Weakly Einstein conformal products

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Abstract. One says that a Riemannian four-manifold is *weakly Einstein* if the three-index contraction of its curvature tensor against itself equals a function times the metric. Since this includes all four-manifolds that are Einstein, or conformally flat and scalar-flat, the term *proper* may be used for weakly Einstein manifolds (or metrics) not belonging to the latter two classes. We establish two classification-type results about proper weakly Einstein metrics conformal to Riemannian products. This includes constructions of new examples, among them – some of (local) cohomogeneity two, in contrast with the two previously known narrow classes of examples, having cohomogeneity zero and one. We also exhibit a simple coordinate description of one of the known examples, the EPS space, which shows that it is a conformal product and constitutes a single local-homothety type. Finally, we prove that there exist no proper weakly Einstein manifolds with harmonic curvature.

1. Introduction

One calls a Riemannian four-manifold *weakly Einstein* [**13**, p. 112] when the triple contraction of its curvature tensor against itself is a functional multiple of the metric. According to formula (2.1) below, this follows if the four-manifold in question is Einstein, or conformally flat and scalar-flat. Weakly Einstein manifolds (or metrics) not belonging to these two classes will be referred to as *proper*.

Known examples of proper weakly Einstein manifolds consist of the EPS space [**12**, p. 602], and a very narrow class of Kähler surfaces [**7**, Sect. 12]. In Sect. 4 we provide a simple coordinate description of the EPS space, which shows that it represents *a single local-homothety type*. This is a surprising conclusion, since the original description [**12**] of the EPS space realizes it in the form of left-invariant metrics on Lie groups corresponding to infinitely many non-isomorphic Lie algebras.

As another consequence of its coordinate description, the EPS space admits two kinds of warped-product decompositions, with the base/fibre dimensions $2+2$ (a unique one) and $3+1$ (infinitely many), which raises the question whether there exist other proper weakly Einstein warped products and – more generally – *conformal products* (that is, manifolds/metrics conformal to Riemannian products).

The present paper answers this question in the affirmative, which at the same time leads to new examples of proper weakly Einstein four-manifolds. Those constructed in Sect. 15 have, as a consequence of Remark 15.1, local cohomogeneity

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two, which makes them fundamentally different (see Sect. 21) from all the currently known examples, mentioned above.

Our main result consists of two classification-type theorems for proper weakly Einstein conformal products. We start from some constructions of examples, phrased as Theorems 9.1 and 9.2, and followed by four sections in which we show that the resulting metrics are in fact weakly Einstein, and prove the existence of the geometric structures used in the constructions. The classification itself is provided by Theorems 16.1 and 16.2, stating that, locally, at generic points, the constructions just mentioned yield all proper weakly Einstein conformal products, except possibly some nongeneric ones of type $3+1$.

Sect. 21 clarifies how the new examples of proper weakly Einstein four-manifolds, arising from Theorems 9.1 and 9.2, differ from the previously known ones, and where they stand within the classification, established in [**8**], of algebraic curvature tensors having the weakly Einstein property. Finally, in Sect. 22, we combine our classification theorems with some results of [**6**] to show that a proper weakly Einstein Riemannian four-manifold cannot have harmonic curvature.

2. Preliminaries

Manifolds are by definition connected, all manifolds and tensor fields smooth; $R, W, \mathrm{r}, \mathrm{e}$ stand for the curvature, Weyl, Ricci and Einstein tensors of a Riemannian metric g in dimension n, and s for its scalar curvature, R having the sign such that $\mathrm{r}_{ij} = g^{pq}R_{ipjq}$. Thus, $\mathrm{e} = \mathrm{r} - \mathrm{s}g/n$, while, if $n \geq 3$,

$$W_{ijpq} = R_{ijpq} - \frac{1}{n-2}\,(g_{ip}\mathrm{r}_{jq} + g_{jq}\mathrm{r}_{ip} - g_{jp}\mathrm{r}_{iq} - g_{iq}\mathrm{r}_{jp}) + \frac{\mathrm{s}}{(n-1)(n-2)}\,(g_{ip}g_{jq} - g_{jp}g_{iq}).$$

As already mentioned in the Introduction, a Riemannian four-manifold (M, g) is said to be weakly Einstein when the three-index contraction of its curvature tensor R equals some function ϕ times the metric (in coordinates: $R_{ikpq}R_j{}^{kpq} = \phi g_{ij}$). According to [**7**, formula (4.7)],

$$(M, g) \text{ is weakly Einstein if and only if } 6W\mathrm{e} = -\mathrm{s}\mathrm{e}, \tag{2.1}$$

with $W\mathrm{e}$ denoting here the usual action of algebraic curvature tensors on symmetric $(0,2)$ tensors: $[W\mathrm{e}]_{ij} = W_{ipjq}\mathrm{e}^{pq}$.

For a torsion-free connection ∇ with the Ricci tensor r on a manifold M, every vector field v satisfies the *Bochner identity*

$$\delta\nabla v \,=\, \mathrm{r}(\cdot, v) \,+\, d\delta v, \tag{2.2}$$

δ being the divergence. In fact, the coordinate form $v^k{}_{,ik} = \mathrm{r}_{ik}v^k + v^k{}_{,ki}$ of (2.2) arises via contraction from the Ricci identity $v^p{}_{,ij} - v^p{}_{,ji} = R_{ijk}{}^p v^k$. For functions $\alpha : M \to \mathrm{I\!R}$ and $\varphi : M \to \mathrm{I\!R} \smallsetminus \{0\}$ on a Riemannian manifold (M, g),

$$\begin{array}{ll} \text{a)} & dQ = 2[\nabla d\alpha](\nabla\alpha, \cdot), \quad \text{where} \quad Q = g(\nabla\alpha, \nabla\alpha), \\ \text{b)} & \varphi^3\Delta\varphi^{-1} = \,2g(\nabla\varphi, \nabla\varphi) \,-\, \varphi\Delta\varphi, \end{array} \tag{2.3}$$

(2.3-a) obvious since, in local coordinates, $Q_{,i} \,=\, (\alpha^{,k}\alpha_{,k})_{,i} \,=\, 2\alpha_{,ik}\alpha^{,k}$. In an oriented Riemannian four-manifold (M, g), we denote by $W^{\pm} : \Lambda^{\pm}M \to \Lambda^{\pm}M$ the vector-bundle morphisms arising when one restricts the Weyl tensor W acting on bivectors to the subbundles $\Lambda^{\pm}M$ of self-dual and anti-self-dual bivectors.

Remark 2.1. We always denote by $\nabla, \overline{\nabla}, \widehat{\nabla}, \widetilde{\nabla}$ the gradient operators and Levi-Civita connections of the metrics $g, \bar{g}, \widehat{g}, \widetilde{g}$, and similarly for the scalar curvatures s, Ricci/Einstein tensors r, e and Laplacians Δ. We will also write

$$\begin{array}{lll} \overline{Y} = \overline{\Delta}\varphi, & \widehat{Y} = \widehat{\Delta}\varphi, & \widetilde{Y} = \widetilde{\Delta}\varphi, \\ \bar{Q} = \bar{g}(\overline{\nabla}\varphi, \overline{\nabla}\varphi), & \widehat{Q} = \widehat{g}(\widehat{\nabla}\varphi, \widehat{\nabla}\varphi), & \widetilde{Q} = \widetilde{g}(\widetilde{\nabla}\varphi, \widetilde{\nabla}\varphi). \end{array} \tag{2.4}$$

Given a product metric $\bar{g} = \widehat{g} + \widetilde{g}$ on a product manifold $\widehat{M} \times \widetilde{M}$, we interpret $\widehat{Y}, \widetilde{Y}, \widehat{Q}, \widetilde{Q}$ and $\widehat{\nabla} d\varphi, \widetilde{\nabla} d\varphi$ as the results of "partial" operations, involving the restrictions of $\varphi : \widehat{M} \times \widetilde{M} \to \mathrm{I\!R}$ to submanifolds of the form $\widehat{M} \times \{z\}$ and $\{y\} \times \widetilde{M}$, where $(y,z) \in \widehat{M} \times \widetilde{M}$, so that $\overline{\Delta}\varphi = \widehat{Y} + \widetilde{Y}$ and $\bar{g}(\overline{\nabla}\varphi, \overline{\nabla}\varphi) = \widehat{Q} + \widetilde{Q}$.

Remark 2.2. On a Kähler surface, $W^+ : \Lambda^+ M \to \Lambda^+ M$, for the standard orientation, has the spectrum $(\mathrm{s}/6, -\mathrm{s}/12, -\mathrm{s}/12)$, the eigenvalue function $\mathrm{s}/6$ being realized by the Kähler form treated as a bivector [**5**, p. 459].

Remark 2.3. In a Riemannian product (M, g) of two surfaces, which is locally Kähler for two complex structures corresponding to opposite orientations, Remark 2.2 implies that both $W^\pm : \Lambda^\pm M \to \Lambda^\pm M$ have the same spectrum $(\mathrm{s}/6, -\mathrm{s}/12, -\mathrm{s}/12)$. Thus, conformal flatness of (M, g) is equivalent to the vanishing of its scalar curvature.

Remark 2.4. A proper weakly Einstein manifold cannot be a Riemannian product. Namely, the triple contraction of the curvature tensor against itself behaves "multiplicatively" under Riemannian products, is equal to 0 in dimension one, and to $2K^2 g$ for a surface metric g with Gaussian curvature K. Thus, for a $3+1$ (or, $2+2$) product, being weakly Einstein is equivalent to flatness or, respectively, to being Einstein or conformally flat. (The last two options correspond to equal/opposite Gaussian curvatures of the factor metrics, cf. Remark 2.3.)

Remark 2.5. Vanishing of se implies, even without assuming real-analyticity, that one of s, e is identically zero [**7**, Remark 2.1]. Thus, any conformally flat weakly Einstein manifold has, by (2.1), either $\mathrm{e} = 0$, or $W = 0$ and $\mathrm{s} = 0$.

Remark 2.6. Given several self-adjoint endomorphisms A_j of vector bundles over a manifold M endowed with Riemannian fibre metrics, we say that a point $x \in M$ is *generic relative to the spectra of* (all) A_j if, for some neighborhood U of x, every A_j restricted to U has a constant number of distinct eigenvalues. It is clear that such generic points form a dense open subset M' of M, and on each connected component of M' the eigenvalues of each A_j constitute smooth *eigenvalue functions*, while the corresponding eigenspaces form smooth *eigenspace subbundles* of the vector bundles in question.

Remark 2.7. Eigenvectors of a linear endomorphism D of a real vector space $\mathcal{E}$, corresponding to mutually different eigenvalues, are linearly independent (or else in a D-invariant subspace $\mathcal{E}'$ spanned by such eigenvectors, D would have more than $\dim \mathcal{E}'$ eigenvalues). Consequently, the space of polynomial functions $\mathcal{E} \times \mathcal{E} \to \mathrm{I\!R}$ is the direct sum of the subspaces $\mathcal{P}_{i,j}$, each formed by polynomials of bidegree (i,j). Namely, each $\mathcal{P}_{i,j}$ is the $(i+qj)$-eigenspace of the directional derivative operator d_w, for the linear vector field w with $w_{(y,z)} = (y, qz)$, where q is any fixed irrational number.

REMARK 2.8. If P, S are C^∞ functions on a manifold M and 0 is a regular value of S, while $P = 0$ along $\Sigma = S^{-1}(0)$, then $P/S : M \smallsetminus \Sigma \to \mathrm{I\!R}$ has a C^∞ extension to M. In fact, identifying a neighborhood of $z \in \Sigma$ with a convex neighborhood U of $z = 0$ in $\mathrm{I\!R}^n$, $n = \dim M$, so that S is the coordinate function x^1, one has $P(x^1, \dots, x^n) = P(x^1, \dots, x^n) - P(0, x^2, \dots, x^n) = x^1 \int_0^1 [\partial_1 P](tx^1, x^2, \dots, x^n)\, dt$.

3. Warped products

The *warped product* of Riemannian manifolds (Σ, γ) and (Π, δ) with the *warping function* $f : \Sigma \to (0, \infty)$ is the Riemannian manifold

$$(M, g) = (\Sigma \times \Pi,\, \gamma + f^2 \delta), \tag{3.1}$$

γ, δ, f standing here for also the pullbacks of γ, δ, f to the product $M = \Sigma \times \Pi$. One calls (Σ, γ) the *base* and (Π, δ) the *fibre* of (3.1). As $\gamma + f^2\delta = f^2[f^{-2}\gamma + \delta]$,

(3.2) a warped product is just a Riemannian manifold conformal to a Riemannian product via multiplication by a positive function which is constant along one of the factor manifolds.

We will need the easy and well-known observation – see, e.g., [**9**, formula (A.3)] – is that, in any warped product with the Ricci tensor r,

(3.3)
a) the base and fibre factor distributions are r-orthogonal,
b) if the fibre dimension is less than three, all nonzero vectors tangent to the fibre distribution are eigenvectors of r.

REMARK 3.1. It is well known [**5**, Lemma 19.2], [**9**, Remark 3.1] that a one-dimensional distribution on a Riemannian manifold is, locally, the fibre distribution of a warped-product decomposition if and only if it is spanned, locally, by a Killing field v without zeros having an integrable orthogonal complement $v^\perp$.

REMARK 3.2. If the fibre (Π, δ) of (3.1) a Riemannian product, for instance, $(\Pi, \delta) = (\Pi' \times \Pi'', \delta' + \delta'')$, then (3.1) (Π, δ) is, obviously, also a warped product with the base $(\Sigma \times \Pi',\, \gamma + f^2\delta')$ and fibre (Π'', δ'').

Thus, any warped product having the base and fibre dimensions $m, 2$ and a flat fibre is also, locally, a warped product with the dimensions $m+1$ and 1.

REMARK 3.3. By (3.2), isometries of the fibre act isometrically on the warped-product manifold, and so do isometries of the base preserving the warping function.

4. The EPS space

The *EPS space* [**13**, p. 112] is an example of a proper weakly Einstein manifold (M, g) arising as follows: M is a Lie group which carries left-invariant g-orthonormal vector fields $u_1, \dots, u_4$, trivializing TM, and having the Lie brackets

$$\begin{array}{l} [u_1, u_2] = au_2, \quad [u_1, u_3] = -au_3 - bu_4, \quad [u_1, u_4] = bu_3 - au_4, \\ [u_2, u_3] = [u_2, u_4] = [u_3, u_4] = 0, \text{ with constants } a \neq 0 \text{ and } b. \end{array} \tag{4.1}$$

The acronym 'EPS' was introduced in [**1**]. The Lie algebras $\mathfrak{h}$ defined by (4.1)

(4.2) form infinitely many Lie-algebra isomorphism types,

as pointed out in [**1**, Theorem 8.1]. In fact, the ad action on $[\mathfrak{h}, \mathfrak{h}] = \mathrm{span}(u_2, u_3, u_4)$ by the coset of u_1 spanning $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ has the complex characteristic roots a and $-a \pm bi$, which makes $|b/a|$ an algebraic invariant of $\mathfrak{h}$, ranging over $[0, \infty)$.

From [**12**, formula (3.14)] it is immediate that the only nonzero components of the Ricci tensor r in the frame $u_1,\dots,u_4$ are equal to

$$\text{the eigenvalues}\quad \mathrm{r}_{11}=-3a^2,\quad \mathrm{r}_{22}=a^2,\quad \mathrm{r}_{33}=\mathrm{r}_{44}=-a^2. \tag{4.3}$$

The 1-form ζ on M with $(\zeta(u_1),\zeta(u_2),\zeta(u_3),\zeta(u_4))=(2a,0,0,0)$ is closed due to (4.1), so that, locally, $\nabla\tau=2au_1$ for some function τ. It is also immediate from (4.1) that the vector fields $\partial_1,\dots,\partial_4$ given by

$$\begin{array}{l} 2a\partial_1=u_1,\qquad 2a\partial_2=e^{-\tau/2}u_2,\\ 2a\partial_3=e^{\tau/2}[\cos(b\tau/2a)u_3\,+\,\sin(b\tau/2a)u_4],\\ 2a\partial_4=e^{\tau/2}[-\sin(b\tau/2a)u_3\,+\,\cos(b\tau/2a)u_4]\end{array} \tag{4.4}$$

all commute. They are thus, locally, the coordinate vector fields for some local coordinates τ,ξ,η,ζ, where the first coordinate may be chosen equal to our function τ, as $d\tau$ sends $\partial_1,\dots,\partial_4$ to $1,0,0,0$. Clearly, in the coordinates τ,ξ,η,ζ,

$$\begin{array}{l} 4a^2g\,=\,d\tau^2+e^{-\tau}d\xi^2+e^{\tau}(d\eta^2+d\zeta^2),\ \text{that is, the component}\\ \text{functions of the metric } 4a^2g \text{ form the matrix } \mathrm{diag}(1,\,e^{-\tau},\,e^{\tau},\,e^{\tau}).\end{array} \tag{4.5}$$

This has two immediate consequences. First,

$$\text{the EPS space represents a single local-homothety type.} \tag{4.6}$$

Secondly, the EPS space admits both $2+2$ and $3+1$ warped-product decompositions (cf. the final clause of Remark 3.2), with f and δ in (3.1) given by

$$\begin{array}{ll} \text{a)} & f=e^{\tau/2}\ \text{and}\ \delta=d\eta^2+d\zeta^2,\ \text{or}\\ \text{b)} & f=e^{-\tau/2}\ \text{and}\ \delta=d\xi^2,\ \text{or}\\ \text{c)} & f=e^{\tau/2}\ \text{and}\ \delta=d\hat\zeta^2,\end{array} \tag{4.7}$$

where $\hat\zeta$ in (4.7-c) belongs to a pair $\hat\eta,\hat\zeta$ arising from η,ζ via any fixed rotation (so that $d\hat\eta^2+d\hat\zeta^2=d\eta^2+d\zeta^2$). Thus, even locally, among the warped-product decompositions of the EPS space,

$$\begin{array}{rl} \text{i)} & \text{the } 2+2 \text{ decomposition is unique, and given by (4.7-a),}\\ \text{ii)} & \text{there are infinitely many decompositions of the type } 3+1,\end{array} \tag{4.8}$$

'unique' meaning *up to multiplying* f *and* δ *by mutually inverse constants.* In fact, (4.8-i) is obvious from (3.3-b) and (4.3), while (4.7-c) yields (4.8-ii).

