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THE TOPOLOGY OF COMPACT RANK-ONE ECS MANIFOLDS

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Abstract Pseudo-Riemannian manifolds with parallel Weyl tensor that are not conformally flat or locally symmetric, also known as essentially conformally symmetric (ECS) manifolds, have a natural local invariant, the rank, which equals 1 or 2, and is the rank of a certain distinguished null parallel distribution $\mathcal{D}$. All known examples of compact ECS manifolds are of rank one and have dimensions greater than 4. We prove that a compact rank-one ECS manifold, if not locally homogeneous, replaced when necessary by a twofold isometric covering, must be a bundle over the circle with leaves of $\mathcal{D}^\perp$ serving as the fibres. The same conclusion holds in the locally homogeneous case if one assumes that $\mathcal{D}^\perp$ has at least one compact leaf. We also show that in the pseudo-Riemannian universal covering space of any compact rank-one ECS manifold, the leaves of $\mathcal{D}^\perp$ are the factor manifolds of a global product decomposition.

Keywords: conformally symmetric manifold; parallel Weyl tensor

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1. Introduction

Pseudo-Riemannian manifolds (or metrics) in dimensions $n \geq 4$ with parallel Weyl tensor W are often called *conformally symmetric* [1]. One speaks of *ECS manifolds/metrics* [4] when, in addition, the metric is neither conformally flat nor locally symmetric, 'ECS' being short for *essentially conformally symmetric*.

ECS metrics exist in every dimension $n \geq 4$, as shown by Roter [17, Corollary 3], who also proved that they are all indefinite [3, Theorem 2]. The local structure of all ECS metrics is described in [6].

Given an ECS manifold (M, g), we define its *rank* [8] to be the rank $d \in \{1, 2\}$ of its *Olszak distribution* $\mathcal{D}$, which is a null parallel distribution on M discovered by Olszak [16]. The sections of $\mathcal{D}$ are the vector fields v having the property that $\mathsf{g}(v, \cdot) \wedge [W(v', v'', \cdot, \cdot)] = 0$ for all vector fields v', v''. Every Lorentzian ECS manifold has rank one, as the Lorentzian signature limits the ranks of null distributions to at most 1. For more details, see [6, p. 119].

Compact rank-one ECS manifolds are known to exist in all dimensions $n \geq 5$, where they represent all indefinite metric signatures [7, 8]. There are also non-compact locally homogeneous ECS manifolds [2] of every dimension $n \geq 4$. More recently, in [10], we constructed examples of compact locally homogeneous ECS manifolds of all odd dimensions $n \geq 5$.

Our main result can be phrased as follows.

Theorem A. *Every non-locally homogeneous compact rank-one ECS manifold with transversally orientable distribution $\mathcal{D}^\perp$ is diffeomorphic to a bundle over the circle in such a way that the fibres coincide with the leaves of $\mathcal{D}^\perp$. This conclusion remains valid in the locally homogeneous case, as long as $\mathcal{D}^\perp$ is assumed to have at least one compact leaf.*

The assertion of Theorem A obviously implies that the leaves of $\mathcal{D}^\perp$ are all compact and mutually diffeomorphic. Note that transversal orientability of $\mathcal{D}^\perp$ can always be achieved by replacing the manifold in question, if necessary, with a twofold isometric covering.

Theorem A generalizes Theorem B of [5] from the Lorentzian case to any indefinite metric signature. The assumption in [5, Theorem B] does not include rank one or exclude local homogeneity, since a Lorentzian ECS manifold necessarily has rank one (see above) and cannot be locally homogeneous (Remark 7.4). The Appendix explains how our proof of Theorem A differs from that used for [5, Theorem B].

The examples of [10], mentioned earlier, show that the final clause of Theorem A is non-vacuous, at least in odd dimensions.

Triviality of the pullback to $\mathbb{R}$ of a bundle over S^1 makes the next result an obvious consequence of Theorem A *except when* $(\widehat{M}, \mathrm{g})$ *is locally homogeneous.*

Theorem B. *The leaves of $\mathcal{D}^\perp$ in the pseudo-Riemannian universal covering space $(\widehat{M}, \mathrm{g})$ of any compact rank-one ECS manifold are the factor manifolds of a global product decomposition of $\widehat{M}$. More precisely, every leaf L of $\mathcal{D}^\perp$ in $\widehat{M}$ is simply connected, and $\widehat{M}$ is diffeomorphic to $\mathbb{R} \times L$.*

We prove both theorems in § 11.

2. Outline of the main argument

We fix a compact rank-one ECS manifold (M, g) of dimension $n \geq 4$, and denote by $(\widehat{M}, \mathrm{g})$ its pseudo-Riemannian universal covering and by $\pi : \widehat{M} \to M = \widehat{M}/\Gamma$ the covering projection. Here $\Gamma \approx \pi_1 M$ is a group of isometries of $(\widehat{M}, \mathrm{g})$ acting on $\widehat{M}$ freely and properly discontinuously. Also, $\mathcal{D}$ stands for the Olszak distribution (see the Introduction), with the same symbols $\mathcal{D}$ and $\mathcal{V} = \mathcal{D}^\perp$ denoting objects in $\widehat{M}$ and their projections onto M. According to Equations (6.3)–(6.7), on $\widehat{M}$ there exists a function t with a parallel gradient ∇t spanning $\mathcal{D}$, so that $\mathcal{D}^\perp = \operatorname{Ker} dt$, and the Ricci tensor of $(\widehat{M}, \mathrm{g})$ equals $(2 - n) f(t)\, dt \otimes dt$, where $f : \widehat{M} \to \mathbb{R}$ is locally a function of t. If a C^1 function $\chi : \widehat{M} \to \mathbb{R}$ is locally a function of t, one may define its t-derivative $\dot{\chi} : \widehat{M} \to \mathbb{R}$

by $d\chi = \dot{\chi}\,dt$. It easily follows – cf. Equation (6.8) – that

$$\begin{array}{l}\text{the action of } \Gamma \text{ multiplies } \nabla t \text{ by non-zero constants,}\\ \text{implying } \Gamma\text{-invariance of both } |f|^{1/2}\,dt \text{ and } |\dot{f}|^{1/3}\,dt.\end{array} \tag{2.1}$$

We now assume transversal orientability of $\mathcal{V}$ and proceed to summarize the steps, leading to the main conclusion of Theorem A: that, unless g is locally homogeneous, $\mathcal{V} = \mathcal{D}^{\perp}$ must be the vertical distribution of a fibration $M \to S^1$.

This is achieved by showing that $\mathcal{V}$, in addition to being a transversally orientable codimension-one foliation on the compact manifold M, also has what we call property (4.1): *for every compact leaf L of $\mathcal{V}$, the nearby leaves are either all non-compact – with the exception of L – or they are all compact and there exists a product-like $\mathcal{V}$-saturated tubular neighbourhood of L in M.* Furthermore, *some compact leaf of $\mathcal{V}$ then realizes the second case in the either–or clause of* Equation (4.1).

The reason why the main claim in Theorem A follows from the two conditions italicized above is that, even within the general context of foliations, with no reference to ECS geometry, these conditions imply that all leaves of $\mathcal{V}$ are compact, which forces M to be a bundle over the circle (Theorem 4.1).

Returning to our ECS case, we prove property (4.1) for $\mathcal{V} = \mathcal{D}^{\perp}$ by first observing, in § 10, that the rank-one Olszak distribution $\mathcal{D}$ on $\widehat{M}$ is spanned by the parallel gradient ∇t, and so the Levi-Civita connection of the compact ECS manifold (M, g) induces flat linear connections both in $\mathcal{D}$ (over M) and in the line bundle $\mathcal{D}_L^*$ over any leaf L of $\mathcal{D}^{\perp}$, dual to the line bundle $\mathcal{D}_L$ arising as the restriction of $\mathcal{D}$ to L. At the same time, $\mathcal{D}_L^*$ is canonically isomorphic to the normal bundle of L in M. Assuming compactness of L, we then show in Theorem 10.1 (see the next paragraph) that, under a suitable diffeomorphic identification Ψ of a neighbourhood U of L in M with a neighbourhood U' of the zero section L in the line bundle $\mathcal{D}_L^*$, the distribution $\mathcal{D}^{\perp}$ on U corresponds to the restriction to U' of the horizontal distribution of the flat linear connection in $\mathcal{D}_L^*$, mentioned above. By Equation (2.1), the holonomy group H_L of the latter connection consists of multiplications by positive real constants ('non-zero' in Equation (2.1) becoming 'positive' due to transversal orientability of $\mathcal{V} = \mathcal{D}^{\perp}$), and the dichotomy required in Equation (4.1) comes from the obvious fact that H_L is either infinite or trivial.

The identification $\Psi : U \to U'$, given by formula (10.2), uses a fixed smooth vector field on M, nowhere tangent to $\mathcal{D}^{\perp}$, to provide the curve segments forming the fibres of the tubular neighbourhood U, and along these segments, pulled back to $\widehat{M}$, we define Ψ so that it sends local t-levels to local sections parallel relative to the flat linear connection. Due to Equation (2.1), this construction is Γ-equivariant, and hence projects into M.

Finally, still in the ECS case, we establish the second option of property (4.1) for some compact leaf of $\mathcal{V} = \mathcal{D}^{\perp}$ by considering, in § 8, the vector space $\mathcal{F}$ of all continuous functions $\chi : \widehat{M} \to \mathrm{I\!R}$ such that the 1-form $\chi\,dt$ is closed (i.e., locally exact) and projectable onto M, with the linear operator $P : \mathcal{F} \to H^1(M, \mathrm{I\!R})$ sending χ to the cohomology class of the projected 1-form on M.

First, let $\dim \mathcal{F} = m < \infty$. By Equation (2.1), $|f|^{1/2}, |\dot{f}|^{1/3} \in \mathcal{F}$, and $\mathcal{F}$ is closed under the m-argument operation assigning $|\psi_1 \cdots \psi_m|^{1/m}$ to $\psi_1, \ldots, \psi_m$. Simple set-theoretical reasons then cause $|\dot{f}|^{1/3}$ to be a constant multiple of $|f|^{1/2}$ (see the proof of Theorem 9.1).

