The local structure of conformally symmetric manifolds

Andrzej Derdzinski   Witold Roter

Abstract

This is a final step in a local classification of pseudo-Riemannian manifolds with parallel Weyl tensor that are not conformally flat or locally symmetric.

Introduction

The present paper provides a finishing touch in a local classification of essentially conformally symmetric pseudo-Riemannian metrics.

A pseudo-Riemannian manifold of dimension $n \geq 4$ is called *essentially conformally symmetric* if it is *conformally symmetric* [2] (in the sense that its Weyl conformal tensor is parallel) without being conformally flat or locally symmetric.

The metric of an essentially conformally symmetric manifold is always indefinite [4, Theorem 2]. *Compact* essentially conformally symmetric manifolds are known to exist in all dimensions $n \geq 5$ with $n \equiv 5 \pmod 3$, where they represent all indefinite metric signatures [8], while examples of essentially conformally symmetric pseudo-Riemannian metrics on open manifolds of all dimensions $n \geq 4$ were first constructed in [16].

On every conformally symmetric manifold there is a naturally distinguished parallel distribution $\mathcal{D}$, of some dimension d, which we call the *Olszak distribution*. As shown by Olszak [13], for an essentially conformally symmetric manifold one has $d \in \{1, 2\}$.

Received by the editors April 2007.
Communicated by L. Vanhecke.
2000 *Mathematics Subject Classification* : 53B30.
Key words and phrases : Parallel Weyl tensor, conformally symmetric manifold.

In [7] we described the local structure of all conformally symmetric manifolds with $d = 2$. See also Section 3. This paper establishes an analogous result (Theorem 4.1) for the case $d = 1$.

In both cases, some of the metrics in question are locally symmetric. In Remark 4.2 we explain why a similar classification result cannot be valid just for *essentially* conformally symmetric manifolds.

Essentially conformally symmetric manifolds with $d = 1$ are all *Ricci-recurrent*, in the sense that, for every tangent vector field v, the Ricci tensor ρ and the covariant derivative $\nabla_v \rho$ are linearly dependent at each point. The local structure of essentially conformally symmetric Ricci-recurrent manifolds at points with $\rho \otimes \nabla\rho \neq 0$ has already been determined by the second author [16]. Our new contribution settles the one case still left open in the local classification problem, namely, that of essentially conformally symmetric manifolds with $d = 1$ at points where $\rho \otimes \nabla\rho = 0$.

The literature dealing with conformally symmetric manifolds includes, among others, [9, 10, 12, 15, 17, 18] and the papers cited above. A local classification of *homogeneous* essentially conformally symmetric manifolds can be found in [3].

1 Preliminaries

Throughout this paper, all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞. A manifold is, by definition, connected. Unless stated otherwise, a mapping is always a C^∞ mapping between manifolds.

Given a connection ∇ in a vector bundle $\mathcal{E}$ over a manifold M, a section ψ of $\mathcal{E}$, and vector fields u, v tangent to M, we use the sign convention

$$R(u,v)\psi \;=\; \nabla_v\nabla_u\psi \,-\, \nabla_u\nabla_v\psi \,+\, \nabla_{[u,v]}\psi \tag{1}$$

for the curvature tensor $R = R^\nabla$.

The Levi-Civita connection of a given pseudo-Riemannian manifold (M, g) is always denoted by ∇. We also use the symbol ∇ for connections induced by ∇ in various ∇-parallel subbundles of TM and their quotients.

The Schouten tensor σ and Weyl conformal tensor W of a pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ are given by $\sigma = \rho - (2n-2)^{-1}\,\mathrm{s}g$, with ρ denoting the Ricci tensor, $\mathrm{s} = \mathrm{tr}_g\rho$ standing for the scalar curvature, and

$$W \;=\; R \,-\, (n-2)^{-1} g \wedge \sigma. \tag{2}$$

Here $\wedge$ is the exterior multiplication of 1-forms valued in 1-forms, which uses the ordinary $\wedge$ as the valuewise multiplication; thus, $g \wedge \sigma$ is a 2-form valued in 2-forms.

Let $(t,s) \mapsto x(t,s)$ be a fixed *variation of curves* in a pseudo-Riemannian manifold (M, g), that is, an M-valued C^∞ mapping from a rectangle (product of intervals) in the ts-plane. By a *vector field w along the variation* we mean, as usual, a section of the pullback of TM to the rectangle (so that $w(t,s) \in T_{x(t,s)}M$). Examples are x_t and x_s, which assign to (t,s) the velocity of the curve $t \mapsto x(t,s)$

or $s \mapsto x(t,s)$ at t or s. Further examples are provided by restrictions to the variation of vector fields on M. The partial covariant derivatives of a vector field w along the variation are the vector fields w_t, w_s along the variation, obtained by differentiating w covariantly along the curves $t \mapsto x(t,s)$ or $s \mapsto x(t,s)$. Skipping parentheses, we write w_{ts}, w_{stt}, etc., rather than $(w_t)_s$, $((w_s)_t)_t$ for higher-order derivatives, as well as x_{ss}, x_{st} instead of $(x_s)_s$, $(x_s)_t$. One always has $w_{ts} = w_{st} + R(x_t, x_s)w$, cf. [11, formula (5.29) on p. 460], and, since the Levi-Civita connection ∇ is torsionfree, $x_{st} = x_{ts}$. Thus, whenever $(t,s) \mapsto x(t,s)$ is a variation of curves in M,

$$x_{tss} \;=\; x_{sst} \,+\, R(x_t, x_s)x_s\,. \tag{3}$$

2 The Olszak distribution

The *Olszak distribution* of a conformally symmetric manifold (M,g) is the parallel subbundle $\mathcal{D}$ of TM, the sections of which are the vector fields u with the property that $\xi \wedge \Omega = 0$ for all vector fields v, v' and for the differential forms $\xi = g(u,\,\cdot\,)$ and $\Omega = W(v,v',\,\cdot\,,\,\cdot\,)$. The distribution $\mathcal{D}$ was introduced, in a more general situation, by Olszak [13], who also proved the following lemma.