THEOREM 4.1. *The simply connected complete model of an EPS space consists of* $\mathrm{I\!R}^4$ *with the metric* $g=d\tau^2+e^{-\tau}d\xi^2+e^{\tau}(d\eta^2+d\zeta^2)$ *in the coordinates* τ,ξ,η,ζ. *Its full isometry group is five-dimensional, acts on* $\mathrm{I\!R}^4$ *transitively, and has the identity component* G *formed by the mappings*

$$\mathrm{I\!R}^4=\mathrm{I\!R}^2\times\mathbb{C}\ni(\tau,\xi,\eta+i\zeta)\mapsto(\tau+2c,\,e^c\xi+p,\,e^{-c}w(\eta+i\zeta)+q) \tag{4.9}$$

depending on the parameters $c,p\in\mathrm{I\!R}$ *and* $w,q\in\mathbb{C}$ *with* $|w|=1$.

PROOF. The mappings (4.9) are easily seen to form a group acting on $\mathrm{I\!R}^4$ effectively and transitively, via isometries of g, as they separately preserve $d\tau^2$, $e^{-\tau}d\xi^2$ and $e^{\tau}(d\eta^2+d\zeta^2)$. Our assertion now follows: the dimension of the full isometry group equals 5 due to the fact that, by (4.3), the Lie algebra of the isotropy group acting in any given tangent space is of dimension less than 2. □

The conclusion (4.6) above may seem surprising in the light of (4.2). The puzzle is resolved by Theorem 4.1: G has infinitely many four-dimensional subgroups, acting on $\mathrm{I\!R}^4$ simply transitively, with mutually nonisomorphic Lie algebras.

For (4.7-b) and (4.7-c), the ξ or $\hat{\zeta}$ coordinate vector field is, obviously, a Killing field with an integrable orthogonal complement, as required in Remark 3.1.

The EPS space has natural higher-dimensional generalizations [**11**, Remark 3.1].

5. Weakly Einstein curvature tensors

In the following theorem $\mathcal{T}$ is an oriented Euclidean 4-space, $\mathrm{s}, \mathrm{e}, W$ denote the scalar curvature, Einstein (traceless Ricci) and Weyl tensors of the algebraic curvature tensor in question, $W^\pm : \Lambda^\pm \to \Lambda^\pm$ being the restrictions of W to the spaces $\Lambda^\pm$ of self-dual and anti-self-dual bivectors in $\mathcal{T}$. Any positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$ leads to the length $\sqrt{2}$ orthogonal bases of $\Lambda^\pm$ given by

$$(5.1) \qquad u_1 \wedge u_2 \pm u_3 \wedge u_4, \quad u_1 \wedge u_3 \pm u_4 \wedge u_2, \quad u_1 \wedge u_4 \pm u_2 \wedge u_3,$$

so that $\Lambda^\pm$ are both canonically oriented. Cf. [**8**, the lines following (7.4)].

THEOREM 5.1. *Given a non-Einstein, weakly Einstein algebraic curvature tensor in* $\mathcal{T}$ *such that, respectively,*

$$(5.2) \qquad \begin{array}{rl} \text{a)} & \mathrm{s} = 0 \text{ and } \mathrm{e} \text{ does not have two distinct double eigenvalues, or} \\ \text{b)} & \mathrm{s} \neq 0 \text{ and } \mathrm{e} \text{ does not have two distinct double eigenvalues, or} \\ \text{c)} & \mathrm{e} \text{ has two distinct double eigenvalues,} \end{array}$$

there exists a positive orthonormal basis $u_1, \dots, u_4$ *of* $\mathcal{T}$*, consisting of eigenvectors of* e*, for which the bases* (5.1) *of* $\Lambda^\pm$ *diagonalize* $W^\pm$*, with the corresponding ordered quadruple and two triples of eigenvalues having the respective form*

$$(5.3) \qquad \begin{array}{rl} \text{a)} & (\mu_1, \mu_2, \mu_3, \mu_4) \text{ and } (\pm c_2, \pm c_3, \pm c_4), \\ \text{b)} & (-\lambda, -\mu, \mu, \lambda) \text{ and } (\pm c_2 - \mathrm{s}/12,\ \pm c_3 - \mathrm{s}/12,\ \pm c_4 + \mathrm{s}/6), \\ \text{c)} & (-\lambda, -\lambda, \lambda, \lambda) \text{ and } (\pm c_2 - \mathrm{s}/12,\ \pm c_3 + \xi - \mathrm{s}/12,\ \pm c_4 - \xi + \mathrm{s}/6), \end{array}$$

the parameters $\mu_1, \mu_2, \mu_3, \mu_4, c_2, c_3, c_4, \mathrm{s}, \lambda, \mu, \xi \in \mathrm{I\!R}$ *having*

$$(5.4) \qquad \begin{array}{rl} \text{i)} & c_2 + c_3 + c_4 = 0 \text{ in all cases, as well as} \\ \text{ii)} & \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mathrm{s} = 0 \text{ in case (5.3-a),} \\ \text{iii)} & \lambda > \mu \geq 0 \neq \mathrm{s} \text{ in (5.3-b), } \lambda > 0 \text{ in (5.3-c).} \end{array}$$

PROOF. See [**8**, Theorems 1.3–1.5]. □

REMARK 5.2. Conversely, according to [**8**, Theorems 1.3–1.5], each of the choices (5.3), for $\mu_1, \mu_2, \mu_3, \mu_4, c_2, c_3, c_4, \mathrm{s}, \lambda, \mu, \xi$ with (5.4-i) and (5.4-ii), defines, via the Singer-Thorpe theorem [**2**, Sect. 1.128], an algebraic curvature tensor R which is weakly Einstein. Even though [**8**] assumes (5.4-iii), this assumption can be relaxed for the following reasons. First, we may assume that $\lambda \neq 0$ in (5.3-c), for otherwise R is Einstein and hence, by (2.1), weakly Einstein. Similarly, the cases $\lambda < 0$ in (5.3-c) and $\mu = \lambda$ (or, $\mu = -\lambda$) in (5.3-b) can be excluded, as the former amounts to the second part of (5.4-iii), with $(u_3, u_4, u_1, u_2, -\lambda)$ used instead of $(u_1, \dots, u_4, \lambda)$, and the latter reduces (5.3-b) to (5.3-c) for $\xi = 0$ (or, respectively, to (5.3-c) with (u_1, u_3, u_4, u_2) instead of $(u_1, \dots, u_4)$, and $\xi = \mathrm{s}/4$). Also, (5.3-b) with $\mathrm{s} = 0$ is (5.3-a) for $(\mu_1, \mu_2, \mu_3, \mu_4) = (-\lambda, -\mu, \mu, \lambda)$.

The next four possibilities amount to four lines, each being a condition imposed on λ, μ in (5.3-b), followed by a nonuple replacing $(u_1, \dots, u_4, c_2, c_3, c_4, \lambda, \mu)$, so that the first part of (5.4-iii) is satisfied by the new data:

$$\begin{array}{ll} -\lambda < -\mu \leq 0, & (u_1, u_2, u_3, u_4, c_2, c_3, c_4, \lambda, \mu), \\ -\lambda < \mu \leq 0, & (u_1, u_3, u_2, -u_4, c_3, c_2, c_4, \lambda, -\mu), \\ \lambda < -\mu \leq 0, & (u_4, u_2, u_3, -u_1, c_3, c_2, c_4, -\lambda, \mu), \\ \lambda < \mu \leq 0, & (u_4, u_3, u_2, u_1, c_2, c_3, c_4, -\lambda, -\mu). \end{array} \tag{5.5}$$

There are four more options left: $-\mu < -\lambda \leq 0$, $-\mu < \lambda \leq 0$, $\mu < -\lambda \leq 0$ and $\mu < \lambda \leq 0$. We reduce them to (5.5) by replacing $(u_1, u_2, u_3, u_4, c_2, c_3, c_4, \lambda, \mu)$ with $(u_2, u_1, u_4, u_3, c_2, c_3, c_4, \mu, \lambda)$.

Remark 5.3. It is well known – see, e.g., [**8**, Lemma 7.1] – that every pair of length $\sqrt{2}$ positive orthogonal bases of $\Lambda^{\pm}$ has the form (5.1) for some positive orthonormal basis $u_1, \dots, u_4$ of $\mathcal{T}$, which is unique up to an overall sign change.

Remark 5.4. Each of the following nine choices of the data $c_2, c_3, c_4, \xi, \mathrm{s}$ clearly leads, via (5.3-c) (as well as (5.3-b), for the three quadruples with $\xi = 0$), to the ordered spectra of $24W^+$ and $24W^-$ appearing at the end of each line below:

$$\begin{array}{rlll} \text{i)} & (c_2, c_3, c_4, \xi) = (0, 0, 0, \mathrm{s}/8), & (-2\mathrm{s}, \mathrm{s}, \mathrm{s}), & (-2\mathrm{s}, \mathrm{s}, \mathrm{s}), \\ \text{ii)} & (c_2, c_3, c_4, \xi) = (-\mathrm{s}/4, \mathrm{s}/4, 0, 0), & (-8\mathrm{s}, 4\mathrm{s}, 4\mathrm{s}), & (4\mathrm{s}, -8\mathrm{s}, 4\mathrm{s}), \\ \text{iii)} & (c_2, c_3, c_4, \xi) = (-\mathrm{s}/4, 0, \mathrm{s}/4, \mathrm{s}/4), & (-8\mathrm{s}, 4\mathrm{s}, 4\mathrm{s}), & (4\mathrm{s}, 4\mathrm{s}, -8\mathrm{s}), \\ \text{iv)} & (c_2, c_3, c_4, \xi) = (\mathrm{s}/4, -\mathrm{s}/4, 0, 0), & (4\mathrm{s}, -8\mathrm{s}, 4\mathrm{s}), & (-8\mathrm{s}, 4\mathrm{s}, 4\mathrm{s}), \\ \text{v)} & (c_2, c_3, c_4, \xi) = (0, 0, 0, \mathrm{s}/4), & (-2\mathrm{s}, 4\mathrm{s}, -2\mathrm{s}), & (-2\mathrm{s}, 4\mathrm{s}, -2\mathrm{s}), \\ \text{vi)} & (c_2, c_3, c_4, \xi) = (0, \mathrm{s}/8, -\mathrm{s}/8, \mathrm{s}/8), & (-2\mathrm{s}, 4\mathrm{s}, -2\mathrm{s}), & (-2\mathrm{s}, -2\mathrm{s}, 4\mathrm{s}), \\ \text{vii)} & (c_2, c_3, c_4, \xi) = (\mathrm{s}/4, 0, -\mathrm{s}/4, \mathrm{s}/4), & (4\mathrm{s}, 4\mathrm{s}, -8\mathrm{s}), & (-8\mathrm{s}, 4\mathrm{s}, 4\mathrm{s}), \\ \text{viii)} & (c_2, c_3, c_4, \xi) = (0, -\mathrm{s}/8, \mathrm{s}/8, \mathrm{s}/8), & (-2\mathrm{s}, -2\mathrm{s}, 4\mathrm{s}), & (-2\mathrm{s}, 4\mathrm{s}, -2\mathrm{s}), \\ \text{ix)} & (c_2, c_3, c_4, \xi) = (0, 0, 0, 0), & (-2\mathrm{s}, -2\mathrm{s}, 4\mathrm{s}), & (-2\mathrm{s}, -2\mathrm{s}, 4\mathrm{s}). \end{array} \tag{5.6}$$

In each of the above nine cases it immediately follows that

$$\begin{array}{ll} \text{a)} & \text{both } W^{\pm} \text{ have the same unordered spectrum } \{\sigma, -\sigma/2, -\sigma/2\}, \text{ and} \\ \text{b)} & \text{the unique simple eigenvalue } \sigma \text{ is one of } -\mathrm{s}/3, -\mathrm{s}/12, \mathrm{s}/6, \text{ if } \mathrm{s} \neq 0. \end{array} \tag{5.7}$$

Remark 5.5. Let there be given a non-Einstein, weakly Einstein algebraic curvature tensor in $\mathcal{T}$ such that $W^{\pm}$, both nonzero, have the same unordered spectrum with a repeated eigenvalue. Using Theorem 5.1, we then get one of the nine cases of (5.6). In fact, (5.3-a) is excluded: $\{\sigma, -\sigma/2, -\sigma/2\} = \{-\sigma, \sigma/2, \sigma/2\}$ only if $\sigma = 0$. Thus, (5.3-b) or (5.3-c) follows and, focusing just on the spectra of $W^{\pm}$, we may treat the former as a subcase of the latter, with $\xi = 0$. There are now nine possibilities, based on choosing which eigenvalue in Λ^+ and which in Λ^- is simple. Listed in the order first-first, first-second, ..., third-second, third-third, the nine cases easily lead to (5.6).

Remark 5.6. Given a proper weakly Einstein oriented four-manifold (M, g) and a point $x \in M$, Theorem 5.1 allows us to form an ordered string of eleven scalars, consisting of the scalar curvature $\mathrm{s}(x)$ followed by the ordered quadruple and two triples in the respective line of (5.3), representing the spectra of the Einstein tensor e and $W^{\pm}$ at x in a suitable positive orthonormal basis $u_1, \dots, u_4$ of T_xM and the corresponding bases (5.1) of self-dual and anti-self-dual bivectors. When treated as defined only up to rescaling and certain specific permutations, this eleven-scalar string is clearly *a local-homothety invariant* of the metric g at x.

6. Permutation groups

Any positive basis of an oriented real vector space can be subjected to

(6.1) permutations, possibly combined with some sign changes, so as to preserve the orientation.

Given an oriented Euclidean 4-space $\mathcal{T}$, let G be the finite matrix group transforming positive orthonormal bases $v_1,\dots,v_4$ of $\mathcal{T}$ by (6.1). Each element of G acts on the corresponding pair, analogous to (5.1), of ordered bases of $\Lambda^\pm$, again by (6.1), in both Λ^+ and Λ^-, in such a way that

(6.2) both bases in the pair undergo the same permutation,

along with some possible sign changes. The reason is that the permutation group of $\{1,2,3,4\}$ naturally acts on $2+2$ *partitions* of $\{1,2,3,4\}$, by which we mean

(6.3) the three sets $\{\{i,j\},\{k,l\}\}$, where $\{i,j,k,l\}=\{1,2,3,4\}$,

and each $v_i\wedge v_j\pm v_k\wedge v_l$ is associated with a unique partition $\{\{i,j\},\{k,l\}\}$.

Remark 6.1. In Theorem 5.1, suppose that e *has four distinct eigenvalues*, or *each of* $W^\pm$ *has three distinct eigenvalues.* Let $v_1,\dots,v_4$ be a positive orthonormal basis of $\mathcal{T}$. If, in the former case, $v_1,\dots,v_4$ consists of eigenvectors of e, or, in the latter, the corresponding bases of type (5.1) in $\Lambda^\pm$ diagonalize $W^\pm$, then a basis $u_1,\dots,u_4$ realizing the respective conclusions (5.3-a) – (5.3-c) arises from $v_1,\dots,v_4$ via (6.1). This is clear from Remark 5.3, as distinctness of the eigenvalues makes the basis/bases in question unique up to (6.1).

As a consequence of (6.2), the ordered spectra of $W^\pm$ realized in (5.1), for $u_1,\dots,u_4$, and those in the analog of (5.1) for $v_1,\dots,v_4$, *differ by a permutation which is the same for both signs* $\pm$.

7. Conformal changes of product metrics

The scalar curvatures and Einstein tensors of two conformally related metrics $\bar g$ and $g=\bar g/\varphi^2$ in dimension n are themselves related by

$$\text{(7.1)}\qquad \begin{aligned} &\text{i)}\quad \mathrm{s}\,=\,\varphi^2\bar{\mathrm{s}}+2(n-1)\varphi\overline{\Delta}\varphi-n(n-1)\,\bar g(\overline{\nabla}\varphi,\overline{\nabla}\varphi),\\ &\text{ii)}\quad \mathrm{e}\,=\,\bar{\mathrm{e}}\,+\,\frac{n-2}{\varphi}[\overline{\nabla}d\varphi\,-\,\frac{\overline{\Delta}\varphi}{n}\bar g],\end{aligned}$$

since – see, e.g., [**5**, p. 529] – for the Ricci tensors one has

$$\text{(7.2)}\qquad \mathrm{r}\,=\,\bar{\mathrm{r}}\,+\,(n-2)\varphi^{-1}\,\overline{\nabla}d\varphi\,+\,[\varphi^{-1}\overline{\Delta}\varphi\,-\,(n-1)\varphi^{-2}\,\bar g(\overline{\nabla}\varphi,\overline{\nabla}\varphi)]\bar g.$$

When $\bar g=\widehat g+\widetilde g$ is a product metric on a product manifold $\widehat M\times\widetilde M$,

(7.3) the factor distributions are e-orthogonal if and only if φ has additively separated variables: $\varphi=\widehat\alpha+\widetilde\alpha$, where $\widehat\alpha:\widehat M\to\mathrm{I\!R}$ and $\widetilde\alpha:\widetilde M\to\mathrm{I\!R}$,

This is clear from (7.1-ii), as both conditions amount to $\partial_i\partial_a\varphi=0$ in product coordinates x^i,x^a. With the notation of Remark 2.1, for such a product metric

$\bar{g} = \widehat{g} + \widetilde{g}$ and $g = \bar{g}/\varphi^2$, (7.1) yields

$$(7.4)\quad \begin{array}{ll} \text{a)} & \mathrm{s} = \varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}}) + 2(n-1)\varphi(\widehat{Y} + \widetilde{Y}) - n(n-1)(\widehat{Q} + \widetilde{Q}),\\ \text{b)} & \mathrm{e} = \hat{\mathrm{e}} + (n-2)\varphi^{-1}\widehat{\nabla}d\varphi + \hat{\xi}\widehat{g} \ \text{ along } \widehat{M}, \text{ as well as}\\ \text{c)} & \mathrm{e} = \tilde{\mathrm{e}} + (n-2)\varphi^{-1}\widetilde{\nabla}d\varphi + \tilde{\xi}\widetilde{g} \ \text{ along } \widetilde{M}, \text{ where}\\ \text{d)} & \hat{\xi} = p^{-1}\hat{\mathrm{s}} - n^{-1}[\hat{\mathrm{s}} + \tilde{\mathrm{s}} + (n-2)\varphi^{-1}(\widehat{Y} + \widetilde{Y})],\\ \text{e)} & \tilde{\xi} = q^{-1}\tilde{\mathrm{s}} - n^{-1}[\hat{\mathrm{s}} + \tilde{\mathrm{s}} + (n-2)\varphi^{-1}(\widehat{Y} + \widetilde{Y})],\\ \text{f)} & \text{for } p = \dim\widehat{M} \text{ and } q = \dim\widetilde{M}, \text{ with } n = p + q. \end{array}$$

