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Rank-one ECS manifolds of dilational type

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Abstract. We study ECS manifolds, that is, pseudo-Riemannian manifolds with parallel Weyl tensor which are neither conformally flat nor locally symmetric. Every ECS manifold has rank 1 or 2, the rank being the dimension of a distinguished null parallel distribution discovered by Olszak, and a rank-one ECS manifold may be called translational or dilational, depending on whether the holonomy group of a natural flat connection in the Olszak distribution is finite or infinite. Some such manifolds are in a natural sense generic, which refers to the algebraic structure of the Weyl tensor. Various examples of compact rank-one ECS manifolds are known: translational ones (both generic and nongeneric) in every dimension $n \geq 5$, as well as odd-dimensional nongeneric dilational ones, some of which are locally homogeneous. As we show, generic compact rank-one ECS manifolds must be translational or locally homogeneous, provided that they arise as isometric quotients of a specific class of explicitly constructed "model" manifolds. This result is relevant since the clause starting with "provided that" may be dropped: according to a theorem which we prove in another paper, the models just mentioned include the isometry types of the pseudo-Riemannian universal coverings of all generic compact rank-one ECS manifolds. Consequently, all generic compact rank-one ECS manifolds are translational.

1. Introduction

By *ECS manifolds* [3] one means those pseudo-Riemannian manifolds of dimensions $n \geq 4$ which have parallel Weyl tensor, but not for one of two obvious reasons: conformal flatness or local symmetry. Both their existence, for every $n \geq 4$, and indefiniteness of their metrics, are results of Roter [13, Corollary 3], [2, Theorem 2]. Their local structure has been completely described in [4].

The acronym "ECS" stands for *essentially conformally symmetric*. On every ECS manifold (M, g) there exists a naturally distinguished null parallel distribution $\mathcal{D}$, known as the *Olszak distribution* [12], [4, p. 119]. Its dimension, necessarily equal to 1 or 2, is referred to as the *rank* of (M, g). We call a rank-one ECS manifold

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translational, or *dilational*, when the holonomy group of the flat connection in $\mathcal{D}$, induced by the Levi-Civita connection, is finite or, respectively, infinite.

Examples of *compact* rank-one ECS manifolds are known [5, 6] to exist for every dimension $n \geq 5$. They are all geodesically complete, translational, and none of them is locally homogeneous. Quite recently [8] we constructed dilational type compact rank-one ECS manifolds, including locally-homogeneous ones, in all odd dimensions $n \geq 5$. It remains an open question whether a compact ECS manifold may have rank two, or be of dimension four.

In Section 4 we describe specific rank-one ECS *model manifolds* [13, p. 93], representing all dimensions $n \geq 4$ and all indefinite metric signatures. Some of them are *generic*, which refers to a self-adjoint linear endomorphism A of a pseudo-Euclidean vector space used in constructing the model manifold, and means that there are only finitely many linear isometries commuting with A. (In Remark 4.4 we point out that this genericity is an intrinsic geometric property of the metric, and not just a condition imposed on the construction.)

The dilational examples of [8], mentioned earlier, are all nongeneric, while among the translational ones in [5, 6], some are generic, and others are not, which raises an obvious question: Can a dilational type compact rank-one ECS manifold be generic? Theorem C of the present paper, combined with results of [9] mentioned below, answers this question in the negative:

$$\text{all generic compact rank-one ECS manifolds are translational.} \tag{1.1}$$

Here are some details. Since the Olszak distribution $\mathcal{D}$ is a *real line bundle* over the compact rank-one ECS manifold in question, the holonomy group K of the flat connection in $\mathcal{D}$ induced by the Levi-Civita connection is a countable multiplicative subgroup of $\mathbb{R} \smallsetminus \{0\}$ (see Section 2), and we will repeatedly refer to

$$\text{the positive holonomy group } K_+ = K \cap (0, \infty) \text{ of the flat connection in } \mathcal{D}. \tag{1.2}$$

Our first main result, established in Section 9, can be stated as follows.

Theorem A. *In a generic compact isometric quotient of a rank-one ECS model manifold, the group K_+ in (1.2) is not infinite cyclic.*

The next fact, which we prove at the very end of Section 3, holds in a more abstract setting, with no reference to either genericity or model manifolds.

Theorem B. *Given a compact rank-one ECS manifold (M, g), with K_+ in (1.2) not infinite cyclic, K_+ may be trivial, which makes (M, g) translational, or else K_+ is dense in $(0, \infty)$, and then (M, g) must be locally homogeneous.*

The third result trivially follows from Theorems A and B.

Theorem C. *Every generic compact isometric quotient of a rank-one ECS model manifold is either translational or locally homogeneous.*

In the locally-homogeneous case the group (1.2) *is dense in* $(0, \infty)$.

According to the final clause of our Theorem 3.3, compact locally homogeneous rank-one ECS manifolds are necessarily dilational. Theorem C thus has the following consequence.

Corollary D. *For a generic compact rank-one ECS manifold arising as an isometric quotient of a model manifold, the property of being dilational is equivalent to local homogeneity.*

Both Theorem C and Corollary D do not really require assuming that the manifold is an isometric quotient of a model. Namely, as we show in [9, Corollary D], *the pseudo-Riemannian universal covering of any generic compact rank-one ECS manifold is necessarily isometric to one of the model manifolds.*

Furthermore, according to another result of the same paper [9, Theorem E], a generic compact rank-one ECS manifold cannot be locally homogeneous. Thus, the final clause of our Theorem C is actually vacuous, and (1.1) follows. However, Theorem C, precisely as stated here, is a crucial step in the arguments of [9].