LEMMA 2.1. *The following conclusions hold for the dimension d of the Olszak distribution $\mathcal{D}$ in any conformally symmetric manifold (M,g) with* $\dim M = n \ge 4$.

i. *$d \in \{0,1,2,n\}$, and $d = n$ if and only if (M,g) is conformally flat.*

ii. *$d \in \{1,2\}$ if (M,g) is essentially conformally symmetric.*

iii. *$d = 2$ if and only if* $\operatorname{rank} W = 1$, *in the sense that W, as an operator acting on exterior 2-forms, has rank 1 at each point.*

iv. *If $d = 2$, the distribution $\mathcal{D}$ is spanned by all vector fields of the form $W(u,v)v'$ for arbitrary vector fields u, v, v' on M.*

Proof. See Appendix I. ■

In the next lemma, parts (a) and (d) are due to Olszak [13, 2° and 3° on p. 214].

LEMMA 2.2. *If $d \in \{1,2\}$, where d is the dimension of the Olszak distribution $\mathcal{D}$ of a given conformally symmetric manifold (M,g) with* $\dim M = n \ge 4$, *then*

a. *$\mathcal{D}$ is a null parallel distribution,*

b. *at any $x \in M$ the space $\mathcal{D}_x$ contains the image of the Ricci tensor ρ_x treated, with the aid of g_x, as an endomorphism of T_xM,*

c. *the scalar curvature is identically zero and $R \;=\; W + (n-2)^{-1} g \wedge \rho$,*

d. *$W(u,\,\cdot\,,\,\cdot\,,\,\cdot\,) \;=\; 0$ whenever u is a section of $\mathcal{D}$,*

e. *$R(v,v',\,\cdot\,,\,\cdot\,) = W(v,v',\,\cdot\,,\,\cdot\,) \;=\; 0$ for any sections v and v' of $\mathcal{D}^\perp$,*

f. *of the connections in $\mathcal{D}$ and $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$, induced by the Levi-Civita connection of g, the latter is always flat, and the former is flat if $d = 1$.*

Proof. Assertion (e) for W is immediate from the definition of $\mathcal{D}$. Namely, at any point $x \in M$, every 2-form Ω_x in the image of W_x (for W_x acting on 2-forms at x) is $\wedge$-divisible by $\xi = g_x(u, \cdot\,)$ for each $u \in \mathcal{D}_x \smallsetminus \{0\}$, and so $\Omega_x(v, v') = 0$ if $v, v' \in \mathcal{D}_x^\perp$.

We now proceed to prove (a), (b), (c) and (d).

First, let $d = 2$. By Lemma 2.1(iii), this amounts to the condition $\operatorname{rank} W = 1$, so that (a), (b) and (c) follow from Lemma 2.1(iv) combined with [7, Lemma 17.1(ii) and Lemma 17.2]. Also, for a nonzero 2-form Ω_x chosen as in the last paragraph, $\mathcal{D}_x$ is the image of Ω_x, that is, Ω_x equals the exterior product of two vectors in $\mathcal{D}_x$ (treated as 1-forms, with the aid of g_x). Now (d) follows since, by (a), $\Omega_x(u_x, \cdot\,) = 0$ if u is a section of $\mathcal{D}$.

Next, suppose that $d = 1$. Replacing M by a neighborhood of any given point, we may assume that $\mathcal{D}$ is spanned by a vector field u. If u were not null, we would have $W(u, v, u, v') = 0$ for any sections v, v' of $\mathcal{D}^\perp$, as one sees contracting the twice-covariant tensor field $W(\,\cdot\,, v, \,\cdot\,, v')$, at any point x, in an orthogonal basis containing the vector u_x. (We have already established (e) for W.) Combined with (e) for W and the symmetries of W, the relation $W(u, v, u, v') = 0$ for v, v' in $\mathcal{D}^\perp$ would then give $W = 0$, contrary to the assumption that $d = 1$. Thus, u is null, which yields (a). Now

$$\text{we choose, locally, a null vector field } u' \text{ with } g(u, u') = 1. \tag{4}$$

For any section v of $\mathcal{D}^\perp$ one sees that $W(u, \,\cdot\,, u', v) = 0$ by contracting the tensor field $W(\,\cdot\,, \,\cdot\,, \,\cdot\,, v)$ in the first and third arguments, at any point x, in

$$\text{a basis of } T_xM \text{ formed by } u_x, u'_x \text{ and } n-2 \text{ vectors orthogonal to them,} \tag{5}$$

and using (e) for W, along with the inclusion $\mathcal{D} \subset \mathcal{D}^\perp$, cf. (a). Since u' and $\mathcal{D}^\perp$ span TM, assertion (e) for W thus implies (d).

To obtain (b) and (c) when $d = 1$, we distinguish two cases: (M, g) is either essentially conformally symmetric, or locally symmetric. For (c), it suffices to establish vanishing of the scalar curvature s (cf. (2)). Now, in the former case, $\mathrm{s} = 0$ according to [5, Theorem 7], while (b) follows since, as shown in [6, Theorem 7 on p. 18], for arbitrary vector fields v, v' and v'' on an essentially conformally symmetric pseudo-Riemannian manifold, $\xi \wedge \Omega = 0$, where $\xi = \rho(v, \cdot\,)$ and $\Omega = W(v', v'', \,\cdot\,, \cdot\,)$. In the case where g is locally symmetric, (b) and (c) are proved in Appendix II.