We will use the obvious fact that, in (7.4), if $p = q$,

$$(7.5)\qquad n\hat{\xi} = \hat{\mathrm{s}} - \tilde{\mathrm{s}} - (n-2)\varphi^{-1}(\widehat{Y} + \widetilde{Y}), \quad n\tilde{\xi} = \tilde{\mathrm{s}} - \hat{\mathrm{s}} - (n-2)\varphi^{-1}(\widehat{Y} + \widetilde{Y}).$$

For a conformal change $g = \bar{g}/\chi^2$ of metrics in dimension n, and any function φ, one has – see, e.g. [**5**, p. 528] – the well-known relations

$$(7.6)\quad \begin{array}{ll} \text{a)} & \nabla d\varphi = \overline{\nabla}d\varphi + \chi^{-1}[d\chi \otimes d\varphi + d\varphi \otimes d\chi - \bar{g}(\overline{\nabla}\chi, \overline{\nabla}\varphi)\bar{g}],\\ \text{b)} & \Delta\varphi = \chi^2\overline{\Delta}\varphi - (n-2)\chi\bar{g}(\overline{\nabla}\chi, \overline{\nabla}\varphi), \quad g(\nabla\varphi, \nabla\varphi) = \chi^2\bar{g}(\overline{\nabla}\varphi, \overline{\nabla}\varphi). \end{array}$$

8. Conformal products and the Weyl tensor

Throughout this section we assume that $\widehat{M} \times \widetilde{M}$ is an oriented product manifold, $\varphi : \widehat{M} \times \widetilde{M} \to \mathrm{I\!R} \smallsetminus \{0\}$, while $g = (\widehat{g} + \widetilde{g})/\varphi^2$ on $\widehat{M} \times \widetilde{M}$, and either

$$(8.1)\qquad \text{a)} \ \dim\widehat{M} = \dim\widetilde{M} = 2, \quad \text{or} \quad \text{b)} \ \dim\widehat{M} = 3 \ \text{ and } \ \dim\widetilde{M} = 1.$$

At any given point $x \in \widehat{M} \times \widetilde{M}$ we may choose a positive g-orthonormal basis $v_1, \dots, v_4$ of the tangent space, so that, depending on which case of (8.1) occurs, for some even permutation (i, j, k, l) of $\{1, 2, 3, 4\}$,

$$(8.2)\quad \begin{array}{ll} \text{a)} & v_i, v_j \text{ are tangent to } \widehat{M} \text{ and } v_k, v_l \text{ tangent to } \widetilde{M}, \text{ or}\\ \text{b)} & v_l \text{ is tangent to } \widetilde{M} \text{ and } v_i, v_j, v_k, \text{ tangent to } \widehat{M}, \text{ diago-}\\ & \text{nalize the Einstein tensor of } \widehat{g} \text{ with some eigenvalues } \theta_i, \theta_j, \theta_k. \end{array}$$

In case (8.2-a), as $v_i \wedge v_j \pm v_k \wedge v_l$ are the two Kähler forms in Remarks 2.2 – 2.3,

$$(8.3)\ \text{both } 12W^\pm \text{ have the same spectrum } (2\varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}}), -\varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}}), -\varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}})),$$

in $\Lambda^\pm M$, at x, while, if (8.2-b) holds, then the only possibly-nonzero components of $2W$ at x are those algebraically related to $2W_{jkjk} = 2W_{ilil} = -\varphi^2\theta_i$, so that

$$(8.4)\qquad \text{both } 2\varphi^{-2}W^\pm \text{ have the same eigenvalues } (-\theta_k, -\theta_j, -\theta_i),$$

and the ordered spectrum in (8.3), as well as that in (8.4), is realized

$$(8.5)\qquad \text{by the bases } v_i \wedge v_j \pm v_k \wedge v_l,\ v_i \wedge v_k \pm v_l \wedge v_j,\ v_i \wedge v_l \pm v_j \wedge v_k.$$

Suppose now that, at the given point $x = (y, z) \in \widehat{M} \times \widetilde{M}$, one has

$$(8.6)\qquad \text{either (8.1-a), or (8.1-b) and } \theta_i = \theta_j.$$

In other words, we want to lump together (8.1-a) with a subcase of (8.1-b) having a repeated eigenvalue in (8.2-b), and we restrict the choice of permutations (i, j, k, l) by requiring that $\theta_i = \theta_j$. From (8.3) or, respectively, (8.4) we now get

$$(8.7)\qquad \text{the condition (5.7-a) with } 6\sigma = \varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}}), \text{ or } 2\sigma = -\varphi^2\theta_k,$$

which makes σ the simple-eigenvalue function of $W^\pm$.

Lemma 8.1. *If* $\sigma \neq 0$ *in* (8.6) – (8.7) *and* $\zeta^\pm$ *are any length* $\sqrt{2}$ *eigenbivectors of* $W^\pm$ *at* $x = (y,z) \in \widehat{M} \times \widetilde{M}$ *for the simple eigenvalue* σ*, then* $\zeta^+ \pm \zeta^-$ *both have rank two, and their images coincide, as an unordered pair, with*

(i) *the direct summands* $T_y\widehat{M}$ *and* $T_z\widetilde{M}$ *of* $T_x[\widehat{M} \times \widetilde{M}]$*, in the case* (8.1-a)*,*

(ii) *the eigenspace of the Einstein tensor of* $\widehat{g}$ *at* y *for the eigenvalue* θ_i*, and its orthogonal complement in* $T_x[\widehat{M} \times \widetilde{M}]$*, for* (8.1-b) *with* $\theta_i = \theta_j$*.*

Proof. Up to a sign: each of $\zeta^\pm$ is unique, and hence equals $v_i \wedge v_j \pm v_k \wedge v_l$ in (8.5). Now our claim follows from (8.2). □

We will refer to the two subspaces of $T_x[\widehat{M} \times \widetilde{M}]$ forming the unordered pair in (i) or (ii) above as the *summand planes* at x.

In Lemma 8.1, if $\zeta^\pm = u_i \wedge u_j \pm u_k \wedge u_l$, or $\zeta^+ = u_i \wedge u_j + u_k \wedge u_l$ and $\zeta^- = u_i \wedge u_k - u_l \wedge u_j$, for a positive g-orthonormal basis $u_1, \dots, u_4$ of $T_x[\widehat{M} \times \widetilde{M}]$ and an even permutation (i,j,k,l) of $\{1,2,3,4\}$, then

(8.8) one summand plane is spanned by u_i, u_j, the other by u_k, u_l or, respectively, one by $u_i - u_l, u_j + u_k$, the other by $u_i + u_l, u_j - u_k$,

as one sees evaluating $\zeta^+ \pm \zeta^-$, with $u_i \wedge u_j + u_k \wedge u_l \pm (u_i \wedge u_k - u_l \wedge u_j) = (u_i \mp u_l) \wedge (u_j \pm u_k)$.

For the remainder of this section we also *assume that* g *is a weakly Einstein metric, and* $\sigma \neq 0$ *in* (8.6) – (8.7).

If the Einstein tensor of g at x is nonzero, Remark 5.5 yields the existence of a positive g-orthonormal basis $u_1, \dots, u_4$ of $T_x[\widehat{M} \times \widetilde{M}]$ satisfying (5.3-c) or (5.3-b), with one of the nine options in (5.6). Now, due to (8.8), each of the following nine ordered orthogonal bases of $T_x[\widehat{M} \times \widetilde{M}]$, corresponding to the nine cases of (5.6), consists of a basis of one summand plane followed by a basis of the other:

(8.9)
$$
\begin{array}{rl}
\text{i)} & (u_1, u_2, u_3, u_4), \quad \text{ii)}\ (u_1 - u_4, u_2 + u_3, u_1 + u_4, u_2 - u_3),\\
\text{iii)} & (u_1 + u_3, u_2 + u_4, u_1 - u_3, u_2 - u_4),\\
\text{iv)} & (u_1 + u_4, u_2 + u_3, u_1 - u_4, u_2 - u_3),\\
\text{v)} & (u_1, u_3, u_2, u_4), \quad \text{vi)}\ (u_1 - u_2, u_3 + u_4, u_1 + u_2, u_3 - u_4),\\
\text{vii)} & (u_1 - u_3, u_2 + u_4, u_1 + u_3, u_2 - u_4),\\
\text{viii)} & (u_1 + u_2, u_3 + u_4, u_1 - u_2, u_3 - u_4), \quad \text{ix)}\ (u_1, u_4, u_2, u_3).
\end{array}
$$

The third, fourth and eighth bases arise here from the following analog of the second line in (8.8): if $\zeta^+ = u_i \wedge u_j + u_k \wedge u_l$ and $\zeta^- = u_i \wedge u_l - u_j \wedge u_k$, then one summand plane is spanned by $u_i + u_k, u_j + u_l$, and the other by $u_i - u_k, u_j - u_l$, which is obvious since $u_i \wedge u_j + u_k \wedge u_l \pm (u_i \wedge u_l - u_j \wedge u_k) = (u_i \pm u_k) \wedge (u_j \pm u_l)$.

Lemma 8.2. *In the cases* (8.9-ii)*,* (8.9-iii)*,* (8.9-iv) *and* (8.9-vii)*, the restrictions of the Einstein tensor* e *of* g *to both summand planes equal zero, while in the remaining five cases the summand planes are* e*-orthogonal. When* (5.3-c) *and* (8.9-i) *hold, the summand planes are the eigenspaces of* e*.*

Proof. By (5.3), the endomorphism E of $T_x[\widehat{M} \times \widetilde{M}]$ corresponding via g to e assigns $-\lambda u_1, -\mu u_2, \mu u_3, \lambda u_4$ to u_1, u_2, u_3, u_4, with μ standing for λ in (5.3-c). Consequently, in the former four cases, E sends each summand plane into the other. In the latter five, E leaves both summand planes invariant. For (8.9-i), (8.9-v) and (8.9-ix), this last claim is immediate, For (8.9-vi) and (8.9-viii), it follows since

$\mu=\lambda$. Namely, our assumption that $\sigma\ne 0$ in (8.6) – (8.7) gives $\mathrm{s}\ne 0$ in (5.6), and hence $\xi\ne 0$, leading to (5.3-c) rather than (5.3-b). ☐

REMARK 8.3. In the eight cases of (8.9) other than (8.9-i), the restrictions of e to the summand planes are both g-traceless. For the fifth, sixth, eighth and ninth cases this follows since, by (5.3), the corresponding orthogonal basis in (8.9) consists of eigenvectors of e in $T_x[\widehat{M}\times\widetilde{M}]$ for the eigenvalues $(-\lambda,\lambda,-\mu,\mu)$ (the ninth case) or $(-\lambda,\lambda,-\lambda,\lambda)$ (the other three). For the remaining four (second, third, fourth, seventh) our claim is obvious from Lemma 8.2.

9. Examples

The following two theorems provide constructions of weakly Einstein conformal products. Their proofs, and explicit realizations of the geometric structures used in the constructions, are given in Sect. 10 – 15.

THEOREM 9.1. *Given Riemannian surfaces (Σ,g) and (Π,h) with Gaussian curvatures K and c such that c is constant, and a function $\varphi:\Sigma\to\mathrm{I\!R}\smallsetminus\{0\}$, for Δ and ∇ referring to the metric g, let on Σ either*

(i) $\Delta\varphi^{-1}=0$ *and* $2g(\nabla\varphi,\nabla\varphi)=(c-K)\varphi^2$, *or*
(ii) $2\nabla d\varphi=(\Delta\varphi)g$ *and* $(K+c)\varphi^2+\varphi\Delta\varphi=2g(\nabla\varphi,\nabla\varphi)$.

Then the metric $(g+h)/\varphi^2$ on $\Sigma\times\Pi$ is weakly Einstein. Also, $(g+h)/\varphi^2$ is

(iii) *Einstein in the case* (i) *if and only if* $2\nabla d\varphi=(\Delta\varphi)g$,
(iv) *Einstein in the case* (ii) *if and only if* $g(\nabla\varphi,\nabla\varphi)=c\varphi^2$,
(v) *conformally flat if and only if* $K=-c$.

More precisely, the equalities in (iii) – (v) *characterize points at which the Einstein or, respectively, Weyl tensor of the metric $(g+h)/\varphi^2$ vanishes. Wherever both these tensors are nonzero, the local-homothety invariant of Remark* 5.6 *is given by*

$$\begin{array}{ll}\bar{\mathrm{s}},\ (-\lambda,0,0,\lambda),\ (-\bar{\mathrm{s}}/12,-\bar{\mathrm{s}}/12,\,\bar{\mathrm{s}}/6),\ (-\bar{\mathrm{s}}/12,-\bar{\mathrm{s}}/12,\,\bar{\mathrm{s}}/6) & \text{in the case (i)},\\ \bar{\mathrm{s}},\ (-\lambda,-\lambda,\lambda,\lambda),\ (-\bar{\mathrm{s}}/12,\,\bar{\mathrm{s}}/24,\,\bar{\mathrm{s}}/24),\ (-\bar{\mathrm{s}}/12,\,\bar{\mathrm{s}}/24,\,\bar{\mathrm{s}}/24) & \text{in the case (ii)},\end{array}\tag{9.1}$$

with the scalar curvature $\bar{\mathrm{s}}\ne 0$ of the metric $(g+h)/\varphi^2$ and a parameter $\lambda\ne 0$.

The next theorem employs our usual notations: $\mathrm{r},\mathrm{e},\mathrm{s}$ for the Ricci/Einstein tensors and scalar curvature of the metric g, and ∇ for its Levi-Civita connection as well as the g-gradient, while the barred symbols $\bar{\mathrm{e}},\bar{\mathrm{s}}$ and $\overline{W}$ correspond to $\bar{g}$.

THEOREM 9.2. *Let there be given an open interval $I\subseteq\mathrm{I\!R}$ with the coordinate τ and the metric $d\tau^2$, a Riemannian three-manifold (M,g), a vector field v on M and functions $\varphi,\theta:M\to\mathrm{I\!R}$ such that $\varphi\ne 0$ everywhere. If, moreover,*

$$\begin{array}{rl}\text{i)} & 2\nabla d\varphi+\varphi\mathrm{r}\ \text{has the } g\text{-spectrum } (-|v|^2,-|v|^2,0),\\ \text{ii)} & (2\nabla d\varphi+\varphi\mathrm{r})(v,\,\cdot\,)=0,\\ \text{iii)} & [(\mathrm{s}-6\theta)\varphi+6\Delta\varphi]\varphi=12g(\nabla\varphi,\nabla\varphi),\\ \text{iv)} & \mathrm{e}(v,\,\cdot\,)=\theta g(v,\,\cdot\,),\end{array}\tag{9.2}$$

then $\bar{g}=(g+d\tau^2)/\varphi^2$ is a weakly Einstein metric on $M\times I$. Also, $\bar{g}$ is Einstein, or conformally flat, if and only if v vanishes identically on M or, respectively, g has constant sectional curvature. More precisely, $\bar{\mathrm{e}}=0$ or $\overline{W}=0$ at precisely those points at which $v=0$ or, respectively, $\mathrm{e}=0$. Wherever $v\ne 0$ and $\mathrm{e}\ne 0$, the local-homothety invariant of Remark 5.6 *is given by*

$$\bar{\mathrm{s}},\quad(-\lambda,-\lambda,\lambda,\lambda),\quad(-\bar{\mathrm{s}}/12,\,\xi-\bar{\mathrm{s}}/12,-\xi+\bar{\mathrm{s}}/6),\quad(-\bar{\mathrm{s}}/12,\,\xi-\bar{\mathrm{s}}/12,-\xi+\bar{\mathrm{s}}/6),$$

$\bar{\mathrm{s}} \neq 0$ *being the scalar curvature of* $\bar{g}$, *with parameters* $\lambda \neq 0$ *and* ξ, *while* $\xi = \bar{\mathrm{s}}/8$ *in the case where* e *has, at every point, the* g-*spectrum* $\{\theta, -\theta/2, -\theta/2\}$.

Remark 9.3. The conditions i) – ii) in (9.2), imposed on a function φ and a vector field v on a Riemannian three-manifold (M, g), are equivalent to

$$2\nabla d\varphi + \varphi \mathrm{r} - (\Delta\varphi + \mathrm{s}\varphi/2)g = \omega \otimes \omega \text{ for the 1-form } \omega = g(v, \,\cdot\,), \text{ which also implies that } \mathrm{b} = 2\nabla d\varphi + \varphi \mathrm{r} \text{ has } \mathrm{tr}_g \mathrm{b} = 2(\Delta\varphi + \mathrm{s}\varphi/2) = -2|v|^2. \tag{9.3}$$

In fact, i) – ii) obviously yield the second line of (9.3), and so does the first line of (9.3). Now assume i) – ii). At points x where $v_x = 0$, (9.3) follows: the tensor field $\mathrm{b} = 2\nabla d\varphi + \varphi\mathrm{r}$, having the spectrum $(-|v|^2, -|v|^2, 0)$, vanishes at x along with its g-trace $2(\Delta\varphi + \mathrm{s}\varphi/2) = -2|v|^2$. Wherever $v \neq 0$, the left-hand side in (9.3), with the spectrum $(0, 0, |v|^2)$, clearly equals the right-hand side: treated as $(1, 1)$ tensors, both send v to $|v|^2 v$, and annihilate vectors orthogonal to v. Conversely, (9.3) easily gives $\mathrm{b}(v, \,\cdot\,) = 0$ and $\mathrm{b}(v', \,\cdot\,) = -|v|^2 g(v', \,\cdot\,)$ if $g(v, v') = 0$.

10. Proof of Theorem 9.1

Let us adopt the notation of (2.4) and (7.1), with $(\widehat{\alpha}, \widetilde{\alpha}) = (\varphi, 0)$. Now our

$$\Sigma, g, \nabla, \Delta, 2K, \Pi, h, 2c \text{ become } \widehat{M}, \widehat{g}, \widehat{\nabla}, \widehat{\Delta}, \hat{\mathrm{s}}, \widetilde{M}, \widetilde{g}, \tilde{\mathrm{s}}, \text{ and } \widetilde{Y} = \widetilde{Q} = 0, \text{ while } g \text{ is now the conformal-product metric } g = \bar{g}/\varphi^2, \text{ for } \bar{g} = \widehat{g} + \widetilde{g}. \tag{10.1}$$

Except for (iii) – (v), all of our assertion will be obvious from Remark 5.2 if we prove the following two claims, for some positive g-orthonormal basis $u_1, \ldots, u_4$ of the tangent space at any point:

(I) Assuming (i) we get (5.3-b) with $c_2 = c_3 = c_4 = 0$ and $\mu = 0$.
(II) Similarly, (ii) leads to (5.3-c) with $c_2 = c_3 = c_4 = 0$ and $\xi = \mathrm{s}/8$.

