) to the sum of the cycles \( \rho(P) \) and \( \sigma(Q) \).

Lemma 14.4. Given real numbers \( R, \varepsilon \) with \( 0 < 4\varepsilon < R^2 \), formula
\[
(14.2) \quad f(t, e^{i\theta}) = (x, y) \quad \text{with} \quad x = t + iR \cos \theta, \quad y = R t \sin \theta + \varepsilon i(1 - t^2) \sin 2\theta,
\]
defines a totally real immersion \( f : [-1,1] \times S^1 \to \mathbb{C}^2 \) which has a single transverse self-intersection due to the relation \( f(0, \pm i) = (0, 0) \) and sends the boundary circles \( \{\pm 1\} \times S^1 \) onto the radius \( R \) circles centered at \((\pm 1, 0)\) in the real planes given by \( \text{Im} z_2 = \text{Re} z_1 \mp 1 = 0 \).
Proof. Let us denote by $J$ the value, at any given $(t, \theta)$, of the Jacobian $J(x, y) = x_1 y_0 - x_0 y_1$, where the subscripts stand for partial derivatives (see Example 7.1). Thus, $\text{Re} J = R t (1 + 4 t^2 \sin^2 \theta) \cos \theta$, $\text{Im} J = R^2 \sin^2 \theta + 2 \varepsilon (1 - t^2) \cos 2 \theta$. If $\text{Re} J = 0$, we have $t = 0$ (and so $\text{Im} J = R^2 \sin^2 \theta + 2 \varepsilon \cos 2 \theta (2 \sin^2 \theta + \cos 2 \theta) = 2 \varepsilon \neq 0$), or $\cos \theta = 0$ (and hence $\text{Im} J = R^2 + 2 \varepsilon (t^2 - 1) > 0$). In any case, $J \neq 0$, so that $(x, y)$ is a totally real embedding (cf. Example 7.1), and one easily verifies the assertion about the self-intersection. This completes the proof. □

15. TOPOLOGY OF $E(M)$

For some specific almost complex manifolds $M$, the manifolds $E(M), E^+(M)$ defined in §3 can be described more directly:

**Proposition 15.1.** There exist diffeomorphic identifications

$$E = M \times S^1 \quad \text{with} \quad E = E(M) \quad \text{or} \quad E = E^+(M),$$

whenever $M$ is $C^n$, a torus $T^n$ treated as a complex $n$-manifold, a Hopf manifold $S^1 \times S^{2n-1}$, or a $K3$ surface. Also

$$E^+(\mathbb{C}P^n) = S^{2n+1}/\mathbb{Z}_{n+1}, \quad E^+(\mathbb{C}P^n) = S^{2n+1}/\mathbb{Z}_{2n+1},$$

$$E(\mathbb{C}P^1 \times \mathbb{C}P^1) = S^5 \times \mathbb{R}P^3 = S^5 \times [S^3/\mathbb{Z}_2],$$

$$E(\mathbb{C}P^1 \times \mathbb{C}P^1) = [S^2 \times S^2]/\mathbb{Z}_4,$$

where the $\mathbb{Z}_q$ group actions on the unit odd-dimensional spheres $S^{2k-1} \subset C^k$ all come from the obvious action of the circle $U(1)$ on $C^k$, while a generator of the action of $\mathbb{Z}_4$ on $S^2 \times S^2$ assigns $(x, qx)$ to a pair $(x, q) \in H \times H$ of quaternions with $|x| = |q| = 1$ and $\text{Re} x = 0$.

**Proof.** Relations (15.1) are obvious since, for $M$ as in (15.1), the determinant bundle $\det TM$ is trivial. When $M = \mathbb{C}P^n$, $\det TM$ coincides with the tensor power $[L^*]^{\otimes (n+1)}$ of the dual $L^*$ to the tautological line bundle $L$ over $\mathbb{C}P^n$, while the unit circle bundle of $L$ or $L^*$ has the total space $S^{2n+1}$. This proves (15.2) for $M = \mathbb{C}P^n$. Next, let $M = \mathbb{C}P^1 \times \mathbb{C}P^1 \approx S^5 \times S^2$. Now $\det TM$ is the tensor product of the complex line bundles obtained as pull-backs of $T[\mathbb{C}P^1]$ under the factor projections $M \to \mathbb{C}P^1$ (since $TM$ is the direct sum of those line bundles). Thus, in real terms, the fibre of the unit circle bundle $E^+(M)$ of $\det M$ over $(x, y) \in S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ consists of all orientation-reversing linear isometries $A$ of the plane $x^+ \subset \mathbb{R}^3$ onto $y^+$. If we extend every such $A$ to an orientation-preserving linear isometry $\mathbb{R}^3 \to \mathbb{R}^3$ by requiring that $Ax = -y$, we obtain a diffeomorphism $(x, y, A) \mapsto (x, A)$ of $E^+(M)$ (with $M = \mathbb{C}P^1 \times \mathbb{C}P^1$) onto $S^2 \times SO(3)$, which proves the second line of (15.2). Finally, the prove the last relation in (15.2), let us set $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ and note that $E^+(M) = S^2 \times SO(3)$ has the universal covering space $S^2 \times S^3$. Specifically, we identify $SO(3)$ with $S^3/\mathbb{Z}_2$ by letting the sphere $S^3$ of unit quaternions act on the sphere $S^2$ of pure unit quaternions by $(q, x) \mapsto qxq$. Using the fact that $xy = -yx$ and $x^2 = -1$ for any two orthogonal pure quaternions $x, y$ with $|x| = 1$, we now easily verify that our $\mathbb{Z}_2$ action generator is in fact of order 4 and its square generates the covering projection $S^3 \times S^2 \to E^+(M) = S^2 \times [S^3/\mathbb{Z}_2]$, while the generator itself descends to $E^+(M)$ producing a generator of the antipodal $\mathbb{Z}_2$ action with the quotient $E(M)$ (cf. (1.2)). This completes the proof. □
Proposition 15.2. Let a simply connected almost complex surface $M$ admit a pseudoholomorphic embedding $f$ of the sphere $S^2$ with the self-intersection number $-1$. Then $E^+(M)$ is simply connected and $\pi_1[E(M)] = \mathbb{Z}_2$.

For instance, $E^+(M)$ is simply connected and $\pi_1[E(M)] = \mathbb{Z}_2$ whenever $M$ is obtained by blowing up a point in a simply connected complex surface.

Proof. The connecting homomorphism in (1.1) is $\pi_2 M \to H_2(M, \mathbb{Z}) \to \mathbb{Z}$, the composite of the Hurewicz homomorphism with the first Chern class of the line bundle $\det_C TM$ (for $E = E^+(M)$) or $[\det_C TM]^\otimes 2$ (for $E = E(M)$). Since $M$ is simply connected, the image of the connecting homomorphism, that is, the kernel of $\mathbb{Z} \to \pi_1 E$ in (1.1), is $\frac{1}{2q} \mathbb{Z}$ for $E = E^+(M)$ and $q \mathbb{Z}$ for $E = E(M)$, with $q$ as in (1.4) (cf. also (1.3)). For any pseudoholomorphic immersion $f : S^2 \to M$ we have $f^* \det_C TM = [\tau \otimes \nu]^{\otimes 2} = \tau \otimes \nu$, with $\tau, \nu$ as in (5.1). Consequently, $\langle c_1(M), f_*[S^2] \rangle = \langle c_1(\tau) + c_1(\nu), [S^2] \rangle = 2 + \langle c_1(\nu), [S^2] \rangle$. If, in addition, $f$ is an embedding with $f_*[S^2] \cdot f_*[S^2] = -1$, this gives $\langle c_1(M), f_*[S^2] \rangle = 1$, as one has $\langle c_1(\nu), [S^2] \rangle = f_*[S^2] \cdot f_*[S^2]$ for all embeddings $f : S^2 \to M$. Our assumption thus yields $1 \in \frac{1}{2q} \mathbb{Z}$, that is, $q = 2$. By (1.4) and (1.3), this completes the proof. □

Remark 15.3. When $M$ is one of the complex surfaces (0.1), relations (15.1), (15.2) and (1.4) give $q = \infty$ (for $M = \mathbb{C}^2$), or $q = 6$ (for $M = \mathbb{C}P^2$), or $q = 4$ (for $M = \mathbb{C}P^1 \times \mathbb{C}P^1$), or, finally, $q = 2$ (for $M = \mathbb{C}P^2 \# k \mathbb{C}P^2$, $k \geq 1$). Thus, by (2.7), if $f : \Sigma \to M$ is a totally real immersion of a closed real surface $\Sigma$, then

$$
\begin{align*}
\text{a)} & \quad i(f) \in \mathcal{I}_0(\Sigma) \subset H^1(\Sigma, \mathbb{Z}) \quad \text{for} \quad M = \mathbb{C}^2, \\
\text{b)} & \quad i(f) \in \mathcal{I}_0(\Sigma) \subset H^1(\Sigma, \mathbb{Z}_6) \quad \text{for} \quad M = \mathbb{C}P^2, \\
\text{c)} & \quad i(f) \in \mathcal{I}_0(\Sigma) \subset H^1(\Sigma, \mathbb{Z}_4) \quad \text{for} \quad M = \mathbb{C}P^1 \times \mathbb{C}P^1, \\
\text{d)} & \quad i(f) = \omega_1(\Sigma) \in H^1(\Sigma, \mathbb{Z}_2) \quad \text{for} \quad M = \mathbb{C}P^2 \# k \mathbb{C}P^2, \quad k \geq 1.
\end{align*}
$$

Here (d) follows since, by (2.10), $\mathcal{I}_0(\Sigma) = \{\omega_1(\Sigma)\}$.

16. Totally real tori and Klein bottles in $\mathbb{C}^2$

We begin this section with some simple examples of totally real immersions and embeddings of the torus $T^2$ and Klein bottle $K^2$ in $\mathbb{C}^2$.

For $C^1$ functions $x, y : U \to \mathbb{C}$ on an open set $U$ in the $(s, t)$-plane $\mathbb{R}^2$, we denote by $J(x, y)$ the complex Jacobian determinant of $(x, y)$, that is, the function $J(x, y) : U \to \mathbb{C}$ with $J(x, y) = x_ay_b - x_by_a$ (cf. Example 7.1). The subscripts, here and in the following example, stand for the partial derivatives.

Example 16.1. Let $x, y, h : \mathbb{R}^2 \to \mathbb{C}$ be doubly $2\pi$-periodic $C^1$ functions of the variables $s, t$ such that $|J(x, y)|$ is bounded on $\mathbb{R}^2$ and $h_s = 0$ identically, while $|x_s|^2 + |y_s|^2 \geq \varepsilon$ for some real $\varepsilon > 0$. The mapping $(x, y + rh) : \mathbb{R}^2 \to \mathbb{C}^2$, with any constant $r \in \mathbb{R}$, then descends to the torus $T^2 = [\mathbb{R}/2\pi \mathbb{Z}] \times [\mathbb{R}/2\pi \mathbb{Z}]$ and, for large $|r|$, it produces a totally real immersion $f : T^2 \to \mathbb{C}^2$. If, in addition, $x, y, h$ are all invariant under the transformation $(s, t) \to (-s, t + \pi)$ of $\mathbb{R}^2$, then $(x, y + rh)$ further descends to a totally real immersion $f : K^2 \to \mathbb{C}^2$ of the Klein bottle $K^2 = \mathbb{R}^2/\Gamma = T^2/\mathbb{Z}_2$, where $\Gamma$ is the transformation group generated by $\Phi$ and $\Psi$ with $\Phi(s, t) = (-s, t + \pi)$ and $\Psi(s, t) = (s + 2\pi, t)$.

In fact, $(x, u) : \mathbb{R}^2 \to \mathbb{C}^2$ is a totally real immersion if and only if $J(x, u) \neq 0$ everywhere in $\mathbb{R}^2$ (see Example 7.1). Since, for any $r \in \mathbb{R}$, we obviously have

$$J(x, y + rh) = J(x, y) + r x_s h_t,$$
it follows that $|J(x, y + rh)| \to \infty$ as $r \to \infty$, uniformly on $U$. Therefore, if $|r|$ is sufficiently large, $J(x, y + rh) \neq 0$ everywhere in $\mathbb{R}^2$.

**Example 16.2.** The assumptions listed in Example 16.1 are obviously satisfied by the functions $x(s, t) = e^{ikt}(\sin s + i \sin 2s)$, $y(s, t) = e^{ilt} \cos s$, $h(s, t) = e^{ilt}$, where $k$ and $l$ are fixed integers with $l \neq 0$. The mapping $(x, y + rh) : \mathbb{R}^2 \to \mathbb{C}^2$ thus descends, for large $r$, to a totally real immersion $f = f^{k,l} : T^2 \to \mathbb{C}^2$ of the 2-torus; in the case where $k$ is odd and $l$ is even, it similarly descends to a totally real immersion $f = f^{k,l} : K^2 \to \mathbb{C}^2$ of the Klein bottle.

Consequently, if $k, l$ are integers and either

(a) $k$ is a multiple of $l$ and $\Sigma$ is the 2-torus $T^2$, or
(b) $l = 2$, while $k$ is odd and $\Sigma$ is the Klein bottle $K^2$,

then, for all sufficiently large $r > 1$, the mapping $f^{k,l} : \Sigma \to \mathbb{C}^2$ defined above is a totally real embedding.

In fact, injectivity of $f^{k,l}$ follows since $(x, u) = (x(s, t), y(s, t) + rh(s, t))$ determines $(\alpha, \beta) = (e^{i\alpha}, e^{i\beta})$ either uniquely (case (a)), or up to the involution $(\alpha, \beta) \mapsto (\alpha, -\beta)$ (case (b)). Specifically, $\cos s = |u| - r$, $e^{ilt} = u/|u|$ and $\sin s = xe^{-ikt}(1 + 2i \cos s)^{-1}$ (while, in case (a), $e^{-ikt} = (e^{ilt})^{-k/l}$).

It will be shown later in §24 that the totally real embeddings $f^{k,l} : T^2 \to \mathbb{C}^2$ and $f^{k,3} : K^2 \to \mathbb{C}^2$ (if odd), described above, when composed with reparametrizations of $T^2$ or $K^2$, realize all possible $\sim_{tri}$ equivalence classes (see §9) of totally real immersions of these surfaces in $\mathbb{C}^2$.