Assertion (e) for R is now obvious from (e) for W and (c), since, by (b), $\rho(v, \cdot\,) = 0$ for any section v of $\mathcal{D}^\perp$. The claim about $\mathcal{E}$ in (f) is in turn immediate from (1) and (e) for R, which states that $R(w, w')v$, for arbitrary vector fields w, w' and any section v of $\mathcal{D}^\perp$, is orthogonal to all sections of $\mathcal{D}^\perp$ (and hence must be a section of $\mathcal{D}$). Finally, to prove (f) for $\mathcal{D}$, with $d = 1$, let us fix a section u of $\mathcal{D}$, a vector field v, and define a differential 2-form ζ by $\zeta(w, w') = (n-2)R(w, w', u, v)$ for any vector fields w, w'. By (c) and (e), $\zeta = g(u, \cdot\,) \wedge \rho(v, \cdot\,)$, as $\mathcal{D} \subset \mathcal{D}^\perp$ (cf. (a)), and so $\rho(u, \cdot\,) = 0$ in view of (b) and symmetry of ρ. However, by (b), both $g(u, \cdot\,)$ and $\rho(v, \cdot\,)$ are sections of the subbundle of T^*M corresponding to $\mathcal{D}$ under the bundle isomorphism $TM \to T^*M$ induced by g, so that $\zeta = 0$ since the distribution $\mathcal{D}$ is one-dimensional. ∎

3 The case $d = 2$

For more details of the construction described below, we refer the reader to [7].

Let there be given a surface Σ, a projectively flat torsionfree connection D on Σ with a D-parallel area form α, an integer $n \geq 4$, a sign factor $\varepsilon = \pm 1$, a real vector space V with $\dim V = n-4$, and a pseudo-Euclidean inner product $\langle\,,\rangle$ on V.

We also assume the existence of a twice-contravariant symmetric tensor field T on Σ with $\mathrm{div}^{\mathrm{D}}(\mathrm{div}^{\mathrm{D}} T) + (\rho^{\mathrm{D}}, T) = \varepsilon$ (in coordinates: $T^{jk}{}_{,jk} + T^{jk}R_{jk} = \varepsilon$). Here $\mathrm{div}^{\mathrm{D}}$ denotes the D-divergence, ρ^{D} is the Ricci tensor of D, and $(\,,)$ stands for the obvious pairing. Such T always exists locally in Σ. In fact, according to [7, Theorem 10.2(i)] combined with [7, Lemma 11.2], T exists whenever Σ is simply connected and noncompact.

For T chosen as above, we define a twice-covariant symmetric tensor field τ on Σ, that is, a section of $[T^*\Sigma]^{\odot 2}$, by requiring τ to correspond to the section T of $[T\Sigma]^{\odot 2}$ under the vector-bundle isomorphism $T\Sigma \to T^*\Sigma$ which acts on vector fields v by $v \mapsto \alpha(v,\,\cdot\,)$. In coordinates, $\tau_{jk} = \alpha_{jl}\alpha_{km}T^{lm}$.

Next, we denote by h^{D} the *Patterson-Walker Riemann extension metric* [14] on the total space $T^*\Sigma$, obtained by requiring that all vertical and all D-horizontal vectors be h^{D}-null, while $h^{\mathrm{D}}_x(\zeta, w) = \zeta(d\pi_x w)$ for $x \in T^*\Sigma$, an arbitrary vector $w \in T_x T^*\Sigma$, any vertical vector $\zeta \in \mathrm{Ker}\, d\pi_x = T^*_{\pi(x)}\Sigma$, and the bundle projection $\pi : T^*\Sigma \to \Sigma$.

Finally, let γ and θ be the constant pseudo-Riemannian metric on V corresponding to the inner product $\langle\,,\rangle$, and the function $V \to \mathbf{R}$ with $\theta(v) = \langle v, v\rangle$.

Our $\Sigma, \mathrm{D}, \alpha, n, \varepsilon, V, \langle\,,\rangle$ now give rise to the pseudo-Riemannian manifold

$$(T^*\Sigma \times V,\ h^{\mathrm{D}} - 2\tau + \gamma - \theta\rho^{\mathrm{D}}), \tag{6}$$

of dimension n, with the metric $h^{\mathrm{D}} - 2\tau + \gamma - \theta\rho^{\mathrm{D}}$, where the function θ and covariant tensor fields $\tau, \rho^{\mathrm{D}}, h^{\mathrm{D}}, \gamma$ on Σ, $T^*\Sigma$ or V are identified with their pullbacks to $T^*\Sigma \times V$. (Thus, for instance, $h^{\mathrm{D}} - 2\tau + \gamma$ is a product metric.)

We have the following local classification result, in which d stands for the dimension of Olszak distribution $\mathcal{D}$.

THEOREM 3.1. *The pseudo-Riemannian manifold* (6) *obtained as above from any data $\Sigma, \mathrm{D}, \alpha, n, \varepsilon, V, \langle\,,\rangle$ with the stated properties is conformally symmetric and has $d = 2$. Conversely, in any conformally symmetric pseudo-Riemannian manifold such that $d = 2$, every point has a connected neighborhood isometric to an open subset of a manifold* (6) *constructed above from some such data $\Sigma, \mathrm{D}, \alpha, n, \varepsilon, V, \langle\,,\rangle$.*

The manifold (6) *is never conformally flat, and it is locally symmetric if and only if the Ricci tensor ρ^{D} is* D*-parallel.*

Proof. See [7, Section 22]. Note that, in view of Lemma 2.1(iii), the condition $\mathrm{rank}\, W = 1$ used in [7] is equivalent to $d = 2$. ∎

The objects $\Sigma, \mathrm{D}, \alpha, n, \varepsilon, V, \langle\,,\rangle$ are treated as parameters of the above construction, while T is merely assumed to exist, even though the metric g in (6) clearly depends on τ (and hence on T). This is justified by the fact that, with fixed

$\Sigma, \mathrm{D}, \alpha, n, \varepsilon, V, \langle\,,\rangle$, the metrics corresponding to two choices of T are, locally, isometric to each other, cf. [7, Remark 22.1].

The metric signature of (6) is clearly given by $--\ldots++$, with the dots standing for the sign pattern of $\langle\,,\rangle$.