This makes f globally a function of t, of the form $f = \varepsilon(t-b)^{-2}$ with real constants $\varepsilon \neq 0$ and b, which combined with a result from algebraic geometry – Whitney's theorem – implies local homogeneity of g (a precise cross-reference being Lemma 7.1 invoked in the proof of Theorem 7.3).

On the other hand, when $\mathcal{F}$ is infinite-dimensional, P must be non-injective due to compactness of M and, given any $\chi \in \mathcal{F} \smallsetminus \{0\}$ with $P\chi = 0$, we see that $\chi\, \mathrm{d}t$ projects onto an exact 1-form on M, and hence onto $\mathrm{d}\mu$ for some (non-constant) C^1 function $\mu : M \to \mathrm{I\!R}$. As $\mathcal{D}^\perp = \operatorname{Ker} \mathrm{d}t$ on $\widehat{M}$, this μ is constant along $\mathcal{D}^\perp$.

Even though μ is only guaranteed to be of class C^1, Sard's theorem still applies (Remark 9.2), and any connected component L of a regular level of μ clearly realizes the second case of Equation (4.1).

3. Preliminaries

Manifolds are (usually) connected, pseudo-Riemannian metrics and vector fields are assumed C^∞-differentiable, while functions may be of lower regularity. The terms 'foliation' and (integrable) 'distribution' will be used interchangeably; by their 'leaves', we always mean *maximal connected integral manifolds*.

The following four facts will be used in § 4, 7 and 9.

Remark 3.1. Let points x, x' of codimension-one submanifolds L, L' of a manifold M be joined by an integral curve C_x of a complete C^∞ vector field v on M, which is transverse to L at x and to L' at x'. Then C_x belongs to a smooth variation $\widetilde{U} \ni y \mapsto C_y$ of integral curves of v, parametrized by a neighbourhood $\widetilde{U}$ of x in M such that, for some neighbourhoods U, U' of x in L and x' in L' with $U \subseteq \widetilde{U}$, each C_y joins $y \in \widetilde{U}$ to a point $y' \in U'$ and the resulting mapping $y \mapsto y'$ is a submersion $\widetilde{U} \to U'$, while its restriction to U is a diffeomorphism $U \to U'$. The word 'smooth' also applies here to the domain intervals of the integral curves.

Namely, let $\mathrm{I\!R} \times M \ni (\tau, y) \mapsto \phi(\tau, y) \in M$ be the flow of v, so that C_x is parametrized by $[0, \tau_*] \ni \tau \mapsto \phi(\tau, x)$. We now fix a neighbourhood $\widetilde{U}'$ of x' in M and a C^∞ function $\theta : \widetilde{U}' \to \mathrm{I\!R}$ with $L' \cap \widetilde{U}' = \theta^{-1}(0)$, having 0 as a regular value. The equation $\theta(\phi(\tau, y)) = 0$, imposed on $(\tau, y) \in \mathrm{I\!R} \times \widetilde{U}$, is satisfied when $(\tau, y) = (\tau_*, x)$. The implicit function theorem [12, p. 18] applied to this equation yields a C^∞ function $y \mapsto \tau(y)$, defined near x in M, with $\theta(\phi(\tau(y), y)) = 0$ and $\tau(x) = \tau_*$. Consequently, $y' = \phi(\tau(y), y)$ lies in L'. The submersion/diffeomorphism property of the mapping $\widetilde{U} \to U'$ or $U \to U'$ arising in this way, with $U = L \cap \widetilde{U}$, is immediate since $\widetilde{U} \to U'$ has a right inverse $U' \to U$ obtained by using the same principle for $-v$ instead of v.

Remark 3.2. For a (possibly disconnected) codimension-one submanifold L of a manifold M and the flow $\mathrm{I\!R} \times M \ni (\tau, x) \mapsto \phi(\tau, x) \in M$ of a complete C^∞ vector field on M, which is nowhere tangent to L, the restriction $\phi : \mathrm{I\!R} \times L \to M$ is locally diffeomorphic (a codimension-zero immersion). If, in addition, L is also compact, $\phi : (-\varepsilon, \varepsilon) \times L \to M$ is an embedding for all $\varepsilon > 0$ close to 0.

In fact, the first claim follows, since $\phi : \mathrm{I\!R} \times L \to M$ has smooth local inverses as an easy consequence of Remark 3.1. Next, if the embedding assertion failed, there would

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exist two termwise distinct sequences in $\mathbb{R} \times L$ having the $\mathbb{R}$ components tending to 0, with the same sequence of ϕ-values. Since L is now compact, so are $\phi([-\varepsilon, \varepsilon] \times L)$ for all $\varepsilon > 0$, and hence both sequences have subsequences converging to the same limit $x \in L$. Injectivity of our mapping on a neighbourhood of $(0, x)$ now leads to a contradiction.

By an m-argument operation Π in the following lemma, we mean any mapping associating, with any ordered m-tuple $(\psi_1, \ldots, \psi_m)$ of functions $X \to \mathbb{R}$, a function $\Pi(\psi_1, \ldots, \psi_m) : X \to \mathbb{R}$.

Lemma 3.3. *Let a vector space $\mathcal{F}$ of functions $X \to \mathbb{R}$ on a set X have a positive finite dimension m. If X is closed both under the absolute-value operation $\psi \mapsto |\psi|$ and under some m-argument operation Π sending any $\psi_1, \ldots, \psi_m$ to a non-negative function $\Pi(\psi_1, \ldots, \psi_m)$, which has the same zeros as the product $\psi_1, \ldots, \psi_m$, then there exists a proper subset X_0 of X, a partition $\{X_j\}_{j=1}^m$ of $X \smallsetminus X_0$ and a basis $\chi_1, \ldots, \chi_m$ of $\mathcal{F}$ such that $\chi_j > 0$ on X_j and $\chi_j = 0$ on $X \smallsetminus X_j$ for every $j = 1, \ldots, m$. In other words, some basis $\chi_1, \ldots, \chi_m$ of $\mathcal{F}$ consists of non-negative functions with pairwise disjoint supports, the word 'support' meaning here the complement of the zero set.*

Proof. Choose $x_1, \ldots, x_m \in X$ with the m evaluations δ_{x_j} forming a basis of the dual space $\mathcal{F}^*$, so that some $\sigma_1, \ldots, \sigma_m \in \mathcal{F}$ have $\sigma_j(x_i) < 0 < \sigma_j(x_j)$ whenever $i \neq j$. The positive-part and negative-part operations $()^{\pm} : \mathcal{F} \to \mathcal{F}$ are given by $\sigma^{\pm} = (|\sigma| \pm \sigma)/2$. We set $\chi_j = \Pi(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_m)$, where $\widetilde{\sigma}_j = \sigma_j^+$ and $\widetilde{\sigma}_i = \sigma_i^-$ for $i \neq j$. The supports $X_j = \chi_j^{-1}((0, \infty))$ are non-empty (as $x_j \in X_j$) and pairwise disjoint (since, if $i \neq j$, one has $\sigma_i < 0 < \sigma_j$ on X_j), which trivially implies linear independence of $\chi_1, \ldots, \chi_m$. Our claim thus follows if we declare X_0 to be $X \smallsetminus \bigcup_{j=1}^m X_j$, that is, the simultaneous zero set of $\chi_1, \ldots, \chi_m$. □

Remark 3.4. A locally diffeomorphic C^∞ mapping from a compact manifold into a connected one is necessarily surjective, and so the latter manifold must be compact as well. In fact, the image of the mapping is both compact and open. (We do not need the well-known stronger conclusion that the mapping is then also a covering projection.)

Remark 3.5. A non-empty proper subset K of $(0, \infty)$, different from $\{1\}$, and closed under the mappings $q \mapsto q^r$ for all $r \in \mathbb{Z}$, must have infinitely many connected components: if $q \in (1, \infty) \smallsetminus K$ is fixed, choosing a connected component K_r of each non-empty intersection $K \cap (q^r, q^{r+1})$, $r \in \mathbb{Z}$, we obtain, due to unboundedness of K, an infinite family of such components K_r.

4. Codimension-one foliations

The results of this section are the most crucial steps in proving Theorem A.

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Here is a property of a codimension-one foliation $\mathcal{V}$ on a manifold M:

$$\begin{aligned}&\text{every compact leaf } L \text{ of } \mathcal{V} \text{ has a neighbourhood } U \text{ in } M \text{ such that}\\ &\text{the leaves of } \mathcal{V} \text{ intersecting } U \smallsetminus L \text{ are either all non-compact, or}\\ &\text{they are all compact and some neighbourhood of } L \text{ in } M \text{ is a union}\\ &\text{of compact leaves of } \mathcal{V} \text{ and may be diffeomorphically identified}\\ &\text{with } \mathrm{I\!R} \times L \text{ so as to make } \mathcal{V} \text{ appear as the } L \text{ factor foliation.}\end{aligned} \tag{4.1}$$

The final clause of property (4.1) means that some diffeomorphism of a neighbourhood of L in M onto $\mathrm{I\!R} \times L$ pushes $\mathcal{V}$ forward onto the L *factor foliation* of $\mathrm{I\!R} \times L$, which has the leaves $\{\tau\} \times L$, for $\tau \in \mathrm{I\!R}$.

Theorem 4.1. *Let a transversally-orientable codimension-one foliation $\mathcal{V}$ on a compact manifold M satisfy condition (4.1). If, in addition, some compact leaf L of $\mathcal{V}$ realizes the second possibility in property (4.1), so as to have a product-like $\mathcal{V}$-saturated neighbourhood in M formed by compact leaves, then the leaves of $\mathcal{V}$ are all compact and they constitute the fibres of a bundle projection $M \to S^1$.*

Proof. Transversal orientability of $\mathcal{V}$ allows us to fix a C^∞ vector field v on M, nowhere tangent to $\mathcal{V}$. We also fix a compact leaf L of $\mathcal{V}$ satisfying the second option in the either–or clause of property (4.1). With $\mathrm{I\!R} \times M \ni (\tau, x) \mapsto \phi(\tau, x) \in M$ denoting the flow of v and z a given point of L, the 'second option' guarantees the existence of an open interval (a', b') containing 0 such that, for all $\tau \in (a', b')$, the leaf $L_{\phi(\tau,z)}$ of $\mathcal{V}$ passing through $\phi(\tau, z)$ is compact. Let (a, b) be the maximal open interval with this property (i.e., the union of all such intervals).