**17. Proofs of Theorem 2.3 and Corollaries 2.4 - 2.9**

Let $M$ be an almost complex surface. Corollary 9.3 implies that $S^2$ admits a totally real immersion in $M$, while totally real embeddings $T^2 \to M$ and $K^2 \to M$ exist in view of Proposition 8.1 combined with Example 16.2.

The claim made in Theorem 2.3 about immersions and connected sums is immediate from Lemma 14.4 along with an obvious smoothing argument and Lemma 8.3.

The assertion about embeddings follows in turn from Lemma 14.4 and Lemma 14.2 (the latter used to remove the resulting self-intersection). This proves Theorem 2.3.

Corollaries 2.4 - 2.6 now follow immediately, and Corollary 2.7 is obvious from Corollary 2.6, due to the existence of a “real form” $\mathbb{R}P^2 \subset \mathbb{C}P^2$ (see (vi) in §7).

Corollary 2.8 for $T^2$ and $S^2$ is clear from Theorem 2.3, Remark 11.5 and Lemma 10.1(a). To prove Corollary 2.8 for nonorientable closed surfaces $\Sigma$, note that the operation $\Sigma \to \Sigma \# T^2 \# K^2$ (where $K^2$ is the Klein bottle) reduces the Euler characteristic by 4, and so, in view Theorem 2.3, one needs only to show the existence of a totally real embedding $\Sigma \to M = \mathbb{C}P^2 \# k \mathbb{C}P^2$, $k \geq 1$, under the additional assumption that $\chi(\Sigma) \in \{-2, -1, 0, 1\}$. Now, if $\chi(\Sigma) = 0$, this follows from Corollary 2.5. For $\chi(\Sigma) = -2$, we may use the (already established) embeddability of $S^2$, as $\Sigma = S^2 \# T^2 \# K^2$. Finally, if $\chi(\Sigma) = \pm 1$, we have $\Sigma = \mathbb{R}P^2$ or $\Sigma = 3\mathbb{R}P^2$, and our claim is obvious from Corollary 10.3 and Lemma 10.1(a).

Corollary 2.9 is in turn immediate from Corollary 2.4 along with (5.3,a) and Wu’s formula (1.7).

**18. Obstructions for Embedded Orientable Surfaces**

Let $M$ be a compact almost complex surface. If a closed oriented real surface $\Sigma$ admits a totally real embedding $f : \Sigma \to M$ and $\sigma \in H^2(M, \mathbb{R})$ denotes the
(real) Poincaré dual of $f_*[\Sigma] \in H_2(M, \mathbb{R})$, setting $\chi = \chi(\Sigma)$ and $c_1 = c_1(M)$, we can rewrite (5.4) for $n = 2$ and (5.3b) as

\begin{equation}
\sigma \sim \sigma = -\chi, \quad \sigma \sim c_1 = 0.
\end{equation}

For any closed oriented 4-manifold $M$, the cup product $\sim$ (treated as a real-valued symmetric bilinear form in $H^2(M, \mathbb{R})$) is nondegenerate, and its sign pattern consists of $b_+^2$ pluses and $b_-^2$ minuses, so that $b_2(M) = b_+^2 + b_-^2$.

**Proposition 18.1.** Let $M$ be a closed almost complex surface for which $b_+^2 = 1$, $c_1 \sim c_1 \geq 0$ and $c_1 \neq 0$ in $H^2(M, \mathbb{R})$, and let a closed orientable surface $\Sigma$ admit a totally real embedding $f : \Sigma \to M$.

(a) $\Sigma$ must then be diffeomorphic to the torus $T^2$ or the sphere $S^2$.
(b) If, in addition, $M$ has $b_-^2 = 0$, then $\Sigma$ is diffeomorphic to $T^2$.
(c) If $\Sigma$ is diffeomorphic to $T^2$, then either $c_1 \sim c_1 > 0$ and $f_*[\Sigma] = 0$ in $H_2(M, \mathbb{R})$, or $c_1 \sim c_1 = 0$ and $f_*[\Sigma]$ is a real multiple of the dual of $c_1$ in $H_2(M, \mathbb{R})$.

**Proof.** If $\chi \leq 0$, relations (18.1) imply that the cup-product form $\sim$ is positive semidefinite on the subspace $W \subset H_2(M, \mathbb{R})$ spanned by $\sigma$ and $c_1$. Since $\sim$ has the Lorentzian sign pattern $+ - \ldots -$, this shows that $\dim W = 1$. Using (18.1), we now obtain (c). Also, as the inequality $\chi < 0$ would, by (18.1), make $\sigma$ and $c_1$ linearly independent (and hence yield $\dim W = 2$), we see that $\chi = \chi(\Sigma) \geq 0$, which proves (a). Finally, condition $b_-^2 = 0$ implies that $c_1$ spans $H^2(M, \mathbb{R})$, so that (18.1) gives $\sigma = 0$ and $\chi = 0$, which yields (b). This completes the proof.

**Corollary 18.2.** The torus $T^2$ is the only closed orientable real surface that admits a totally real embedding in $\mathbb{C}^2$, $\mathbb{CP}^2$ or the complex surface obtained by blowing up a point in $\mathbb{CP}^2$.

**Proof.** A totally real embedding of $T^2$ exists according to (v) in §7. Concerning its nonexistence for other real surfaces, the case of $\mathbb{C}^2$ follows from that of $\mathbb{CP}^2$ via the inclusion $\mathbb{C}^2 \subset \mathbb{CP}^2$ (or, directly from (5.4) with $H_2(\mathbb{C}^2, \mathbb{Z}) = \{0\}$). As for $M = \mathbb{CP}^2$ or $M = \mathbb{CP}^2 \# \mathbb{CP}^2$, the only other possibility left by Proposition 18.1 is that of a totally real 2-sphere embedded in $M$. This in turn is excluded by Proposition 5.2. (For $\mathbb{CP}^2$, we may also use Proposition 18.1(b)).

**Proofs of Corollaries 2.10 and 2.11.** In both corollaries, the nonexistence part is immediate from Proposition 18.1 and Corollary 2.9. (Note that condition $k \leq 9$ in Corollary 2.11 amounts to $c_1 \sim c_1 \geq 0$, cf. §27) Corollary 2.11 now is immediate from Corollary 2.8. As for Corollary 2.10, the existence assertion for $T^2$ is clear from (v) in §7, and for $S^3$ it is provided by (vii) in §7. Finally, let $\Sigma$ be a nonorientable closed surface. Since the operation $\Sigma \to \Sigma \# T^2 \# K^2$ (where $K^2$ is the Klein bottle) reduces the Euler characteristic of any closed surface by 4, by Theorem 2.3 we just need to show the existence of a totally real embedding $\Sigma \to M = \mathbb{CP}^1 \times \mathbb{CP}^1$ under the additional assumption $\chi(\Sigma) \in \{-2, 0\}$. Now, if $\chi(\Sigma) = 0$, this follows from Corollary 2.5. For $\chi(\Sigma) = -2$, we have $\Sigma = S^2 \# T^2 \# K^2$, and so we may use the already-established embeddability of $S^2$. This completes the proof.
19. Embeddings of nonorientable surfaces

In contrast with the orientable case, the self-intersection formula (5.5) is often too crude to detect non-embeddability of nonorientable surfaces. For instance, the genus 3 surface $3\mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ admits no totally real embedding in $\mathbb{C}P^2$, yet this fact cannot be concluded from (5.5). Developing useful obstructions for totally real embeddings of nonorientable closed manifolds requires more subtle intersection-theoretic tools. One such device is the Pontryagin square operation, applicable to this case via a result of Massey, as described below. Our presentation follows [9].

The Pontryagin square for a closed oriented manifold $M$ and $k = 0, 1, 2, \ldots$ (see [9]) is a natural cohomology operation $H^k(M, \mathbb{Z}_2) \to H^{2k}(M, \mathbb{Z}_2)$ which, when applied to mod 2 reductions of integral classes, assigns $[\xi \sim \xi \mod 4]$ to $[\xi \mod 2]$ for $\xi \in H^k(M, \mathbb{Z})$. We will use the symbol $H^\sigma(M, \mathbb{Z}_2) \ni \sigma \mapsto \sigma^2 \in \mathbb{Z}_2$ for the Pontryagin square in the case where $k = n$ and $\dim \mathbb{R} M = 2n$. Our main interest will be for the case where $M$ is an almost complex manifold of even complex dimension $n$.

Any embedding $f : \Sigma \to M$ gives rise to the associated mapping from $M$ into the Thom space $\text{Th}(\nu)$ (that is, a one-point compactification) of the normal bundle $\nu$ of $f$. The Thom space is an oriented pseudomanifold, and with the usual orientation conventions, this mapping $M \to \text{Th}(\nu)$ induces the identity homomorphism between the top (co)homology groups.

The Thom space carries the Thom class $\mathcal{U} = \mathcal{U}(\nu) \in H^n(\text{Th}(\nu), \infty; \mathbb{Z})$, where the $\mathbb{Z}$ coefficients are twisted by the orientation homomorphism of $\nu$. Consequently, $\mathcal{U}$ has a well-defined mod 2 reduction $[\mathcal{U} \mod 2]$ in the cohomology group $H^n(\text{Th}(\nu), \infty; \mathbb{Z}_2)$ with constant (untwisted) $\mathbb{Z}_2$ coefficients, whose Pontryagin square $[\mathcal{U} \mod 2]^2$ obeys the following formula due to Massey [9]:

\begin{equation}
[\mathcal{U} \mod 2]^2 = \mathcal{U} \sim ([e(\nu) \mod 4] + 2[w_1 \sim w_{n-1}])
\end{equation}

Here $[e(\nu) \mod 4]$ denotes the mod 4 reduction of the twisted Euler class of $\nu$, and $2[w_1 \sim w_{n-1}]$ is the image of the cup product of the Stiefel-Whitney classes $w_1, w_{n-1}$ of $\Sigma$ under the nontrivial coefficient homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_4$. Both sides of (19.1) thus are elements of $H^{2n}(\text{Th}(\nu), \infty; \mathbb{Z}_4)$, and the right-hand side is the image of an element of $H^{2n}(\Sigma, \mathbb{Z}_4)$ under the Thom isomorphism

\begin{equation}
H^n(\Sigma, \mathbb{Z}_4) \to H^{2n}(\text{Th}(\nu), \infty; \mathbb{Z}_4),
\end{equation}

which operates via the cup product with the Thom class. (The first group in (19.2) has mod 4 coefficients twisted by the orientation of $\nu$, and the second has constant mod 4 coefficients.)

Using Massey’s formula (19.1) and the fact that the pullback of $[\mathcal{U} \mod 2]$ is dual to the fundamental class $f_*[\Sigma]$ in $H_2(M, \mathbb{Z}_2)$, we obtain

**Lemma 19.1** (Massey [9]). *Given an embedding $f : \Sigma \to M$ of a closed manifold $\Sigma$ in a closed oriented manifold $M$ with $\dim \Sigma = n$ and $\dim \mathbb{R} M = 2n$ for some even integer $n \geq 2$, let $\sigma \in H^n(M, \mathbb{Z}_2)$ stand for the Poincaré dual of $f_*[\Sigma] \in H_n(M, \mathbb{Z}_2)$. Denoting by $\chi(\nu) \in \mathbb{Z}$ and $\sigma^2 \in \mathbb{Z}_2$ the Euler number of the normal bundle $\nu$ of $f$ and the Pontryagin square of $\sigma$, we then have*

\begin{equation}
\sigma^2 = [\chi(\nu) \mod 4] + 2[w_1 \sim w_{n-1}].
\end{equation}
In fact, by naturality of the Pontryagin square we can compute the right-hand side either in the cohomology of \( TH(\nu) \) or in the cohomology of \( M \). (Note that we can view this formula as equality of two numbers mod 4: both cohomology classes lie in the top cohomology group of \( TH(\nu) \), isomorphic to \( \mathbb{Z}_4 \).

For totally real embeddings, relation (5.1) (and Wu's formula (1.7) in the case of surfaces) now give

**Corollary 19.2.** Let \( f: \Sigma \to M \) be a totally real embedding of a closed real manifold \( \Sigma \) in a closed almost complex manifold \( M \) with \( \dim \mathbb{R} \Sigma = \dim \mathbb{C} \Sigma = n \), where \( n \) is even, and let \( \sigma \in H^n(M, \mathbb{Z}_2) \) be the Poincaré dual of the cycle \( f_*[\Sigma] \in H_n(M, \mathbb{Z}_2) \). If we set \( \chi = \chi(\Sigma) \), then the Pontryagin square \( \sigma^2 \in \mathbb{Z}_4 \) is characterized by (19.3). In the case where \( n = 2 \), this becomes

\[
(19.4) \quad \sigma^2 = [\chi \mod 4].
\]

A further consequence is

**Corollary 19.3.** Suppose that a closed real surface \( \Sigma \) admits a totally real embedding in the complex surface \( M = \mathbb{C}P^2 \) or \( M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \). Defining \( \chi_4 \in \{0, 1, 2, 3\} \) by \( \chi_4 \equiv \chi(\Sigma) \mod 4 \), we then have \( \chi_4 \in \{0, 1, 3\} \) if \( M = \mathbb{C}P^2 \) and \( \chi_4 \in \{0, 1, 3\} \) if \( M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \).

In fact, as in the argument for Proposition 5.2, this is immediate from (19.4), since, for integers \( p, q \), the mod 4 congruence class of \( p^2 \) must contain 0 or 1, while that of \( p^2 - q^2 \) must contain 0, 1 or \(-1\).