4 The case $d = 1$

Let there be given an open interval I, a C^∞ function $f : I \to \mathbf{R}$, an integer $n \geq 4$, a real vector space V of dimension $n-2$ with a pseudo-Euclidean inner product $\langle\,,\rangle$, and a nonzero traceless linear operator $A : V \to V$, self-adjoint relative to $\langle\,,\rangle$. As in [16], we then define an n-dimensional pseudo-Riemannian manifold

$$(I \times \mathbf{R} \times V,\ \kappa\, dt^2 + dt\, ds + \gamma)\,, \tag{7}$$

where products of differentials represent symmetric products, t, s denote the Cartesian coordinates on the $I \times \mathbf{R}$ factor, γ stands for the pullback to $I \times \mathbf{R} \times V$ of the flat pseudo-Riemannian metric on V that corresponds to the inner product $\langle\,,\rangle$, and the function $\kappa : I \times \mathbf{R} \times V \to \mathbf{R}$ is given by $\kappa(t,s,\psi) = f(t)\langle\psi,\psi\rangle + \langle A\psi,\psi\rangle$.

The manifolds (7) are characterized by the following local classification result, analogous to Theorem 3.1. As before, d denotes the dimension of the Olszak distribution.

THEOREM 4.1. *For any $I, f, n, V, \langle\,,\rangle, A$ as above, the pseudo-Riemannian manifold* (7) *is conformally symmetric and has $d = 1$. Conversely, in any conformally symmetric pseudo-Riemannian manifold such that $d = 1$, every point has a connected neighborhood isometric to an open subset of a manifold* (7) *constructed from some such $I, f, n, V, \langle\,,\rangle, A$.*

The manifold (7) *is never conformally flat, and it is locally symmetric if and only if f is constant.*

A proof of Theorem 4.1 is given at the end of the next section.

Obviously, the metric $\kappa\, dt^2 + dt\, ds + \gamma$ in (7) has the sign pattern $-\ldots+$, where the dots stand for the sign pattern of $\langle\,,\rangle$.

REMARK 4.2. A classification result of the same format as Theorem 4.1 cannot be true just for *essentially* conformally symmetric manifolds with $d = 1$. Namely, such manifolds do not satisfy a principle of unique continuation: formula (7) with f which is nonconstant on I, but constant on some nonempty open subinterval I' of I, defines an essentially conformally symmetric manifold with a locally symmetric open submanifold $U = I' \times \mathbf{R} \times V$. At points of U, the local structure of (7) does not, therefore, arise from a construction that, locally, produces all essentially conformally symmetric manifolds and nothing else.

As explained in [7, Section 24], an analogous situation arises when $d = 2$.

5 Proof of Theorem 4.1

The following assumptions will be used in Lemma 5.1.

a. (M, g) is a conformally symmetric manifold $\dim M = n \geq 4$ and $y \in M$.

b. The Olszak distribution $\mathcal{D}$ of (M, g) is one-dimensional.

c. u is a global parallel vector field spanning $\mathcal{D}$.

d. $t : M \to \mathbf{R}$ is a C^∞ function with $g(u, \,\cdot\,) = dt$ and $t(y) = 0$.

e. $\dim V = n - 2$ for the space V of all parallel sections of $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$.

f. $\rho = (2 - n) f(t)\, dt \otimes dt$ for some C^∞ function $f : I' \to \mathbf{R}$ on an open interval I', where ρ is the Ricci tensor and $f(t)$ stands for $f \circ t$.

For local considerations, only (a) and (b) are essential. In fact, condition (e) (in which 'parallel' refers to the connection in $\mathcal{E}$ induced by the Levi-Civita connection of g), as well (c) and (d) for some u and t, follow from (a) – (b) if M is simply connected. See Lemma 2.2(f). On the other hand, (c) – (d), Lemma 2.2(b) and symmetry of ρ give $\nabla dt = 0$ and $\rho = \chi\, dt \otimes dt$ for some function $\chi : M \to \mathbf{R}$, so that $\nabla\rho = d\chi \otimes dt \otimes dt$. However, $\nabla\rho$ is totally symmetric (that is, ρ satisfies the Codazzi equation): our assumption $\nabla W = 0$ implies the condition $\operatorname{div} W = 0$, well known [11, formula (5.29) on p. 460] to be equivalent to the Codazzi equation for the Schouten tensor σ, while $\sigma = \rho$ by Lemma 2.2(c). Thus, $d\chi$ equals a function times dt, and so χ is, locally, a function of t, which (locally) yields (f).

For any section v of $\mathcal{D}^\perp$, we denote by $\underline{v}$ the image of v under the quotient-projection morphism $\mathcal{D}^\perp \to \mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$.

The data for the construction in Section 4 consist of f, n, V appearing in (a) – (f), of an open subinterval $I \subset I'$ chosen as described below, of the pseudo-Euclidean inner product $\langle\,,\rangle$ in V, induced in an obvious way by g (cf. Lemma 2.2(f)), and of $A : V \to V$ with $\langle A\psi, \psi'\rangle = W(u', v, v', u')$, for $\psi, \psi' \in V$, where a vector field u' and sections v, v' of $\mathcal{D}^\perp$ are chosen, locally, so that $g(u, u') = 1$, $\psi = \underline{v}$ and $\psi' = \underline{v'}$. (The bilinear form $(\psi, \psi') \mapsto \langle A\psi, \psi'\rangle$ on V then is well-defined, that is, unaffected by the choices of u', v or v', in view of Lemma 2.2(d),(e), while the function $W(u', v, v', u')$ is in fact constant, by Lemma 2.2(d), as ones sees differentiating it via the Leibniz rule and noting that, since $\underline{v}$ and $\underline{v'}$ are parallel, the covariant derivatives of v and v' in the direction of any vector field are sections of $\mathcal{D}$.) That A is traceless and self-adjoint is immediate from the symmetries of W. Finally, $A \neq 0$ since, otherwise, W would vanish. (Namely, in view of Lemma 2.2(d),(e), W would yield 0 when evaluated on any quadruple of vector fields, each of which is either u' or a section of $\mathcal{D}^\perp$.)