All $L_{\phi(\tau,z)}$ with $\tau \in (a, b)$ satisfy the second option in condition (4.1), as their compactness obviously precludes the first one. The resulting product structure of a neighbourhood of each $L_{\phi(\tau,z)}$ has three immediate consequences. First, the set $E = \{(\tau, y) \in (a, b) \times M : y \in L_{\phi(\tau,z)}\}$ is open in $(a, b) \times M$. Second, the mapping $E \ni (\tau, y) \mapsto \tau \in (a, b)$ constitutes a bundle projection. Finally,

$$\text{the mapping } E \ni (\tau, y) \mapsto y \in M \text{ is locally diffeomorphic.} \tag{4.2}$$

The pullback of v under this last mapping is a vector field on the total space E of the bundle in (b), transverse to the fibres, so that it spans a nonlinear connection (horizontal distribution) in E.

For any fixed $x \in L$, let $(a, b) \ni \tau \mapsto (\tau, \lambda(\tau, x)) \in E$ be the horizontal lift, relative to this connection, of the curve $\tau \mapsto \tau$ in the base manifold (a, b), with the initial value $(0, x)$ at $\tau = 0$. Such a horizontal lift clearly exists on some neighbourhood (c, d) of 0 in (a, b), as it constitutes a solution to an ordinary differential equation. Compactness of the fibres $E_\tau = \{\tau\} \times L_{\phi(\tau,z)}$ guarantees in turn that the maximal such (c, d) equals (a, b). Namely, if we had $c > a$, or $d < b$, a neighbourhood of E_c or E_d in E forming a union of horizontal curves would have the property that a horizontal lift entering it can be extended so as to reach E_c or E_d and beyond, contrary to maximality of (c, d).

Note that $(a,b) \ni \tau \mapsto \lambda(\tau,x)$, for any given $x \in L$, is a reparametrization of the integral curve $\tau \mapsto \phi(\tau,x)$ of v. Thus, for any $(\tau,x) \in (a,b) \times L$,

$$C_x = \{\lambda(\tau,x) : \tau \in (a,b)\} \text{ is an unparametrized integral curve of } v \tag{4.3}$$

passing through x for $\tau = 0$ and, for some C^∞ function $\sigma : (a,b) \times L \to \mathbb{R}$,

$$\lambda(\tau,x) = \phi(\sigma(\tau,x),x), \qquad \lambda(\tau,z) = \phi(\tau,z) \tag{4.4}$$

whenever $(\tau,x) \in (a,b) \times L$. Also,

$$\sigma(0,x) = x, \quad \sigma(\tau,z) = \tau, \quad \phi(\sigma(\tau,x),x) \in L_{\phi(\tau,z)}, \quad \mathrm{d}[\sigma(\tau,x)]/\mathrm{d}\tau > 0, \tag{4.5}$$

$\mathrm{d}[\sigma(\tau,x)]/\mathrm{d}\tau$ being positive as it is non-zero everywhere and $\sigma(\tau,z) = \tau$.

We now proceed to establish the following conclusion:

$$\text{there exist } c,d \text{ with } a < c < d < b \text{ and } L_{\phi(c,z)} = L_{\phi(d,z)}. \tag{4.6}$$

We will achieve this by deriving a contradiction from the assumption that

$$L_{\phi(\tau,z)}, \text{ for } \tau \in (a,b), \text{ are all mutually distinct.} \tag{4.7}$$

First, $\lambda(\tau,x) \in L_{\phi(\tau,z)}$ by Equations (4.4)–(4.5), so that Equations (4.7) and (4.3) give

$$C_x \cap L_{\phi(\tau,z)} = \{\lambda(\tau,x)\} \text{ for any } (\tau,x) \in (a,b) \times L. \tag{4.8}$$

Positivity of $\mathrm{d}[\sigma(\tau,x)]/\mathrm{d}\tau$ – see Equation (4.5) – implies, whenever $x \in L$, that $\sigma(\tau,x)$ has a limit $\sigma(b,x) \leq \infty$ as $\tau \to b$. As a further consequence of Equation (4.7), $\sigma(b,x) < \infty$ for every $x \in L$ (and so, by Equation (4.5), $b = \sigma(b,z) < \infty$). Otherwise, we may fix $x \in L$ and a strictly increasing sequence $\tau_j > a$ with $\tau_j \to b$ and $\sigma(\tau_j,x) \to \infty$ as $j \to \infty$. The sequence $\lambda(\tau_j,x) = \phi(\sigma(\tau_j,x),x)$ lies in the single integral curve C_x and, passing to a subsequence, we may assume that it converges to some point $y \in M$, which has a neighbourhood in M forming a union of 'short' unparametrized segments of integral curves of v intersecting a neighbourhood of y in the leaf L_y. Since $\lambda(\tau_j,x) \to y$ while its parameter $\sigma(\tau_j,x)$ increases towards an infinite limit and, due to Equations (4.7)–(4.8), $(a,b) \ni \tau \mapsto \lambda(\tau,x)$ is injective, C_x must contain infinitely many of the 'short' integral-curve segments and, as a result, intersect some leaves $L_{\phi(\tau,z)}$, with $\tau = \tau_j$, at infinitely many points, contrary to Equation (4.8). Thus, $\sigma(b,x) < \infty$ whenever $x \in L$.

Next, let us set $\lambda(b,x) = \phi(\sigma(b,x),x)$ and denote by $L_{b,x} = L_{\lambda(b,x)}$ the leaf of $\mathcal{V}$ through $\lambda(b,x)$. Now

$$\text{the mapping } L \ni x \mapsto L_{b,x} \text{ is locally constant,} \tag{4.9}$$

and $L \ni x \mapsto \sigma(b,x)$ is C^∞-differentiable. In fact, given $x \in L$, the integral curve (4.3) joins x to $\lambda(b,x) \in L_{b,x}$. Remark 3.1 applied to our L and $L' = L_{b,x}$ yields a smooth variation of integral curves C_y of v, for all points y from a neighbourhood U of

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x in L, with each C_y joining y to a point in $L_{b,x}$. Each of these C_y is parametrized by $\tau \mapsto \phi(\tau, x)$ with τ ranging over an interval $[0, \tau_*]$, where τ_* depends on y. As before, $\lambda(b, x)$ has a neighbourhood in M constituting a union of 'short' unparametrized integral-curve segments intersecting a neighbourhood of $\lambda(b, x)$ in $L_{b,x}$. One of these segments, the one passing through $\lambda(b, x)$, contains the portion $\{\lambda(\tau, x) : b-\varepsilon \le \tau \le b\}$ of C_x, with some $\varepsilon > 0$. The third equality of Equation (4.5) along with Equation (4.8) for $y \in L$ near x (rather than x itself) show that each nearby C_y similarly contains $\{\lambda(\tau, y) : b-\varepsilon \le \tau < b\}$ and hence also the limit $\lambda(b, x)$. However, by Equation (4.8), all $\lambda(\tau, y)$ lie in $L_{\phi(\tau,z)}$, just as $\lambda(\tau, x)$ does, and so $\lambda(b, y) \in L_{b,x}$, which gives $L_{b,y} = L_{b,x}$ and thus proves Equation (4.9). At the same time, Remark 3.1 yields smoothness of the mapping $L \ni y \mapsto \sigma(b, y)$.

Since the leaf L is connected, local constancy in assertion (4.9) amounts to constancy, so that $L_{\lambda(b,x)} = L_{\phi(b,z)}$ for all $x \in L$. Thus, for every $x \in L$, the integral curve $[0, \sigma(b, x)] \ni \tau \mapsto \phi(\tau, x) \in M$ joins x to the point $\lambda(b, x)$ in $L_{\phi(b,z)}$. Remark 3.1 also implies that the mapping $L \ni x \mapsto \lambda(b, x) \in L_{\phi(b,z)}$ is locally diffeomorphic. Compactness of L and Remark 3.4 now yield compactness of $L_{\phi(b,z)}$ along with the second possibility in Equation (4.1), for $L_{\phi(b,z)}$, since the first one is precluded by compactness of the nearby leaves $L_{\phi(\tau,z)}$ with $a < \tau < b$. This in turn also implies compactness of $L_{\phi(\tau,z)}$ for $\tau \ge b$, close to b, contradicting maximality of (a, b) and, consequently, proving Equation (4.6).

Let us now fix c, d with the property (4.6). The image, under the mapping in Equation (4.2), of the set $E_{[c,d]} = \{(\tau, y) \in [c, d] \times M : y \in L_{\phi(\tau,z)}\}$ is then clearly compact, but also open in M. In fact, it suffices to verify that $L_{\phi(c,z)}$ is contained in the interior of this image – which trivially follows since the image contains both kinds of sufficiently small one-sided neighbourhoods of $L_{\phi(c,z)} = L_{\phi(d,z)}$ in M. The image thus coincides with M, which proves that all leaves of $\mathcal{V}$ are compact.