In both cases, by (7.3), for the Einstein tensor $\mathrm{e} = \mathrm{r} - \mathrm{s}g/4$,

$$\text{the factor distributions } T\widehat{M} \text{ and } T\widetilde{M} \text{ are e-orthogonal.} \tag{10.2}$$

We will choose $(u_1, \ldots, u_4) = (v_1, \ldots, v_4)$ and (i, j, k, l) so as to have (8.2-a), (8.3) and (8.5). In the case (i), we begin by observing that

(a) $\varphi^2(\tilde{\mathrm{s}} - \hat{\mathrm{s}}) = 4\widehat{Q}$, (b) $\varphi\widehat{Y} = 2\widehat{Q}$,
(c) $\mathrm{s} = \varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}})$ for the scalar curvature s of g,
(d) e restricted to $T\widehat{M}$ is g-traceless, and on $T\widetilde{M}$ it equals 0.

Namely, (i) yields (a) – (b), as $\varphi\widehat{Y} - 2\widehat{Q} = \varphi\widehat{\Delta}\varphi - 2\widehat{g}(\widehat{\nabla}\varphi, \widehat{\nabla}\varphi) = -\varphi^3\widehat{\Delta}\varphi^{-1} = 0$ by (2.3-b). Now (b) and (7.4-a), with $\widetilde{Y} = \widetilde{Q} = 0$, imply (c), while (10.2) leads to (d) for $T\widehat{M}$, since (d) for $T\widetilde{M}$ is obvious from (a) – (b), (7.4-c) and (7.5).

In view of (d) and (10.2) we may choose an orientation and a positive g-orthonormal basis $(u_1, \ldots, u_4) = (v_1, \ldots, v_4)$ of T_xM at any given point x, realizing for e a spectrum of the form $(-\lambda, 0, 0, \lambda)$, which yields (8.2-a) with $(i, j, k, l) = (1, 4, 2, 3)$.

Now (8.3), (8.5) and (c) imply (5.3-b) for $c_2 = c_3 = c_4 = 0$, as required in (I).

Next, assume (ii), which has the following obvious consequences:

(e) $2\varphi\widehat{Y} - 4\widehat{Q} = -\varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}})$,
(f) e restricted to either factor distribution equals a function times g,
(g) $\mathrm{s} = -2\varphi^2(\hat{\mathrm{s}} + \tilde{\mathrm{s}})$ for the scalar curvature s of g,

(g) being immediate from (e) and (17.1-iii) with $\widetilde{Y} = \widetilde{Q} = 0$.

Just as (d) did before, (f) now allows us to choose a positive g-orthonormal basis $u_1, \dots, u_4$ of of any given oriented tangent space consisting of eigenvectors of e with eigenvalues having, this time, the form $(-\lambda, -\lambda, \lambda, \lambda)$, leading to (8.2-a), where $(i, j, k, l) = (1, 2, 3, 4)$. Then (8.3) and (g) give (5.3-c) with $c_2 = c_3 = c_4 = 0$ and $\xi = \mathrm{s}/8$, proving (II).

For the conformal-product metric $g = \bar{g}/\varphi^2$, since $\widetilde{\alpha} = \widetilde{Y} = \widetilde{Q} = 0$,

(10.3) the right-hand sides of (7.4-b) and (7.4-c) have g-traces differing by $2\varphi\widehat{Y} + \varphi^2(\hat{\mathrm{s}} - \tilde{\mathrm{s}})$.

The 'only if' claim in (iii) is obvious from (7.4-b). The 'if' part now follows: assuming that $2\widehat{\nabla} d\varphi = \widehat{Y}\widehat{g}$, we get $2\varphi\widehat{Y} + \varphi^2(\hat{\mathrm{s}} - \tilde{\mathrm{s}}) = 0$ in (10.3) by combining (a) and (b). Similarly, in the case of (ii), the Einstein condition – that is, vanishing of $2\varphi\widehat{Y} + \varphi^2(\hat{\mathrm{s}} - \tilde{\mathrm{s}})$ in (10.3) – is, by (e), equivalent to having $2\widehat{Q} = \varphi^2\tilde{\mathrm{s}}$, which yields (iv). Finally, (v) is obvious from Remark 2.3.

11. Proof of Theorem 9.2

The tensor field $\mathrm{b} = 2\nabla d\varphi + \varphi\mathrm{r}$ has the spectrum $(-|v|^2, -|v|^2, 0)$. In the interior of the subset of $M \times I$ on which $v = 0$, one thus has $\mathrm{b} = 0$, and (7.2), applied to $n = 4$ and $g, \bar{g}$ replaced with our $\bar{g}$ and the product metric $g + dt^2$, implies that $\bar{\mathrm{r}}$ is a functional multiple of $\bar{g}$, which makes $\bar{g}$ an Einstein metric, proving our first claim in this case.

We may therefore restrict our discussion to points where $v \neq 0$, further assuming that they are generic relative to the spectrum of e (see Remark 2.6), and M is oriented. Thus, by (9.2-iv), locally, some positive $\bar{g}$-orthonormal frame $u_1, \dots, u_4$ has u_1, u_2, u_3 tangent to the M factor and diagonalizing e, with our v equal to a positive functional multiple of u_3.

Next, $u_1, \dots, u_4$ also diagonalize $\bar{\mathrm{e}}$ with the ordered $\bar{g}$-spectrum $(-\lambda, -\lambda, \lambda, \lambda)$, for $\lambda = \varphi|v|^2/2$. Namely, we have (7.4) with $\tilde{\mathrm{s}}, \widetilde{Y}, \widetilde{Q}, \tilde{\mathrm{e}}, \widetilde{\nabla} d\varphi$ equal to zero, $(n, p, q) = (4, 3, 1)$, and $M, I, \bar{\mathrm{s}}, \bar{\mathrm{e}}, g, \nabla, Y, Q, dt^2$ replacing $\widehat{M}, \widetilde{M}, \mathrm{s}, \mathrm{e}, \widehat{g}, \widehat{\nabla}, \widehat{Y}, \widehat{Q}, \widetilde{g}$, so that

(11.1)
a) $\bar{\mathrm{s}} = \varphi^2\mathrm{s} + 6\varphi Y - 12Q$, where $Q = g(\nabla\varphi, \nabla\varphi)$ and $Y = \Delta\varphi$,
b) $12\bar{\mathrm{e}} = 12\mathrm{e} + 24\varphi^{-1}\nabla d\varphi + (\mathrm{s} - 6\varphi^{-1}Y)g$ along M,
c) $4\bar{\mathrm{e}} = -(\mathrm{s} + 2\varphi^{-1}Y)\, dt^2$ along I.

As $\mathrm{e} = \mathrm{r} - \mathrm{s}g/3$, (11.1-b) combined with (9.3) gives $\varphi\bar{\mathrm{e}} = \mathrm{b} + |v|^2 g/2$, along M, which – according to (9.2) – is diagonalized by u_1, u_2, u_3 with the ordered g-spectrum $(-|v|^2/2, -|v|^2/2, |v|^2/2)$, that is, the $\bar{g}$-spectrum $(-\varphi\lambda, -\varphi\lambda, \varphi\lambda)$. At the same time, by (11.1-c) and (9.3), along I (the span of u_4) $\bar{\mathrm{e}}$ equals dt^2 times $\varphi^{-1}|v|^2/2$, which is nothing else than $\bar{g}$ times λ. Consequently,

(11.2) $\bar{\mathrm{e}}$ has the $\bar{g}$-spectrum $(-\lambda, -\lambda, \lambda, \lambda)$ with $\lambda = \varphi|v|^2/2$, realized by our $(u_1, \dots, u_4)$.

On the other hand, with the new notation, in view of (8.4),

(11.3) $2\varphi^{-2}\overline{W}^{\pm}$ have the $\bar{g}$-spectrum $(-\theta_3, -\theta_2, -\theta_1)$, realized as in (8.5),

for the Weyl tensor $\overline{W}$ of $\bar{g}$, where θ_i are the eigenvalue functions of e, associated with u_i, $i = 1, 2, 3$. Consequently, our choice of u_3, combined with (9.2-iv) gives $\theta = \theta_3$. The condition (9.2-iii), which now reads $[(\mathrm{s} - 6\theta)\varphi + 6Y]\varphi = 12Q$, turns

(11.1-a) into the equality $\bar{\mathrm{s}} = 6\varphi^2\theta$. We thus obtain (5.3-c) for $\bar{\mathrm{e}}$ and $\overline{W}$, with $c_2 = c_3 = c_4 = 0$ and ξ given by $12\xi = \bar{\mathrm{s}} - 6\varphi^2\theta_2 = 2\bar{\mathrm{s}} + 6\varphi^2\theta_1$ (both equalities yield the same ξ, since $\theta_1 + \theta_2 = -\theta$), along with the final clause of the theorem: if $\theta_1 = \theta_2 = -\theta/2$, the preceding sentence shows that $\xi = \bar{\mathrm{s}}/8$.

Remark 5.2 now implies that $\bar{g}$ is a weakly Einstein metric, while the claim about the Einstein and conformally flat cases is obvious from (11.2) and (11.3).

12. Euclidean spheres and Lorentzian pseudospheres

Throughout this section, $\langle\,,\rangle$ denotes a Euclidean or Lorentzian $(-\,+\,\ldots+)$ inner product $\langle\,,\rangle$ in a real vector space $\mathcal{V}$ of dimension $n+1 \geq 3$, and Σ stands for the sphere $S^{-1}(0)$ or one sheet of the two-sheeted pseudosphere $S^{-1}(0)$, where $S(v) = \langle v, v\rangle - a$, and $a \in (0,\infty)$ or, respectively, $a \in (-\infty, 0)$. The submanifold metric g of Σ thus has constant sectional curvature $K = a^{-1}$.

A function φ on a Riemannian manifold (M, g) is called *concircular* [**14**] if

$$\text{(12.1)} \qquad \begin{array}{l} \nabla d\varphi = \sigma g \text{ for some function } \sigma, \text{ obviously given by } n\sigma = \Delta\varphi, \\ \text{for } n = \dim M, \text{ or, equivalently, } \nabla\varphi \text{ is a conformal vector field.} \end{array}$$

Remark 12.1. In (12.1), if $\sigma = F(\varphi)$ is a function of φ, then the gradient $\nabla\varphi$, at points where $\nabla\varphi \neq 0$, is an eigenvector of the Ricci tensor for the eigenvalue $(1-n)F'(\varphi)$. At such points, g has the Gaussian curvature $-F'(\varphi)$ when $n = 2$. (This is obvious from (2.2) applied to $v = \nabla\varphi$.)

Lemma 12.2. *Concircular functions on any nonempty connected open subset U of Σ, up to additive constants, coincide with restrictions to U of linear functionals $\mathcal{V} \to \mathrm{I\!R}$, and σ in* (12.1) *then equals $-a^{-1}\varphi$.*

This is immediate from the Bochner identity (2.2) applied to (12.1), as one sees differentiating restrictions of linear functionals twice along the g-geodesics, and using the trigonometric or hyperbolic descriptions of the latter.

Let $u \in \mathcal{V}$. For the Laplacian Δ and gradient ∇ associated with the submanifold metric g of Σ, and the function $\chi : \Sigma \to \mathrm{I\!R}$ obtained by restricting to Σ the linear functional $\langle u, \cdot\,\rangle$, one then has

$$\text{(12.2)} \qquad \text{a) } \Delta\chi = -na^{-1}\chi, \qquad \text{b) } g(\nabla\chi, \nabla\chi) = -a^{-1}\chi^2 + \langle u, u\rangle.$$

In fact, Lemma 12.2 yields (12.2-a), while (12.2-b) is also immediate: at any $y \in \Sigma$, the g-gradient of χ is the component of u tangent to Σ, with the norm squared $\langle u, u\rangle - a^{-1}\langle u, y\rangle^2$.

On $M = \Sigma \times \Sigma$, let $\overline{\Delta}$ and $\overline{\nabla}$ be the $\bar{g}$-Laplacian and $\bar{g}$-gradient, for the Riemannian product $\bar{g}$ of two copies of the metric g (which is also the submanifold metric of M within $\mathcal{V} \times \mathcal{V}$). Then, for the function $\psi : M \to \mathrm{I\!R}$ arising as the restriction of B to M of any bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathrm{I\!R}$,

$$\text{(12.3)} \qquad \text{a) } \overline{\Delta}\psi = -2na^{-1}\psi, \qquad \text{b) } \bar{g}(\overline{\nabla}\psi, \overline{\nabla}\psi) = -2a^{-1}\psi^2 + F,$$

where $F : M \to \mathrm{I\!R}$ is given by $F(y, z) = \langle Ay, Ay\rangle + \langle A^*z, A^*z\rangle$ for the linear endomorphism A of V such that $B = \langle A\cdot\,, \cdot\,\rangle$. Namely, our claim trivially follows from (12.2) applied to $\chi = \psi(\,\cdot\,, z)$, or $\chi = \psi(y, \cdot\,)$, with fixed z, or y, and $u = A^*z$ or, respectively, $u = Ay$.

Remark 12.3. Any polynomial function $P : \mathcal{V} \to \mathrm{I\!R}$ vanishing on Σ is polynomially divisible by S. To see this, rather than invoking Hilbert's *Nullstellensatz*, let us fix a line $t \mapsto x + tv$ intersecting Σ at two points. Along the line, P/S is a polynomial, and so $d^k[P/S]/dt^k = 0$ for some k. The same holds for all nearby lines; thus, P/S (which is a smooth function on $\mathcal{V}$ according to Remark 2.8) has the kth directional derivative along v equal to zero at x for all pairs (v, x) from a nonempty open set, and our conclusion is a consequence of real-analyticity.

LEMMA 12.4. *If* $(\widehat{M}\times\widetilde{M}, \widehat{g}+\widetilde{g})$ *is a Riemannian product of surfaces of constant scalar curvatures* $\tilde{\mathrm{s}}$ *and* $\hat{\mathrm{s}}$ *with* $\hat{\mathrm{s}}+\tilde{\mathrm{s}} \neq 0$ *and, for a function* $\varphi : \widehat{M} \times \widetilde{M} \to \mathrm{I\!R}$*,*

(i) $\widehat{\nabla}d\varphi$ *and* $\widetilde{\nabla}d\varphi$ *are functional multiples of* $\widehat{g}$ *and* $\widetilde{g}$*,*
(ii) $2(\widehat{Y} - \widetilde{Y}) = \varphi(\tilde{\mathrm{s}} - \hat{\mathrm{s}})$*,*
(iii) $(1-\varepsilon)(\hat{\mathrm{s}}+\tilde{\mathrm{s}})\varphi^2 + 6\varphi(\widehat{Y}+\widetilde{Y}) = 12(\widehat{Q}+\widetilde{Q})$*, where* $\varepsilon \in \mathrm{I\!R} \smallsetminus \{1\}$*,*

then $\varphi = \widehat{\alpha} + \widetilde{\alpha}$ *with some* $\widehat{\alpha} : \widehat{M} \to \mathrm{I\!R}$ *and* $\widetilde{\alpha} : \widetilde{M} \to \mathrm{I\!R}$*.*

PROOF. By (i) Lemma 12.2, $\widehat{Y} = \widehat{\Delta}\varphi = -\hat{\mathrm{s}}\varphi + \eta_1$ and $\widetilde{Y} = \widetilde{\Delta}\varphi = -\tilde{\mathrm{s}}\varphi + \eta_2$, with η_1 (or, η_2) constant along $\widehat{M}$ (or, $\widetilde{M}$), which is clearly also true when $\hat{\mathrm{s}}\tilde{\mathrm{s}} = 0$. Now (ii) gives $\varphi(\hat{\mathrm{s}} - \tilde{\mathrm{s}}) = 2(\eta_1 - \eta_2)$. If $\hat{\mathrm{s}} \neq \tilde{\mathrm{s}}$, our claim follows. In the remaining case, $\hat{\mathrm{s}} = \tilde{\mathrm{s}} = 2c$ and $\eta_1 = \eta_2 = 2c\gamma$ for some real $c \neq 0$ and γ.

Locally, we may identify both $(\widehat{M}, \widehat{g})$ and $(\widetilde{M}, \widetilde{g})$ with (Σ, g) defined above, for $a = c^{-1}$. Using (i) and Lemma 12.2 again, we see that $\varphi(y, z) - \gamma$ has, for each fixed $z \in \Sigma$ (or, $y \in \Sigma$) an extension from Σ to a linear functional in the variable $y \in \mathcal{V}$ (or, $z \in \mathcal{V}$), and this unique extension $(y, z) \mapsto B(y, z)$, being linear in y and z separately, is bilinear, leading to (12.3) for $\psi = \varphi - \gamma$, with $\overline{\Delta}\psi = \widehat{Y} + \widetilde{Y}$ and $\bar{g}(\overline{\nabla}\psi, \overline{\nabla}\psi) = \widehat{Q} + \widetilde{Q}$. Now (iii) states that

$$(1-\varepsilon)c[B(y,z) + 2\gamma]B(y,z) - 3[\langle Ay, Ay\rangle + \langle A^*z, A^*z\rangle] - 6c\gamma \tag{12.4}$$

equals 0 for all $y, z \in \Sigma \subseteq S^{-1}(0)$, cf. the beginning of this section. Remark 12.3 for each fixed $z \in \Sigma$ (or, $y \in \Sigma$) shows that, on $\mathcal{V}$, (12.4) simultaneously equals $\langle y, y\rangle - a$ times a function of z, and $\langle z, z\rangle - a$ times a function of y. Equating the latter two products and separating variables, we get (12.4) equal to a constant p times $[\langle y, y\rangle - a][\langle z, z\rangle - a]$. Applying Remark 2.7 to the components of bidegree $(2, 2)$, we get $(1-\varepsilon)c[B(y,z)]^2 = p\langle y, y\rangle\langle z, z\rangle$ for all $y, z \in \mathcal{V}$. If p were nonzero, choosing y, z with $B(y, z) = 0 \neq \langle y, y\rangle\langle z, z\rangle$ we would get a contradiction. Thus, p equals 0, and hence so does B, making φ constant, as $\varphi(y, z) = \gamma + B(y, z)$, which again yields our assertion. □

13. Surface metrics

In this section (Σ, g) is always a Riemannian surface, assumed oriented whenever we mention its complex-structure tensor J. Its Kähler form $g(J\cdot\,, \cdot)$ then equals the area form of g. The following three conditions are mutually equivalent:

(i) g is, locally, a warped-product metric,
(ii) there exists a nowhere-zero g-Killing field w on Σ,
(iii) locally, (Σ, g) admits a concircular function α without critical points, cf. (12.1), and then we may choose w in (ii) to be Jv, for $v = \nabla\varphi$.