Under the assumptions (a) – (f), with $f = f(t)$, we then have

$$R(u', v)v' \;=\; [f g(v, v') + \langle A\underline{v}, \underline{v'}\rangle] g(u', u) u \tag{8}$$

for any sections v, v' of $\mathcal{D}^\perp$ and any vector field u'. In fact, $\rho(v, \,\cdot\,) = \rho(v', \,\cdot\,) = 0$ from symmetry of ρ and Lemma 2.2(b), so that, by Lemma 2.2(c), $R(u', v)v' = W(u', v)v' - (n-2)^{-1} g(v, v')\rho u'$, where $\rho u'$ denotes the unique vector field with

$g(\rho u',\,\cdot\,) = \rho(u',\,\cdot\,)$. Thus, (8) follows: due to (d), (f) and the definition of A, both sides have the same g-inner product with u', and are orthogonal to $u^\perp = \mathcal{D}^\perp$ (since $R(u',v)v'$ is orthogonal to $\mathcal{D}^\perp$ as a consequence of Lemma 2.2(e)).

We may now fix an open subinterval I of I', containing 0, and a null geodesic $I \ni t \mapsto x(t)$ in M with $x(0) = y$, parametrized by the function t (in the sense that the function t restricted to the geodesic coincides with the geodesic parameter). Namely, since $\nabla dt = 0$, the restriction of t to any geodesic is an affine function of the parameter; thus, by (d), it suffices to prescribe the initial data formed by $x(0) = y$ and a null vector $\dot{x}(0) \in T_yM$ with $g(\dot{x}(0), u_y) = 1$.

As $g(\dot{x}(0), u_y) = 1$, the plane P in T_yM, spanned by the null vectors $\dot{x}(0)$ and u_y (cf. Lemma 2.2(a)) is g_y-nondegenerate, and so $T_yM = P \oplus \widetilde{V}$, for $\widetilde{V} = P^\perp$. Let $\mathrm{pr} : T_yM \to \widetilde{V}$ be the orthogonal projection. Since $\mathrm{pr}(\mathcal{D}_y) = \{0\}$, the restriction of pr to $\mathcal{D}_y^\perp$ descends to an isomorphism $\mathcal{E}_y = \mathcal{D}_y^\perp/\mathcal{D}_y \to \widetilde{V}$, also denoted by pr. Finally, for $\psi \in V$, we let $t \mapsto \tilde{\psi}(t) \in T_{x(t)}M$ be the parallel field with $\tilde{\psi}(0) = \mathrm{pr}\,\psi_y$, and set $\kappa(t,s,\psi) = f(t)\langle\psi,\psi\rangle + \langle A\psi,\psi\rangle$, as in Section 4.

The formula $F(t,s,\psi) = \exp_{x(t)}(\tilde{\psi}(t) + s u_{x(t)}/2)$ now defines a C^∞ mapping F from an open subset of $\mathbf{R}^2 \times V$ into M.

LEMMA 5.1. *Under the above hypotheses,* $F^*g = \kappa\, dt^2 + dt ds + h$.

Proof. The F-images $w, w', F_*\psi$ of the constant vector fields $(1,0,0), (0,1,0)$ and $(0,0,\psi)$ in $\mathbf{R}^2 \times V$, for $\psi \in V$, are vector fields tangent to M along F (sections of F^*TM). Since $\mathcal{D}^\perp$ is parallel, its leaves are totally geodesic and, by Lemma 2.2(e), the Levi-Civita connection of g induces on each leaf a flat torsionfree connection. Thus, w' and each $F_*\psi$ are parallel along each leaf of $\mathcal{D}^\perp$, as well as tangent to the leaf, and parallel along the geodesic $t \mapsto x(t)$. Therefore, $w' = u/2$, while the functions $g(w', F_*\psi)$ and $g(F_*\psi, F_*\psi')$, for $\psi, \psi' \in V$, are constant, and hence equal to their values at y, that is, 0 and $\langle\psi,\psi'\rangle$. It now remains to be shown that $g(w,w) = \kappa \circ F$, $g(w, u/2) = 1/2$ and $g(w, F_*\psi) = 0$. To this end, we consider the variation $x(t,s) = F(t, sa, s\psi)$ of curves in M, with any fixed $a \in \mathbf{R}$ and $\psi \in V$. Clearly, $w = x_t$ along the variation (notation of Section 1). Next, $x_{ts} = x_{st}$ is tangent to $\mathcal{D}^\perp$, since so is x_s, while $\mathcal{D}^\perp$ is parallel. Consequently, $[g(x_t, u)]_s = 0$, as u is parallel and tangent to $\mathcal{D}$. Thus, $g(w,u) = g(x_t,u) = 1$. (Note that $g(x_t,u) = 1$ at $s = 0$, due to (d), as the geodesic $t \mapsto x(t)$ is parametrized by the function t.) However, $x_{ss} = 0$ and x_s is tangent to $\mathcal{D}^\perp$, so that (3) and (8) now give $x_{tss} = [fg(x_s,x_s) + \langle A\underline{x_s}, \underline{x_s}\rangle]u$, which is parallel in the s direction, while $x_{ts} = x_{st} = 0$ at $s = 0$. Hence $x_{ts} = s[fg(x_s,x_s) + \langle A\underline{x_s}, \underline{x_s}\rangle]u$, and so $g(x_{ts}, x_{ts}) = 0$ (cf. (c) above and Lemma 2.2(a)). This further yields $[g(x_t,x_t)]_{ss}/2 = g(x_t, x_{tss}) = fg(x_s,x_s) + \langle A\underline{x_s}, \underline{x_s}\rangle$. The last function is constant in the s direction, while $g(x_t,x_t) = [g(x_t,x_t)]_s = 0$ at $s = 0$, and so $g(w,w) = g(x_t,x_t) = s^2[fg(x_s,x_s) + \langle A\underline{x_s}, \underline{x_s}\rangle] = \kappa$. Finally, being proportional to u at each point, x_{ts} is orthogonal to $\mathcal{D}^\perp$, and hence to $F_*\psi$, which implies that $[g(x_t, F_*\psi)]_s = 0$, and, as $g(w, F_*\psi) = g(x_t, F_*\psi) = 0$ at $s = 0$, we get $g(w, F_*\psi) = 0$ everywhere. ∎