Therefore, there exists a bundle projection $M \to S^1$ with the leaves of $\mathcal{V}$ serving as the fibres [15, Exercise 2.29(3)(i) on p. 49]. $\square$

5. Rank-one ECS metrics

Let the data $f, I, n, V, \langle\cdot,\cdot\rangle, A$ consist of

$$\begin{aligned}&\text{a non-constant } C^\infty \text{ function } f : I \to \mathrm{I\!R} \text{ on an open interval}\\&I \subseteq \mathrm{I\!R},\ \text{an integer } n \ge 4,\ \text{a real vector space } V \text{ of dimension}\\&n-2,\ \text{a pseudo-Euclidean inner product } \langle\cdot,\cdot\rangle \text{ on } V,\ \text{and a non-zero,}\\&\text{traceless, } \langle\cdot,\cdot\rangle\text{-self-adjoint linear endomorphism } A \text{ of } V.\end{aligned} \tag{5.1}$$

Following [17], one then defines a rank-one ECS metric [6, Theorem 4.1]

$$g = \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \delta \tag{5.2}$$

on the n-dimensional manifold $I \times \mathrm{I\!R} \times V$. The products of differentials denote here symmetric products, t, s are the Cartesian coordinates on $I \times \mathrm{I\!R}$ treated, with the aid of the projection $I \times \mathrm{I\!R} \times V \to I \times \mathrm{I\!R}$, as functions on $I \times \mathrm{I\!R} \times V$, and δ is the pullback to $I \times \mathrm{I\!R} \times V$ of the flat pseudo-Riemannian metric on V corresponding to the inner

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product $\langle\cdot,\cdot\rangle$. Finally, $\kappa : I \times \mathrm{I\!R} \times V \to \mathrm{I\!R}$ is the function given by

$$\kappa(t, s, x) = f(t)\langle x, x\rangle + \langle Ax, x\rangle. \tag{5.3}$$

If we let i, j range over $2, \ldots, n-1$, fix linear coordinates x^i on V and use them, along with $x^1 = t$ on I and $x^n = s/2$ on $\mathrm{I\!R}$, to form a global coordinate system on $I \times \mathrm{I\!R} \times V$, then the possibly non-zero components of the metric g and the Levi-Civita connection ∇ are [17, p. 93] those algebraically related to

$$\begin{aligned} &g_{11} = \kappa, \quad g_{1n} = g_{n1} = 1 \quad \text{and (constants) } g_{ij}, \\ &\Gamma^n_{11} = \partial_1\kappa/2, \quad \Gamma^i_{11} = -g^{ij}\partial_j\kappa/2, \quad \Gamma^n_{1i} = \partial_i\kappa/2. \end{aligned} \tag{5.4}$$

Remark 5.1. Conversely [6, Theorem 4.1], in any n-dimensional rank-one ECS manifold, some neighbourhood of any given point is isometric to an open subset of a manifold of type (5.2), where one has Equation (5.1) with one possible exception: f may be constant. More precisely, $\mathrm{d}f/\mathrm{d}t = 0$ precisely at those points at which the covariant derivative ∇R of the curvature tensor vanishes. Thus, if f is constant on a subinterval I' of I, the metric (5.2) will have $\nabla R = 0$ on $I' \times \mathrm{I\!R} \times V$.

In other words, a rank-one ECS manifold may have locally symmetric open submanifolds, and then the function f appearing in the local-coordinate form of [6, Theorem 4.1] is constant.

6. Assumptions and notation

Our (M, g) is always a rank-one ECS manifold, often compact, and $(\widehat{M}, \mathrm{g})$ denotes its pseudo-Riemannian universal covering space, which leads to

$$\text{the universal covering projection } \pi : \widehat{M} \to M = \widehat{M}/\Gamma, \tag{6.1}$$

$\Gamma \approx \pi_1 M$ being a group of isometries of $(\widehat{M}, \mathrm{g})$ acting on $\widehat{M}$ freely and properly discontinuously. Most of the time, the same symbols will stand for objects in $\widehat{M}$ and their projections in M, such as the metric g, and the (rank-one) Olszak distribution $\mathcal{D}$ described in the Introduction. In any rank-one ECS manifold, the distribution $\mathcal{D}$, being parallel, carries a linear connection induced by the Levi-Civita connection of g, and this induced connection is flat since, locally, $\mathcal{D}$ is spanned by the parallel gradient ∇t of the coordinate function t appearing in Equation (5.2) – see [8, the lines following formula (3.6)]. (As stated in Remark 5.1, Equation (5.2) is a general local description of all rank-one ECS metrics.) Simple connectivity of $\widehat{M}$ allows us to choose a global parallel vector field w on $\widehat{M}$ spanning $\mathcal{D}$, and then the 1-form $\mathrm{g}(w, \cdot)$, being parallel, is closed, and hence exact, so that we may also fix a C^∞ function $t : \widehat{M} \to \mathrm{I\!R}$ with $\mathrm{d}t = \mathrm{g}(w, \cdot)$. This determines t uniquely up to affine substitutions, that is, replacements of t by $qt + p$ with $q, p \in \mathrm{I\!R}$ and $q > 0$. Positivity of q reflects the fact that we always assume transversal orientability of the orthogonal complement $\mathcal{D}^\perp$, which can be achieved by replacing (M, g) with a twofold isometric covering and amounts to requiring that Γ be a subgroup of the group

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$\mathrm{Iso}^+(\widehat{M}, \mathrm{g})$ of all self-isometries of $(\widehat{M}, \mathrm{g})$ preserving a fixed transversal orientation of $\mathcal{D}^\perp$. Thus, for every $\gamma \in \mathrm{Iso}^+(\widehat{M}, \mathrm{g})$,

$$\text{there exist unique } q, p \in \mathrm{I\!R} \text{ with } q > 0 \text{ and } t \circ \gamma = qt + p. \tag{6.2}$$

The assignment $\gamma \mapsto q$ is a group homomorphism $\mathrm{Iso}^+(\widehat{M}, \mathrm{g}) \to (0, \infty)$, and we refer to q as the *q-image* of γ. Summarizing,

$$w = \nabla t \text{ is a parallel vector field on } \widehat{M}, \text{ spanning } \mathcal{D}, \text{ and } \mathrm{d}t = \mathrm{g}(w, \cdot\,). \tag{6.3}$$

As a consequence of Equation (6.3),

$$\mathcal{D}^\perp = \mathrm{Ker}\,\mathrm{d}t \text{ on } \widehat{M}. \tag{6.4}$$

Since Equation (5.2) remains unchanged when t and s are replaced by $qt + p$ and $q^{-1}s$,

$$t \text{ in Equation (5.2) can always be made equal to our } t \text{ chosen as above,} \tag{6.5}$$

cf. Remark 5.1. According to [17, p. 93], where the curvature tensor has the sign opposite to ours, the metric (5.2) has the Ricci tensor $\mathrm{Ric} = (2 - n) f(t)\, \mathrm{d}t \otimes \mathrm{d}t$, for $n = \dim M$. Therefore, by Remark 5.1, with our t chosen as above,

$$\text{on } (\widehat{M}, \mathrm{g}) \text{ one has } \mathrm{Ric} = (2 - n) f\, \mathrm{d}t \otimes \mathrm{d}t \tag{6.6}$$

for a unique (non-constant) function $f : \widehat{M} \to \mathrm{I\!R}$ and, again by Remark 5.1,

$$f \text{ is locally a function of } t. \tag{6.7}$$

Changing the notational convention so as to absorb the factor $2 - n$ into the function f does not seem to be a good option, since the inverse of that factor then would have to appear in Equation (5.3). Also, the word 'locally' in Equation (6.7) cannot in general be skipped: it means that f is constant on the connected components of the level sets of t, and the level sets themselves may be disconnected.

If $\chi : \widehat{M} \to \mathrm{I\!R}$ is of class C^1 and, locally, a function of t (which amounts to $\mathrm{d}\chi$ being a functional multiple of $\mathrm{d}t$), we define the derivative $\dot{\chi} : \widehat{M} \to \mathrm{I\!R}$ by $\mathrm{d}\chi = \dot{\chi}\, \mathrm{d}t$, so that, in terms of a local expression of g in Equations (5.2)–(5.4), $\dot{\chi} = \mathrm{d}\chi/\mathrm{d}t$. For f in Equations (6.6)–(6.7), any $\gamma \in \Gamma$, any C^1 function $\chi : \widehat{M} \to \mathrm{I\!R}$, which is locally a function of t, and $q, p, a \in \mathrm{I\!R}$ with $q > 0$,

$$\begin{aligned} &\text{if} \quad t \circ \gamma = qt + p, \quad \text{then} \quad \gamma^* \mathrm{d}t = q\, \mathrm{d}t \quad \text{and} \quad f \circ \gamma = q^{-2} f, \\ &\text{if} \quad \chi \circ \gamma = q^a \chi, \quad \text{then} \quad \dot{\chi} \circ \gamma = q^{a-1} \chi, \end{aligned} \tag{6.8}$$

which is clear from Equation (6.6) and the fact that the pullback of differential forms commutes with exterior differentiation. By Equations (6.3) and (6.8), for any $\gamma \in \mathrm{Iso}^+(\widehat{M}, \mathrm{g})$,

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$$\gamma \text{ pulls } w \text{ back to } qw, \text{ where } q \in (0, \infty) \text{ is the } q\text{-image of } \gamma. \tag{6.9}$$

Let us point out that the choices of t and w made above are convenient but not canonical, and so, rather than being preserved by isometries, t and w are transformed by them via Equations (6.8) and (6.9).

7. Local homogeneity

We adopt here the assumptions and notation of §6. First, note that

$$\text{in any ECS manifold, Ric} \neq 0 \text{ somewhere,} \tag{7.1}$$

or else the curvature and Weyl tensors would coincide, implying local symmetry.