About (i) – (ii), see Remark 3.1. In the case of (ii) – (iii), the g-Killing property of w (skew-adjointness of $B = \nabla w : TM \to TM$) is, for dimensional reasons, nothing else than $B = \sigma J$ for some function σ, which in turn amounts to $g(A\cdot\,, \cdot) = \sigma g$ for $A = -JB = \nabla v$, with $v = -Jw$ being locally a gradient, as A is self-adjoint.

Assuming (i), we get, in suitable local coordinates $(x^1, x^2) = (t, y)$, for some function χ of t, the Gaussian curvature K of g, the gradient $\nabla\varphi$, Hessian $\nabla d\varphi$

and Laplacian $\Delta\varphi$ of any function φ depending only on t, with $(\ )^{\cdot} = d/dt$,

$$(13.1)\quad \begin{array}{lll} \text{a)}\ \ g = dt^2 + e^{2\chi}dy^2, & \text{b)}\ \ K = -(\ddot\chi + \dot\chi^2), & \text{c)}\ \ g(\nabla\varphi,\nabla\varphi) = \dot\varphi^2, \\ \text{d)}\ \ \nabla d\varphi = \ddot\varphi\, dt^2 + \dot\varphi\dot\chi e^{2\chi}dy^2, & & \text{e)}\ \ \Delta\varphi = \ddot\varphi + \dot\varphi\dot\chi, \\ \text{f)}\ \ 2\nabla d\varphi = (\Delta\varphi)g \text{ if and only if } \ddot\varphi = \dot\varphi\dot\chi, & & \\ \text{g)}\ \ K\dot\varphi = -\dddot\varphi \text{ if } 2\nabla d\varphi = (\Delta\varphi)g. & & \end{array}$$

In fact, (13.1-a) – (13.1-e) follow, since one easily verifies that

$$\begin{array}{l} g_{11} = g^{11} = 1, \quad g_{22} = e^{2\chi}, \quad g^{22} = e^{-2\chi}, \quad g_{12} = g^{12} = 0, \\ \Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0, \qquad \Gamma_{12}^2 = \dot\chi, \qquad \Gamma_{22}^1 = -\dot\chi e^{2\chi}, \\ R_{121}{}^2 = -(\ddot\chi + \dot\chi^2), \quad \varphi_{,11} = \ddot\varphi, \quad \varphi_{,12} = 0, \quad \varphi_{,22} = \dot\varphi\dot\chi e^{2\chi}. \end{array}$$

Now (13.1-d) gives (13.1-f) and, by (13.1-b), d/dt applied to (13.1-f) yields (13.1-g).

Remark 13.1. Let α be a concircular function on (Σ, g). Then, locally, at points where $d\alpha$ is nonzero, $Y = \Delta\alpha$ and $Q = g(\nabla\alpha,\nabla\alpha)$ are functions of α. Furthermore, in the open set where $dK \neq 0$, such α is unique up to affine replacements ($p\alpha + q$ with constants $p \neq 0$ and q).

In fact, by (iii), $d_w Y = d_w Q = 0$ for the Killing field $w = Jv$, where $v = \nabla\varphi$, while two concircular functions not satisfying an affine relation lead to two linearly independent Killing fields, making K constant.

Lemma 13.2. *If α is a concircular function on (Σ, g), then, locally, at points where $\nabla\alpha \neq 0$, in suitable coordinates x^1, x^2, for $(\ )^{\cdot} = \partial/\partial x^1$ and the Gaussian curvature K of g, one has $\partial\alpha/\partial x^2 = 0$ and*

$$(13.2)\qquad \Delta\alpha = 2\ddot\alpha, \quad g(\nabla\alpha,\nabla\alpha) = \dot\alpha^2, \quad K = -\frac{\dddot\alpha}{\dot\alpha}.$$

This is obvious from (i) – (iii), (13.1-c), and (13.1-e) – (13.1-g) for $\varphi = \alpha$.

Lemma 13.3. *Let $\Delta\alpha = 2p\alpha + 2q$ and $g(\nabla\alpha,\nabla\alpha) = p\alpha^2 + 2q\alpha + q'$ for a function α on a Riemannian surface (M, g) and constants p, q, q'. Then, wherever $\nabla\alpha$ is nonzero, $\nabla d\alpha = (p\alpha + q)g$ and g has the constant Gaussian curvature $K = -p$.*

Proof. The Hessian $\nabla d\alpha$ has the trace $2(p\alpha + q)$ and, wherever $\nabla\alpha \neq 0$, (2.3-a) shows that $\nabla\alpha$ is an eigenvector of $\nabla d\alpha$ for the eigenvalue $p\alpha + q$. Thus $\nabla d\alpha = (p\alpha + q)g$, and our claim follows from Remark 12.1. □

14. Geometric realizations with Killing fields

We now show that Theorems 9.1 and 9.2 actually yield examples of proper weakly Einstein metrics, using Riemannian surfaces (Σ, g) which admit nowhere-zero Killing fields. This last condition is necessary for (ii) in Theorem 9.1 (see (ii) – (iii) in Sect. 13) but, seemingly, not for (i).

In terms of (13.1), the assumptions of Theorem 9.1 imposed on χ and φ are

$$(14.1)\quad \begin{array}{rl} \text{i)} & (\ddot\varphi + \dot\varphi\dot\chi)\varphi = 2\dot\varphi^2 = (c + \ddot\chi + \dot\chi^2)\varphi^2 \text{ due to } (2.3-\text{b}), \text{ in case (i)}, \\ \text{ii)} & \ddot\varphi = \dot\varphi\dot\chi, \quad (c - \ddot\chi - \dot\chi^2)\varphi^2 + (\ddot\varphi + \dot\varphi\dot\chi)\varphi = 2\dot\varphi^2 \text{ in case (ii)}. \end{array}$$

In (14.1-i), or (14.1-ii), using the first equality to eliminate $\dot\chi$, we get

$$(14.2)\quad \begin{array}{rl} \text{i)} & \dot\chi + \ddot\varphi/\dot\varphi = 2\dot\varphi/\varphi \text{ wherever } \varphi\dot\varphi \neq 0, \text{ case (i)} \\ \text{ii)} & \dot\varphi = \ddot\varphi/\dot\chi \text{ wherever } \dot\varphi \neq 0 \text{ in case (ii)}. \end{array}$$

Consequently, in the coordinates $(x^1,x^2)=(t,y)$, (i) or, respectively, (ii) amounts to the following third-order ordinary differential equation, imposed on φ alone:

$$\text{(14.3)}\quad \text{i) } \varphi\dot\varphi\dddot\varphi = 2(\varphi\ddot\varphi-\dot\varphi^2)\ddot\varphi + c\varphi\dot\varphi^2, \text{ or } \text{ ii) } \varphi^2\dddot\varphi = (2\varphi\ddot\varphi-2\dot\varphi^2+c\varphi^2)\dot\varphi.$$

Solving (14.3) with initial conditions such that $\varphi\dot\varphi\neq 0$ (which reflects Remark 2.4), we thus get examples of weakly Einstein metrics $\bar g=(g+h)/\varphi^2$ on $\Sigma\times\Pi$. To evaluate the scalar curvature $\bar{\mathrm{s}}$ of $\bar g$ and the spectrum of its Einstein tensor $\bar{\mathrm{e}}$, we apply (7.4) – (7.5) and (13.1-b) – (13.1-e) to $\widehat g = g = dt^2+e^{2\chi}dy^2$ and

$$\text{(14.4)}\qquad \begin{aligned}&(n,p,q,\widehat M,\widetilde M,\widetilde g,\hat{\mathrm{s}},\tilde{\mathrm{s}},\widehat Y,\widehat Q,\widetilde Y,\widetilde Q)\\ &\quad= (4,2,2,\Sigma,\Pi,h,2K,2c,\Delta\varphi,g(\nabla\varphi,\nabla\varphi),0,0),\end{aligned}$$

concluding, from (14.2) and (14.1), that, in case (i) of Theorem 9.1

(a) $(c-K)\varphi^2=\varphi\Delta\varphi=2\dot\varphi^2$ and $\nabla d\varphi=\ddot\varphi\,dt^2-(\ddot\varphi-2\dot\varphi^2/\varphi)e^{2\chi}dy^2$,
(b) $\bar{\mathrm{s}}=4(c\varphi^2-\dot\varphi^2)$, while $\hat\xi\varphi^2=-2\dot\varphi^2$ and $\tilde\xi=0$,
(c) $\varphi^2\bar{\mathrm{e}}=2(\varphi\ddot\varphi-\dot\varphi^2)(dt^2-e^{2\chi}dy^2)$ along Σ and $\bar{\mathrm{e}}=0$ along Π,

and, similarly, in case (ii),

(d) $(K+c)\varphi^2=2(\dot\varphi^2-\varphi\ddot\varphi)$ and $\Delta\varphi=2\ddot\varphi$, while $\nabla d\varphi=\ddot\varphi(dt^2+e^{2\chi}dy^2)$,
(e) $\bar{\mathrm{s}}=8(\varphi\ddot\varphi-\dot\varphi^2)$, while $\hat\xi\varphi^2=\dot\varphi^2-c\varphi^2-2\varphi\ddot\varphi$ and $\tilde\xi=c\varphi^2-\dot\varphi^2$,
(f) $\varphi^2\bar{\mathrm{e}}=(\dot\varphi^2-c\varphi^2)(dt^2+e^{2\chi}dy^2)$ along Σ and $\varphi^2\bar{\mathrm{e}}=(c\varphi^2-\dot\varphi^2)h$ along Π.

In both cases, by (7.3), the factor distributions are $\bar{\mathrm{e}}$-orthogonal.

Now, due to (v) in Theorem 9.1, (b), (f), and (13.1-b) or, respectively, (13.1-g), the conformal flatness of $\bar g=(g+h)/\varphi^2$ is equivalent to

$$\text{(14.5)}\qquad \dot\varphi^2=c\varphi^2 \text{ in case (i)},\quad \varphi\ddot\varphi=\dot\varphi^2 \text{ in case (ii)}.$$

Theorem 9.1(iii)-(iv), along with (c) and (f) imply, in turn, that $\bar g=(g+h)/\varphi^2$ is Einstein if and only if

$$\text{(14.6)}\qquad \varphi\ddot\varphi=\dot\varphi^2 \text{ in case (i)},\quad \dot\varphi^2=c\varphi^2 \text{ in case (ii)}.$$

The special solutions with (14.5) or (14.6) form, within the three-dimensional manifold of solutions to (14.3) having $\varphi\dot\varphi\neq0$, submanifolds of positive codimensions.

The remaining "generic" solutions *lead to proper weakly Einstein metrics.*

Remark 14.1. According to (c), (f) and the line following (f), the unordered spectrum of $\bar{\mathrm{e}}$ consists of $\pm2(\varphi\ddot\varphi-\dot\varphi^2),0,0$ in case (i), and $\pm(\dot\varphi^2-c\varphi^2)$, each repeated twice, in case (ii).

Remark 14.2. The proper weakly Einstein $2+2$ conformal-product metrics $\bar g$ constructed above have the form $(dt^2+e^{2\chi}dy^2+h)/\varphi^2$, where h has the constant Gaussian curvature c and χ,φ are functions of t. Thus they obviously constitute conformal $3+1$ products, with the factor metrics $e^{-2\chi}(dt^2+h)$ and dy^2, the Riemannian product of which is divided by the square of $\varphi e^{-\chi}$.

If $c=0$, then – as in the final clause of Remark 3.2 – writing, locally, $h=d\eta^2+d\zeta^2$, we obtain a further $3+1$ conformal-product decomposition of $\bar g$, the factor metrics being this time $dt^2+e^{2\chi}dy^2+d\eta^2$ and $d\zeta^2$.

15. Geometric realizations via conformal changes

We again use (14.4) and replace g by $\widehat g$ to simplify references to Sect. 7.

In the case (i) of Theorem 9.1, rather than insisting on the existence of a nontrivial Killing field, we can also proceed by expressing $\widehat{g}$, locally, as $\widehat{g} = e^{2\kappa}g$, where $g = dx^2 + dy^2$ is now a flat metric. By (7.1-i) with $\varphi = e^{-\kappa}$ and (7.6-b),

$$\text{(15.1)} \qquad \text{i) } \hat{\mathrm{s}} = -2e^{-2\kappa}\Delta\kappa, \qquad \text{ii) } \widehat{\Delta} = e^{-2\kappa}\Delta.$$

Thus, the condition (i) states that $\psi = \varphi^{-1}$ is a harmonic function without zeros, while, due to (7.6-b) and (15.1-i),

$$\text{(15.2)} \qquad \Delta\kappa + ce^{2\kappa} = 2g(\nabla\log|\psi|, \nabla\log|\psi|).$$

Proper weakly Einstein metrics $\bar{g} = (\widehat{g} + \widetilde{g})/\varphi^2$ will arise if one chooses ψ to be harmonic and then finds κ with (15.2), making sure, according to (iii) – (v) in Theorem 9.1, that

$$\text{(15.3)} \qquad \text{the vector field } e^{-2\kappa}\nabla\psi^{-1} \text{ is not conformal, and } \Delta\kappa \neq ce^{2\kappa}.$$

Let us now set $\psi(x, y) = e^x \cos y$ in Cartesian coordinates x, y, so that ψ is harmonic. We solve (15.2) for κ assumed to be a function of y, which amounts to the second-order ordinary differential equation

$$\text{(15.4)} \qquad \kappa''(y) + ce^{2\kappa(y)} = 2\sec^2 y.$$

Note that, in the case where

$$\text{(15.5)} \qquad \kappa'(y) = \tan y \text{ or, equivalently, } y \mapsto \kappa(y) + \log|\cos y| \text{ is constant,}$$

equation (15.4) fails to hold if $c \leq 0$, but does hold when $c > 0$, provided that the constant in (15.5) is $-\log\sqrt{c}$. For $\zeta(y) = \kappa'(y) - \tan y$, (15.4) yields

$$\text{(15.6)} \qquad \begin{array}{rl} \text{i)} & \zeta'(y) = \sec^2 y - ce^{2\kappa(y)}, \quad \text{ii) } \zeta''(y) = 2\zeta'(y)\tan y - 2ce^{2\kappa(y)}\zeta(y), \\ \text{iii)} & \text{so that } \zeta \text{ is nonconstant as long as we exclude the case (15.5),} \end{array}$$

since constancy of ζ and (15.6-i) would give $c > 0$ and (15.5) with the constant $-\log\sqrt{c}$. Solutions κ of (15.4), except those with (15.5), lead to proper weakly Einstein metrics: namely, we have (15.3). In fact, (15.4) with $\kappa''(y) = \Delta\kappa = ce^{2\kappa}$ would give (15.5), while if $w = e^{-2\kappa}\nabla\psi^{-1}$ were a conformal vector field, the component equality $w_{1,2} + w_{2,1} = 0$ for its covariant derivative, in the coordinates x, y, would read $2[\kappa'(y) - \tan y]e^{-x-2\kappa(y)}\sec y = 0$, again implying (15.5).

The surface metrics g obtained here still admit nontrivial Killing fields, provided by the coordinate vector field $\partial/\partial x$, since the conformal factor, multiplied by the standard flat metric so as to yield g, does not depend on x. However, in contrast with the examples described in Sect. 14, $\varphi = \psi^{-1}$ and he conformal factor are now functionally independent.

As $\varphi = e^{-x}\sec y$, using subscripts for partial derivatives we get

(a) $(\varphi_x, \varphi_y, \varphi_{xx}, \varphi_{xy}, \varphi_{yy}) = \varphi(-1, \tan y, 1, -\tan y, 1 + 2\tan^2 y)$.

With the notation of Remark 2.1, (a) and (15.1-ii) give $\widehat{\Delta}\varphi = 2e^{-x-2\kappa(y)}\sec^3 y$, so that, by (7.6-a) with $\chi = e^{-\kappa}$, (7.4-a) and (7.5),

(b) $\widehat{\nabla}d\varphi = \nabla d\varphi - d\kappa \otimes d\varphi - d\varphi \otimes d\kappa + g(\nabla\kappa, \nabla\varphi)g$,
(c) $\bar{\mathrm{s}} = 4(c - e^{-2\kappa(y)}\sec^2 y)e^{-2x}\sec^2 y$,
(d) $\hat{\xi} = -2e^{-2\kappa(y)}\sec^2 y$ and $\tilde{\xi} = 0$.

Due to (a) and (b), $\widehat{\nabla}d\varphi$, in the coordinates x, y, equals φ times the symmetric matrix with the diagonal $(1 + \kappa'(y)\tan y, 1 + 2\tan^2 y - \kappa'(y)\tan y)$ and the off-diagonal entry $\kappa'(y) - \tan y$. In view of (d), (7.4-b) and (7.4-c), the analogous data for

$\bar{\mathrm{e}}/2$ along $\widehat{M}$ are: diagonal $(\kappa'(y)\tan y - \tan^2 y, \tan^2 y - \kappa'(y)\tan y)$, off-diagonal $\kappa'(y) - \tan y$, while $\bar{\mathrm{e}} = 0$ along $\widetilde{M}$. As the matrix of $\bar{\mathrm{e}}/2$ along $\widehat{M}$ is traceless, with the determinant $-[\kappa'(y) - \tan y]^2 \sec^2 y$, the g-spectrum of $\bar{\mathrm{e}}/2$ along $\widehat{M}$ consists of $\pm[\kappa'(y) - \tan y]\sec y$.