We are now in a position to prove Theorem 4.1. First, (7) is conformally symmetric and has $d = 1$, as one can verify by a direct calculation, cf. [16, Theorem 3]. Conversely, if conditions (a) and (b) above are satisfied, we may also assume (c) – (f). (See the comment following (f).) Our assertion is now immediate from Lemma 5.1.

Appendix I: Proof of Lemma 2.1

We prove Lemma 2.1 here, since Olszak's paper [13] may be difficult to obtain.

The condition $d = n$ is equivalent to conformal flatness of (M, g), since $n > 2$ and so $\Omega = 0$ is the only 2-form $\wedge$-divisible by all nonzero 1-forms ξ. At a fixed point x, the metric g_x allows us to treat the Ricci tensor ρ_x and any 2-form Ω_x as endomorphisms of T_xM, so that we may consider their images (which are subspaces of T_xM). If $W \neq 0$, fixing a nonzero 2-form Ω_x in the image of W_x acting on 2-forms at x we see that, for every $u \in \mathcal{D}_x$, our Ω_x is $\wedge$-divisible by $\xi = g_x(u, \cdot\,)$, and so the image of Ω_x contains $\mathcal{D}_x$. Thus, $d \leq 2$, and (i) follows. (Being nonzero and decomposable, Ω_x has rank 2.) As shown in [6, Theorem 7 on p. 18], if (M, g) is essentially conformally symmetric, the image of ρ_x is a subspace of $\mathcal{D}_x$, so that (i) yields (ii), since g in (ii) cannot be Ricci-flat. Next, if $d = 2$, the image of our Ω_x coincides with $\mathcal{D}_x$ (as $\operatorname{rank}\Omega_x = 2$). Every 2-form in the image of W_x thus is a multiple of Ω_x, being the exterior product of two vectors in $\mathcal{D}_x$, identified, via g_x, with 1-forms. Hence $\operatorname{rank} W = 1$. Conversely, if $\operatorname{rank} W = 1$, all nonzero 2-forms Ω_x in the image of W_x are of rank 2, as W_x, being self-adjoint, is a multiple of $\Omega_x \otimes \Omega_x$, and so the Bianchi identity for W gives $\Omega_x \wedge \Omega_x = 0$. All such Ω_x are therefore $\wedge$-divisible by $\xi = g_x(u, \cdot\,)$, for every nonzero vector u in the common 2-dimensional image of such Ω_x, which shows that $d = 2$. Finally, (iv) follows if one chooses $\Omega_x \neq 0$ equal to $W_x(v, v', \cdot\,, \cdot\,)$ for some $v, v' \in T_xM$.

Appendix II: Lemma 2.2(b),(c) in the locally symmetric case

Parts (b) and (c) of Lemma 2.2 for locally symmetric manifolds with $d = 1$ could, in principle, be derived from Cahen and Parker's classification [1] of pseudo-Riemannian symmetric manifolds. We prove them here directly, for the reader's convenience. Our argument uses assertions (a), (d) in Lemma 2.2, along with (e) for W, which were established in the proof of Lemma 2.2 before Appendix II was mentioned.

Suppose that $\nabla R = 0$ and $d = 1$. Replacing M by an open subset, we also assume that the Olszak distribution $\mathcal{D}$ is spanned by a vector field u. By (1),

$$\text{i)}\quad R(\,\cdot\,,\,\cdot\,)u \;=\; \Omega \otimes u \qquad \text{or, in coordinates,} \qquad \text{ii)}\quad u^l R_{jkl}{}^s = \Omega_{jk} u^s, \tag{9}$$

for some differential 2-form Ω, which obviously does not depend on the choice of u. (It is also clear from (1) that Ω is the curvature form of the connection in the line bundle $\mathcal{D}$, induced by the Levi-Civita connection of g.) Being unique, Ω is parallel, and so are ρ and W, which implies the Ricci identities $R \cdot \Omega = 0$,

$R\cdot\rho=0$, and $R\cdot W=0$. Equivalently, $R_{mlj}{}^s\tau_{sk}+R_{mlk}{}^s\tau_{js}=0$ for $\tau=\Omega$ or $\tau=\rho$, and

$$R_{qpj}{}^sW_{sklm}+R_{qpk}{}^sW_{jslm}+R_{qpl}{}^sW_{jksm}+R_{qpk}{}^sW_{jkls}=0. \tag{10}$$