Just as was the case with $\mathcal{D}$, the distribution $\mathcal{D}^{\perp}$ is parallel, and so it inherits a linear connection from the Levi-Civita connection ∇ of g. As $\mathcal{D} \subseteq \mathcal{D}^{\perp}$, a natural connection arises in the quotient bundle $\mathcal{E} = \mathcal{D}^{\perp}/\mathcal{D}$ over $\widehat{M}$, or M, and

$$\text{the connection induced by } \nabla \text{ in } \mathcal{E} = \mathcal{D}^{\perp}/\mathcal{D} \text{ is flat.} \tag{7.2}$$

This was established in [6, Lemma 2.2-f], but we need to justify it here, again, to draw additional conclusions. Namely, by Equations (5.4) and (6.3), the coordinate vector field $\partial_n = w = \nabla t$, spanning $\mathcal{D}$, is parallel, while ∂_n and ∂_i, for $i, j = 2, \ldots, n-1$, span $\mathcal{D}^{\perp}$. Now Equation (7.2) follows since, in Equation (5.4), $\Gamma_{1n}^{\bullet} = \Gamma_{1i}^{\bullet} = \Gamma_{in}^{\bullet} = \Gamma_{ij}^{\bullet} = \Gamma_{nn}^{\bullet} = \Gamma_{ni}^{\bullet} = 0$, with $\bullet$ denoting any index other than n, so that ∂_i, $i = 2, \ldots, n-1$, project onto parallel local trivializing sections of $\mathcal{E}$. As ∂_i are also constant vector fields on the space V in Equations (5.1)–(5.2), we may identify V with the space of parallel sections of $\mathcal{E}$, defined on the coordinate domain. The parallel vector-bundle morphism $[\mathcal{D}^*]^{\otimes 2} \to \mathcal{E}^{\otimes 2}$, over any rank-one ECS manifold, defined in [5, formula (6)] (where it was denoted by Φ, and built from the Weyl tensor W, cf. the Appendix), is easily seen to be valued in $\mathcal{E}^{\odot 2}$. Let the parallel section ξ of the line bundle $\mathcal{D}^*$ over $\widehat{M}$ be

$$\text{dual to } w \text{ in the sense that } \xi(w) = 1 \tag{7.3}$$

for the trivializing parallel section w of $\mathcal{D}$ appearing in Equation (6.3). Thus, over $\widehat{M}$, the morphism $[\mathcal{D}^*]^{\otimes 2} \to \mathcal{E}^{\otimes 2}$ sends $\xi \otimes \xi$ to a parallel section of $\mathcal{E}^{\odot 2}$, which may also be treated as a parallel vector-bundle endomorphism of $\mathcal{E}$, due to the presence in $\mathcal{E}$ of the parallel fibre metric induced by g. This last endomorphism acts on local parallel sections of $\mathcal{E}$ and, when the space of these sections is identified with V (see above), it becomes the endomorphism $A : V \to V$ of Equation (5.1). (One sees this comparing formula (6) in [5] with the expression, in [17, p. 93], for the only possibly non-zero essential component W_{1i1j} of the Weyl tensor W.) We will therefore use the symbols V and $A : V \to V$ for the space of parallel sections of $\mathcal{E}$ over $\widehat{M}$ and for the endomorphism just described. Note that, due to simple connectivity of $\widehat{M}$ and Equation (7.2), the bundle $\mathcal{E}$ is trivialized by global parallel sections.

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By Equations (6.2) and (6.9), any $\gamma \in \mathrm{Iso}^+(\widehat{M}, \mathrm{g})$ pulls w, or ξ appearing in Equation (7.3), back to qw or, respectively, $q^{-1}\xi$, for some $q \in (0, \infty)$. Since $A : V \to V$ as interpreted above was the image of $\xi \otimes \xi$ under a natural vector-bundle morphism, the isometry in question pulls A back to $q^{-2}A$, while it also acts as a linear isometry B of the space V of parallel sections. In terms of push-forwards rather than pullbacks,

$$BAB^{-1} = q^2 A. \tag{7.4}$$

Lemma 7.1. *Let A be a non-zero linear endomorphism of a pseudo-Euclidean vector space V. If, for some $q \in (0, \infty) \smallsetminus \{1\}$ there exists a linear isometry B of V satisfying Equation (7.4) then, with our fixed A, such B exists for every $q \in (0, \infty)$.*

Proof. For the space End V of all linear endomorphisms $V \to V$, the formula $\mathcal{J} = \{(q, B) \in \mathbb{R} \times \mathrm{End}\, V : (BB^*, BAB^*) = (\mathrm{Id}, q^2 A)\}$ defines an algebraic variety $\mathcal{J} \subseteq \mathbb{R} \times \mathrm{End}\, V$. By Whitney's classical result [18, Theorem 3], $\mathcal{J}$ has finitely many connected components, and hence so does the intersection $K = K' \cap (0, \infty)$ for the image K' of $\mathcal{J}$ under the projection $(q, B) \mapsto q$. As $q^r \in K$ whenever $q \in K$ and $r \in \mathbb{Z}$, Remark 3.5 now gives $K = (0, \infty)$. □

We have the following immediate consequence of the conclusion (7.1):

$$\text{local homogeneity of } (M, \mathrm{g}) \text{ or } (\widehat{M}, \mathrm{g}) \text{ implies that } \mathrm{Ric} \neq 0 \text{ everywhere.} \tag{7.5}$$

If $\mathrm{Ric} \neq 0$ everywhere, it follows from Equations (6.6)–(6.8) that

$$|f|^{1/2}\, \mathrm{d}t \text{ is a closed } \Gamma\text{-invariant } C^\infty \text{ 1-form without zeros on } \widehat{M}. \tag{7.6}$$

Closedness is here due to Equation (6.7). Thus, in view of Equation (6.3), the C^∞ vector field $w' = |f|^{1/2} w$ is π-projectable onto a vector field without zeros on $M = \widehat{M}/\Gamma$, also denoted by w' and, from Equations (6.3), (6.4) and (6.7),

$$\text{on } M, \text{ if } \mathrm{Ric} \neq 0 \text{ everywhere, } w' \text{ spans } \mathcal{D} \text{ and is parallel along } \mathcal{D}^\perp. \tag{7.7}$$

Lemma 7.2. *Let $(\widehat{M}, \mathrm{g})$ be the pseudo-Riemannian universal covering space of a compact rank-one ECS manifold. If $\widehat{M}$ admits a closed Γ-invariant C^∞ 1-form without zeros, which is a functional multiple of $\mathrm{d}t$, for the function $t : \widehat{M} \to \mathbb{R}$ introduced in §6, then the leaves of $\mathcal{D}^\perp$ in $\widehat{M}$ are the factor manifolds of a global product decomposition of $\widehat{M}$ and coincide with the levels of t.*

Proof. Up to a change of sign, the 1-form in question may be expressed as $\psi\, \mathrm{d}t$, where $\psi : \widehat{M} \to (0, \infty)$ is locally a function of t. Choosing $\chi : \widehat{M} \to \mathbb{R}$ with $\mathrm{d}\chi = \psi\, \mathrm{d}t$, and denoting by $D\chi$ the h-gradient of χ, where h is the π-pullback of a Riemannian metric on M, we see that $D\chi$ and $u = D\chi/[h(D\chi, D\chi)]$ are both non-zero everywhere and complete (the latter since $\mathrm{d}\chi = h(D\chi, \cdot)$, with $\mathrm{d}\chi$ and h both π-projectable onto M). By Equation (6.4), the (possibly disconnected) levels of χ are unions of leaves of $\mathcal{D}^\perp$.

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We now use Milnor's argument [14, p. 12] involving the flow $(\tau, x) \mapsto \phi(\tau, x)$ of u. Remark 3.2 may be applied to a fixed (possibly disconnected) level L of χ and the restriction of the flow ϕ to $\mathbb{R} \times L$. The restriction is not only locally diffeomorphic, but bijective as well, since $d_u\chi = 1$ and so the parameter τ of each integral curve differs from χ by a constant. Since $\mathbb{R} \times L$, diffeomorphic to $\widehat{M}$, must be connected, the assertion follows, the levels of t being the same as those of χ (and thus equal to the leaves of $\mathcal{D}^\perp$) due to positivity of $\psi = \mathrm{d}\chi/\mathrm{d}t$. $\square$

Theorem 7.3. *Let $\widehat{M}, \mathrm{g}, \Gamma, f, t, (\,)^{\dot{}} = \mathrm{d}/\mathrm{d}t$ and $M = \widehat{M}/\Gamma$ be as in §6. If M is compact, the following three conditions are mutually equivalent.*

(i) *$(\widehat{M}, \mathrm{g})$, or (M, g), is locally homogeneous.*
(ii) *$f \neq 0$ everywhere and $(|f|^{-1/2})^{\ddot{}} = 0$.*
(iii) *$(|f|^{-1/2})^{\ddot{}} = 0$ wherever $f \neq 0$.*

Proof. First, (i) yields $f \neq 0$ everywhere due to Equations (7.5) and (6.6). By Remark 5.1, g has, locally, the form (5.4) on some coordinate domain U, and then, as shown in [8, formula (3.3)], the existence of a Killing field v on U with an everywhere-non-zero component in the t coordinate direction (which follows from local homogeneity) gives $(|f|^{-1/2})^{\ddot{}} = 0$ on U, that is, (ii), while (ii) trivially leads to (iii). Assuming (iii), we now obtain (ii): $f \neq 0$ everywhere, since otherwise we could choose local coordinates as above around a boundary point of the zero set of f, with t ranging over an open interval I'. (By Equations (6.6) and (7.1), $f \neq 0$ somewhere.) A maximal open subinterval I'' of I' with $|f| > 0$ on I'' must equal I', or else I'' would have a finite endpoint lying in I', at which $|f|^{-1/2}$, being a linear function, would have a finite limit, contrary to vanishing of f at the endpoint due to maximality of I''. This contradiction shows that the zero set of f has no boundary points, proving (ii).

Suppose now that (ii) holds. According to Equation (7.6), the 1-form $|f|^{1/2}\,\mathrm{d}t$ satisfies the assumption, and hence the assertion, of Lemma 7.2, so that the levels of t are connected. We may thus drop the word 'locally' in Equation (6.7) and, with $I \subseteq \mathbb{R}$ denoting the range of t, view f as a function $f : I \to \mathbb{R}$ which, since $(|f|^{-1/2})^{\ddot{}} = 0$ and f is non-constant (Remark 5.1), has the form $f(t) = \varepsilon(t-b)^{-2}$ for some real constants $\varepsilon \neq 0$ and b. Thus, $I \subseteq (-\infty, b)$ or $I \subseteq (b, \infty)$. Now Equation (6.2) leads to a homomorphism $\Gamma \ni \gamma \mapsto (q, p) \in \mathrm{Aff}^+(\mathbb{R})$ into the group of increasing affine transformations of $\mathbb{R}$ mapping I onto itself. As $I \subseteq (-\infty, b)$ or $I \subseteq (b, \infty)$, the image of this homomorphism cannot contain a non-trivial translation and must be infinite (or else it would be trivial, causing t to descend to a function without critical points on the compact manifold $M = \widehat{M}/\Gamma$). Consequently, some $\gamma \in \Gamma$ has $q \in (0, \infty) \smallsetminus \{1\}$ in Equation (6.2). Then Equation (7.4) and Lemma 7.1 imply that

$$\begin{aligned}&\text{for every } q \in (0,\infty) \smallsetminus \{1\} \text{ there exists a linear}\\ &\langle\cdot,\cdot\rangle \text{ isometry } B : V \to V \text{ with } BAB^{-1} = q^2 A,\end{aligned} \tag{7.8}$$

where A, V and $\langle\cdot,\cdot\rangle$ are a part of the data (5.1) representing the metric g in suitable coordinates around any point of $\widehat{M}$. (See the lines preceding Equation (7.4).) Also, instead

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of $f(t) = \varepsilon(t-b)^{-2}$, we may require that f have the form

$$f(t) = \varepsilon t^{-2} \text{ for all } t \in I = (0,\infty), \tag{7.9}$$

as we are free to modify our choice of t via the affine substitution replacing t by $|t-b|$. (The equality $I = (0,\infty)$, rather than just the inclusion $I \subseteq (0,\infty)$, follows since an infinite group of increasing affine transformations maps I onto itself.)