Remark 15.1. According to the preceding two sentences and (c), the local-homothety invariant (9.1-i), for the metrics $\bar{g}$ constructed above, has $\pm\lambda$ equal to $2[\kappa'(y) - \tan y]e^{-2x-2\kappa(y)}\sec^3 y$ and $\bar{\mathrm{s}} = 4(c - e^{-2\kappa(y)}\sec^2 y)e^{-2x}\sec^2 y$, so that, by (15.6-i), $\bar{\mathrm{s}} = -4\zeta' e^{-2\kappa}\varphi^2$ and $\pm\lambda = 2\zeta e^{-2\kappa}\varphi^2 \sec y$. Thus, due to (15.6-iii), $\bar{\mathrm{s}}\lambda \neq 0$. Next, $\bar{\mathrm{s}}$ *and* λ *are functionally independent*, which follows since the ratio $\bar{\mathrm{s}}/\lambda = \mp 2\zeta^{-1}\zeta'\cos y$, a function of y, is nonconstant (allowing us to express y, and then also x, in terms of $\bar{\mathrm{s}}$ and λ). Namely, if $2p = \zeta^{-1}\zeta'\cos y$ were constant, with $p \neq 0$ by (15.6-iii), differentiating the equality $\zeta' = 2p\zeta\sec y$ we would obtain $\zeta'' = 4p^2\zeta\sec^2 y + 2p\zeta\sec y\tan y$ while, at the same time, from (15.6-ii), $\zeta'' = 4p\zeta\sec y\tan y - 2ce^{2\kappa}\zeta$. Equating the two expressions for ζ'' we get $ce^{2\kappa} = p\sec y\tan y - 2p^2\sec^2 y$. Thus, (15.6-i) reads $\zeta' = (2p^2+1)\sec^2 y - p\sec y\tan y$, and so ζ equals $(2p^2+1)\tan y - p\sec y$ plus a constant. As $\zeta' = 2p\zeta\sec y$, it now follows that $4p^2 + 1 - p(4p^2+3)\sin y$ is equal to a constant times $\cos y$, which is the required contradiction.

16. Classification-type theorems

Theorem 16.1. *Any weakly Einstein metric in dimension four, conformal to a product of surface metrics, arises, locally at points where* $\mathrm{e} \neq 0$ *and* $W \neq 0$*, from one of the constructions described in Theorem* 9.1*.*

To provide some context for the next theorem, let us recall two facts, one stated as (8.3) – (8.4), the other consisting of Remark 5.5 and (5.7-a). They refer to what happens at every point x of an oriented conformal-product Riemannian four-manifold with the metric $\bar{g}$, scalar curvature $\bar{\mathrm{s}}$, and Weyl tensor $\overline{W}$. First,

(16.1) both $\overline{W}^{\pm}$ have the same unordered spectrum at x.

Secondly, *if* $\bar{g}$ *is weakly Einstein, while* $\overline{W}^{+}$ *is nonzero and has a repeated eigenvalue at* x*, then, at* x,

(16.2) the unique simple eigenvalue of $\overline{W}^{+}$ equals $-\bar{\mathrm{s}}/3$, or $-\bar{\mathrm{s}}/12$, or $\bar{\mathrm{s}}/6$.

Theorem 16.2. *Let* $\bar{g}$ *be a weakly Einstein* $3+1$ *conformal-product metric on* $M \times I$*, where* $I \subseteq \mathrm{I\!R}$ *is an open interval. If, for a fixed local orientation and a point* $x \in M \times I$*, either*

(i) $\overline{W}^{+}$ *has three distinct eigenvalues at* x*, or*
(ii) $\overline{W}^{+} \neq 0$ *at* x *and* $\overline{W}^{+}$ *has a repeated eigenvalue at all points of some neighborhood of* x*, with its simple-eigenvalue function equal to* $-\bar{\mathrm{s}}/12$*,*

then $\bar{g}$ *arises near* x *as in Theorem* 9.2*, from some data* g, v, φ, θ*. In both cases,*

(16.3) at every point, $-\bar{\mathrm{s}}/12$ is an eigenvalue of $\overline{W}^{\pm}$.

Under the hypothesis preceding (i) in Theorem 16.2, at points with $\overline{W}^{+} \neq 0$ that are generic relative to the spectrum of $\overline{W}^{+}$, in the sense of Remark 2.6, one must – due to (16.2) – have either (i), or a version of (ii) in which the simple eigenvalue $-\bar{\mathrm{s}}/12$ is replaced, possibly, by $-\bar{\mathrm{s}}/3$ or $\bar{\mathrm{s}}/6$.

Using a different (and more standard) notion of genericity, one may thus say that Theorem 16.2 covers the generic situation, in (i), as well as "one-third" – represented by (ii) – of the nongeneric case.

17. Three lemmas needed for Theorem 16.1

Whenever $(\widehat{M}\times\widetilde{M},\widehat{g}+\widetilde{g})$ is a Riemannian product of oriented surfaces and $g=(\widehat{g}+\widetilde{g})/\varphi^2$ on $\widehat{M}\times\widetilde{M}$, where $\varphi:\widehat{M}\times\widetilde{M}\to\mathrm{I\!R}\smallsetminus\{0\}$, (7.4) – (7.5) for $(n,p,q)=(4,2,2)$ give, for the Einstein tensor $\mathrm{e}=\mathrm{r}-\mathrm{s}g/4$,

$$\text{(17.1)}\qquad \begin{array}{rll} \text{i)} & 4\mathrm{e}=8\varphi^{-1}\widehat{\nabla}d\varphi+\varphi[\varphi(\hat{\mathrm{s}}-\tilde{\mathrm{s}})-2(\widehat{Y}+\widetilde{Y})]g & \text{along } \widehat{M},\\ \text{ii)} & 4\mathrm{e}=8\varphi^{-1}\widetilde{\nabla}d\varphi+\varphi[\varphi(\tilde{\mathrm{s}}-\hat{\mathrm{s}})-2(\widehat{Y}+\widetilde{Y})]g & \text{along } \widetilde{M},\\ \text{iii)} & \mathrm{s}=\varphi^2(\hat{\mathrm{s}}+\tilde{\mathrm{s}})+6\varphi(\widehat{Y}+\widetilde{Y})-12(\widehat{Q}+\widetilde{Q}), & \text{with (2.4).}\end{array}$$

We will repeatedly assume that, for $\widehat{M},\widetilde{M},\widehat{g},\widetilde{g}$ and φ as above,

$$\text{(17.2)}\qquad \begin{array}{l} g=(\widehat{g}+\widetilde{g})/\varphi^2 \text{ is a weakly Einstein metric on the oriented}\\ \text{four-manifold } M=\widehat{M}\times\widetilde{M}, \text{ where } \dim\widehat{M}=\dim\widetilde{M}=2.\end{array}$$

As before, s is the scalar curvature, and r,e,W the Ricci, Einstein and Weyl tensors of the metric g, the obvious modified versions of these symbols denote their analogs for $\widehat{g}$ and $\widetilde{g}$, while $\widehat{Y},\widetilde{Y},\widehat{Q},\widetilde{Q}$ equal $\widehat{\Delta}\varphi$, $\widetilde{\Delta}\varphi$, $\widehat{g}(\widehat{\nabla}\varphi,\widehat{\nabla}\varphi)$ and $\widetilde{g}(\widetilde{\nabla}\varphi,\widetilde{\nabla}\varphi)$.

Lemma 17.1. *Under the assumption* (17.2)*, in every connected component U of the open set in M where $\mathrm{e}\neq 0$ and $W\neq 0$, for some constant $\varepsilon\in\{-2,-\frac{1}{2},1\}$,*

(a) $\mathrm{s}=\varepsilon\varphi^2(\hat{\mathrm{s}}+\tilde{\mathrm{s}})$ *and* $\hat{\mathrm{s}}+\tilde{\mathrm{s}}\neq 0$ *everywhere in* U,
(b) $(1-\varepsilon)\varphi^2(\hat{\mathrm{s}}+\tilde{\mathrm{s}})+6\varphi(\widehat{Y}+\widetilde{Y})=12(\widehat{Q}+\widetilde{Q})$,
(c) $2(\widehat{Y}-\widetilde{Y})=\varphi(\tilde{\mathrm{s}}-\hat{\mathrm{s}})$ *if* $\varepsilon\in\{-\frac{1}{2},1\}$,
(d) $\widehat{\nabla}d\varphi$ *and* $\widetilde{\nabla}d\varphi$ *are functional multiples of* $\widehat{g}$ *and* $\widetilde{g}$ *if* $\varepsilon\in\{-2,-\frac{1}{2}\}$,
(e) $\varepsilon\in\{-2,1\}$ *if and only if* $T\widehat{M}$ *and* $T\widetilde{M}$ *are* e*-orthogonal,*
(f) $\varepsilon=-\frac{1}{2}$ *if and only if* e *restricted to both* $T\widehat{M}$ *and* $T\widetilde{M}$ *yields* 0.

Let $\widehat{d}\varphi$, $\widetilde{d}\varphi$ be the "partial differentials" of φ, and $\widetilde{\widehat{d}d}\varphi$ a $(0,2)$ tensor, with the components in product coordinates x^i,x^a given by $\partial_i\varphi$ and $\partial_a\varphi$ for $\widehat{d}\varphi$ and $\widetilde{d}\varphi$, as well as $\mathrm{d}_{ia}=\mathrm{d}_{ai}=\partial_i\partial_a\varphi$ and $\mathrm{d}_{ij}=\mathrm{d}_{ab}=0$ for $\mathrm{d}=\widetilde{\widehat{d}d}\varphi$.

We will eventually show, in Lemma 17.3, that Lemma 17.1(f) only holds vacuously, since $\varepsilon\neq-\frac{1}{2}$. As an intermediate step, we first establish some conclusions, valid when $\varepsilon=-\frac{1}{2}$, so as to use them later in deriving a contradiction.

Lemma 17.2. *For $M,g,U,\varphi,\hat{\mathrm{s}},\tilde{\mathrm{s}}$ and ε as in Lemma 17.1, if $\varepsilon\in\{-2,1\}$, then $\widehat{d}\varphi\otimes\widetilde{d}\varphi=0$ identically on U, while if $\varepsilon=-\frac{1}{2}$, then $d\hat{\mathrm{s}}\otimes d\tilde{\mathrm{s}}=0$ and $\widetilde{\widehat{d}d}\varphi\neq 0$ everywhere in U.*

Lemma 17.3. *Under the hypotheses of Lemma 17.1, $\varepsilon\neq-\frac{1}{2}$.*

18. Proofs of the first two lemmas

Proof of Lemma 17.1. By (8.3), at any point of U, we have (5.7) and – consequently, in view of Remark 5.5 – one of the nine cases in (5.6) where, at the end of each line, we also provide the resulting ordered spectra of $24W^+$ and

$24W^-$. Equating each simple eigenvalue in (5.6) with the one in (8.3), we obtain the assertion (a) where, with the same order as in (5.6) and (8.9),

$$\varepsilon \text{ equals, respectively, } -2, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, 1, 1, -\tfrac{1}{2}, 1, 1. \tag{18.1}$$

Now (18.1) and Lemma 8.2 yield (e) – (f), the 'if' parts obvious for logical reasons as $\mathrm{e} \neq 0$. Next, (d) for $\varepsilon \in \{-\frac{1}{2}, -2\}$ follows from (f) and (17.1) or, respectively, from (18.1), (8.9), (17.1) and (5.3-c). Finally, we get (c) from Remark 8.3, evaluating either g-trace in (17.1), while (a) and (17.1-iii) imply (b). □

We prove Lemma 17.2 by cases: (i) with $\varepsilon \in \{-2, 1\}$, and (ii) with $\varepsilon = -\frac{1}{2}$.

PROOF OF LEMMA 17.2(i). We derive a contradiction from the assumption that the subset U' of U on which $\widehat{d}\varphi$ and $\widetilde{d}\varphi$ are both nonzero is nonempty.

By Lemma 17.1(e) and (7.3), $\varphi : U \to \mathrm{I\!R} \smallsetminus \{0\}$ has additively separated variables: $\varphi = \widehat{\alpha} + \widetilde{\alpha}$ with $\widetilde{d}\widehat{\alpha} = \widehat{d}\widetilde{\alpha} = 0$.

If $\varepsilon = 1$, (b) and (c) in Lemma 17.1 easily imply, via a separation-of-variables argument, that, for some constants $p_1, p_2, q_1, \ldots, q_5$, locally in U',

$$\begin{aligned} &\widehat{Y} = 2p_1\widehat{\alpha} + 2q_1, \quad \widetilde{Y} = -2p_1\widetilde{\alpha} + 2q_2, \quad \hat{\mathrm{s}} = p_2\widehat{\alpha} + q_3, \quad \tilde{\mathrm{s}} = p_2\widetilde{\alpha} + q_4, \\ &\widehat{Q} = p_1\widehat{\alpha}^2 + (q_1 + q_2)\widehat{\alpha} + q_5, \qquad \widetilde{Q} = -p_1\widetilde{\alpha}^2 + (q_1 + q_2)\widetilde{\alpha} - q_5. \end{aligned}$$

Plugging this into the equality of Lemma 17.1(c) we see that, as $d\widehat{\alpha}, d\widetilde{\alpha}$ are nonzero, $p_2 = 4p_1 + q_3 - q_4 = 4(q_1 - q_2) = 0$, and the above displayed formula yields

$$\begin{aligned} \widehat{Y} &= 2p_1\widehat{\alpha} + 2q_1, \qquad & \widehat{Q} &= p_1\widehat{\alpha}^2 + 2q_1\widehat{\alpha} + q_5 \\ \widetilde{Y} &= -2p_1\widetilde{\alpha} + 2q_1, \qquad & \widetilde{Q} &= -p_1\widetilde{\alpha}^2 + 2q_1\widetilde{\alpha} - q_5. \end{aligned}$$

Lemma 13.3 applied first to $(g, \alpha, p, q, q') = (\widehat{g}, \widehat{\alpha}, p_1, q_1, q_5)$, then to $(g, \alpha, p, q, q') = (\widetilde{g}, \widetilde{\alpha}, -p_1, q_1, -q_5)$, thus gives $(\hat{\mathrm{s}}, \tilde{\mathrm{s}}) = (-2p_1, 2p_1)$, and so, by (8.3), $W = 0$, contrary to the definition of U, which proves our claim in the case $\varepsilon = 1$.

Suppose now that $\varepsilon = -2$. Locally in U', applying Lemma 13.2 to both pairs $(\widehat{g}, \widehat{\alpha})$ and $(\widetilde{g}, \widetilde{\alpha})$, which is allowed by Lemma 17.1(d), we get coordinates x^1, x^2 for $\widehat{M}$ and x^3, x^4 for $\widetilde{M}$ such that, by Lemma 17.1(b) and (13.2),

$$(\alpha + \beta)^2(\dot{\beta}\dddot{\alpha} + \dot{\alpha}\dddot{\beta}) = 2\dot{\alpha}\dot{\beta}[(\alpha + \beta)(\ddot{\alpha} + \ddot{\beta}) - \dot{\alpha}^2 - \dot{\beta}^2]. \tag{18.2}$$

Our notation here is simplified: α stands for $\widehat{\alpha}$ and β for $\widetilde{\alpha}$, while $(\;)^{\dot{}}$ denotes $\partial/\partial x^1$ when applied to functions of x^1, and $\partial/\partial x^3$ for functions of x^3.

Note that, due to the definition of U', as $\varphi = \alpha + \beta : U \to \mathrm{I\!R} \smallsetminus \{0\}$,

$$\dot{\alpha}\dot{\beta}(\alpha + \beta) \neq 0 \text{ everywhere in } U'. \tag{18.3}$$

We can consequently rewrite (18.2) as

$$\frac{\dddot{\alpha}}{2\dot{\alpha}} - \frac{\ddot{\alpha}}{\alpha + \beta} + \frac{\dot{\alpha}^2}{(\alpha + \beta)^2} = \frac{\ddot{\beta}}{\alpha + \beta} - \frac{\dot{\beta}^2}{(\alpha + \beta)^2} + \frac{\theta}{2}, \quad \text{where } \theta = -\frac{\dddot{\beta}}{\dot{\beta}},$$

and apply $\dot{\beta}^{-1}(\alpha + \beta)^2\partial/\partial x^3$ twice in a row, first obtaining

$$\ddot{\alpha} - \frac{2\dot{\alpha}^2}{\alpha + \beta} = \frac{\dot{\theta}}{2\dot{\beta}}(\alpha + \beta)^2 - \theta(\alpha + \beta) + \frac{2\dot{\beta}^2}{\alpha + \beta} - 3\ddot{\beta},$$

and then

$$2\dot{\alpha}^2 = \frac{(\dot{\theta}/\dot{\beta})^{\dot{}}}{2\dot{\beta}}(\alpha + \beta)^4 + 2\theta(\alpha + \beta)^2 + 4\ddot{\beta}(\alpha + \beta) - 2\dot{\beta}^2.$$

Now, applying $2(\alpha+\beta)^{-2}\partial/\partial x^3$, we see that

$$[(\dot\theta/\dot\beta)/\dot\beta]^{\dot{}}(\alpha+\beta)^2+4(\dot\theta/\dot\beta)^{\dot{}}(\alpha+\beta)+4\dot\theta=0.$$

It follows that θ is constant, or else $\alpha+\beta$, being a root of a nontrivial quadratic equation with coefficients that are functions of x^3, would itself be a function of x^3, even though $d\widehat\alpha\neq 0$. Thus, in (18.2), $2\dddot\beta=-\mathrm{s}_1\dot\beta$ for some constant s_1.

Since (18.2) involves α and β symmetrically, we also have $2\dddot\alpha=-\mathrm{s}_0\dot\alpha$ for some $\mathrm{s}_0\in\mathrm{I\!R}$, while, by (13.2),

(18.4) s_0 and s_1 are the scalar curvatures of $\widehat g$ and $\widetilde g$.

With $2\dddot\alpha, 2\dddot\beta$ replaced by $-\mathrm{s}_0\dot\alpha$ and $-\mathrm{s}_1\dot\beta$, (18.2) reads

$$(\mathrm{s}_0+\mathrm{s}_1)(\alpha+\beta)^2\,=\,4[\dot\alpha^2+\dot\beta^2-(\alpha+\beta)(\ddot\alpha+\ddot\beta)],$$

as (18.3) allows us to divide by $\dot\alpha\dot\beta$. In other words,

$$\ddot\alpha-\frac{\dot\alpha^2}{\alpha+\beta}\,=\,-\frac{\mathrm{s}_0+\mathrm{s}_1}{4}(\alpha+\beta)+\frac{\dot\beta^2}{\alpha+\beta}-\ddot\beta.$$

Applying $\dot\beta^{-1}(\alpha+\beta)^2\partial/\partial x^3$ to the last equality, we obtain

$$\dot\alpha^2\,=\,\frac{\mathrm{s}_1-\mathrm{s}_0}{4}(\alpha+\beta)^2+2\ddot\beta(\alpha+\beta)-\dot\beta^2$$

and, further applying $4\partial/\partial x^3$, we get $0=-2(\mathrm{s}_0+\mathrm{s}_1)(\alpha+\beta)\dot\beta$. Hence $W=0$ according to (18.3) – (18.4) and (8.3), which is the required contradiction. □

Proof of Lemma 17.2(ii). Now $\varepsilon=-\frac{1}{2}$. First, $\widetilde{\widehat{dd}}\varphi\neq 0$ everywhere in U since, by Lemma 17.1(f) and (7.1-ii), e would vanish at points with $\widetilde{\widehat{dd}}\varphi=0$.