Summing $R_{mlj}{}^s\Omega_{sk}+R_{mlk}{}^s\Omega_{js}=0$ against u^l, we obtain $\Omega\circ\Omega=0$, where the metric g is used to treat Ω as a bundle morphism $TM\to TM$ that sends each vector field v to the vector field Ωv with $g(\Omega v,v')=\Omega(v,v')$ for all vector fields v'. Lemma 2.2(d) and (9.i) give $W(\,\cdot\,,\,\cdot\,,u,v)=R(\,\cdot\,,\,\cdot\,,u,v)=0$ for our fixed vector field u, spanning $\mathcal{D}$, and any section v of $\mathcal{D}^\perp$. Hence, by (2), $g(u,\,\cdot\,)\wedge\sigma(v,\,\cdot\,)=g(v,\,\cdot\,)\wedge\sigma(u,\,\cdot\,)$. Thus, $\sigma u=cu$ for the Schouten tensor σ and some constant c, with σu defined analogously to Ωv. (Otherwise, choosing v such that $u,\sigma u$ and v are linearly independent at a given point x, we would obtain a contradiction with the equality between planes in T_xM, corresponding to the above equality between exterior products.) Now, $g(u,\,\cdot\,)\wedge(\sigma+cg)(v,\,\cdot\,)=0$, and so $\sigma v+cv$ is a section of $\mathcal{D}$ whenever v is a section of $\mathcal{D}^\perp$. Let us also fix u' as in (4). Symmetry of σ gives $g(\sigma u',u)=c$. In a suitably ordered basis with (5), at any point x, the endomorphism of T_xM corresponding to σ_x thus has an upper triangular matrix with the diagonal entries $c,-c,\dots,-c,c$, so that $\mathrm{tr}_g\sigma=(4-n)c$. Consequently, $(n-2)\,\mathrm{s}=2(n-1)(4-n)c$, for the scalar curvature s, and $(n-2)\rho u=2cu$. However, contracting (9.ii) in $k=s$, we get $\rho u=-\Omega u$, and so $(n-2)\Omega u=-2cu$. The equality $\Omega\circ\Omega=0$ that we derived from the Ricci identity $R\cdot\Omega=0$ now gives $c=0$. Hence $\mathrm{s}=0$ (which yields Lemma 2.2(c)), and $\rho u=0$.

As $c=0$ and $\sigma=\rho$, the assertion about $\sigma v+cv$ obtained above means that ρv is a section of $\mathcal{D}$ whenever v is a section of $\mathcal{D}^\perp$. Let λ,μ,ξ be the 1-forms with $\lambda=g(u,\,\cdot\,)$, $\mu=g(u',\,\cdot\,)$, $\xi(u')=0$, and $\rho v=\xi(v)u$ for sections v of $\mathcal{D}^\perp$. Transvecting (9.ii) with μ_s, we get the relation $\Omega=R(\,\cdot\,,\,\cdot\,,u,u')=(n-2)^{-1}\lambda\wedge\rho(u',\,\cdot\,)$ from Lemma 2.2(c) with $\rho u=0$ and Lemma 2.2(d). However, evaluating $\rho(u',\,\cdot\,)$ on u',u and sections v of $\mathcal{D}^\perp$, we see that $\rho(u',\,\cdot\,)=h\lambda+\xi$, with $h=\rho(u',u')$. (Note that $\xi(u)=0$ since $\rho u=0$, while $\mathcal{D}\subset\mathcal{D}^\perp$ by Lemma 2.2(a).) Therefore,

$$\text{i)}\;\;(n-2)\Omega=\lambda\wedge\xi,\qquad \text{ii)}\;\;\rho=h\lambda\otimes\lambda+\lambda\otimes\xi+\xi\otimes\lambda. \tag{11}$$

In addition, if v' denotes the unique vector field with $g(v',\,\cdot\,)=\xi$, then u and v' are null and orthogonal, or, equivalently,

$$\text{the 1-forms } \lambda \text{ and } \xi \text{ are null and mutually orthogonal.} \tag{12}$$

In fact, $g(u,u)=0$ by Lemma 2.2(a), $g(u,v')=0$ as $\xi(u)=0$, and v' is null since (11) yields $(n-2)[\rho(\Omega u')-\Omega(\rho u')]=2g(v',v')u$, while, transvecting the Ricci identity $R_{mlj}{}^sR_{sk}+R_{mlk}{}^sR_{js}=0$ with u^l and using (9.ii), we see that ρ and Ω commute as bundle morphisms $TM\to TM$.

Furthermore, transvecting with $\mu^k\mu^m$ the coordinate form $R_{mlj}{}^s\tau_{sk}+R_{mlk}{}^s\tau_{js}=0$ of the Ricci identity $R\cdot\tau=0$ for the parallel tensor field $\tau=(n-2)\Omega+\rho=h\lambda\otimes\lambda+2\lambda\otimes\xi$ (cf. (11)), we get $2\lambda_jb_{ls}\xi^s=0$, where $b=W(u',\,\cdot\,,u',\,\cdot\,)$. Namely, $R=W+(n-2)^{-1}g\wedge\rho$ by Lemma 2.2(c), $W_{mlj}{}^s\tau_{sk}=0$ in view of Lemma 2.2(d), $\mu^k\mu^mW_{mlk}{}^s\tau_{js}=2\lambda_jb_{ls}\xi^s$ since $b(u,\,\cdot\,)=0$ (again from Lemma 2.2(d)), and the remaining terms, related to $g \wedge \rho$, add up to 0 as a consequence of (12), (11.ii) and the formula for τ. (Note that (12) gives $R_j{}^s \tau_{sk} = R_j{}^s \tau_{ks} = 0$, and so four out of the eight remaining terms vanish individually.) However, $u \neq 0$, and so $\lambda \neq 0$, which gives $b(\,\cdot\,, v') = 0$, where v' is the vector field with $g(v', \cdot\,) = \xi$. Thus, $W(u', \cdot\,, u', v') = 0$. As a result, the 3-tensor $W(\,\cdot\,, \cdot\,, \cdot\,, v')$ must vanish: it yields the value 0 whenever each of the three arguments is either u' or a section of $\mathcal{D}^\perp$. (Lemma 2.2(e) for W is already established.)

The relation $W(\,\cdot\,, \cdot\,, \cdot\,, v') = 0$ implies in turn that $W(\,\cdot\,, \cdot\,, \cdot\,, \rho v) = 0$ (in coordinates: $W_{jkl}{}^s R_{sp} = 0$). In fact, by (11.ii), the image of ρ is spanned by u and v', while $W(\,\cdot\,, \cdot\,, \cdot\,, u) = 0$ according to Lemma 2.2(d).