Local homogeneity of g now follows: by Equation (7.9), the local-coordinate expression (5.4) of g amounts to that of the metric g^P in [2, top of p. 170], with our coordinate $x^1 = t$ denoted there by u^1. Our Equation (7.8) now becomes formula (10) in [2, p. 172] which, as stated there, guarantees homogeneity of the metric g^P on $I \times \mathrm{I\!R} \times V$, for $I = (0,\infty)$ and $V = \mathrm{I\!R}^{n-2}$.

We have thus shown that (ii) implies (i), completing the proof. □

Remark 7.4. A Lorentzian ECS manifold must have rank one (see the Introduction) and is never locally homogeneous: Equation (7.4) with $q \neq \pm 1$ implies nilpotency of A, while in the Lorentzian case, the fibre metric in $\mathcal{E}$ induced by g, which corresponds to $\langle\cdot,\cdot\rangle$ in (5.1), is positive definite, and so the non-zero $\langle\cdot,\cdot\rangle$-self-adjoint linear endomorphism $A : V \to V$, being diagonalizable, cannot be nilpotent.

8. Function spaces and the first real cohomology

We refer to a continuous 1-form ζ on a manifold M as *closed* if it is locally exact in the sense that every point of M has a neighbourhood U with $\zeta = d\psi$ on U for some C^1 function $\psi : U \to \mathrm{I\!R}$.

Due to the universal coefficient theorem [13, p. 378, Theorem 13.43], for any manifold M, one has an isomorphic identification $H^1(M, \mathrm{I\!R}) = \mathrm{Hom}(\pi_1 M, \mathrm{I\!R})$.

As in the case of smooth 1-forms, a closed continuous 1-form ζ on M thus gives rise to the cohomology class $[\zeta] \in H^1(M, \mathrm{I\!R})$, which, as a homomorphism $\pi_1 M \to \mathrm{I\!R}$, assigns to each homotopy class of piecewise C^1 loops at a fixed base point the integral of ζ over a representative loop. Clearly, $[\zeta] = 0$ if and only if $\zeta = \mathrm{d}\psi$ for some C^1 function $\psi : M \to \mathrm{I\!R}$.

The following lemma uses the notation $\pi, \widehat{M}, t, f, \Gamma$ introduced in §6.

Lemma 8.1. *Let $\mathcal{F}$ be the real vector space formed by all continuous functions $\chi : \widehat{M} \to \mathrm{I\!R}$ such that the 1-form $\chi\,\mathrm{d}t$ is closed and*

$$\chi \circ \gamma = q^{-1}\chi \text{ whenever } \gamma \in \Gamma \text{ and } q \in (0,\infty) \text{ is the } q \text{ image of } \gamma. \tag{8.1}$$

Then $|f|^{1/2} \in \mathcal{F}$ and $|\dot{f}|^{1/3} \in \mathcal{F}$. Furthermore, if $\chi \in \mathcal{F}$, then the 1-form $\chi\,\mathrm{d}t$ is Γ-invariant, and hence π-projectable onto a closed 1-form on $M = \widehat{M}/\Gamma$, also denoted by $\chi\,\mathrm{d}t$, which gives rise to a linear operator

$$\mathcal{F} \ni \chi \mapsto [\chi\,\mathrm{d}t] \in H^1(M, \mathrm{I\!R}). \tag{8.2}$$

This is a trivial consequence of Equation (6.8).

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9. Existence of special functions

Theorem 9.1. *Let (M, g) be a compact rank-one ECS manifold such that $\mathcal{D}^\perp$ is transversally orientable. If (M, g) is not locally homogeneous, then there exists a non-constant C^1 function $\mu : M \to \mathrm{I\!R}$, constant along $\mathcal{D}^\perp$.*

Proof. We will show that, for $\mathcal{F}$ defined in Lemma 8.1, either $\dim \mathcal{F} < \infty$ and (M, g) is locally homogeneous, or $\dim \mathcal{F} = \infty$ and such μ exists.

First, let $\dim \mathcal{F} < \infty$. In this case, even without using compactness of M, we can apply Lemma 3.3 to $X = M$, our vector space $\mathcal{F}$, and the m-argument operation Π sending $\psi_1, \ldots, \psi_m$ to the product of powers of the absolute values $|\psi_1|, \ldots, |\psi_m|$ with any fixed positive exponents adding up to 1 (for instance, the geometric mean of the absolute values). As $|f|^{1/2}, |\dot{f}|^{1/3}$ lie in $\mathcal{F}$ (see Lemma 8.1), on each of the open sets $X_j \subseteq M$ obtained in Lemma 3.3, $|f|^{1/2}$ and $|\dot{f}|^{1/3}$ are constant multiples of the jth function χ_j from the basis $\chi_1, \ldots, \chi_m$ of $\mathcal{F}$, which clearly gives $(|f|^{-1/2})^{\cdot\cdot} = 0$ wherever $f \neq 0$. (Note that $|f|^{1/2}$, being a linear combination of the functions $\chi_1, \ldots, \chi_m$, vanishes on their simultaneous zero set $X \smallsetminus \bigcup_{j=1}^m X_j$.) Local homogeneity is now immediate from Theorem 7.3.

Finally, suppose that $\dim \mathcal{F} = \infty$. Due to compactness of M, Equation (8.2) is non-injective and we may fix $\chi \in \mathcal{F} \smallsetminus \{0\}$ lying in its kernel, so that $\chi\, \mathrm{d}t$ treated as a 1-form on M equals $\mathrm{d}\mu$ for some (non-constant) C^1 function $\mu : M \to \mathrm{I\!R}$. By Equation (6.4), this completes the proof. $\square$

Remark 9.2. Sard's theorem [11, Theorem 1.3 on p. 69] normally applies to C^k mappings from an n-manifold into an m-manifold, with $k \geq \max(n - m + 1, 1)$, guaranteeing that the critical values form a set of zero measure. In our case, even though $\mu : M \to \mathrm{I\!R}$ is only of class C^1, and M can have any dimension $n \geq 4$, the same conclusion remains valid, and so, due to compactness of M,

$$\mu(M) \text{ contains an open interval consisting of regular values of } \mu, \tag{9.1}$$

$\mu(M)$ being the range of μ. In fact, Equation (6.3) gives $\mathrm{d}t \neq 0$ everywhere in $\widehat{M}$, and so M is covered by finitely many connected open sets U each of which can be diffeomorphically identified with an open set $\widehat{U} \subseteq \widehat{M}$ such that the levels of $t : \widehat{U} \to \mathrm{I\!R}$ are all connected. This turns μ restricted to U into a function of t, allowing us to use Sard's theorem as stated above for $k = n = m = 1$.

10. Holonomy of compact leaves

Let (M, g) be a rank-one ECS manifold. Its rank-one Olszak distribution $\mathcal{D}$ (see the Introduction), being parallel, carries a linear connection induced from the Levi-Civita connection of g, and this connection is flat since, locally, $\mathcal{D}$ is spanned by the parallel gradient ∇t, cf. Equation (6.3).

For any leaf L of $\mathcal{D}^\perp$, this flat connection gives rise to one in the line bundle $\mathcal{D}_L$ arising as the restriction of $\mathcal{D}$ to L and, consequently, also to what we call

$$\text{the flat connection in the line bundle } \mathcal{D}_L^* \text{ dual to } \mathcal{D}_L. \tag{10.1}$$

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Note that $\mathcal{D}_L^*$ is canonically isomorphic to the normal bundle of L in M, since $\mathcal{D}$ is isomorphic to the dual of $TM/\mathcal{D}^\perp$, via the isomorphism assigning to $v \in \mathcal{D}_x$, at any $x \in M$, the linear functional $\mathrm{g}_x(v,\cdot) : T_xM \to \mathrm{I\!R}$, vanishing on $\mathcal{D}_x^\perp$.

The next result remains valid without transversal orientability of $\mathcal{D}^\perp$ or compactness of M. We make these assumptions here just to simplify the discussion.

Theorem 10.1. *Let L be a compact leaf of $\mathcal{D}^\perp$ in a compact rank-one ECS manifold (M,g). If $\mathcal{D}^\perp$ is transversally orientable, then some neighbourhood U of L in M can be diffeomorphically identified with a neighbourhood U' of the zero section L in the line bundle $\mathcal{D}_L^*$ so as to make the distribution $\mathcal{D}^\perp$ on U correspond to the restriction to U' of the horizontal distribution of the flat connection in $\mathcal{D}_L^*$.*

Proof. Let $U = \phi((-\varepsilon,\varepsilon)\times L)$ for the flow ϕ of a fixed C^∞ vector field v on M, which is nowhere tangent to $\mathcal{D}^\perp$, and for ε chosen as in Remark 3.2. We define the required diffeomorphism $\Psi : U \to U'$ by declaring how $\Psi(\phi(\tau,x))$ depends on $(\tau,x)\in(-\varepsilon,\varepsilon)\times L$, cf. Remark 3.2. For any point $y \in \pi^{-1}(x)$, with π as in Equation (6.1), the flow $\hat\phi$ of the vector field $\hat v$ on $\widehat{M}$ projecting under π onto v, and the parallel section ξ of the line bundle $\mathcal{D}^*$ over $\widehat{M}$ appearing in Equation (7.3), we set

$$\Psi(\phi(\tau,x)) = [t(\hat\phi(\tau,y)) - t(y)]\xi_y \circ (d\pi_y)^{-1} \in \mathcal{D}_x^* \subseteq \mathcal{D}_L^*. \tag{10.2}$$

Here $\xi_y \in \mathcal{D}_y^*$ is a linear functional on $\mathcal{D}_y \subseteq T_y\widehat{M}$, and we compose it with the inverse of the isomorphism $d\pi_y : T_y\widehat{M} \to T_xM$, so that the result is a functional on $\mathcal{D}_x \subseteq T_xM$, which we then multiply by the scalar $t(\hat\phi(\tau,y)) - t(y)$. The fibres $\mathcal{D}_x$ of the line bundle $\mathcal{D}_L^*$ over L are treated here as pairwise disjoint subsets of the total space, also denoted by $\mathcal{D}_L^*$.