To prove the other claim, suppose that, on the contrary, the subset U' of U on which $d\hat{\mathrm{s}}$ and $d\tilde{\mathrm{s}}$ are both nonzero is nonempty.

Thus, $\varphi:U'\to\mathrm{I\!R}\smallsetminus\{0\}$ has the form $\varphi=\widehat\alpha\widetilde\alpha+\gamma$ with $\widetilde d\widehat\alpha=\widehat d\widetilde\alpha=0$ and a constant γ. In fact, we obtain $\widehat\alpha$ by restricting φ to $[\widehat M\times\{z\}]\cap U'$ with a fixed $z\in\widetilde M$, and as z varies, the "affine" form $\varphi=\widehat\alpha\widetilde\alpha+\widetilde\gamma$ follows from Lemma 17.1(d) and Remark 13.1, where $\widetilde\gamma$ may still vary with z, but $\widehat d\widetilde\gamma=0$. As our assumptions involve $\widehat M$ and $\widetilde M$ symmetrically, we similarly have $\varphi=\widehat\beta\widetilde\beta+\widehat\delta$ with $\widetilde d\widehat\beta=\widehat d\widetilde\beta=\widetilde d\widehat\delta=0$. Thus, applying $\partial_i\partial_a$ to both expressions for φ, in product coordinates x^i,x^a, and noting that $\widetilde{\widehat{dd}}\varphi\neq 0$, we get $d\widehat\beta=p\,d\widehat\alpha$ and $d\widetilde\beta=p^{-1}d\widetilde\alpha$ with a constant $p\neq 0$. Writing $\widehat\beta=p\widehat\alpha+q_1$ and $\widetilde\beta=p^{-1}\widetilde\alpha+q_2$, where $q_1,q_2\in\mathrm{I\!R}$, we now get $\widehat\alpha\widetilde\alpha+\widetilde\gamma=\varphi=\widehat\alpha\widetilde\alpha+pq_2\widehat\alpha+p^{-1}q_1\widetilde\alpha+\widehat\delta$, and so $\gamma=\widetilde\gamma-p^{-1}q_1\widetilde\alpha=pq_2\widehat\alpha+\widehat\delta$ is constant, and our claim follows if we replace $\widehat\alpha$ by $\widehat\alpha+p^{-1}q_1$.

Locally in U', Lemma 13.2 applied to both $(\widehat g,\widehat\alpha)$ and $(\widetilde g,\widetilde\alpha)$ yields coordinates x^1,x^2 for $\widehat M$ and x^3,x^4 for $\widetilde M$ such that, by Lemma 17.1(b) and (13.2),

$$\text{(18.5)}\quad \frac{\dddot\alpha}{\dot\alpha}-\frac{4\ddot\alpha\beta}{\alpha\beta+\gamma}+\frac{4\dot\alpha^2\beta^2}{(\alpha\beta+\gamma)^2}\,=\,\theta+\frac{4\alpha\ddot\beta}{\alpha\beta+\gamma}-\frac{4\alpha^2\dot\beta^2}{(\alpha\beta+\gamma)^2},\quad\text{where }\ \theta=-\frac{\dddot\beta}{\dot\beta},$$

Our notation here is simplified: α stands for $\widehat\alpha$ and β for $\widetilde\alpha$, while $(\ )^{\dot{}}$ denotes $\partial/\partial x^1$ when applied to functions of x^1, and $\partial/\partial x^3$ for functions of x^3. The divisions in (18.5) are allowed: as $\varphi=\alpha\beta+\gamma:U\to\mathrm{I\!R}\smallsetminus\{0\}$ and $\widetilde{\widehat{dd}}\varphi\neq 0$.

(18.6) $\dot\alpha\dot\beta(\alpha\beta+\gamma)\,\neq\,0$ everywhere in U'.

We now apply $\dot\beta^{-1}(\alpha\beta+\gamma)^2\partial/\partial x^3$ to (18.5) twice in a row, first obtaining

$$-4\gamma\ddot\alpha+\frac{8\gamma\dot\alpha^2\beta}{\alpha\beta+\gamma}=\frac{\dot\theta}{\dot\beta}(\alpha\beta+\gamma)^2-4\theta\alpha(\alpha\beta+\gamma)-12\alpha^2\ddot\beta+\frac{8\alpha^3\dot\beta^2}{\alpha\beta+\gamma}$$

and then

$$8\gamma^2\dot\alpha^2=\frac{(\dot\theta/\dot\beta)^{\boldsymbol\cdot}}{\dot\beta}(\alpha\beta+\gamma)^4-\frac{2\dot\theta}{\dot\beta}\alpha(\alpha\beta+\gamma)^3+8\theta\alpha^2(\alpha\beta+\gamma)^2+16\alpha^3\ddot\beta(\alpha\beta+\gamma)-8\alpha^4\dot\beta^2.$$

Next, applying $\alpha^{-2}(\alpha\beta+\gamma)^{-2}\partial/\partial x^3$, we see that

$$[(\dot\theta/\dot\beta)/\dot\beta]^{\boldsymbol\cdot}\,(\beta+\gamma/\alpha)^2+2(\dot\theta/\dot\beta)^{\boldsymbol\cdot}\,(\beta+\gamma/\alpha)+2\dot\theta\,=\,0.$$

It follows that $\gamma=0$. Otherwise, θ would be constant, as $\beta+\gamma/\alpha$, being a root of a nontrivial quadratic equation with coefficients that are functions of x^3, would itself be a function of x^3, even though $d\widehat\alpha\neq 0$. However, by (13.2) and (18.5), $2\theta=\tilde{\mathrm s}$, while $d\tilde{\mathrm s}=0$ on U'.

With $\gamma=0$, (18.5) becomes

$$\frac{\dddot\alpha}{\dot\alpha}-\frac{4\ddot\alpha}{\alpha}+\frac{4\dot\alpha^2}{\alpha^2}\,=\,-\frac{\dddot\beta}{\dot\beta}+\frac{4\ddot\beta}{\beta}-\frac{4\dot\beta^2}{\beta^2}. \tag{18.7}$$

At the same time, Lemma 17.1(c) and (13.2) give

$$\frac{\dddot\alpha}{\dot\alpha}-\frac{2\ddot\alpha}{\alpha}\,=\,\frac{\dddot\beta}{\dot\beta}-\frac{2\ddot\beta}{\beta}. \tag{18.8}$$

An obvious separation-of-variables argument shows that both sides in (18.7) and (18.8) are constant, so that

$$\text{i)}\ \ \frac{\dddot\alpha}{\dot\alpha}-\frac{4\ddot\alpha}{\alpha}+\frac{4\dot\alpha^2}{\alpha^2}\,=\,b_0,\qquad \text{ii)}\ \ \frac{\dddot\alpha}{\dot\alpha}-\frac{2\ddot\alpha}{\alpha}\,=\,b_1 \tag{18.9}$$

for some $b_0,b_1\in\mathrm{I\!R}$. Equation (18.9-ii) states that $3b_2=\alpha^{-2}(\ddot\alpha+b_1\alpha)$ is constant, and multiplying the equality $\ddot\alpha+b_1\alpha=3b_2\alpha^2$ by $2\dot\alpha$ we get $\dot\alpha^2+b_1\alpha^2=2b_2\alpha^3+b_3$ with some constant b_3.

Subtracting (18.9-i) from (18.9-ii), and then replacing $\ddot\alpha$ by $-b_1\alpha+3b_2\alpha^2$ (see the last paragraph), we obtain $4\dot\alpha^2=(b_0-3b_1)\alpha^2+6b_2\alpha^3$. while, as we just saw, $4\dot\alpha^2=-4b_1\alpha^2+8b_2\alpha^3+4b_3$. As α is nonconstant by (18.6), $b_1=-b_0$ and $b_2=b_3=0$. Thus, $\ddot\alpha=b_0\alpha$ and $\dot\alpha^2=b_0\alpha^2$. However, (18.7) and (18.8) for β yield the analog of (18.9) with $(-b_0,b_1)$ instead of (b_0,b_1), and so our conclusion that $b_1=-b_0$, which also applies to the new pair, now amounts to $b_0=b_1=0$. The resulting constancy of α and β contradicts (18.6). □

19. Proofs of Lemma 17.3 and Theorem 16.1

Assuming that $\varepsilon=-\frac12$ in Lemma 17.1, we will derive a contradiction.

Since the hypotheses of Lemma 17.1 involve the two factor manifolds symmetrically, Lemma 17.2 allows us to restrict our discussion to a product-type connected open set $U'\subseteq U$ on which either $\tilde{\mathrm s}$ is constant and $d\hat{\mathrm s}\neq 0$ everywhere, or both $\hat{\mathrm s}$ and $\tilde{\mathrm s}$ are constant.

In the former case, $\varphi:U'\to\mathrm{I\!R}\smallsetminus\{0\}$ equals $\alpha\beta+\gamma$ with $\widetilde d\alpha=\widehat d\beta=\widehat d\gamma=0$. In fact, we obtain α by restricting φ to $[\widehat M\times\{z\}]\cap U'$ with a fixed $z\in\widetilde M$, and as z varies, Lemma 17.1(d) and Remark 13.1 imply the “affine” form $\varphi = \alpha\beta + \gamma$. Since $\widetilde{d}\widetilde{d}\varphi \neq 0$ everywhere (see Lemma 17.2),

$$d\alpha \text{ and } d\beta \text{ are both nonzero everywhere in } U'. \tag{19.1}$$

Due to (19.1) for α and Lemma 17.1(d), the $\widetilde{g}$-Hessians of β and γ are both functional multiples of the metric $\widetilde{g}$, and so, in view of Lemma 12.2 and (12.2-a),

$$\widetilde{\Delta}\beta = -\tilde{\mathrm{s}}\beta + b_1 \text{ and } \widetilde{\Delta}\gamma = -\tilde{\mathrm{s}}\gamma + b_2 \text{ for some constants } b_1, b_2. \tag{19.2}$$

Thus, $(\alpha\beta+\gamma)(\tilde{\mathrm{s}}-\hat{\mathrm{s}}) = 2(\beta\widehat{\Delta}\alpha - \alpha\widetilde{\Delta}\beta - \widetilde{\Delta}\gamma) = 2[\beta\widehat{\Delta}\alpha + \tilde{\mathrm{s}}(\alpha\beta+\gamma) - b_1\alpha - b_2]$ by Lemma 17.1(c), that is,

$$[\alpha(\hat{\mathrm{s}}+\tilde{\mathrm{s}}) + 2\widehat{\Delta}\alpha]\beta \,+\, (\hat{\mathrm{s}}+\tilde{\mathrm{s}})\gamma \,=\, 2(b_1\alpha + b_2).$$

Hence, as $\hat{\mathrm{s}}+\tilde{\mathrm{s}} \neq 0$ in Lemma 17.1(a), along $[\widehat{M}\times\{z\}]\cap U'$ for a fixed $z \in \widetilde{M}$, we get $\gamma = b_3\beta + b_4$, where $b_3, b_4 \in \mathrm{I\!R}$. Replacing α by $\alpha + b_3$ we may now assume that γ is constant. For this new α, the above displayed formula and (19.1) yield

$$2\widehat{\Delta}\alpha \,=\, -\alpha(\hat{\mathrm{s}}+\tilde{\mathrm{s}}). \tag{19.3}$$

Also, by Lemma 12.2 and (12.2-b), $2\widetilde{Q} = -\tilde{\mathrm{s}}(\alpha\beta+\gamma+b_5)^2 + b_6$, where $b_5, b_6 \in \mathrm{I\!R}$. As $\varphi = \alpha\beta+\gamma$, Lemma 17.1(b) with $\varepsilon = -\frac{1}{2}$ and $\widehat{Y} + \widetilde{Y} = \beta\widehat{\Delta}\alpha + \alpha\widetilde{\Delta}\beta$ gives

$$\begin{aligned}(\hat{\mathrm{s}}+\tilde{\mathrm{s}})(\alpha\beta+\gamma)^2 + 4(\alpha\beta+\gamma)[\beta\widehat{\Delta}\alpha + b_1\alpha + \tilde{\mathrm{s}}\gamma + 2\tilde{\mathrm{s}}b_5]&\\ =\, 4[2\beta^2\widehat{g}(\widehat{\nabla}\alpha,\widehat{\nabla}\alpha) - \tilde{\mathrm{s}}b_5^2 + b_6]&\end{aligned}$$

due to (19.2) and (19.3). Thus, β is a root of a quadratic equation with coefficients that are constant along $T\widetilde{M}$, so that the leading coefficient must equal zero:

$$\alpha^2(\hat{\mathrm{s}}+\tilde{\mathrm{s}}) + 4\alpha\widehat{\Delta}\alpha - 8\widehat{g}(\widehat{\nabla}\alpha,\widehat{\nabla}\alpha) \,=\, 0, \tag{19.4}$$

or else β, already constant along $T\widehat{M}$, would be constant, contradicting (19.1).

Locally in U', Lemma 17.1(d) makes Lemma 13.2 applicable to $\widehat{g}$ and α, leading to coordinates x^1, x^2 for $\widehat{M}$ with (13.2), where $(\,)^{\boldsymbol{\cdot}}$ denotes $\partial/\partial x^1$. Setting $c = \tilde{\mathrm{s}}/2$, we now rewrite (19.3) and (19.4) as

$$\text{i) } \alpha\dddot{\alpha} = \dot{\alpha}(2\ddot{\alpha} + c\alpha), \qquad \text{ii) } \alpha^2\dddot{\alpha} = \dot{\alpha}(4\alpha\ddot{\alpha} - 4\dot{\alpha}^2 + c\alpha^2). \tag{19.5}$$

Subtracting (19.5-i) times α from (19.5-ii) we get $\alpha\ddot{\alpha} = 2\dot{\alpha}^2$ which, differentiated, yields $\alpha\dddot{\alpha} = 3\dot{\alpha}\ddot{\alpha}$, turning (19.5-i), due to (19.1), into $\ddot{\alpha} = c\alpha$. Thus, $\dddot{\alpha} = c\dot{\alpha}$, and (13.2) gives $\hat{\mathrm{s}} = 2K = -2c$, contrary to the assumption that $\hat{\mathrm{s}}$ is nonconstant.

This leaves us with the remaining case: $\varepsilon = -\frac{1}{2}$ and $\hat{\mathrm{s}}, \tilde{\mathrm{s}}$ are both constant. Combining parts (b) – (d) of Lemma 17.1 with Lemma 12.4, we get $\widetilde{d}\widetilde{d}\varphi = 0$ everywhere, which contradicts Lemma 17.2, thus completing the proof of Lemma 17.3.

Proof of Theorem 16.1. Due to Lemma 17.3, in Lemma 17.1, $\varphi = \widehat{\alpha} + \widetilde{\alpha}$ with $\widehat{\alpha}: \widehat{M} \to \mathrm{I\!R}$ and $\widetilde{\alpha}: \widetilde{M} \to \mathrm{I\!R}$, cf. Lemma 17.1(e) and (7.3), and either

$$\varepsilon = 1, \quad \varphi(\widehat{Y} + \widetilde{Y}) = 2(\widehat{Q} + \widetilde{Q}), \quad 2(\widehat{Y} - \widetilde{Y}) = (\tilde{\mathrm{s}} - \hat{\mathrm{s}})\varphi, \tag{19.6}$$

or $\varepsilon = -2$, and then, with $\widehat{\alpha}, \widetilde{\alpha}$ chosen as in (7.3),

$$\begin{aligned}&(\widehat{\alpha}+\widetilde{\alpha})^2\,(\hat{\mathrm{s}}+\tilde{\mathrm{s}}) + 2(\widehat{\alpha}+\widetilde{\alpha})(\widehat{Y}+\widetilde{Y}) = 4(\widehat{Q}+\widetilde{Q}),\\ &\widehat{\nabla}d\widehat{\alpha},\ \widetilde{\nabla}d\widetilde{\alpha} \text{ are functional multiples of } \widehat{g} \text{ and } \widetilde{g}.\end{aligned} \tag{19.7}$$

By Lemma 17.2, locally, one of $\widehat{\alpha}, \widetilde{\alpha}$ is constant and, switching them if necessary, we may assume constancy of $\widetilde{\alpha}$, so that φ is constant in the $\widetilde{M}$ direction. Either of (19.6) – (19.7) now implies, via separation of variables, that $\tilde{\mathrm{s}}$ is constant, and (i) or (ii) in Theorem 9.1 follows, as a consequence of (2.3-b). □

20. Proof of Theorem 16.2

We will denote by $M\times I$ various sufficiently small product-type neighborhoods of x in the original product manifold, where $I\subseteq \mathrm{I\!R}$ is an open interval with the coordinate t and the metric dt^2. We are thus given an oriented Riemannian three-manifold (M,g) and a function $\varphi : M\times I \to \mathrm{I\!R}\smallsetminus\{0\}$, such that $\bar{g}=(g+dt^2)/\varphi^2$ is a weakly Einstein metric on $M\times I$. Let us begin by showing that, under either of the assumptions (i) and (ii), φ has additively separated variables:

$$\varphi = \alpha + \beta \quad \text{with} \quad \alpha : M\to\mathrm{I\!R} \text{ and } \beta : I\to\mathrm{I\!R}, \tag{20.1}$$

while, for the Einstein tensor $\bar{\mathrm{e}}=\bar{\mathrm{r}}-\bar{\mathrm{s}}\bar{g}/4$ of $\bar{g}$ and its Weyl tensor $\overline{W}$, near x,

$$\begin{array}{ll} \text{a)} & \text{the } M \text{ and } I \text{ factor distributions are } \bar{\mathrm{e}}\text{-orthogonal,}\\ \text{b)} & \overline{W} \text{ and } \bar{\mathrm{e}} \text{ satisfy (5.3-c) with } c_2=c_3=c_4=0. \end{array} \tag{20.2}$$

More precisely, we will use (20.3) – (20.6) below to establish (20.1) – (20.2) in each connected component of the open dense set of points generic relative to the spectra of e and $\overline{W}^{+}$, in the sense of Remark 2.6. This will imply that (20.1) and (20.2-a) hold everywhere, as they are equalities. (The former reads $\widetilde{dd}\varphi=0$, cf. Sect. 17.)