As in [13, 1° on p. 214], we have $W = (\lambda \otimes \lambda) \wedge b$ (notation of (2)), where, again, $b = W(u', \cdot\,, u', \cdot\,)$. Namely, by Lemma 2.2(e) for W, both sides agree on any quadruple of vector fields, each of which is either u' or a section of $\mathcal{D}^\perp$.

Finally, transvecting (10) with $\mu^k \mu^m$ and replacing R by $W + (n-2)^{-1} g \wedge \rho$, we obtain two contributions, one from W and one from $g \wedge \rho$, the sum of which is zero. Since $W = (\lambda \otimes \lambda) \wedge b$, the W contribution vanishes: its first two terms add up to 0, and so do its other two terms. (As we saw, $b(u, \cdot\,) = 0$, while, obviously, $b(u', \cdot\,) = 0$.) Out of the sixteen terms forming the $g \wedge \rho$ contribution, eight are separately equal to zero since $W_{jkl}{}^s R_{sp} = 0$, and so, in view of (11.ii) and the relation $W = (\lambda \otimes \lambda) \wedge b$, vanishing of the $g \wedge \rho$ contribution gives $\lambda_p S_{jlq} = \lambda_q S_{jlp}$, for $S_{jlq} = 2b_{jl}\xi_q - b_{ql}\xi_j - b_{qj}\xi_l$. Thus, $S_{jlq} = \eta_{jl}\lambda_q$ for some twice-covariant symmetric tensor field η, which, summed cyclically over j, l, q, yields 0 (due to the definition of S_{jlq} and symmetry of b). As $\lambda \neq 0$ and the symmetric product has no zero divisors, we get $\eta = 0$ and $S_{jlq} = 0$. The expression $b_{jl}\xi_q - b_{ql}\xi_j$ is, therefore, skew-symmetric in j, l. As it is also, clearly, skew-symmetric in j, q, it must be totally skew-symmetric and hence equal to one-third of its cyclic sum over j, l, q. That cyclic sum, however, is 0 in view of symmetry of b, so that $b_{jl}\xi_q = b_{ql}\xi_j$. Thus, $\xi = 0$, for otherwise the last equality would yield $b = \varphi \xi \otimes \xi$ for some function φ, and hence $W = (\lambda \otimes \lambda) \wedge b = \varphi(\lambda \otimes \lambda) \wedge (\xi \otimes \xi)$, which would clearly imply that the vector field v' with $g(v', \cdot\,) = \xi$ is a section of the Olszak distribution $\mathcal{D}$, not equal to a function times u (as $\xi(u') = 0$, while $g(u, u') = 1$), contradicting one-dimensionality of $\mathcal{D}$. Therefore, $\rho = h\lambda \otimes \lambda$ by (11.ii) with $\xi = 0$, which proves assertion (b) of Lemma 2.2 in our case.

References

[1] CAHEN, M. & PARKER, M., Pseudo-riemannian symmetric spaces. *Mem. Amer. Math. Soc.* **229** (1980), 1–108.

[2] CHAKI, M. C. & GUPTA, B., On conformally symmetric spaces. *Indian J. Math.* **5** (1963), 113–122.

[3] DERDZIŃSKI, A., On homogeneous conformally symmetric pseudo-Riemannian manifolds. *Colloq. Math.* **40** (1978), 167–185.

[4] DERDZIŃSKI, A. & ROTER, W., On conformally symmetric manifolds with metrics of indices 0 and 1. *Tensor* (N. S.) **31** (1977), 255–259.

[5] DERDZIŃSKI, A. & ROTER, W., Some theorems on conformally symmetric manifolds. *Tensor* (N. S.) **32** (1978), 11–23.

[6] DERDZIŃSKI, A. & ROTER, W., Some properties of conformally symmetric manifolds which are not Ricci-recurrent. *Tensor* (N. S.) **34** (1980), 11–20.

[7] DERDZINSKI, A. & ROTER, W., Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds. *Tohoku Math. J.* **59** (2007), 565–602.

[8] DERDZINSKI, A. & ROTER, W., Compact pseudo-Riemannian manifolds with parallel Weyl tensor. Preprint, math.DG/0702491.

[9] DESZCZ, R., On hypercylinders in conformally symmetric manifolds. *Publ. Inst. Math.* (Beograd) (N. S.) **51**(65) (1992), 101–114.

[10] DESZCZ, R. & HOTLOŚ, M., On a certain subclass of pseudosymmetric manifolds. *Publ. Math. Debrecen* **53** (1998), 29–48.

[11] DILLEN, F. J. E. & VERSTRAELEN, L. C. A. (eds.), *Handbook of Differential Geometry, Vol. I.* North-Holland, Amsterdam, 2000.

[12] HOTLOŚ, M., On conformally symmetric warped products. *Ann. Acad. Paedagog. Cracov. Stud. Math.* **4** (2004), 75–85.

[13] OLSZAK, Z., On conformally recurrent manifolds, I: Special distributions. *Zesz. Nauk. Politech. Śl., Mat.-Fiz.* **68** (1993), 213–225.

[14] PATTERSON, E. M. & WALKER, A. G., Riemann extensions. *Quart. J. Math. Oxford* Ser. (2) **3** (1952), 19–28.

[15] RONG, J. P., On ${}^2K_n^*$ space. *Tensor* (N. S.) **49** (1990), 117–123.

[16] ROTER, W., On conformally symmetric Ricci-recurrent spaces. *Colloq. Math.* **31** (1974), 87–96.

[17] SHARMA, R., Proper conformal symmetries of conformal symmetric spacetimes. *J. Math. Phys.* **29** (1988), 2421–2422.

[18] SIMON, U., Compact conformally symmetric Riemannian spaces. *Math. Z.* **132** (1973), 173–177.

Department of Mathematics
The Ohio State University
Columbus, OH 43210
USA
email:andrzej@math.ohio-state.edu

Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław
Poland
email:Witold.Roter@pwr.wroc.pl