**Proofs of Corollaries 2.12 - 2.14.** The nonexistence assertion for orientable \( \Sigma \) other than the torus is immediate from Corollary 18.2, while the case of \( T^2 \) is covered by Corollary 2.5. Now, let \( \Sigma \) be nonorientable. The operation \( \Sigma \to \Sigma \# T^2 \# K^2 \) (with \( K^2 \) denoting the Klein bottle) reduces \( \chi(\Sigma) \) by \( 4 \); thus, by Theorem 2.3, for the existence assertions in Corollaries 2.12 - 2.14 it suffices to show that a totally real embedding \( \Sigma \to M \) exists if \( \chi(\Sigma) \) is zero, or \( \chi(\Sigma) \in \{0, 1\} \) or, respectively, \( \chi(\Sigma) \in \{0, 1, 3\} \). For \( \chi(\Sigma) = 0 \) in all three corollaries, or \( \chi(\Sigma) = 1 \) in Corollary 2.13, or \( \chi(\Sigma) = \pm 1 \) in Corollary 2.14, this existence statement is clear from Corollary 2.5, or (vi) in §7 or, respectively, (vi) in §7 combined with Lemma 10.1(a) and Theorem 2.3 (as \( \chi(\Sigma) = 3 \) for \( \Sigma = \mathbb{R}P^2 \# T^2 \# K^2 \)).

Finally, the nonexistence statements for nonorientable \( \Sigma \) can be established as follows. Let \( \Sigma \) admit a totally real embedding in \( M \). Assume first that \( M = \mathbb{C}P^2 \) or \( M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \). By Corollary 19.3, \( \chi(\Sigma) \) has the required remainder mod 4. Finally, if \( M = \mathbb{C}^2 \), this last relation along with the inclusion \( \mathbb{C}^2 \subset \mathbb{C}P^2 \) yields \( \chi(\Sigma) \equiv 0 \mod 4 \), since \( \chi(\Sigma) \) is even by Corollary 2.4.

20. Complex curves and small deformations

We now introduce some definitions needed, in §21, to prove Theorem 2.16.

Let \( W, V \) be complex vector spaces. For \( A \in \text{Hom}_\mathbb{R}(W, V) \) we let \( A^+, A^- \) stand for the unique operators \( W \to V \) such that \( A = A^+ + A^- \), \( A^+ \) is \( \mathbb{C} \)-linear, and \( A^- \) is antilinear. Thus, \( A^+, A^- \) are the components of \( A \) relative to the decomposition of \( \text{Hom}_\mathbb{R}(W, V) \) into the \( \pm 1 \)-eigenspaces of the involution \( A \mapsto -i \circ A \circ i \), where \( i \) stands for multiplication by \( i \). In other words,

\[
(20.1) \quad 2A^\pm = A \mp i \circ A \circ i.
\]
Consider now a complex vector bundle \( \eta \) with a fixed connection \( \nabla \) over an almost complex manifold \( \Sigma \). The Cauchy-Riemann operator \( \overline{\partial} \) of \( \nabla \) is the first order linear differential operator that takes any \( C^1 \) section \( \psi \) of \( \eta \) to the section \( \overline{\partial}\psi \) of \( \text{Hom}_\mathbb{C}(T\Sigma, \eta) \), where \( T\Sigma \) is the complex conjugate bundle of \( T\Sigma \), with

\[
\overline{\partial}\psi = [\nabla \psi]^-, 
\]

for \( [\ ]^- \) as in (20.1), so that its value \( [\overline{\partial}\psi]_x \) at any \( x \in \Sigma \) is \( ([\nabla \psi]_x)^- \). (Note that \( \nabla \psi \) is a section of \( \text{Hom}_\mathbb{R}(T\Sigma, \eta) \) sending \( v \in T_x \Sigma \), for any \( x \in \Sigma \), to \( \nabla_v \psi \in \eta_x \). Thus, \( [\overline{\partial}\psi]_x : T_x \Sigma \to \eta_x \), with \( 2[\overline{\partial}\psi]_x v = \nabla_v \psi + i\nabla_{\overline{\partial}\psi} \psi \) for \( v \in T_x \Sigma \).)

Given a real \( k \)-dimensional manifold \( \Sigma \) and an almost complex manifold \( M \), let \([TM]^\wedge k\) denote the \( k \)th complex exterior power of \( TM \), and let \( \det_R T\Sigma \) be as in (5.2.i). Any \( C^\infty \) mapping \( f : \Sigma \to M \) then gives rise to the vector bundle morphism \( \det df : \det_R T\Sigma \to f^*([TM]^\wedge k) \) uniquely characterized by the relation \( (\det df)(v_1 \wedge \ldots \wedge v_k) = [df_xv_1] \wedge \ldots \wedge [df_xv_k] \) for \( x \in \Sigma \) and \( v_1, \ldots, v_k \in T_x \Sigma \). Obviously, \( f \) is a totally real immersion if and only if \( \det df \) is nonzero everywhere (as a section of \( \text{Hom}_\mathbb{R}(\det_R T\Sigma, f^*([TM]^\wedge k)) \)). On the other hand, \( \det df = 0 \) identically whenever \( f \) is a pseudoholomorphic immersion, that is, it is an immersion and each \( df_x(T_x \Sigma) \) is a complex subspace of \( T_{f(x)} M \). (This may happen only if \( k \) is even, and \( \Sigma \) then acquires an obvious almost complex structure.) Note that, when \( k = 2 \) and \( f \) is an immersion, condition \( \det df = 0 \) is not only necessary, but also sufficient for \( f \) to be pseudoholomorphic.

Suppose now that \( \Sigma \) is a real surface and \( M \) is an almost complex surface (\( \dim_R \Sigma = \dim_C M = 2 \)), and let \( f^t : \Sigma \to M \) be a \( C^\infty \) curve (homotopy) of mappings parametrized by \( t \in I \) for some interval \( I \) containing \( 0 \), such that \( f^0 = f \) is a pseudoholomorphic immersion/embedding. Furthermore, let there be a fixed complex normal bundle of \( f \), that is, a complex vector subbundle \( \nu \) of \( f^*TM \) with \( f^*TM = \tau \oplus \nu \) (where \( \tau = T\Sigma \)), and a fixed connection \( D \) in the complex bundle \( TM \). The pull-back of \( D \) under \( f \), when projected onto the \( \nu \) summand, is a connection \( \nabla \) in the complex bundle \( \nu \), and so it gives rise to the corresponding Cauchy-Riemann operator \( \overline{\partial} \). For every \( x \in \Sigma \), denote by \( \omega_x \) the curve \( t \mapsto \omega_x(t) = (\det df^t)(x) \in \text{Hom}_\mathbb{R}([T_x \Sigma]^\wedge k, [T_{f(t,x)} M]^\wedge k) \) (where \( f^t(x) = f(t,x) \)). As \( f^t = f \) is pseudoholomorphic, it is clear from the definition of \( \det df \) that \( \omega_x = 0 \). Thus, at \( t = 0 \) and any \( x \in \Sigma \),

\[
\frac{D}{dt} \frac{\det df^t}{dt} \bigg|_{t=0} = 2 df \wedge \overline{\partial}\psi, \quad \text{and} \quad \det df^t \bigg|_{t=0} = 0, 
\]

where \( D[\det df^t](x)/dt = D \omega_x(t)/dt \) is the \( D \)-covariant derivative, while \( \psi \) is the section of \( \nu \) given by \( \psi = [(df^t)/dt]_0 \) norm, \( [\ ] \) norm being the projection morphism \( f^*TM \to \nu \). (Our notational convention is such that \( (df \wedge \overline{\partial}\psi)_x(v,w) = (df_xv) \wedge ([\overline{\partial}\psi]_x w) \) for \( x \in \Sigma \) and \( v, w \in T_x \Sigma \).) In fact, the first equality in (20.3) is easily verified in local coordinates.

21. PROOFS OF THEOREM 2.16 AND COROLLARY 2.17

**Lemma 21.1.** Let \( dx \) be a fixed positive measure density of class \( C^\infty \) on a compact manifold \( \Sigma \) with \( \dim \Sigma = n \geq 1 \).

(i) For any finite-dimensional vector space \( X \) of real-valued continuous functions on \( \Sigma \), there exist \( C^\infty \) functions \( f, h : \Sigma \to \mathbb{R} \), both \( L^2 \)-orthogonal to \( X \), such that \( |f| + |h| > 0 \) everywhere in \( \Sigma \).
(ii) For any finite-dimensional vector space $\mathcal{W}$ of complex-valued continuous functions on $\Sigma$, there exists a $C^\infty$ function $\varphi : \Sigma \to \mathbb{C}$ which is $L^2$-orthogonal to $\mathcal{W}$ and nonzero everywhere in $\Sigma$.

Proof. To prove (i), set $m = \dim X$, and let $\delta : \Sigma \to X^*$ be the $C^\infty$ mapping sending each $x$ to the evaluation functional (Dirac delta) $\delta[x] : X \to \mathbb{R}$. The image $\{\delta[x] : x \in \Sigma\}$ spans $X^*$, as otherwise it would lie in a proper subspace, that is, some $f \in X \setminus \{0\}$ would vanish at all $x \in \Sigma$. Thus, we may choose $2m$ distinct points $x_1, \ldots, x_m, y_1, \ldots, y_m \in \Sigma$ such that both $\delta[x_1], \ldots, \delta[x_m]$ and $\delta[y_1], \ldots, \delta[y_m]$ are bases of $X^*$, by first picking the $x_a$, and then selecting each $y_a$ near the corresponding $x_a$. Let us also select pairwise disjoint open sets $U_1, \ldots, U_m, U'_1, \ldots, U'_m$ in $M$ with $x_a \in U_a$ and $y_a \in U'_a$ for $a = 1, \ldots, m$.

There must exist a $C^\infty$ function $f : \Sigma \to \mathbb{R}$ which is $L^2$-orthogonal to $X$ and such that $f = 1$ on $M \setminus U$, where $U = U_1 \cup \ldots \cup U_m$. In fact, let $\phi_1, \ldots, \phi_m$ form a basis of $X$ dual to the $\delta[x_1], \ldots, \delta[x_m]$ in $X^*$. Thus, $\phi_a(x_b) = \delta_{ab}$ for $a, b = 1, \ldots, m$. The functions $\phi_1, \ldots, \phi_m$ are linearly independent, when treated as linear functionals acting on the $L^2$ inner product, on the space $\mathcal{F} = C_0^\infty(U_a, \mathbb{R})$ of all compactly supported $C^\infty$ functions $\Sigma \to \mathbb{R}$ whose supports are contained in $U$. (Otherwise, some nontrivial combination of the $\phi_a$ would be $L^2$-orthogonal to $\mathcal{F}$, and hence would vanish everywhere in $U$, which is impossible as $\phi_a(x_b) = \delta_{ab}$.) Consequently, for any given $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, there exists $\xi \in \mathcal{F}$ with $\int_U \phi_a \xi dx = \lambda_a$ for $a = 1, \ldots, m$. Choosing such $\xi$ for $\lambda_a = -\int_U \phi_a dx$, $a = 1, \ldots, m$, we can now define the required function $f$ by $f = \xi + 1$.

Similarly, since the same argument can be applied to the $y_a$ and $U'_a$ rather than $x_a$ and $U_a$, there exists a $C^\infty$ function $h : \Sigma \to \mathbb{R}$, $L^2$-orthogonal to $X$ and such that $h = 1$ on $M \setminus U'$ with $U' = U'_1 \cup \ldots \cup U'_m$. As $(M \setminus U) \cup (M \setminus U') = M$, assertion (i) follows.

Now (ii) is immediate from (i) if we set $\varphi = f + ih$ with $f, h$ chosen as in (i) for the space $X = \{\chi \in \mathcal{W} : \chi \in \mathcal{W}\}$. This completes the proof. □

Proof of Theorem 2.16. As $f^*TM = \tau \oplus \nu$, we have $f^*[\det M] = \tau \otimes \nu = \text{Hom}(\tau, \nu)$, and so the equivalence of (a) and (b) is obvious. Furthermore, (c) implies (a) in view of (5.1) with $n = 2$. Finally, let $\mathcal{O}$ be the Cauchy-Riemann operator (20.2) in $\nu$, for any fixed Riemannian metric on $M$ compatible with the almost complex structure. Since $\mathcal{O}$ is elliptic, equation $\mathcal{O}\psi = \phi$, imposed on sections $\psi$ of $\nu$, is solvable for $\psi$ if and only if $\phi$ on the right-hand side (which is a fixed section of $\text{Hom}(\tau, \nu)$) is $L^2$-orthogonal to the kernel of the formal adjoint of $\mathcal{O}$. Lemma 21.1 now implies that $\phi$ with this property may be chosen to be nowhere zero; in fact, in view of (a), we may treat $\phi$ as a function $\Sigma \to \mathbb{C}$. Using a deformation $t^4$ of $f$ in the direction of the corresponding solution $\psi$, we obtain (c) as an immediate consequence of (20.3). This completes the proof. □

Proof of Corollary 2.17. Let us choose a nonsingular complex curve $Q$ of degree $d$ in $\mathbb{CP}^2$ (e.g., the one given by $a^d + b^d + c^d = 0$ in the homogeneous coordinates $[a, b, c]$). Blowing up $3d$ points on $Q$ and $k - 3d$ points in $\mathbb{CP}^2 \setminus Q$, we thus obtain a nonsingular complex curve $\Sigma$ embedded in $M = \mathbb{CP}^2 \# k \mathbb{CP}^2$, which is diffeomorphic to $Q$ (and hence has the required genus) and, at the same time satisfies, along with $M$ and the inclusion mapping $f : \Sigma \to M$, condition (b) of Theorem 2.16; in fact, blowing up a point on a closed complex curve $\Sigma$ in a complex surface $M$ reduces by 1 the value of $c_1(M)$ integrated over $\Sigma$, while
for the original degree \( d \) curve that value equals \( 3d \). This implies assertion (c) of Theorem 2.16, and hence completes the proof. \( \square \)

22. The sets \( \mathcal{I}_q(\Sigma) \) and \( \mathcal{D}_q^\pm(M) \) in special cases

Given a real manifold \( \Sigma \), an almost complex manifold \( M \), along with \( \varepsilon \in \{0, 1\} \), a fixed sign \( \pm \), and \( q \in \{2, 4, \ldots, \infty\} \), let \( \mathcal{I}_q(\Sigma) \) and \( \mathcal{D}_q^\pm(M) \) be the sets defined by (2.9)–(2.10), with \( \mathbb{Z}_\infty = \mathbb{Z} \). Thus, as \( H^1(S^n, \mathbb{Z}_q) = \{0\} \) for \( n \geq 2 \),

\[
\mathcal{I}_q(S^n) = \{0\} \quad \text{whenever} \quad n \geq 2 \quad \text{and} \quad q \in \{2, 4, 6, \ldots, \infty\}.
\]