Now, if $u_1,\dots,u_4$ and $v_1,\dots,v_4$ are positive $\bar{g}$-orthonormal frames such that

$$\begin{array}{l} u_1,\dots,u_4 \text{ realizes one of the conclusions (5.3) in}\\ \text{Theorem 5.1 and, in particular, it diagonalizes } \bar{\mathrm{e}}, \end{array} \tag{20.3}$$

while the corresponding assumption in (5.2) is also satisfied, and

$$\begin{array}{l} v_1,v_2,v_3, \text{ tangent to the } M \text{ factor, diagonalize the Einstein}\\ \text{tensor } \mathrm{e}=\mathrm{r}-\mathrm{s}g/3 \text{ of } g \text{ with some eigenvalue functions } \theta_i, \end{array} \tag{20.4}$$

then, due to (8.4), for the Weyl tensor $\overline{W}$ of $\bar{g}$, and both $\overline{W}^{\pm}$,

$$2\varphi^{-2}\overline{W}^{\pm} \text{ have the } \bar{g}\text{-spectrum } (-\theta_3,-\theta_2,-\theta_1), \text{ realized as in (8.5).} \tag{20.5}$$

We will derive (20.1) – (20.2) from the following observation.

$$\begin{array}{l} \text{In case (i), } u_1,\dots,u_4 \text{ and } v_1,\dots,v_4 \text{ arise from each other}\\ \text{via permutations, possibly combined with some sign changes.}\\ \text{In case (ii), we may assume that } (u_1,\dots,u_4)=(v_1,\dots,v_4). \end{array} \tag{20.6}$$

Namely, (20.5) and the first part of Remark 6.1 prove the first part of (20.6). Assuming (i), we get $c_2=c_3=c_4=0$ in (5.3), from (20.5) and the final clause of Remark 6.1, which gives (20.2-b) – and hence (16.3) – as (5.3-a) or (5.3-b) would lead to repeated eigenvalues of $\overline{W}^{\pm}$. In view of (20.3) and (20.6), the frame $v_1,\dots,v_4$ now diagonalizes $\bar{\mathrm{e}}$. Consequently, (i) implies (20.2-a), and then (7.3) yields (20.1). We have thus established (20.1) – (20.2) in the case (i).

Next, let (ii) be satisfied. According to (16.1) and Remark 5.5, our frame $u_1,\dots,u_4$ on $M\times I$ realizes – due to the assumption about $-\bar{\mathrm{s}}/12$ in (ii) – both (5.6-i) and (8.9-i). More precisely, Remark 5.5 excludes the option (5.3-a) in Theorem 5.1 while, in (5.6-i), $8\xi=\bar{\mathrm{s}}\neq 0=c_2=c_3=c_4$, and so one has (5.3-c), that is, (20.2-b), rather than (5.3-b). *The two spectra in* (5.3-c) *thus are* $(-\lambda,-\lambda,\lambda,\lambda)$ *and* $(-\bar{\mathrm{s}}/12,\bar{\mathrm{s}}/24,\bar{\mathrm{s}}/24)$. Rearranging v_1,v_2,v_3 so as to have $\theta_1=\theta_2$, we conclude, from (8.9-i) and the final clause of Lemma 8.2, that $\mathrm{span}\,(v_1,v_2)$ equals one of span (u_1,u_2) and span (u_3,u_4), *which are the eigendistributions of* $\bar{\mathrm{e}}$ *for the eigenvalue functions* $-\lambda$ *and* λ. The two italicized statements remain unaffected, up to a sign change in λ, when the pairs (u_1,u_2) and (u_3,u_4) are switched and/or independently rotated, which leads to the second part of (20.6). By (20.3), this gives (20.2-a) and, again, we get (20.1) from (7.3).

We have thus shown that (20.1) – (20.2) hold in both cases (i) and (ii).

One also has (7.4) with $(n,p,q,\tilde{\mathrm{s}},\widetilde{Y},\widetilde{Q},\tilde{\mathrm{e}},\widetilde{\nabla}d\varphi)=(4,3,1,0,\ddot\beta,\dot\beta^2,0,\ddot\beta\,dt^2)$, and $\widehat{M},\widetilde{M},\mathrm{s},\mathrm{e},\widehat{g},\widehat{\nabla},\widehat{Y},\widehat{Q},\widetilde{g}$ replaced by $M,I,\bar{\mathrm{s}},\bar{\mathrm{e}},g,\nabla,Y,Q,dt^2$, which obviously reads

$$
(20.7)\qquad
\begin{array}{ll}
\text{a)} & \bar{\mathrm{s}}=\varphi^2\mathrm{s}+6\varphi(Y+\ddot\beta)-12(Q+\dot\beta^2)\ \text{ for } (Y,Q)=(\Delta\alpha,\ g(\nabla\alpha,\nabla\alpha)),\\
\text{b)} & 12\bar{\mathrm{e}}=12\mathrm{e}+24\varphi^{-1}\nabla d\alpha+[\mathrm{s}-6\varphi^{-1}(Y+\ddot\beta)]g\ \text{ along } M,\\
\text{c)} & 4\bar{\mathrm{e}}\,=\,-[\mathrm{s}+2\varphi^{-1}(Y-3\ddot\beta)]\,dt^2\quad\text{along } I,\ \text{where } (\ )^\cdot=d/dt.
\end{array}
$$

By (20.2-b) and (20.3), $\bar{\mathrm{e}}$ has, the ordered spectrum $(-\lambda,-\lambda,\lambda,\lambda)$ or $(\lambda,\lambda,-\lambda,-\lambda)$, realized by the frame $u_1,\dots,u_4$. Either of (i) – (ii) implies – cf. (20.6) – that v_i in (20.4) are eigenvectors of $\bar{\mathrm{e}}$, and we are free to assume that v_1 and v_2 correspond to the same $\bar{g}$-eigenvalue function $\mp\lambda$ which, in the case (ii), or (i), is obvious from (20.6) or, respectively, (20.6) coupled with (6.1) applied to v_i, $i=1,2,3$. Consequently, by (20.7), v_i, $i=1,2,3$, are g-eigenvectors of $\nabla d\alpha$ with some eigenvalue functions δ_i, and (20.7) yields four separate expressions for λ/φ, namely

$$
(20.8)\qquad
\begin{array}{ll}
\text{a)} & \mp2\lambda/\varphi=(2\theta_i+\mathrm{s}/6)\varphi\,-Y-\ddot\beta+4\delta_i,\ \text{ for } i=1,2,\\
\text{b)} & \pm2\lambda/\varphi=(2\theta_3+\mathrm{s}/6)\varphi\,-Y-\ddot\beta+4\delta_3,\\
\text{c)} & \pm2\lambda/\varphi=-Y+3\ddot\beta-\mathrm{s}\varphi/2.
\end{array}
$$

Furthermore, for some simple-eigenvalue function θ of e,

$$
(20.9)\qquad
\begin{array}{ll}
\text{a)} & \bar{\mathrm{s}}=6\theta\varphi^2\ \text{ with } i\in\{1,2,3\}\ \text{ such that } \theta=\theta_i,\\
\text{b)} & \text{while, in both cases (i) and (ii), } \beta \text{ is constant.}
\end{array}
$$

Here is how (20.9-a) follows. Assuming (i), we choose $\theta=\theta_i$ in (20.5) so as to realize (16.3), which we already proved in the third line after (20.6). If (ii) is satisfied, we let $\theta=\theta_3$, so that $\theta_1=\theta_2=-\theta/2$, and (ii) combined with (20.5) yields (20.9-a).

In the case (i), $\theta_1\ne\theta_2$ by (20.5), and to obtain (20.9-b), that is, the equality $\partial\varphi/\partial t=0$, it suffices to subtract the $i=1$ and $i=2$ versions of (20.8-a).

Next, assuming (ii), and letting $M\times I$ be a small neighborhood of a point at which $\dot\beta\ne0$, we will arrive at a contradiction. As before, $\theta_3=\theta$ and $\theta_1=\theta_2=-\theta/2$. Adding (20.8-a) for $i=1$ to (20.8-b), we get $(\theta+\mathrm{s}/3)(\alpha+\beta)=2(Y+\ddot\beta)-4(\delta_1+\delta_3)$, and d/dt applied to this, via separation of variables, gives $\dddot\beta=p\dot\beta$, where $2p=\theta+\mathrm{s}/3$ is constant. Thus, $\ddot\beta=p\beta+q$ and $\dot\beta^2=p\beta^2+2q\beta+q_1$ for some $q,q_1\in\mathrm{I\!R}$. Now (20.9-a), (20.1) and (20.7-a) yield

$$(\mathrm{s}-6\theta)(\alpha+\beta)^2+6(\alpha+\beta)(Y+p\beta+q)-12(Q+p\beta^2+2q\beta+q_1)=0,$$

which is a quadratic equation with coefficients that are constant along I, imposed on β. The leading coefficient must therefore vanish: $0=\mathrm{s}-6\theta+6p-12p=-9\theta$, as $2p=\theta+\mathrm{s}/3$. With (20.5) and $(\theta_1,\theta_2,\theta_3)=(-\theta/2,-\theta/2,\theta)$, this gives $\overline{W}=0$, providing the required contradiction and, consequently, proving (20.9-b).

Using (20.9-b) and (20.1), we may set $\varphi=\alpha$, with $\beta=0$.

For θ and i chosen in (20.9-a), and v equal to any positive functional multiple of v_i, we now clearly have (9.2-iv), as well as (9.2-iii), the latter from (20.9-a) and (20.7-a) with $(\alpha,\beta)=(\varphi,0)$.

According to the lines following (20.7), v_1 and v_2 correspond to the same $\bar{g}$-eigenvalue function $\mp\lambda$, and so does one of the pairs (u_1, u_2) and (u_3, u_4). Thus, as a consequence of (20.2-b) and (5.3-c), $v_1 \wedge v_2 \pm v_3 \wedge v_4$ are eigenbivectors of $\overline{W}^+$ for the eigenvalue $-\bar{\mathrm{s}}/12$. Now (20.5) gives (20.9-a) for $i = 3$. Hence $\theta = \theta_3$, and v equals a positive function times v_3. With $\mathrm{b} = 2\nabla d\varphi + \varphi\mathrm{r}$ and $(\alpha, \beta) = (\varphi, 0)$, (20.7-b) reads $\varphi\bar{\mathrm{e}} = \mathrm{b} - (\mathrm{tr}_g\mathrm{b})g/4$, and $\mathrm{tr}_g\mathrm{b} = 2Y + \mathrm{s}\varphi$. Let $\psi = \pm\lambda/\varphi$. Now v_1, v_2 and v diagonalize $\bar{\mathrm{e}}$ and b, with the ordered g-spectrum of $\varphi\bar{\mathrm{e}}$ equal to $(-\psi, -\psi, \psi)$, while (20.8-c) with $\beta = 0$ yields $\mathrm{tr}_g\mathrm{b} = 2Y + \mathrm{s}\varphi = -4\psi$. The resulting equality $\varphi\bar{\mathrm{e}} = \mathrm{b} + \psi g$, with $\varphi\bar{\mathrm{e}}(v, \cdot\,) = \psi g(v, \cdot\,)$ and $\varphi\bar{\mathrm{e}}(v_i, \cdot\,) = -\psi g(v_i, \cdot\,)$ for $i = 1, 2$, gives $\mathrm{b}(v, \cdot\,) = 0$, proving (9.2-ii), and $\mathrm{b}(v_i, \cdot\,) = -2\psi g(v_i, \cdot\,)$. Changing the sign of φ (and hence of b), if necessary, so as to replace the last expression with $-2|\psi|g(v_i, \cdot\,)$, and then, suitably normalizing v, we obtain (9.2-i).

This completes the proof of Theorem 16.2.

21. Properties of the new examples

In Sect. 14 and 15 we described examples of proper weakly Einstein conformal products arising from Theorems 9.1 and 9.2. We now show that, with just one exception, those examples are new, that is, different from the ones known before.

The previously known examples of proper weakly Einstein manifolds formed two narrow classes, consisting of the EPS space (Section 4) and certain Kähler surfaces [**7**, Sect. 12]. The local-homothety invariant of Remark 5.6 is given by

$$\begin{array}{lllll} \mathrm{s}, & (\mathrm{s}/2, 0, 0, -\mathrm{s}/2), & (-\mathrm{s}/12, -\mathrm{s}/12, \mathrm{s}/6), & (-\mathrm{s}/12, -\mathrm{s}/12, \mathrm{s}/6) & \text{for the former,} \\ \mathrm{s}, & (-\lambda, -\lambda, \lambda, \lambda), & (\mathrm{s}/6, -\mathrm{s}/12, -\mathrm{s}/12), & (-\mathrm{s}/3, \mathrm{s}/6, \mathrm{s}/6) & \text{for the latter,} \end{array}$$

at every point, with the scalar curvature $\mathrm{s} \neq 0$ and a parameter $\lambda \neq 0$, as pointed out in [**8**, Sect. 10 and formulae (1.6)–(1.7)]. Thus, none of the Kähler-surface examples in [**7**] is, even locally, a conformal product, since, by (8.3) and (8.4), in a conformal product both $W^\pm$ must have the same spectrum.

The EPS space is, however, a warped product, and hence a conformal product; see (4.7) and (3.2). It is realized by the construction of Sect. 14, case (i), with $c = 0$ and any φ such that $\dot{\varphi} = 2p$ is a nonzero constant. In fact, (14.3) follows, without (14.5) or (14.6), and (14.2-i) gives $\chi = q + 2\log|\varphi|$, where $q \in \mathrm{I\!R}$, so that, by (13.1-a), $\bar{g} = (g+h)/\varphi^2 = p^2[d\tau^2 + e^{-\tau}d\xi^2 + e^\tau(d\eta^2 + d\zeta^2)]$, as required in (4.5), for $(\tau, \xi) = (-2\log|\varphi|, e^q y/p)$ and η, ζ with $h = p^2(d\eta^2 + d\zeta^2)$. Also,

(21.1) this is the only way to obtain the EPS space in Sect. 14,

as (9.1) and the formula displayed above force us to use case (i), while (4.8-i) gives $c = 0$, and (b) in Sect. 14, with $c = 0$ and constant $\bar{\mathrm{s}}$, implies constancy of $\dot{\varphi}$.

By the *local cohomogeneity* of a Riemannian manifold (M, g) we mean the minimum codimension in T_xM, over all $x \in M$, of the subspace of T_xM consisting of the values at x of all Killing fields defined on neighborhoods of x.

The local cohomogeneity of the EPS space is obviously zero. For the Kähler-surface examples in [**7**] it equals one: see [**7**, Remark 12.1]. The proper weakly Einstein metrics described in Sect. 15 are of *local cohomogeneity two*, due to the functional-independence conclusion of Remark 15.1, and Remark 3.3 (the latter applicable here since they are warped products with the fibre provided by a surface of constant Gaussian curvature). Since the EPS space is known [**1**] to be, up to local homothety, the only locally homogeneous proper weakly Einstein manifold, (21.1) combined with Remark 3.3 shows that all the remaining proper weakly Einstein metrics of Sect. 14, other than those of case (i) with $c = 0$ and constant $\dot\varphi$, have local cohomogeneity one. They are thus different from the examples of Sect. 15.

The local-homothety invariants (9.1) of the metrics in Sect. 14 realize, at various points, all pairs $(\bar{\mathrm{s}}, \lambda)$ with $\bar{\mathrm{s}}\lambda \neq 0$, as one sees using (b), (e) in Sect. 14, Remark 14.1, and a suitable choice of initial data for (14.3). The same applies to Sect. 15: see Remark 15.1 and (15.4).

22. Harmonic curvature

One says that a Riemannian manifold or metric has *harmonic curvature* when the divergence of its curvature tensor vanishes identically or, equivalently, its Ricci tensor satisfies the Codazzi equation. Obvious examples are provided by

(22.1) Einstein metrics, conformally flat metrics having constant scalar curvature, and products of manifolds with harmonic curvature.

See [**2**, Sect. 16.33]. In dimension four, some further such metrics have the form

(22.2) $(K+c)^{-2}(g+h)$, for the product $g+h$ of surface metrics g, h with Gaussian curvatures K and c, where both c and the function $\gamma = (K+c)^3 + 3(K+c)\Delta K - 6g(\nabla K, \nabla K)$ are constant,

while $K+c \neq 0$ everywhere on the product four-manifold [**4**, Lemma 3].

As shown by DeTurck and Goldschmidt [**10**], harmonic curvature implies

(22.3) real-analyticity of the metric in suitable local coordinates.

Remark 22.1. According to [**3**, Proposition 3(i)], a conformal change leading from a product $\bar g$ of two surface metric, at points where the scalar curvature $\bar{\mathrm{s}}$ of $\bar g$ is nonzero, to a metric with harmonic curvature, if it exists, is – up to a constant factor – unique, and consists in division by $\bar{\mathrm{s}}^2$.

Lemma 22.2. *No proper weakly Einstein metric arises from* (22.2).

Proof. If it did, Theorem 16.1 would give (i) or (ii) in Theorem 9.1 with φ equal, locally, to a constant multiple of $K+c$ (Remark 22.1). From (2.3-b) it would now follow that, for the constant γ in (22.2), $\gamma - (K+c)^3$ equals either 0 or $-3(K+c)^3$, implying constancy of K, contrary to Remark 2.4. □

Theorem 22.3. *There exists no proper weakly Einstein four-dimensional Riemannian manifold with harmonic curvature.*

Proof. Assuming, on the contrary, the existence of such an oriented manifold, we will show, by considering two separate cases, that its metric must then have the form (22.1) or (22.2), which will in turn contradict Remarks 2.4 – 2.5 or, respectively, Lemma 22.2, and hence complete the proof.

In the first case, the Ricci tensor r has four distinct eigenvalues at some point, and so – by (22.3) – at all points of some dense open set U. For a fixed local orientation in U, as shown in [**4**, the lines following formula (6)], if an orthonormal frame $u_1, \ldots, u_4$ diagonalizes r, then W^+ and W^- are diagonalized, with the some ordered spectra for both, by the corresponding bivectors (5.1). Remark 6.1 now leads to (5.3-a) or (5.3-b) in Theorem 5.1, with $c_2 = c_3 = c_4 = 0$. Due to Remark 2.5, the case (5.3-a), implying conformal flatness, is excluded. In (5.3-b), $W^\pm$ has a repeated eigenvalue, and [**6**, Lemma 5.4] then shows that the metric is of type (22.1) or (22.2). On the other hand, if r has fewer than four eigenvalues at each point, [**6**, Theorem 2.2(c)] yields (22.1) or (22.2). □

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