First, Equation (10.2) does not depend on the choice of $y \in \pi^{-1}(x)$. In fact, replacing y by $\gamma(y)$, with $\gamma \in \Gamma$, cf. Equation (6.1), we get the same right-hand side in Equation (10.2), since

$$t(\hat\phi(\tau,\gamma(y))) - t(\gamma(y)) = q[t(\hat\phi(\tau,y)) - t(y)],\ \xi_{\gamma(y)} \circ (d\pi_{\gamma(y)})^{-1} = q^{-1}\xi_y \circ (d\pi_y)^{-1}$$

for some real $q > 0$. Namely, as γ leaves $\hat v$ invariant, $\hat\phi(\tau,\gamma(y)) = \gamma(\hat\phi(\tau,y))$, and so the relation $t\circ\gamma = qt + p$ in Equation (6.2) yields the first equality displayed above. At the same time, Equations (6.9) and (7.3) give $q^{-1}\xi = \gamma^*\xi$, which, since $\pi = \pi\circ\gamma$, leads to $q^{-1}\xi_y = \xi_{\gamma(y)} \circ d\gamma_y$ and $d\pi_y = d\pi_{\gamma(y)} \circ d\gamma_y$, so that $q^{-1}\xi_y \circ (d\pi_y)^{-1} = \xi_{\gamma(y)} \circ (d\pi_{\gamma(y)})^{-1}$.

Smoothness of Ψ is obvious if one uses y depending on x via a local inverse of π. Also, Ψ is a fibre-preserving mapping from $U = \phi((-\varepsilon,\varepsilon)\times L)$ (viewed, when identified with $(-\varepsilon,\varepsilon)\times L$, as a trivial bundle with one-dimensional fibres) into the line bundle $\mathcal{D}_L^*$, operating as the identity on the base manifold L and constituting an embedding of each fibre separately: by Equation (6.4), $|\mathrm{d}[t(\hat\phi(\tau,y))]/\mathrm{d}t| > 0$. This makes Ψ itself an embedding.

Finally, with y near y_* smoothly depending on $x \in L$ near some fixed x_* as before, $t(y) = t(y_*)$ is constant, by Equation (6.4). Requiring that an assignment $y \mapsto \tau(y)$ give $t(\hat\phi(\tau(y),y)) = t_*$ for a constant t_* near $t(y_*)$, and so $y \mapsto \lambda(y) = \hat\phi(\tau(y),y)$ sends a

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neighbourhood of y_* in the leaf L_y into a single leaf, yields (Remark 3.1) a diffeomorphism λ between neighbourhoods of y_* and $\lambda(y_*)$ in the two leaves. By Equation (10.2), $\Psi(\pi(\lambda(y))) = [t_* - t(y_*)]\xi_y \circ (d\pi_y)^{-1}$ which, due to constancy of $t_* - t(y_*)$, is a parallel local section of $\mathcal{D}^*_L$. This completes the proof. □

The holonomy representation of Equation (10.1) assigns to each $x \in L$ and each homotopy class of piecewise C^1 loops at x in L a linear automorphism of the line $\mathcal{D}_x$, that is, the multiplication by some $q \in \mathbb{R} \smallsetminus \{0\}$. Since this is a multiple of the identity, the *holonomy group* H_L of the flat connection (10.1) in the line bundle $\mathcal{D}^*_L$ over L, formed by all these q, does not depend on x. Obviously,

$$\text{the holonomy group } H_L \text{ is either infinite, or trivial.} \tag{10.3}$$

Theorem 10.2. *In any rank-one ECS manifold (M, g) such that $\mathcal{D}^\perp$ is transversally orientable, condition (4.1) holds for $\mathcal{V} = \mathcal{D}^\perp$. In addition, the two possibilities named in Equation (4.1) correspond precisely to the two cases of Equation (10.3).*

Proof. Theorem 10.1 allows us to treat some neighbourhood U' of the zero section L in the line bundle $\mathcal{D}^*_L$ as a neighbourhood of L in M.

Any fixed fibre metric in the line bundle $\mathcal{D}^*_L$ gives rise to the norm function $N : \mathcal{D}^*_L \to [0, \infty)$ on the total space $\mathcal{D}^*_L$ and to radius ε interval sub-bundles $U_\varepsilon = N^{-1}([0, \varepsilon)) \subseteq \mathcal{D}^*_L$, where $\varepsilon > 0$. As L is compact, $U_\varepsilon \subseteq U'$ for ε near 0, which turns U_ε into a neighbourhood of L in M, and is expressed as $U_\varepsilon \subseteq M$.

First let H_L be trivial. The total space of $\mathcal{D}^*_L$ is thus the union of global parallel sections obtained from one such section (on which the norm function N has some maximum value $r > 0$) via multiplication by constants $a \neq 0$. If $|a| \in (0, \varepsilon/r)$, the resulting parallel sections are thus contained in $U_\varepsilon \subseteq M$, and hence constitute compact leaves of $\mathcal{D}^\perp$. This is the second option in the either–or clause of Equation (4.1).

Next, let H_L be infinite. We fix $q \in H_L \cap (0, 1)$ and, for any $x \in L$, choose a piecewise C^1 loop λ_x at x in L such that the parallel transport along λ_x in the line bundle $\mathcal{D}^*_L$ equals the multiplication by q in the line $\mathcal{D}^*_x$. For any $u \in \mathcal{D}^*_x \cap U_\varepsilon$, the horizontal lift $\widetilde{\lambda}_x$ of λ_x with the initial point u has the terminal point qu. Treating $\widetilde{\lambda}_x$ as a compact subset of the total space $\mathcal{D}^*_L$, on which the norm function N has some maximum value $r_x > 0$, we may form the union $Q_x = \bigcup_{i=k}^{\infty} q^i \widetilde{\lambda}_x$ for the least integer $k \geq 0$ with $q^k r_x < \varepsilon$. Thus, $Q_x \subseteq U_\varepsilon$ is a union of piecewise C^1 horizontal curves in the total space $\mathcal{D}^*_L$, joined end-to-end, and the same is true for aQ_x whenever $a \in (-1, 1) \smallsetminus \{0\}$ which, as $U_\varepsilon \subseteq M$, makes aQ_x a subset of a leaf $L_{(a)}$ of $\mathcal{D}^\perp$ with $L_{(a)} \subseteq U_\varepsilon \smallsetminus L$. Each $L_{(a)}$ contains the sequence $aq^i u$, with integers $i \geq k$, and $aq^i u$ converges as $i \to \infty$ to $x \in L$ (the zero vector in the line $\mathcal{D}^*_x$), so that the leaves $L_{(a)}$ are all non-compact. At the same time, $aq^k u \in L_{(a)}$, with our fixed k and all $a \in (-1, 1) \smallsetminus \{0\}$. Such $aq^k u$ form a neighbourhood of 0, with 0 itself removed, in the line $\mathcal{D}^*_x$. Thus, in the portion of the radius ε interval bundle U_ε over a neighbourhood of x in L on which $\mathcal{D}^*_L$ has a non-zero parallel section, the products of this parallel section by all real numbers sufficiently close to 0 fill a neighbourhood $U(x)$ of x in M, and the leaves of $\mathcal{D}^\perp$ intersecting $U(x) \smallsetminus L$ are all non-compact. Compactness of L allows us to choose finitely many $x \in L$ such that L is contained in the union U

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of the corresponding sets $U(x)$, which yields the first option in the either–or clause of Equation (4.1). $\square$

11. Proofs of Theorems A and B

To establish Theorem A, fix a compact rank-one ECS manifold (M, g). Passing to a two-fold isometric covering of (M, g), if necessary, we may also assume transversal orientability of $\mathcal{D}^\perp$. Theorem 10.2 now implies Equation (4.1) for $\mathcal{V} = \mathcal{D}^\perp$.

In addition, under the hypotheses of Theorem A, there exists a compact leaf L of $\mathcal{D}^\perp$ realizing the second possibility in Equation (4.1), the one where some product-like neighbourhood of L in M is a union of compact leaves of $\mathcal{D}^\perp$.

To prove this, note that either (M, g) is locally homogeneous and $\mathcal{D}^\perp$ has a compact leaf or (M, g) is not locally homogeneous.

In the latter case, Theorem 9.1 allows us to choose a non-constant C^1 function $\mu : M \to \mathrm{I\!R}$, constant along $\mathcal{D}^\perp$, and Remark 9.2 implies Equation (9.1). Letting $\mu(x)$ be a regular value of μ and fixing a one-dimensional submanifold of M containing x and transverse to $\mathcal{D}^\perp$, we see that for every point y in this submanifold lying sufficiently close to x, the connected component, containing y, of the μ-preimage of $\mu(y)$ is a compact leaf of $\mathcal{D}^\perp$. This causes L to satisfy the second option in Equation (4.1) by obviously precluding the first one.

Consider now the former case: local homogeneity along with the existence of a compact leaf L. By Equations (7.5) and (7.7), the line bundle $\mathcal{D}^*_L$ with the connection (10.1) has the global parallel section w' and, consequently, its holonomy group H_L is trivial. This leads again, via Theorem 10.2, to the second option in Equation (4.1).

Theorem A is now immediate from Theorem 4.1 applied to $\mathcal{V} = \mathcal{D}^\perp$.

As pointed out in the Introduction, Theorem B trivially follows from Theorem A except when $(\widehat{M}, \mathrm{g})$ is locally homogeneous. On the other hand, in the locally homogeneous case, Equations (7.5) and (7.6) imply that $|f|^{1/2}\,\mathrm{d}t$ is a closed Γ-invariant C^∞ 1-form without zeros on $\widehat{M}$. Theorem B is now obvious from Lemma 7.2.