Since \( \mathcal{I}_q(\Sigma) \) is either empty or forms a coset of the subgroup of \( H^1(\Sigma, \mathbb{Z}_q) = \text{Hom}(\pi_1(\Sigma, \mathbb{Z}_q)) \) (see (1.5)) consisting of all homomorphisms valued in the even subgroup \( 2\mathbb{Z}_q \) (the image of \( \mathbb{Z}_q \) under the homomorphism \( \xi \to 2\xi \)), we have

\[
\mathcal{I}_q(T^n) = (2\mathbb{Z}_q)^n \subset (\mathbb{Z}_q)^n = H^1(T^n, \mathbb{Z}_q),
\]

where \((\mathbb{Z}_q)^n \subset (\mathbb{Z}_q)^n\) stands for the \( n \)th Cartesian power, and \( H^1(T^n, \mathbb{Z}_q) = (\mathbb{Z}_q)^n \) due to the standard identification resulting from (1.5) with \( \pi_1T^n = \mathbb{Z}^n \). Also, one easily verifies that, for all \( q \in \{2, 4, 6, \ldots, \infty\} \),

\[
\mathcal{I}_q(\mathbb{R}P^n) = \begin{cases} \{\varphi\} & \text{if } q/2 \text{ is finite and odd,} \\ \emptyset & \text{if } q = \infty \text{ or } q/2 \text{ is even,} \end{cases}
\]

where \( \varphi \in H^1(\mathbb{R}P^n, \mathbb{Z}_q) = \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_q) \) (cf. (1.5)) is the unique homomorphism \( \mathbb{Z}_2 = \{0, 1\} \to \{0, \ldots, q - 1\} = \mathbb{Z}_q \) with \( \varphi(1) = q/2 \). (Note that \( \pi_1[\mathbb{R}P^n] = \mathbb{Z}_2 \), and \( \varphi_1(\mathbb{R}P^n) \) with (1.6) is the identity homomorphism \( \mathbb{Z}_2 \to \mathbb{Z}_2 \).)

In the case where \( M = \mathbb{C}P^2 \), (2.9) gives besides \( \mathcal{D}_q^\pm(\mathbb{C}P^2) = \emptyset \),

\[
\mathcal{D}_q^1(\mathbb{C}P^2) = \{0\}, \quad \mathcal{D}_q^0(\mathbb{C}P^2) = \{0\}, \quad \mathcal{D}_q^0(\mathbb{C}P^2) = \{[\mathbb{R}P^2]\},
\]

\( [\mathbb{R}P^2] \) being the \( \mathbb{Z}_2 \)-homology class of the submanifold described in (vi) of \$7 \$ with \( n = 2 \). (In fact, \( c_1(M) : H_2(M, \mathbb{Z}) \to \mathbb{Z} \) and \( w_3(M) : H_2(M, \mathbb{Z}_2) \to \mathbb{Z}_2 \) then are isomorphisms.) Similarly, for \( M = \mathbb{C}^2 \), relation \( H_2(C^2, \mathbb{Z}_{[2]} = 0 \) (cf. (2.4)) yields

\[
\mathcal{D}_q^1(\mathbb{C}^2) = \{0\}, \quad \mathcal{D}_q^0(\mathbb{C}^2) = \{0\}, \quad \mathcal{D}_q^0(\mathbb{C}^2) = \mathcal{D}_q^1(\mathbb{C}^2) = \emptyset.
\]

On the other hand, for any \( \varepsilon \in \{0, 1\} \) and either sign \( \pm \), (2.9) implies that

\[
\mathcal{D}_q^1(M) + \mathcal{D}_q^\pm(M) = \mathcal{D}_q^\pm(M).
\]

The ‘sum of sets’ stands here for the set of all sums, while adding an element \( H_2(M, \mathbb{Z}) \) to an element of \( H_2(M, \mathbb{Z}_2) \) is to be preceded by mod 2 reduction of the former, so that the sum lies in \( H_2(M, \mathbb{Z}_2) \).

23. Closed surfaces and cohomology

In this section we gather some standard facts for easy reference.

Given a closed real surface \( \Sigma \) and a small open 2-disk \( U \) embedded in \( \Sigma \), the inclusion \( \Sigma \smallsetminus U \to \Sigma \) induces a surjective homomorphism \( H_1(\Sigma \smallsetminus U, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}) \) whose kernel \( \Gamma \) is generated by the boundary circle \( \partial U \). We have \( \Gamma = \{0\} \) when \( \Sigma \) is orientable, and \( \Gamma \approx \mathbb{Z} \) if it is not.

Let us now consider two closed surfaces \( \Sigma \) and \( \Sigma' \), with \( \Gamma \subset H_1(\Sigma \smallsetminus U, \mathbb{Z}) \) and \( \Gamma' \subset H_1(\Sigma' \smallsetminus U', \mathbb{Z}) \) obtained as above using disks \( U \subset \Sigma, U' \subset \Sigma' \). The connected sum \( \Sigma \# \Sigma' \) results from gluing \( \Sigma \smallsetminus U \) and \( \Sigma' \smallsetminus U' \) together with the aid of a fixed identification \( \partial U = \partial U' \) of the boundary circles. The kernel of the homomorphism \( \phi : H_1(\Sigma \smallsetminus U, \mathbb{Z}) \to H_1(\Sigma \# \Sigma', \mathbb{Z}) \) induced by the inclusion
mapping is \( \Gamma \) or \( \{0\} \), depending on whether the other surface \( \Sigma' \) is orientable or not, while the images of \( \phi \) and its analogue \( \phi' \) for \( \Sigma' \) generate all of \( H_1(\Sigma \# \Sigma', \mathbb{Z}) \) and have the intersection \( \Gamma_\phi \), which is the subgroup of \( H_1(\Sigma \# \Sigma', \mathbb{Z}) \) generated by the circle \( \partial U = \partial U' \). Then \( \Gamma_\phi = \{0\} \) if at least one of \( \Sigma, \Sigma' \) is orientable, and \( \Gamma_\phi \approx \mathbb{Z} \) otherwise. Consequently,

(i) If both surfaces \( \Sigma, \Sigma' \) are orientable, the obvious inclusion mappings involving \( \Sigma, \Sigma', \Sigma \setminus U, \Sigma' \setminus U' \) and \( \Sigma \# \Sigma' \) provide a natural isomorphic identification

\[
H_1(\Sigma \# \Sigma', \mathbb{Z}) = H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma', \mathbb{Z}) .
\]

(ii) When \( \Sigma \) is orientable but \( \Sigma' \) is not, we still have the identification (23.1). This time, however, \( H_1(\Sigma', \mathbb{Z}) \) is identified with the corresponding subgroup of \( H_1(\Sigma \# \Sigma', \mathbb{Z}) \) since they both are images of \( H_1(\Sigma' \setminus U') \) under two homomorphisms having the same kernel \( \Gamma' \approx \mathbb{Z} \).

Remark 23.1. Let us now consider the Klein bottle \( K^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \). Its fundamental group \( \Gamma = \pi_1 K^2 \) has the generators \( \Phi, \Psi \) defined in Example 16.1, where \( \Gamma \) is treated as a group of deck transformations in \( \mathbb{R}^2 \). Since \( \Psi^2 \) equals the commutator \( \Phi \Psi \Phi^{-1} \Psi^{-1} \), the Abelianization \( H_1(K^2, \mathbb{Z}) \) of the group \( \Gamma \) will from now on be identified with the direct product \( \mathbb{Z} \times \mathbb{Z}_2 \) whose factor groups are generated by \( \Phi \) and, respectively, \( \Psi \). Since \( \Psi \) is orientation-preserving while \( \Phi \) is not, \( w_1(K^2) \) treated as the orientation homomorphism \( \pi_1 K^2 \to \mathbb{Z}_2 \) (cf. (1.6)) sends \( \Phi \) and \( \Psi \) to 1 and 0 in \( \mathbb{Z}_2 = \{0,1\} \). Thus, as a homomorphism \( \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}_2 \), \( w_1(K^2) \) is given by \((k,\varepsilon) \mapsto [k \mod 2]\). Finally, the transformation \((s,t) \mapsto (s,-t)\) in \( \mathbb{R}^2 \) commutes with \( \Psi \) and conjugates \( \Phi \) with \( \Phi^{-1} \), so that it descends to a diffeomorphism \( K^2 \to K^2 \) of \( K^2 = \mathbb{R}^2/\Gamma \) whose action in \( H_1(K^2, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2 \) clearly is \((k,\varepsilon) \mapsto (-k,\varepsilon)\).

Given an Abelian group \( G \), we will use the standard identifications

\[
(\mathbb{Z}_q)_\text{ord}_2 = \begin{cases} \{0,q/2\} & \text{if } q \text{ is finite,} \\ \{0\} & \text{if } q = \infty, \end{cases}
\]

and (22.3) can be rewritten as

\[
J_q(\mathbb{RP}^n) = (\mathbb{Z}_q)_\text{ord}_2 \times 2\mathbb{Z}_q , \quad q \in \{2,4,6,\ldots,\infty\}.
\]

Remark 23.2. Let \( G,G_1,\ldots,G_k \) be Abelian groups and let \( \Sigma \) be a manifold with \( H_1(\Sigma, \mathbb{Z}) = G \times \cdots \times G_k \). Then

(i) \( H^1(\Sigma,G) = \text{Hom}(G_1,G) \times \cdots \times \text{Hom}(G_k,G) \). To obtain such an isomorphic identification, we first identify \( H^1(\Sigma,G) \) with \( \text{Hom}(H_1(\Sigma, \mathbb{Z}), G) \) (see (1.5)) and then assign to any homomorphism \( H_1(\Sigma, \mathbb{Z}) \to G \) the \( k \)-tuple of its restrictions to the factors \( G_1,\ldots,G_k \) treated as subgroups of the direct product \( H_1(\Sigma, \mathbb{Z}) \).

(ii) For any given homomorphism \( h : G \to G' \) of Abelian groups, the corresponding coefficient-reduction homomorphism \( H^1(\Sigma,G) \to H^1(\Sigma,G') \)
acts as the Cartesian product of the analogous homomorphisms for the factor groups (with both $H^1(\Sigma, G)$, $H^1(\Sigma, G')$ decomposed as in (i)).

Any Abelian group $G$ now gives rise to isomorphic identifications

\begin{equation}
(23.5) \quad a) \quad H^1(K^2, G) = G \times G_{\text{ord} 2} \quad b) \quad H^1(\Sigma \not\equiv \Sigma', G) = H^1(\Sigma, G) \times H^1(\Sigma', G),
\end{equation}

for any two closed real surfaces $\Sigma$ and $\Sigma'$ such that $\Sigma$ is orientable. This is clear if we combine Remark 23.2(i) and (23.2) with the relation $H_1(K^2, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2$ (see Remark 23.1) or, respectively, (23.1). Note that the diffeomorphism $K^2 \to K^2$ described at the end of Remark 23.1 now acts in $H^1(K^2, G) = G \times G_{\text{ord} 2}$ by

\begin{equation}
(23.6) \quad (a, b) \mapsto (-a, b).
\end{equation}

Furthermore, in terms of (23.5.b) with $G = \mathbb{Z}_2$ we have

\begin{equation}
(23.7) \quad w_1(\Sigma \not\equiv \Sigma') = (w_1(\Sigma), w_1(\Sigma')) \quad \text{if} \quad \Sigma \text{ is orientable},
\end{equation}

since, by (1.6), $w_1(\Sigma \not\equiv \Sigma')$ (acting on $H_1(\cdots, \mathbb{Z})$ rather than $\pi_1(\cdots)$) coincides with the second-factor projection in (23.1) followed by $w_1(\Sigma')$. Using Remark 23.2(ii) and (23.7) along with (2.10), we now see that

\begin{equation}
(23.8) \quad \mathcal{J}_q(\Sigma \not\equiv \Sigma') = \mathcal{J}_q(\Sigma) \times \mathcal{J}_q(\Sigma'), \quad \Sigma \text{ orientable}, \quad q \in \{2, 4, 6, \ldots, \infty\}.
\end{equation}

Note that for $\Sigma = S^2$ relation (23.8) becomes $\mathcal{J}_q(S^2 \not\equiv \Sigma') = \mathcal{J}_q(\Sigma')$ (cf. (22.1)), as long as we agree to identify $\{0\} \times H^1(\Sigma', \mathbb{Z}_q)$ with $H^1(\Sigma', \mathbb{Z}_q)$. Also,

\begin{equation}
(23.9) \quad \mathcal{J}_q(K^2) = (\mathbb{Z}_q \setminus 2\mathbb{Z}_q) \times ([\mathbb{Z}_q]_{\text{ord} 2} \cap 2\mathbb{Z}_q).
\end{equation}

In fact, combining the description of $w_1(K^2)$ in Remark 23.1 with the identification (23.5.a) for $G = \mathbb{Z}_2$ we get $w_1(K^2) = (1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ (as $(\mathbb{Z}_2)_{\text{ord} 2} = \mathbb{Z}_2$). Now (23.9) is immediate from Remark 23.2(ii) and (23.5.a) for $G = \mathbb{Z}_2$. For a closed real surface $\Sigma$ of genus $g$ we have, up to a diffeomorphism, one of three cases:

(a) $\Sigma$ is orientable and $\Sigma = T^2 \# \ldots \# T^2$ ($g$ summands), with $\Sigma = S^2$ when $g = 0$.

(b) $\Sigma$ is nonorientable, $g$ even, $\Sigma = T^2 \# \ldots \# T^2 \# K^2$, with $(g - 2)/2$ torus summands, so that $\Sigma$ is the Klein bottle $K^2$ when $g = 2$.

(c) $\Sigma$ is nonorientable, $g$ odd, $\Sigma = T^2 \# \ldots \# T^2 \# \mathbb{RP}^2$, with $(g - 1)/2$ torus summands; thus, $\Sigma$ is the projective plane $\mathbb{RP}^2$ when $g = 1$.

The Euler characteristic of $\Sigma$ then is given by

\begin{equation}
(23.10) \quad \chi(\Sigma) = 2 - 2g \quad \text{(case (a))}, \quad \chi(\Sigma) = 2 - g \quad \text{(cases (b), (c))}.
\end{equation}

Therefore, every closed surface either is one of $S^2$, $T^2$, $\mathbb{RP}^2$, $K^2$, or can be obtained by iterated connected summation in which all intermediate connected sums and summands, with a possible exception of the last one, are orientable; this makes formula (23.8) applicable at all steps, and so, using (22.2), (23.9), (23.4) and letting superscripts stand for Cartesian powers, we have

\begin{equation}
(23.11) \quad \mathcal{J}_q(\Sigma) = (2\mathbb{Z}_q)^{2g} \quad \text{in case (a)}, \quad \mathcal{J}_q(\Sigma) = (2\mathbb{Z}_q)^{g - 2} \times (\mathbb{Z}_q \setminus 2\mathbb{Z}_q) \times ([\mathbb{Z}_q]_{\text{ord} 2} \cap 2\mathbb{Z}_q) \quad \text{in case (b)}, \quad \mathcal{J}_q(\Sigma) = (2\mathbb{Z}_q)^{g - 1} \times ([\mathbb{Z}_q]_{\text{ord} 2} \setminus 2\mathbb{Z}_q) \quad \text{in case (c)}.
\end{equation}

Let $q \geq 2$ be finite and even, and let $\Sigma$ be a closed manifold of any dimension. The mod $q$ reduction homomorphism $H^1(\Sigma, \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z}_q)$ sends the set $\mathcal{J}_q(\Sigma) \subset H^1(\Sigma, \mathbb{Z})$ into $\mathcal{J}_q(\Sigma) \subset H^1(\Sigma, \mathbb{Z}_q)$ (with $\mathcal{J}_q(\Sigma) \subset H^1(\Sigma, \mathbb{Z}_q)$ as in (2.10)). This is clear from Remark 1.1, since reduction mod $q$ ($q$ even) followed
by reduction mod 2 gives reduction mod 2. We thus obtain the mod $q$ reduction mapping

$$J_\infty(\Sigma) \xrightarrow{\text{mod } q} J_q(\Sigma).$$

\textbf{Lemma 23.3.} Given a closed real surface $\Sigma$ and an even positive integer $q$, 

(i) The mapping (23.12) is surjective if and only if either $\Sigma$ is orientable, or $\Sigma$ is nonorientable and $\chi(\Sigma) - q/2$ is odd.

(ii) If $q = 4$ and $\Sigma$ is the Klein bottle $K^2$, the image of (23.12) is the 2-element subset $\{1, 3\} \times \{0\}$ of the 4-element set

$$J_4(K^2) = \{(1, 3) \times \{0, 2\},$$

which itself is a subset of $H^1(K^2, \mathbb{Z}_4) = \{0, 1, 2, 3\} \times \{0, 2\} \subset \mathbb{Z}_4 \times \mathbb{Z}_4$, cf. (23.5.a), with $\mathbb{Z}_4 = \{0, 1, 2, 3\}$.

\textbf{Proof.} According to Remark 23.2(ii), (23.12) acts by mod $q$ reduction in each factor set appearing in (23.11). Moreover,

(*) the reduction mappings $\mathbb{Z} \to \mathbb{Z}_q$ and $\mathbb{Z} \setminus 2\mathbb{Z} \to \mathbb{Z}_q \setminus 2\mathbb{Z}_q$ are surjective (since $q$ is even); and

(**) $\mathbb{Z}_{ord^2} \cap 2\mathbb{Z} = \{0\}$ and $\mathbb{Z}_{ord^2} \setminus 2\mathbb{Z} = \emptyset$ (by (23.3) with $\mathbb{Z}_\infty = \mathbb{Z}$).

Thus, surjectivity of (23.12) always holds in case (a) of (23.11), while in cases (b), (c) it is equivalent to $(\mathbb{Z}_q)_{ord^2} \cap 2\mathbb{Z}_q = \{0\}$ and, respectively, $(\mathbb{Z}_q)_{ord^2} \setminus 2\mathbb{Z}_q = \emptyset$. Now (i) is immediate from (23.3) and (23.10). On the other hand, (23.5.a) and (23.3) give $H^1(K^2, \mathbb{Z}_4) = \{0, 1, 2, 3\} \times \{0, 2\}$, so that (23.13) is nothing else than (23.9) for $q = 4$. Assertion (ii) now follows from (*) and (**), as $(\mathbb{Z}_4)_{ord^2} \cap 2\mathbb{Z}_4 = \{0, 2\}$ and $\mathbb{Z}_{ord^2} \cap 2\mathbb{Z}_4 = \{0\}$. This completes the proof. 

\section*{24. More on Tori and Klein Bottles in $\mathbb{C}^2$}

We will now evaluate the Maslov index $i$ and degree $d$ for the totally real embeddings $f^{k,l}$ constructed in Example 16.2.

When $\Sigma$ is the torus $T^2$ or the Klein bottle $K^2$, the identifications (22.2) and (23.9) with $n = 2$ and $q = \infty$ become $J_\infty(T^2) = 2\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z} = H^1(T^2, \mathbb{Z})$ and $J_\infty(K^2) = (\mathbb{Z} \setminus 2\mathbb{Z}) \times \{0\} \subset \mathbb{Z} \times \{0\} = H^1(K^2, \mathbb{Z})$ (cf. (23.5.a), (23.3)).

A totally real immersion $(x, u) : \mathbb{R}^3 \rightarrow \mathbb{C}^2$ which is doubly $2\pi$-periodic (or, in addition, also invariant under the transformation $\Phi$ defined in Example 16.1) descends to a totally real immersion $f : \Sigma \rightarrow \mathbb{C}^2$ with $\Sigma = T^2$ (or, respectively, $\Sigma = K^2$). Let us now set $\ell = 1$ for the torus, $\ell = 2$ for the Klein bottle, and

$$p = \frac{1}{2\pi i} \int_0^{2\pi} \phi_x \, ds, \quad q = \frac{\ell}{2\pi i} \int_0^{2\pi/\ell} \phi_x \, dt,$$

with $\phi = \mathcal{J}(x, u)$, where $\mathcal{J}(x, u)$ is defined as in Example 7.1, and the subscripts represent partial derivatives. Thus, $p, q \in \mathbb{Z}$ and the Maslov index of $f$ (see §2) is

$$i(f) = \begin{cases} (2p, 2q) \in 2\mathbb{Z} \times 2\mathbb{Z} = J_\infty(\Sigma) & \text{if } \Sigma = T^2, \\ (q, 0) \in (\mathbb{Z} \setminus 2\mathbb{Z}) \times \{0\} = J_\infty(K^2) & \text{if } \Sigma = K^2. \end{cases}$$

\textbf{Proposition 24.1.} Let $\Sigma$ stand for the 2-torus $T^2$ or the Klein bottle $K^2$, and let $f : \Sigma \rightarrow \mathbb{C}^2$ be a totally real immersion obtained as in Example 16.1 from some
\[ x(s, t), y(s, t) \text{ and } h(s, t) = h(t) \] satisfying the hypotheses of Example 16.1. Then \( f \) has the Maslov index \((24.2)\) with \( p \) and \( q \) given by
\[
(24.3) \quad p = \frac{1}{2\pi i} \int_0^{2\pi} \frac{x_s}{x_s} ds, \quad q = \frac{\ell}{2\pi} \int_0^{2\pi/\ell} \left[ \frac{x_{st}}{x_s + h_{tt}} \right] dt
\]
for \( \ell \) as in \((24.1)\), the subscripts representing the partial derivatives.

**Proof.** Set \( \phi^{[r]} = r^{-1} J(x, y + rh) \) (notation of Example 7.1). We thus have \((24.1)\) with \( \phi = \phi^{[r]} \). Homotopy invariance of the degree guarantees that the integral \((24.1)\) is constant in \( r \), for large \( r > 0 \). Taking its limit as \( r \to \infty \) and noting that \( \lim_{r \to \infty} \phi^{[r]} = x_s h_t \) by \((16.1)\), we obtain \((24.3)\). In view of Lemma 16.1, this completes the proof. \( \square \)

**Proposition 24.2.** There exist totally real embeddings of the 2-torus \( T^2 \) and the Klein bottle \( K^2 \) in \( C^2 \) which realize any prescribed Maslov index \((2p, 2q)\) or \((q, 0)\) with \((24.2)\).

To exhibit examples of such embeddings, let us first apply Proposition 24.1 to the totally real embeddings \( f = f^{k, 1} : T^2 \to C^2 \) for \( k \in \mathbb{Z} \) and \( f = f^{k, 2} : K^2 \to C^2 \) for odd \( k \), described in Example 16.2. The mapping \( h : R/2\pi Z \to S^1 \) defined by the assignment \( R \ni s \mapsto x_s/|x_s| \) (with any fixed \( t \)) now is of degree zero (since \( x_s(s + \pi, t) = -x_s(s, t) \), and so \( h \) is homotopic to its composite with the conjugation). Therefore, by \((24.3)\), we have \( p = 0 \), that is, \( f^{k, 1} \) and \( f^{k, 2} \) have the Maslov indices \((2p, 2q) = (0, 2k + 2)\) and, respectively, \((q, 0) = (k + 2, 0)\). We thus have realized all index values for the Klein bottle; to obtain an arbitrary Maslov index \((2p, 2q)\) for the torus, we set \( k = -1 \) (when \( p = q = 0 \)) or, when \((p, q) \neq (0, 0)\), find a matrix \( \mathfrak{A} \in SL(2, \mathbb{Z}) \) sending the vector \((p, q)\) onto \((0, d)\) for a suitable integer \( d \) (the greatest common factor of \( p \) and \( q \)), and use the composite of our \( f \), for \( k = a - 1 \), with the group automorphism of \( T^2 \) corresponding to \( \mathfrak{A} \). Specifically, we may choose integers \( a, b \) such that \( ap + bq = d \), and then set
\[
(24.4) \quad \mathfrak{A} = \begin{bmatrix} q/d & -p/d \\ a & b \end{bmatrix}.
\]

**25. Index and Degree After Modifications**

The Maslov index \( i \) and degree \( d \) of a totally real immersion in a simply connected almost complex surface were defined in §2. Here we determine how they are affected by the operations of “zooming” and “connected sum” described in §8 and §14.

**Lemma 25.1.** Let \( f \) be a totally real immersion/embedding of a closed real surface \( \Sigma \) in \( C^2 \), and let \( M \) be any simply connected almost complex surface. For a totally real immersion/embedding \( f' : \Sigma \to M \) obtained from \( f \) using a suitable zooming procedure as described in Proposition 8.1, the Maslov index and degree are given by
\[
(25.1) \quad i(f') = [i(f) \text{ mod } q] \in H^1(\Sigma, \mathbb{Z}_q), \quad d(f') = 0,
\]
with \( q \) as in \((2.7)\), \([i(f) \text{ mod } q]\) being the mod \( q \) reduction of \( i(f) \in H^1(\Sigma, \mathbb{Z}) \), that is, its image under the mapping \( (23.12) \).

**Proof.** Let \( B \subset C^2 \) be a ball containing \( f(\Sigma) \). We have \( f'_q(\Sigma) = 0 \) since \( B \) is contractible. On the other hand, \( i(f') \) is the homomorphism of fundamental groups \((3.5)\) induced by the homotopy class \( \mathfrak{M}(f') \) corresponding to \( f' \), and so \( i(f') \) is
nothing else than $i(f)$ followed by the homomorphism $\pi_1[E(B)] \to \pi_1[E(M)]$ induced by an embedding of $B$ in $M$. (Since the latter homomorphism acts via reduction mod $q$, this completes the proof. \hfill \Box

Remark 25.2. Lemma 25.1 becomes particularly simple when $M$ is a complex surface; instead of zooming, we then may obtain $f'$ as the composite of $f$ with a holomorphic embedding in $M$ of an open ball in $\mathbb{C}^2$ containing $f(\Sigma)$.

Corollary 25.3. Given a simply connected almost complex surface $M$, consider the following condition imposed on a closed real surface $\Sigma$ and an element $i \in \mathcal{I}_q(\Sigma)$, with $q \in \{2, 4, 6, \ldots, \infty\}$ defined by (1.4):

\begin{itemize}
  \item[(*)] some totally real embedding $f : \Sigma \to M$ has $i(f) = i$ and $d(f) = 0$.
\end{itemize}

Condition (*) is satisfied by

\begin{itemize}
  \item[(a)] the torus $\Sigma = T^2$ and every $i \in \mathcal{I}_q(\Sigma)$,
  \item[(b)] the Klein bottle $\Sigma = K^2$ and every $i \in \mathcal{I}_q(\Sigma)$, provided that $q/2$ is either infinite, or finite and odd,
  \item[(c)] the Klein bottle $\Sigma = K^2$ and every $i$ in the subset $\{1, 3\} \times \{0\}$ of $\mathcal{I}_q(\Sigma)$, cf. (23.13), provided that $q = 4$.
\end{itemize}

This is clear from (25.1), Example 24.2 and Lemma 23.3(i). We also have

Lemma 25.4. Let $f, f'$ be totally real immersions of closed real surfaces $\Sigma, \Sigma'$ in a simply connected almost complex surface $M$. If $\Sigma$ is orientable, then a totally real immersion $\Sigma \# \Sigma' \to M$ of the connected sum, obtained from $f$ and $f'$ as in Lemma 14.2 and Remark ..(a), has the Maslov index $i$ and degree $d$ given by

\begin{equation}
(i, d) = (i(f), i(f')) , \quad d = d(f) + d(f') \in H_2(M, \mathbb{Z}_{23}) ,
\end{equation}

in agreement with the notational conventions of (23.5), (22.6) and (2.4).