12. Further consequences

Let $(\widehat{M}, \mathrm{g})$ be the pseudo-Riemannian universal covering space of a compact rank-one ECS manifold (M, g), with the Olszak distribution $\mathcal{D}$, and the universal covering projection $\pi : \widehat{M} \to M = \widehat{M}/\Gamma$, cf. Equation (6.1). As in §6, we assume transversal orientability of the orthogonal complement $\mathcal{D}^\perp$.

For future reference, we state here three consequences of the above assumptions.

First, *$\widehat{M}$ admits a smooth positive function $\psi : \widehat{M} \to (0, \infty)$ for which the 1-form $\psi\,\mathrm{d}t$ is both Γ-invariant* (in other words, π-projectable onto M) *and closed* (which amounts to requiring that ψ be, locally, a function of t).

In fact, in the locally homogeneous case (or, more generally, when $\mathrm{Ric} \neq 0$ everywhere), Equation (7.6) allows us to choose $\psi = |f|^{1/2}$.

If (M, g) is not locally homogeneous, $\mathcal{D}^\perp$, on M, is the vertical distribution of a fibration $M \to S^1$. (This is Theorem A.) We obtain our $\psi\,\mathrm{d}t$, or its opposite, by pulling back from S^1 to M a smooth 1-form without zeros.

Second, *the parallel vector field $w = \nabla t$ on $\widehat{M}$, spanning $\mathcal{D}$, which appears in* Equation (6.3), *is complete*. Namely, for $\psi : \widehat{M} \to (0, \infty)$ as above, Γ-invariance of $\psi\, \mathrm{d}t$ implies the same for ψw (since $w = \nabla t$, that is, $\mathrm{d}t = \mathsf{g}(w, \cdot)$). Completeness of ψw now follows due to its resulting π-projectability onto the compact manifold M. However, our ψ is, locally, a function of t and, by Equation (6.4), $\mathcal{D}^\perp = \operatorname{Ker} \mathrm{d}t$ on $\widehat{M}$. This makes ψ constant along every leaf of $\mathcal{D}^\perp$ and w tangent to the leaf. The integral curves of ψw are thus affine reparametrizations of those of w, and so w is complete as well.

Finally, *the levels of $t : \widehat{M} \to \mathbb{R}$ are all connected and coincide with the leaves of $\mathcal{D}^\perp$ in $\widehat{M}$*. Thus, if $\chi : \widehat{M} \to \mathbb{R}$ is locally a function of t, it must also be one globally, in the sense of being a composite of t with some function $t(\widehat{M}) \to \mathbb{R}$, which applies, in particular, to $\chi = f$ appearing in Equations (6.6)–(6.7).

To see this, use Theorem B: the leaves of $\mathcal{D}^\perp$ in $\widehat{M}$ are the factor manifolds of a global product decomposition of $\widehat{M}$, some open interval $I' \subseteq \mathbb{R}$ being the one-dimensional factor. The leaves are thus connected, and $t : \widehat{M} \to \mathbb{R}$, constant along them due to Equation (6.4), and having a non-zero parallel gradient – see Equation (6.3) – descends to a strictly monotone function $I' \to \mathbb{R}$, the levels of which thus are single points. This makes the levels of t equal to single leaves of $\mathcal{D}^\perp$, and hence connected.

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Appendix: The Lorentzian case

From now on we always assume transversal orientability of $\mathcal{D}^\perp$ for the Olszak distribution $\mathcal{D}$ of the ECS manifold (M, g) in question, which makes $\mathcal{D}$ a trivial line bundle over M.

The purpose of this section is to inform the reader how the proof of Theorem A differs from that for Lorentzian ECS manifolds in [5], and specifically what issues arise in higher signatures and how they are dealt with. We already explained in the Introduction why Theorem B of the paper [5] – the Lorentzian case of our Theorem A – does not require assuming rank one or excluding local homogeneity.

Let us begin by outlining the proof of [5, Theorem B], given in [5]. First – as we pointed out in the lines following Equation (6.1), and in Equation (7.2) – in any rank-one ECS manifold (M, g), the Levi-Civita connection ∇ induces natural flat connections both in the Olszak distribution $\mathcal{D}$ and in the quotient bundle $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$ over M, while – see [5, Sect.4] – the Weyl tensor W leads to a vector-bundle morphism $\Phi : (\mathcal{D}^*)^{\otimes 2} \to (\mathcal{E}^*)^{\otimes 2}$ with $\Phi_x(\lambda \otimes \lambda') : \mathcal{E}_x \times \mathcal{E}_x \to \mathrm{I\!R}$, for any $x \in M$ and $\lambda, \lambda' \in \mathcal{D}_x^*$, equal to the symmetric bilinear form sending the cosets $v + \mathcal{D}_x$ and $v' + \mathcal{D}_x$ of vectors $v, v' \in \mathcal{D}_x^\perp$ to $W_x(v, u, u', v')$, where $u, u' \in T_x M$ are any vectors with $\lambda = g_x(u, \cdot)$ and $\lambda' = g_x(u', \cdot)$ on $\mathcal{D}_x$. As observed in [5, Sect.4], the morphism Φ is well-defined, parallel relative to natural flat the connections in the bundles involved and nonzero (which makes it injective) at every point $x \in M$. So far, the metric signature of g was arbitrary: but when it is Lorentzian, the fibre metric in $\mathcal{E} = \mathcal{D}^\perp/\mathcal{D}$ induced by g is positive definite, leading, via injectivity of Φ, to a parallel fibre norm $||$ in the line bundle $\mathcal{D}$. Cf. [5, the end of Sect.4]. This proves [5, Theorem D]: for the Olszak distribution $\mathcal{D}$ of any Lorentzian ECS manifold, it follows (from transversal orientability of $\mathcal{D}^\perp$) that

(a) $\mathcal{D}$ is spanned by a global parallel section w, namely, one with $|w| = 1$.

We can now paraphrase the remainder of the proof of [5, Theorem B], which consists of the paragraph preceding [5, Remark 5.1], followed by [5, Lemma 1.2], and instead of assuming the Lorentzian signature, use only (a); the symbols ξ, u, ρ, ϕ and $\psi = \theta$ of [5]

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correspond to our $\mathrm{d}t, w, \mathrm{Ric}, (2-n)\dot{f}$ and $(2-n)f$. On the pseudo-Riemannian universal covering space $(\widehat{M}, \mathrm{g})$ of (M, g), the pullback of w with (a), still denoted by w, equals ∇t, and $M = \widehat{M}/\Gamma$, as in Equations (6.3) and (6.1). Milnor's argument [14, p. 12] is used in [5, proof of Lemma 1.2] to show that the levels of t in $\widehat{M}$ are connected: the word 'locally' in Equation (6.7) may be skipped. Due to Γ-invariance of $w = \nabla t$, one has Equation (6.2) with $q = 1$, and the first line of Equation (6.8) gives periodicity of f as a function of t (since a non-trivial element of Γ must thus have $p \neq 0$). Non-constancy of f, cf. Equation (6.6), implies that the values of p arising from Γ form a cyclic subgroup of $\mathrm{I\!R}$ with a unique generator $c > 0$. Thus,

(b) $t : \widehat{M} \to \mathrm{I\!R}$ descends to a bundle projection $M \to \mathrm{I\!R}/c\mathbb{Z} = S^1$,

which completes the proof of [5, Theorem B].

As we already pointed out, the above proof remains valid if the Lorentzian hypothesis is replaced by the weaker condition (a), transversal orientability of $\mathcal{D}^\perp$ being always assumed. However, a compact rank-one ECS manifold does not have to be *translational* in the sense of satisfying (a) – in other words, $q \neq 1$ may occur in Equation (6.8), and then $M = \widehat{M}/\Gamma$, with its ECS metric, is referred to as *dilational*. The existence of dilational-type compact rank-one ECS manifolds, including locally homogeneous ones, was established quite recently in [10, Theorems 6.1 and B.1] – and for them, instead of (b), one gets the different conclusion (c) appearing below.

Compared to the above derivation of (b), the path leading to our proof of Theorem A is rather indirect, and we outline it here by briefly summarizing, in the following five sentences, the five paragraphs (or two-paragraph parts) of Section 2 that begin with the phrases 'This is achieved', 'Returning', 'Finally', 'First' and 'On the other hand'. Namely, we start by showing that $\mathcal{D}^\perp$ satisfies condition (4.1). To derive Equation (4.1) for $\mathcal{D}^\perp$, we use the fact that – since in $\widehat{M}$ the leaves of $\mathcal{D}^\perp$ are the connected components of levels of t and $\mathrm{d}t$ is parallel – the leaf holonomy of any compact leaf of $\mathcal{D}^\perp$ may be diffeomorphically identified with its normal-connection holonomy. Next, we introduce a vector space $\mathcal{F}$ of functions $\widehat{M} \to \mathrm{I\!R}$, obviously having *either a finite or an infinite dimension*. The former italicized case implies local homogeneity. The latter one causes a natural linear operator $\mathcal{F} \to H^1(M, \mathrm{I\!R})$ to be non-injective and a non-trivial function lying in its kernel leads, via Sard's theorem, to a compact leaf of $\mathcal{D}^\perp$, satisfying the second option in the either–or clause of Equation (4.1), thus implying compactness of all leaves of $\mathcal{D}^\perp$ and the conclusion that they form the fibres of a bundle projection $M \to S^1$.

A final remark: our proof of Theorem A does eventually lead to a conclusion analogous to (b), but different from it – namely, in the dilational case, unless (M, g) is locally homogeneous, with $\mathcal{D}^\perp$ still assumed transversally orientable,

(c) $\tau : \widehat{M} \to \mathrm{I\!R}$ descends to a bundle projection $M \to \mathrm{I\!R}/\mathbb{Z} = S^1$.

Here our choice of t has been modified by an affine substitution so that the range $t(\widehat{M})$ equals $(0, \infty)$, and $\tau = (\log t)/(\log q)$ with suitably chosen $q \in (0, \infty) \smallsetminus \{1\}$. Cf. [9, the first proof paragraph of Theorem 2.3].

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