In fact, the first relation is clear since the new immersion coincides with $f$ and $f'$ except in small disks removed from $\Sigma$ and $\Sigma'$. As for the second one, it is nothing else than Remark 14.3. Relations (23.8) and (22.6), along with (25.2), now yield

Corollary 25.5. Given a simply connected almost complex surface $M$, let $X$ be the class of diffeomorphism types of closed real surfaces $\Sigma$ such that for every pair

\begin{equation}
(i, d) \in \mathcal{I}_q(\Sigma) \times \mathcal{D}_q^k(M) ,
\end{equation}

with $q, q \in \mathbb{Z}$ determined by $M$, $\Sigma$ as in (1.4) and (2.8), there exists a totally real immersion $f : \Sigma \to M$ with $i(f) = i$ and $d(f) = d$. The class $X$ then is closed under the connected-sum operation applied to two surfaces, at least one of which is orientable.

For a fixed simply connected almost complex surface $M$ and a closed real surface $\Sigma$, let us consider two conditions (with $q, q \in \mathbb{Z}$ as in (1.4) and (2.8)):

\begin{equation}
\text{there exist totally real immersions } S^2 \rightarrow M \text{ realizing every degree } d \text{ in some set generating the group } \mathcal{D}_q^k(M) = \ker [c_1(M)] \subset H_2(M, \mathbb{Z}) .
\end{equation}

\begin{equation}
\text{Every } i \in \mathcal{I}_q(\Sigma) \text{ equals } i(f) \text{ for some totally real immersion } \Sigma \rightarrow M .
\end{equation}
Remark 25.6. Condition (25.4) is satisfied by $M = \mathbb{C}^2$ and $M = \mathbb{CP}^2$. In fact, a totally real immersion $S^2 \to M$ exists (Corollary 9.3), and its degree is necessarily the unique element of $\mathfrak{D}_q^p(M)$ (see (2.8) and (22.4), (22.5)). On the other hand, for any $M$, Corollary 25.3 shows that (25.5) holds for $\Sigma = T^2$ and, if $q/2$ is odd or infinite, also for $\Sigma = K^2$.

Lemma 25.7. Given a simply connected almost complex surface $M$ satisfying condition (25.4), let $\mathcal{X}$ be the class defined in Corollary 25.5.

(i) If (25.5) holds for $\Sigma = K^2$, then the class $\mathcal{X}$ contains every closed real surface whose Euler characteristic is even.

(ii) The assumption, and hence conclusion, of (i) is satisfied whenever, for $q$ appearing in (2.7), $q/2$ is infinite or finite and odd.

(iii) Under the assumption of (i), if $M$ also admits a totally real immersion of the real projective plane $\Sigma = \mathbb{RP}^2$, then the class $\mathcal{X}$ contains all closed real surfaces.

This is immediate from Corollaries 25.3 and 25.5 along with the fact that, if a totally real immersion $\mathbb{RP}^2 \to M$ exists, condition (25.5) follows for $\Sigma = \mathbb{RP}^2$ (since, by (22.3), $\mathfrak{D}_q(\mathbb{RP}^2)$ then is a one-element set).

26. The main result on totally real immersions

We now show that totally real immersions of closed real surfaces $\Sigma$ in the complex surfaces (0,1) realize, as its Maslov index and degree, every pair $(i, d)$ in a set $Z \subset \mathfrak{D}_q(\Sigma) \times \mathfrak{D}_p^q(M)$ with the properties listed in the lines following Theorem 2.2. Note that, in our case, $Z = \mathfrak{D}_q(\Sigma) \times \mathfrak{D}_p^q(M)$ except when the complex surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\Sigma$ is nonorientable, with an even Euler characteristic. Although this result, as stated, is a special case of Theorem 2.2, we will establish it separately, using – in contrast with our proof of Theorem 2.2 in §6 – only explicit geometric constructions.

In this section we prove Theorem 26.1 for the complex surfaces $\mathbb{C}^2$, $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. The proof in the case of $\mathbb{CP}^2 \# k \mathbb{CP}^2$, $k \geq 1$, will be given in §27.

Theorem 26.1. Let $M$ be one of the complex surfaces (0.1). For any $(i, d) \in Z$ there exists a totally real immersion $f : \Sigma \to M$ such that $i(f) = i$ and $d(f) = d$.

For $M = \mathbb{C}^2$ and $M = \mathbb{CP}^2$, the assertion is obvious from Lemma 25.7(i) and Corollary 25.3, since $q = \infty$ for $\mathbb{C}^2$ and $q/2 = 3$ is odd for $\mathbb{CP}^2$, cf. Remark 15.3.

We will prove Theorem 26.1 for $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ at the end of this section. First, let us introduce the standard isomorphic identification

\begin{equation}
H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z}_{[2]}^p) = \mathbb{Z}_{[2]} \times \mathbb{Z}_{[2]}^p, \text{ with } \mathbb{Z}_{[2]} \text{ as in (2.4),}
\end{equation}

induced by the product decomposition of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The generators $(1,0), (0,1)$ of $\mathbb{Z}_{[2]} \times \mathbb{Z}_{[2]}^p$ thus correspond to cycles of the form $\mathbb{CP}^1 \times \{y\}$ and $\{x\} \times \mathbb{CP}^1$. The $\mathbb{Z}_{[2]}$-valued intersection form $\cdot : H_2(M, \mathbb{Z}_{[2]}^p)$ is therefore given by

\begin{equation}
(a, b) \cdot (c, d) = ad + bc
\end{equation}

for $a, b, c, d \in \mathbb{Z}_{[2]}$. Moreover, the first Chern class of $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ acting as a homomorphism $\mathbb{Z} \times \mathbb{Z} = H_2(M, \mathbb{Z}) \to \mathbb{Z}$ (cf. (26.1)) is given by $(a, b) \mapsto 2(a + b)$. Condition (5.3b), necessary for $d = (a, b)$ to be the degree $d(f) \in H_2(M, \mathbb{Z}) \to \mathbb{Z}$
of some totally real immersion \(f : \Sigma \to M\) of an oriented closed real surface \(\Sigma\), thus reads
\[
a + b = 0.
\]

We have the following immediate generalization of (iv) in §7:

**Lemma 26.2.** Suppose that \(N, P\) are almost complex manifolds, \(\Sigma\) is the total space and \(pr : \Sigma \to Q\) is the projection of a locally trivial fibre bundle over a real manifold \(Q\), and \(h : Q \to N\) is a totally real immersion/embedding. Moreover, let \(\Phi : \Sigma \to P\) be a fibrewise-totally real Gauss mapping, that is, a \(C^\infty\) mapping whose restriction to each fibre \(\Sigma_y\), \(y \in Q\), is a totally real immersion/embedding. Then \(f\) given by \(f(x) = (h(pr(x)), \Phi(x))\), for \(x \in \Sigma\), is a totally real immersion/embedding of \(\Sigma\) into the product almost complex manifold \(M = N \times P\).

This is clear since nonzero horizontal and vertical vectors in \(\Sigma\) have \(df\)-images that are linearly independent over \(C\).

**Lemma 26.3.** A totally real immersion/embedding \(f : K^2 \to \mathbb{C}P^1 \times \mathbb{C}P^1\) obtained by applying Lemma 26.2 to the complex manifolds \(N = P = \mathbb{C}P^1\) and the Klein bottle \(\Sigma = K^2\) treated as a circle bundle over the circle \(Q = S^1\), with any immersion/embedding \(h : Q \to N\) and any given fibrewise-totally real Gauss mapping \(\Phi : K^2 \to \mathbb{C}P^1\), has the degree \((0,1)\) and the Maslov index equal, as an element of the set \((23,13)\), to
\[
i(f) = (1,2) \quad \text{or} \quad i(f) = (3,2).
\]

Furthermore, the values \((1,2)\) and \((3,2)\) become interchanged if we replace \(f\) by its composite with the diffeomorphism \(K^2 \to K^2\) acting in \(H^1(K^2, \mathbb{Z}_4)\) via \((23.6)\).

**Proof.** Specifically, let us identify \(\mathbb{C}P^1\) with the unit sphere \(S^2\) centered at zero in \(\mathbb{R}^3\). To obtain a totally real Klein bottle embedded in \(\mathbb{C}P^1 \times \mathbb{C}P^1\) in the manner described in Lemma 26.3, we fix any embedded circle \(Q \subset \mathbb{C}P^1\) as the base, and choose the fibres to be a family of great circles in \(S^2 = \mathbb{C}P^1\), all containing a fixed pair of antipodal points \(u, -u \in S^2\). Furthermore, as a point \(y\) varies in the base \(Q\), we require the corresponding fibres \(\Sigma_y\) to vary by being rotated about the axis \(Ru\) in \(\mathbb{R}^3\) in such a way that, after \(y\) has traveled all the way around the base circle, the fibre circle will have undergone a total rotation by an angle which is an odd multiple of \(\pi\).

That the degree equals \((0,1)\) is clear if one considers the action in \(\mathbb{Z}_2\) homology of the factor projections \(\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1\).

Lemma 26.2 may be applied since \(h\) is totally real (cf. (i) in §7). According to (15.3.c) and (23.13), \(i(f) = (k, l)\) with \(k \in \{1, 3\}\) and \(l \in \{0, 2\}\). Thus, (26.4) will follow if we show that \(l \neq 0\). To this end, let us note that a fibre circle in \(Klein\) bottle, mapped by \(\mathcal{M}(f)\) into \(E(M)\) (where \(M = \mathbb{C}P^1 \times \mathbb{C}P^1\) and \(f : Klein\) bottleneck \(\to M\) is the inclusion mapping), produces a great semicircle in \(S^3\) (notation as in (15.2)), and hence \(l \neq 0\), as required. The final clause in turn follows since \(3\) is the opposite of \(1\) in \(\mathbb{Z}_4\). This completes the proof. \(\square\)

**Proof of Theorem 26.1 in the case \(M = \mathbb{C}P^1 \times \mathbb{C}P^1\).** Condition (25.4) holds for \(M = \mathbb{C}P^1 \times \mathbb{C}P^1\). Namely, it is satisfied by the generating set for the group \(\text{Ker} [c_1(M)]\) formed by \((\pm 1, \mp 1)\) (cf. (26.3)): writing the degree of an “anti-diagonal” 2-sphere of (vii) in §7 as \(d = (a, b)\) we get \(b = \mp 1 \in \mathbb{Z}\), since the
mapping \( x \mapsto (\pi, x) \), followed by the projection of \( M \) onto the second factor, yields the identity of \( \mathbb{C}P^1 \). Now (26.3) gives \( a = \pm 1 \).

Furthermore, condition (25.5) with \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \) holds for the Klein bottle \( \Sigma = K^2 \). The assertion of Theorem 26.1 for \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \) now follows from Lemma 25.7(i).

27. Surfaces immersed in \( \mathbb{C}P^2 \# k \mathbb{C}P^2 \)

Before proving, at the end of this section, the remaining case of Theorem 26.1, we introduce some notational conventions. Let \( M = \mathbb{C}P^2 \# k \mathbb{C}P^2 \) stand, as usual, for the complex surface obtained by blowing up \( k \geq 1 \) distinct points in \( \mathbb{C}P^2 \). Also, let \( \langle \cdot, \cdot \rangle \) and \( \mathbb{Z}^k \) be the standard Euclidean inner product of \( \mathbb{R}^k \), its associated norm, and, respectively, the additive subgroup of \( \mathbb{R}^k \) generated by the standard basis \( e_1, \ldots, e_k \). We then have the identification

\[
H_2(\mathbb{C}P^2 \# k \mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}^{k+1} = \mathbb{Z} \times \mathbb{Z}^k \subset \mathbb{Z} \times \mathbb{R}^k,
\]

similar to (26.1). Specifically, the homology classes which correspond here to the elements \((1, 0) \in \mathbb{Z} \times \mathbb{R}^k \) and \((0, e_j) \in \mathbb{Z} \times \mathbb{R}^k \), \( j = 1, \ldots, k \), are realized by a projective line in \( \mathbb{C}P^2 \) not containing any of the blown-up points \( x_j \) (with its standard orientation, described in Remark 5.1) and, respectively, by the \( k \) embedded copies of \( \mathbb{C}P^1 \) that replace the \( x_j \), each of them with the opposite of its standard orientation. Therefore, for the intersection form \( \cdot \) in \( H_2(M, \mathbb{Z}) \),

\[
(d, q) \cdot (d', q') = dd' - \langle q, q' \rangle.
\]

Also, the first Chern class of \( M = \mathbb{C}P^2 \# k \mathbb{C}P^2 \) acting as a homomorphism \( H_2(M, \mathbb{Z}) \to \mathbb{Z} \) is given by \((d, q) \mapsto 3d - q_1 - \ldots - q_k\) (cf. (27.1)), where \( q = (q_1, \ldots, q_k) \). Thus, the group \( \mathcal{D}_+^k(M) = \text{Ker} [c_1(M)] \subset H_2(M, \mathbb{Z}) \) (see (2.9)) consists precisely of those \((d, q) = (d; q_1, \ldots, q_k)\) with

\[
q_1 + \ldots + q_k = 3d.
\]

This is nothing else than condition (5.3.b), which holds whenever \((d, q)\) is the degree \( d(f) \in H_2(M, \mathbb{Z}) \) of a totally real immersion \( f : \Sigma \to M \) of an oriented closed real surface \( \Sigma \).

Example 27.1. For the totally real immersion \( f : S^2 \to \mathbb{C}P^2 \# k \mathbb{C}P^2 \) described in Corollary 13.3, \( d(f) = (1, 3) \). In fact, let \( d(f) = (d, q) \). Then \( d = 1 \) due to the intersection equality in Corollary 13.3, and (27.3) with \( k = 1 \) gives \((d, q) = (1, 3)\).

Example 27.2. The totally real embedded 2-sphere \( \Sigma \subset \mathbb{C}P^2 \# 2 \mathbb{C}P^2 \) in (i) of Remark 11.5 has the degree \([\Sigma] = (0, \pm 1, \mp 1)\), with the sign depending on the orientation chosen in \( \Sigma \). In fact, setting \([\Sigma] = (d, q)\) we have \( d = 0 \) from (27.2) applied to the homology class \((d', q') = (1, 0)\), represented by a projective line \( Q \) in \( \mathbb{C}P^2 \) not intersecting the sphere \( \Sigma' \) (for instance, one with \( Q \cap \mathbb{R}^2 = \mathbb{C} \times \{t\} \) for sufficiently large \( t \in \mathbb{R} \)). Since \( \Sigma \) is a totally real embedded 2-sphere, (5.4) (for \( n = 2 \)) and (27.2) yield \( |q|^2 = -\langle d, q \rangle : (d, q) = \chi(\Sigma) = 2 \). The two components of \( q \) thus have the absolute value 1 and (by (27.2)) with \( d = 0 \) and \( k = 2 \) opposite signs, as required.

Proof of Theorem 26.1 in the case \( M = \mathbb{C}P^2 \# k \mathbb{C}P^2 \). Condition (25.4) holds for \( M = \mathbb{C}P^2 \# k \mathbb{C}P^2 \). In fact, the elements \((1, 3e_1)\) and \((0, e_j - e_1)\), \( j = 2, \ldots, k \),
form a generating set for the subgroup \( \ker [c_1(M)] \) of \( H_2(M, \mathbb{Z}) \), and they are all realized by totally real immersions, in view of Examples 27.1, 27.2 and Lemma 10.1(a).

On the other hand, Lemma 26.3, combined with Lemma 23.3(ii), implies that condition (25.5), with \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \), for the Klein bottle \( \Sigma = K^2 \). The assertion on Theorem 26.1 for \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \) now follows from Lemma 25.7(i).

\[ \square \]

28. THE MAIN THEOREM ABOUT EMBEDDINGS

Let \( \Sigma \) and \( M \) be a given a closed real surface and one of the complex surfaces (0.1). The pairs \( (i, d) \) that can be simultaneously realized as the Maslov index and degree of a totally real immersion \( \Sigma \to M \) then are nothing else than all elements of the set \( \mathcal{I}_q(\Sigma) \times \mathcal{D}_\varepsilon(M) \) with some specific \( q, \varepsilon \) and \( \pm \) depending on \( M \) and \( \Sigma \). (See Theorem 26.1 and the discussion preceding it.) This is, however, not the case if one replaces the word ‘immersion’ by ‘embedding’ since, in view of (5.4) with \( n = 2 \) and (19.4), the degree \( d = d(f) \) of any totally real embedding \( f : \Sigma \to M \) satisfies the additional restriction

\[
\begin{align*}
  d \cdot d &= -\chi(\Sigma), \quad \text{if } \Sigma \text{ is orientable}, \\
  d^2 &= [\chi(\Sigma) \mod 4], \quad \text{if } \Sigma \text{ is not orientable},
\end{align*}
\]

(28.1)

\( d^2 \in \mathbb{Z}_4 \) being the Pontryagin square of the cohomology class in \( H^2(M, \mathbb{Z}_4) \) which is the Poincaré dual of \( d \in H_2(M, \mathbb{Z}_4) \).

The following theorem states that there are no further restrictions, as long as \( k \) in (0.1) is less than 9. In other words, conditions (25.3) and (28.1) are both necessary and sufficient in order that the pair \( (i, d) \) correspond to a totally real embedding \( \Sigma \to M \), where \( M \) is \( \mathbb{C}P^2 \), \( \mathbb{C}P^2 \times \mathbb{C}P^1 \), or \( \mathbb{C}P^2 \# k \mathbb{C}P^2 \), \( 1 \leq k \leq 8 \).

In this and the next sections we prove Theorem 28.1 for nonorientable surfaces \( \Sigma \), as well in the case where \( \Sigma \) is orientable, while \( M = \mathbb{C}P^2 \) or \( M = \mathbb{C}P^2 \times \mathbb{C}P^1 \), \( \mathbb{C}P^2 \# k \mathbb{C}P^2 \), \( 1 \leq k \leq 8 \), is postponed until §29.

**Theorem 28.1.** Let \( M \) be one of the complex surfaces (0.1), with \( 1 \leq k \leq 8 \), and let \( \Sigma \) be a closed real surface. Any pair \( (i, d) \in \mathcal{I}_q(\Sigma) \times \mathcal{D}_\varepsilon(M) \) which also satisfies (28.1), with \( q, i(f), d(f), \mathcal{I}_q(\Sigma), \mathcal{D}_\varepsilon(M), \varepsilon \) and \( \pm \) as in (1.4), (2.5), (2.6) and (2.8) – (2.10), then equals \( (i(f), d(f)) \) for some totally real embedding \( f : \Sigma \to M \).

As stated above, we will begin by proving Theorem 28.1 in some special cases. First, the nonorientable case will follow once we establish the assertion for orientable surfaces \( x \), as one sees using Lemmas 25.4 and Theorem 2.3, combined with reparametrizations of \( T^2 \) or \( K^2 \).

Now let \( \Sigma \) be orientable, and let \( M = \mathbb{C}P^2 \) or \( M = \mathbb{C}P^2 \times \mathbb{C}P^1 \). As \( d \in \mathcal{D}_\varepsilon(M) = \{0\} \) by (22.4), (22.5) and (2.8), relation (28.1) shows that \( \Sigma \) must be the torus \( T^2 \). Our assertion now is obvious from Corollary 25.3.

Our next step is to verify Theorem 28.1 in the case where \( \Sigma \) is orientable and \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \), using the notations of (26.1) – (26.2). In view of (26.2) – (26.3), conditions \( d \in \mathcal{D}_\varepsilon(M) = \ker [c_1(M)] \) and (28.1), for any given closed oriented surface \( \Sigma \), now read \( a + b = 0 \) and \( 2ab = -\chi(\Sigma) \). This gives \( \chi(\Sigma) = -2ab = 2a^2 \geq 0 \), that is, \( \Sigma \) can only be either the torus \( T^2 \), with \( a = b = 0 \), or the sphere \( S^2 \), with \( (a, b) = (\pm 1, \mp 1) \). A totally real embedding \( \Sigma \to M \) realizing
the pair \((i, d)\) for any \(i \in \mathcal{I}_4(\Sigma)\) and this particular \(d = (a, b)\) now exists in view of Corollary 25.3(a) (for \(\Sigma = S^2\)); for \(\Sigma = S^2\), the “anti-diagonal” 2-sphere (see (vii) in §7) has \(d = (\pm 1, \mp 1) \in \mathbb{Z} \times \mathbb{Z}\), as one sees intersecting it with the factor submanifolds. Note that, for \(\Sigma = S^2\), every value of \(i \in \mathcal{I}_4(\Sigma)\) is realized, as there is only one possible value, \(i = 0\) (by (22.1)).

29. EMBEDDED ORIENTABLE SURFACES IN \(\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}\)

In this section we will prove Theorem 28.1 in the case where \(\Sigma\) is orientable and \(M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}\), with \(k = 1, 2, \ldots, 8\). Specifically, for any pair \((i, d)\) with (25.3) and (28.1) we will describe a totally real embedding \(f : \Sigma \to M\) such that 
\[i(f) = i \quad \text{and} \quad d(f) = d.\]
However, \(i(f)\) can be safely ignored since, for this choice of \(M\), it has a unique possible value, independent of \(f\), that is, determined solely by \(\Sigma\) (see the last line in (15.3)). As for \(d\), we will use the identification (27.1) to treat it as a pair \((d, q)\) with \(d \in \mathbb{Z}\) and \(q \in \mathbb{Z}^k \subset \mathbb{R}^k\) or, equivalently, as a \((k+1)\)-tuple \((d; q_1, \ldots, q_k)\) of integers. Since (27.2) gives \((d, q) \cdot (d, q) = d^2 - |q|^2\), it is clear that (25.3) and (28.1), as conditions on \(d = (d, q)\), now read
\[(q, 1) = 3d, \quad |q|^2 = d^2 + \chi, \quad d \in \mathbb{Z}, \quad q \in \mathbb{Z}^k\]
(with \(\langle \cdot, \cdot \rangle\) and \(| \cdot |\) as in §27), where \(\chi = \chi(\Sigma)\) and \(1 = (1, \ldots, 1) \in \mathbb{R}^k\).

We will prove Theorem 28.1 in the special case named above by first solving equations (29.1) with \(\chi \leq 2\) and \(1 \leq k \leq 8\) (see Lemmas 29.4, 29.6) and then realizing every solution \((d, q)\), in Example 29.7, as the degree \([\Sigma] = (d, q)\) of a totally real closed oriented surface \(\Sigma\), embedded in \(M\). (Note that \([\Sigma] = d(f)\) for the inclusion mapping \(f : \Sigma \to M\).

**Remark 29.1.** Treating (29.1) as just a system of equations imposed on \(d\) and \(q = (q_1, \ldots, q_k)\), in which \(k, \chi \in \mathbb{Z}\) are fixed parameters with \(k \geq 1\), we can rewrite it as \(q_1 + \cdots + q_k = 3d\) and \(q_1^2 + \cdots + q_k^2 = d^2 + \chi\), the unknown now being \(d, q_1, \ldots, q_k \in \mathbb{Z}\). Thus, since \(q^2 \equiv q \mod 2\) for any \(q \in \mathbb{Z}\), a solution to (29.1) exists only if \(\chi\) is even. Each of the following three families of solutions to (29.1) represents infinitely many values of \(k\) (with \(d\) always denoting an integer):

(i) \((d, q) = (0, 0) = (0; 0, \ldots, 0),\) with any \(k \geq 1\) and \(\chi = 0\),
(ii) \((d, q) = (d, 1) = (d; 1, \ldots, 1),\) for \(d \geq 1\), with \(k = 3d\) and \(\chi = (3 - d)d\),
(iii) \((d, q) = (d; d-1, \ldots, 1),\) for any \(d \geq 0\), with \(k = 2d + 2\) and \(\chi = 2\).

The formula for \(\chi\) in (ii) does not produce the value \(\chi = -2\), and (as shown in Lemma 29.4 below) a solution to (29.1) with \(\chi < 0\) cannot exist unless \(k \geq 10\). When \(k \geq 10\), however, solutions \((d; q_1, \ldots, q_k)\) with \(\chi = -2\) do exist; examples are (7; 3, 3, 2, ..., 2, 1) for \(k = 10\) and (4; 2, 1, ..., 1) for \(k = 11\). Furthermore, (7; 3, 2, ..., 2) is a solution with \(\chi = -4\) for \(k = 10\). There are also solutions with \(\chi = 0\) other than those in (ii) for \(d = 3\) or in (i), such as (4; 2, 2, 1, ..., 1) and (7; 4, 2, ..., 2, 1), both for \(k = 10\).

**Example 29.2.** Equations (29.1) remain satisfied after any permutation of the components \(q_1, \ldots, q_k\) of \(q\), as well as after the signs of \(d\) and all \(q_1, \ldots, q_k\) have been changed; also, a new solution with \(k\) replaced by \(k' > k\) (or, \(k' < k\) arises if one inserts additional \(k' - k\) zeros (or, respectively, deletes existing \(k - k'\) zeros) among the \(q_1, \ldots, q_k\). Successive applications of these operations, repeated in any order, any number of times, lead to what we call **trivial modifications** of the given solution \((d, q)\) to (29.1). Geometrically, the sign change corresponds to re-orienting
the real surface $\Sigma$, while inserting additional $k' - k$ zeros (so as to transform $(d, q)$ into $(d', q')$) amounts to switching from a given totally real oriented closed surface $\Sigma \subset M$ with $[\Sigma] = (d, q)$ to $\Sigma' \subset M'$ with $[\Sigma'] = (d', q')$, where $M'$ arises from $M$ by blowing up $k' - k$ points not lying in $\Sigma$, and $\Sigma'$ stands for $\Sigma$ treated as a real surface in $M'$. (Cf. Lemma 10.1(i).)

**Example 29.3.** Let $M = \mathbb{CP}^2 \# k \mathbb{CP}^2$ be the complex surface obtained by blowing up $k$ distinct points in $\mathbb{CP}^2$. Every solution $(d, q)$ to (29.1) listed under (i), (ii) or (iii) in Remark 29.1 is actually realized as the degree $[\Sigma]$ of a totally real oriented surface $\Sigma$ embedded in $M$, so that $k$ in (29.1) coincide with the number of blown-up points and $\chi = \chi(\Sigma)$. Here is how we construct $\Sigma$.

(i) To obtain the degree $(0; 0, \ldots, 0)$, we may fix a totally real torus in a bounded connected open set $U \subset \mathbb{C}^2$ (cf. (v) in §7), and then embed it in $M$ using a holomorphic embedding $U \to M$.

(ii) The degree $(d; 1, \ldots, 1)$ for any $d \geq 1$, with $k = 3d$ and $\chi = (3 - d)d_s$ is realized by choosing a nonsingular degree $d$ holomorphic curve $Q \subset \mathbb{CP}^2$, then blowing up $k = 3d$ points of $\mathbb{CP}^2$, all of which lie in $Q$, so that $Q$ treated as a submanifold of $M = \mathbb{CP}^2 \# k \mathbb{CP}^2$ has the degree $(d; 1, \ldots, 1)$, and, finally, using Theorem 2.16 to deform $Q$ to an oriented totally real surface $\Sigma \subset M$ of genus $g = (d - 1)(d - 2)/2$, having the same degree as $Q$. (Note that condition (b) in Theorem 2.16 holds, for the inclusion mapping $f : \Sigma \to M$, due to the first equality in (29.1).)

(iii) The degree $(d; d - 1, 1, \ldots, 1)$ for $d \geq 3$, with $k = 2d + 2$, is realized by the totally real 2-sphere $\Sigma$ in $M$, obtained as follows. Equation $x^d = y^{d-1}z$ in homogeneous coordinates $[x, y, z]$ defines a singular degree $d$ curve in $\mathbb{CP}^2$, and blowing up its unique singularity (at $[0, 0, 1]$), we replace it by a nonsingular holomorphic curve $P \subset \mathbb{CP}^2 \# \mathbb{CP}^2$ diffeomorphic to $S^2$, with $[P] = (d; d - 1)$. (In fact, in the holomorphic local coordinates $\xi, \eta$ for $\mathbb{CP}^2 \# \mathbb{CP}^2$, with $\xi = x/z$, $\eta = y/x$, equation $x^d = y^{d-1}z$ reads $\xi = \eta^{d-1}$, while the exceptional divisor is given by $\xi = 0$.) Blowing up a further set of $k - 1 = 2d + 1$ distinct points in $P$, and then using Theorem 2.16 as in (ii), we now get a totally real sphere of degree $(d; d - 1, 1, \ldots, 1)$ in $\mathbb{CP}^2 \# k \mathbb{CP}^2$.

Discussing the cases $d = 0, 1, 2$ in (iii) is not necessary, as the corresponding degrees have already been realized: for $d = 0$, by Example 27.2; for $d = 1$, by (ii) with $d = 1$ (which yields $(1; 1, 1, 1)$ rather than $(1; 0, 1, 1, 1)$, but the extra 0 amounts to a trivial modification, cf. Example 29.2, that is, to blowing up one more point outside of the totally real 2-sphere); and, for $d = 2$, by (ii) with $d = 2$.

If necessary, we will eliminate the freedom of applying trivial modifications to $(k + 1)$-tuples $(d; q_1, \ldots, q_k)$ with (29.1) by imposing on them additional normalizing conditions such as

\begin{equation}
q_1 \geq \ldots \geq q_k; \quad d \geq 0; \quad q_1 \ldots q_k \neq 0. \tag{29.2}
\end{equation}

For any $(d, q)$ with (29.1), the Schwarz inequality $(q, 1)^2 \leq |q|^2|1|^2$ gives

\begin{equation}
(9 - k)d^2 \leq k\chi. \tag{29.3}
\end{equation}

Let us now suppose that $1 \leq k \leq 9$. By (29.3), $\chi \geq 0$. If, in addition, $\chi = 0$, (29.3) yields $(9 - k)d = 0$, and so $q$ is a multiple of $1$ (the equality case in the
Schwarz inequality (29.3)); thus, either \( \chi = 9 - k = 0 \), or \( k < 9 \) and \( \chi = d = 0 \). In view of (29.1), this establishes the following lemma, closely related to a special case of Proposition 18.1.

**Lemma 29.4.** If \( 1 \leq k \leq 9 \), the system (29.1) has no solutions for \( \chi < 0 \). When \( \chi = 0 \), the only solution \((d, q)\) with \( 1 \leq k \leq 8 \) is \((0, 0)\), while the only solutions with \( \chi = 0 \) and \( k = 9 \) are \((3s; s, s, s, s, s, s, s)\) for \( s \in \mathbb{Z} \).

For \((d, q)\), \(k\), \(\chi\) with (29.1), let \(s\) be the greatest integer with \(3s \leq d+1\). Thus,

\[
d = 3s + r, \quad \text{with } s \in \mathbb{Z} \text{ and } r \in \{-1, 0, 1\}.
\]

Setting \(s = s1 = (s, \ldots, s) \in \mathbb{R}^k\) we now have \(|q|^2 = d^2 + \chi = 9s^2 + 6rs + r^2 + \chi\), \(|s|^2 = ks^2\) and \(\langle q, s \rangle = 3sd = 3s(3s + r) = 9s^2 + 3rs\). Hence, as \(r^2 \leq 1\),

\[
|q - s|^2 = (k - 9)s^2 + r^2 + \chi, \quad r^2 + \chi \in \{\chi, \chi + 1\},
\]

and, consequently,

\[
(9 - k)s^2 \leq r^2 + \chi \leq \chi + 1.
\]

**Remark 29.5.** Given \(q = (q_1, \ldots, q_k) \in \mathbb{R}^k \), \(k \geq 1\), let \(\ell = |q - 1|^2\) with \(1 = (1, \ldots, 1) \in \mathbb{Z}^k\). If \(q_1 \ldots q_k \neq 0\) and \(\ell \leq 3\), then \(k - \ell\) of the \(k\) integers \(q_1, \ldots, q_k\) equal 1, and the remaining \(\ell\) of them all equal 2. Therefore,

\[
q_1 + \ldots + q_k = k + \ell.
\]

In fact, the integer \(\ell \leq 3\) written as a sum of positive integer squares must appear as \(1 + \ldots + 1\), so that among the numbers \(q_1 - 1, \ldots, q_k - 1\) there are \(k - \ell\) zeros and \(\ell\) ones (but none equal to \(-1\), since \(q_j \neq 0\)).

**Lemma 29.6.** The only solutions \((d, q) = (d; q_1, \ldots, q_k)\) to (29.1) with \(\chi = 2\) and \(1 \leq k \leq 8\) are

\[
(29.8) \quad (0; 1, -1), \ (1; 1, 1, 1), \ (2; 1, 1, 1, 1, 1, 1), \ (3; 2, 1, 1, 1, 1, 1, 1, 1),
\]

and those obtained from them by trivial modifications, defined as in Example 29.2.

Note that, up to trivial modifications, (29.8) are precisely the solutions (iii) in Remark 29.1 for \(d = 0, 1, 2, 3\).

**Proof.** By (29.6), \(s^2 \leq (9 - k)s^2 \leq \chi + 1 = 3\), and so \(s \in \{-1, 0, 1\}\). To eliminate trivial modifications we may assume all of conditions (29.2). In particular, we have \(d \geq 0\), which gives \(s \in \{0, 1\}\) (cf. (29.4)). Let \(\ell = |q - s|^2\). As \(k \leq 8\), (29.5) with \(\chi = 2\) gives \(\ell \in \{2, 3\}\) (when \(s = 0\), or \(\ell \in \{k - 7, k - 6\}\) (when \(s = 1\)). Since \(\ell \geq 0\), it follows that \(k \in \{6, 7, 8\}\) if \(s = 0\) while \(k \in \{2, 3\}\) if \(s = 1\). Hence, \(q_1 = \ldots = q_k = 1\) if \(s = 0\), as \(\ell = |q|^2 \leq 3\) with \(q_1 \ldots q_k \neq 0\) (cf. (29.2)). The triple \((s, k, \ell)\) thus must assume one of the seven values \((0, 2, 2), (0, 3, 3), (1, 6, 0), (1, 7, 0), (1, 7, 1), (1, 8, 1)\) and \((1, 8, 2)\). However, when \(s = 1\), (29.7) combined with (27.2) (which is a part of (29.1)) gives \(k + \ell = 3d\), thus eliminating three of the seven triples \((s, k, \ell)\), and leaving only those with \(s = 0\) or \(k + \ell\) divisible by \(3\): \((0, 2, 2), (0, 3, 3), (1, 6, 0), (1, 8, 1)\). These four triples lead to the four possibilities listed in (29.8). To see this, let us first consider \((0, 2, 2)\) and \((0, 3, 3)\), with \(k \in \{2, 3\}\), and \(|q_1| = \ldots = |q_k| = 1\). The nonincreasing sequence \((q_1, \ldots, q_k)\) of two or three integers of absolute value 1, whose sum equals \(3d > 0\) (cf. (29.1) and (29.2)), clearly must be either \((1, -1)\) with \(d = 0\), or \((1, 1, 1)\) with \(d = 1\), as required. On the other hand, it is obvious from Remark 29.5 that the triples
$(s, k, \ell) = (1, 6, 0)$, and $(s, k, \ell) = (1, 8, 1)$, with $s = 1$ and $k + \ell = 3d$, correspond to the last two solutions in (29.8). This completes the proof. \qed

**Example 29.7.** The following examples of totally real 2-spheres $\Sigma$ embedded in $\mathbb{CP}^2 \# k \mathbb{CP}^2$, with suitable $k \in \{1, 2, \ldots, 8\}$, realize all degrees $|\Sigma|$ listed in (29.8).

(a) $\Sigma \subset \mathbb{CP}^2 \# 2 \mathbb{CP}^2$ described in Example 27.2 has $|\Sigma| = (0; 1, 1, 1)$.  
(b) $|\Sigma| = (1; 1, 1, 1, 1)$ obtained as in Example 29.3(ii) with $d = 1$.  
(c) $|\Sigma| = (2; 1, 1, 1, 1, 1, 1, 1, 1, 1)$ in the $d = 2$ case of Example 29.3(ii).  
(d) $|\Sigma| = (3; 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, realized as in Example 29.3(iii), with $d = 3$.

**30. Appendix. Spheres and tori in $\mathbb{CP}^2 \# k \mathbb{CP}^2$ for $k = 9, 10$**

As we saw in the lines following Theorem 28.1, the existence of a totally real embedding leads to no restriction on $k \geq 1$, when $\Sigma$ not orientable. This obviously suggests trying to reach beyond the restriction $1 \leq k \leq 8$ in the orientable case as well. Some partial results for $k = 9$ and $k = 10$ are discussed below.

First, we will show, in Example 30.2, that the final clause of Lemma 29.4 describes all the degrees of totally real 2-tori embedded in $\mathbb{CP}^2 \# 9 \mathbb{CP}^2$ (and nothing else). We start with some general remarks.

Since being totally real is an open property, a totally real immersion $f : \Sigma \to M$ with $\Sigma$ compact and $\dim_R \Sigma < \dim_C M$ leads, via a generic small deformation of $f$, to totally real *embeddings*. The following lemma and corollary show how a similar deformation can sometimes be obtained also in the case where $\dim_R \Sigma = \dim_C M$.

Given a closed real manifold $\Sigma$, an integer $d \geq 2$, and a surjective homomorphism $\varphi : \pi_1 \Sigma \to \mathbb{Z}_d$, let $\pi : \hat{\Sigma} \to \Sigma$ be the $d$-fold covering projection corresponding to the subgroup $\ker \varphi$ of $\pi_1 \Sigma$. Thus, $\hat{\Sigma}$ is a principal $\mathbb{Z}_d$-bundle over $\Sigma$, and we will denote by $\lambda$ the complex line bundle associated with it via the standard representation of $\mathbb{Z}_d \subset U(1)$ in $\mathbb{C}$.

**Lemma 30.1.** With $\Sigma, d, \varphi, \pi, \hat{\Sigma}$ and $\lambda$ as above, let us suppose that $\lambda$ is real-isomorphic to a vector subbundle of $T\Sigma$. Then, for any totally real embedding $f : \Sigma \to M$ in an almost complex manifold $M$, there exists a totally real embedding $\hat{\Sigma} \to M$ which is $C^\infty$ homotopic, through totally real immersions, to the composite immersion $f \circ \pi$.

**Proof.** Since $f$ is totally real, our assumption about $\lambda$ allows us to choose an embedding $F : U \to M$, where $U$ is a neighborhood of the zero section $\Sigma$ in the total space of $\lambda$, such that $F = f$ on $\Sigma$. (In fact, we may let $F$ be the composite of an injective real vector-bundle morphism from $\lambda$ to the normal bundle of $f$, followed by exponentiation relative to a fixed metric on $M$.) As $\hat{\Sigma}$ is naturally embedded in the total space of the unit circle bundle of $\lambda$ and $\pi : \hat{\Sigma} \to \Sigma$ is the restriction to $\hat{\Sigma}$ of the bundle projection $\pi : \lambda \to \Sigma$, the mappings given by $\Sigma \ni \xi \mapsto F(t\xi) \in M$, each of them depending on a fixed parameter $t \geq 0$ close to 0, form a $C^\infty$ homotopy between $f \circ \pi$ (with $t = 0$) and an embedding $\hat{\Sigma} \to M$ (with any small $t > 0$). Since being totally real is an open property, for small $t$ such embeddings are totally real, which completes the proof. \qed

Let $M = \mathbb{CP}^2 \# 9 \mathbb{CP}^2$. According to Lemma 29.4, the degree of any totally real embedding $T^2 \to M$ equals $(3s; s, s, s, s, s, s, s, s, s, s)$ for some $s \in \mathbb{Z}$. We
can now show that, conversely, every such degree is realized by some totally real embedding $T^2 \to M$.

**Example 30.2.** Every degree $(3s; s, s, s, s, s, s, s, s, s, s)$, with $s \in \mathbb{Z}$, is realized by some totally real embedding $T^2 \to M$, where $M = \mathbb{CP}^5 \not\# 9 \mathbb{CP}^2$. First, if $s = 1$, such a totally real embedding $f$ is provided by Example 29.2(ii) for $d = 3$. If $s > 1$, we can use Lemma 30.1 for this $f$ and $\Sigma = \Sigma = T^2$, with $d = s$, noting that $\lambda$ is trivial (since so is $\lambda^{\otimes d}$ and $\pi_1 \Sigma$ is free). Finally, the case $s < 0$ is reduced to $s \geq 1$ by re-orienting $\Sigma$ (cf. Example 29.2), and the case $s = 0$ is obvious from Example 29.3(i).

As for the degrees of totally real embedded 2-spheres, we have the following partial results:

**Lemma 30.3.** For every prescribed integer $d$, the system (29.1) with $k = 9$ and $\chi = 2$ has a unique solution $(d, q) = (d; q_1, \ldots, q_k)$ that satisfies the normalizing condition $q_1 \geq \ldots \geq q_k$, cf. (29.2). Explicitly, we have

$$
(30.1) \quad (q_1, \ldots, q_9) =
\begin{cases}
(s + 1, s, s, s, s, s, s, s, s), & \text{if } d = 3s, s \in \mathbb{Z},
(s + 1, s, s, s, s, s, s, s, s), & \text{if } d = 3s + 1, s \in \mathbb{Z},
(s, s, s, s, s, s, s, s, s), & \text{if } d = 3s - 1, s \in \mathbb{Z}.
\end{cases}
$$

In fact, (29.5) with $k = 9$ and $\chi = 2$ yields $|q - s|^2 \in \{2, 3\}$. Setting $q_i = q_j - s$ and then decomposing 2 or 3 into all possible sums $\sum_{j=1}^9 q_j^2$ with $q_j \in \mathbb{Z}$, $q_1 \geq \ldots \geq q_9$ and $\sum_{j=1}^9 q_j = 3(d - 3s)$ (cf. (27.2)) we easily obtain (30.1).

**Example 30.4.** The solutions (30.1) with $d = 4$, $d = 5$ or $d = 6$ are geometrically realized by an algebraic curve in $\mathbb{CP}^5$ whose only singularities are, respectively: three double points; six double points; or, seven double points and a singularity of type $x^3 = y^4$.

**References**


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