The paper is organized as follows. Sections 3 and 4, dealing with rank-one ECS manifolds, are followed by some material from linear algebra and algebraic number theory (genericity of nilpotent self-adjoint linear endomorphisms of pseudo-Euclidean spaces, and the cyclic root-group condition for $\mathrm{GL}(\mathbb{Z})$-polynomials), in Sections 5 and 7. Those two are separated by a section devoted to subspaces of certain spaces $\mathcal{E}$ of vector-valued functions on $(0, \infty)$, invariant under an operator $CT : \mathcal{E} \to \mathcal{E}$ which is relevant to the existence question for generic compact isometric quotients of rank-one ECS model manifolds. After Section 8, presenting a combinatorial argument (Theorem 8.1) needed to establish Theorem A, comes the final Section 9, where we prove Theorem A by contradiction, assuming that its hypotheses hold and yet K_+ in (1.2) is infinite cyclic. Lemma 9.2 provides the first important consequence of this assumption: the existence of a CT-invariant vector subspace, of the type discussed in Section 6, with the additional properties (9.5). Such a subspace necessarily satisfies further conditions, listed in Lemma 9.4, and leading – for reasons stated at the very end of Section 9 – to a combinatorial structure, the existence of which contradicts Theorem 8.1.

2. Preliminaries

Unless stated otherwise, manifolds and mappings are smooth, the former connected. The group $\mathrm{Aff}(\mathbb{R})$ of affine transformations $t \mapsto qt + p$ of $\mathbb{R}$, with real p and $q \neq 0$, has the index-two subgroup $\mathrm{Aff}^+(\mathbb{R}) = \{(q, p) \in \mathrm{Aff}(\mathbb{R}) : q > 0\}$, and

$$\text{nontrivial finite subgroups of } \mathrm{Aff}(\mathbb{R}) \text{ have the form } \{(1,0), (-1, 2c)\} \tag{2.1}$$

with any center $c \in \mathbb{R}$ of the reflection $(-1, 2c)$. In fact, the square of any (q, p) in such a subgroup Ξ lies in the intersection $\Xi \cap \mathrm{Aff}^+(\mathbb{R})$, which due to its finiteness must consist of translations, and hence be trivial.

Every $(q, p) \in \mathrm{Aff}^+(\mathbb{R}) \smallsetminus \{(1,0)\}$ is either a translation $(q = 1)$, or has a unique fixed point c (and then we call it a *dilation* with center c, since by choosing c as the new origin we turn c into 0 and (q, p) into $(q, 0)$). Now,

$$\begin{aligned}&\text{any Abelian subgroup of } \mathrm{Aff}^+(\mathbb{R}) \text{ consists of}\\ &\text{translations, or of dilations with a single center,}\end{aligned} \tag{2.2}$$

as two commuting self-mappings of a set preserve each other's fixed-point sets, and so in $\mathrm{Aff}^+(\mathbb{R}) \smallsetminus \{(1,0)\}$ two dilations with different centers cannot commute with each other or with a translation.

Lemma 2.1. *Let $(\cdot,\cdot)$ be a symmetric bilinear form in a real vector space. If a coset S of a $(\cdot,\cdot)$-null one-dimensional subspace Q is not contained in the $(\cdot,\cdot)$-orthogonal complement of Q, then S contains a unique $(\cdot,\cdot)$-null vector.*

In fact, S is parametrized by $t \mapsto x = v + tu$, where u spans Q and $(v, u) \neq 0$, so that $(x, x) = (v, v) + 2t(v, u)$ vanishes for a unique $t \in \mathbb{R}$.

Let a group Γ act on a manifold $\widehat{M}$ freely by diffeomorphisms. One calls the action of Γ *properly discontinuous* if there exists a locally diffeomorphic surjective mapping $\pi : \widehat{M} \to M$ onto some manifold M such that the π-preimages of points of M coincide with the orbits of the Γ action. One then refers to M as the *quotient* of $\widehat{M}$ under the action of Γ and writes $M = \widehat{M}/\Gamma$.

For $\pi, \widehat{M}, M, \Gamma$ as above and a flat linear connection ∇ in a vector bundle $\mathcal{Z}$ over M, let $\widehat{\mathcal{Z}}$ and $\widehat{\nabla}$ be the π-pullbacks of $\mathcal{Z}, \nabla$ to $\widehat{M}$. If $\widehat{M}$ is also simply connected, the vector space $\mathcal{F}$ of all $\widehat{\nabla}$-parallel sections of $\widehat{\mathcal{Z}}$ trivializes $\widehat{\mathcal{Z}}$, and a homomorphism $\Gamma \to \mathrm{GL}(\mathcal{F})$, known as the *holonomy representation* of ∇, assigns to $\gamma \in \Gamma$ the composite isomorphism

$$\mathcal{F} \to \widehat{\mathcal{Z}}_y \to \mathcal{Z}_x \to \widehat{\mathcal{Z}}_{\gamma(y)} \to \mathcal{F}, \tag{2.3}$$

described with the aid of any given $y \in \widehat{M}$ and $x = \pi(y)$, where the two middle arrows denote the identity automorphism of $\widehat{\mathcal{Z}}_y = \mathcal{Z}_x = \widehat{\mathcal{Z}}_{\gamma(y)}$, and the first/last one is the evaluation operator or its inverse. Note that (2.3) does not depend on the choice of $y \in \widehat{M}$, being locally (and hence globally) constant as a function of y. To see this, we choose connected neighborhoods $\widehat{U}$ of y in $\widehat{M}$ and U of $x = \pi(y)$ in M such that $\mathcal{Z}$ restricted to U is trivialized by the space $\mathcal{F}_U$ of its ∇-parallel sections and π maps $\widehat{U}$ diffeomorphically onto U. The isomorphism $\mathcal{F} \to \mathcal{F}_U$ arising as the restriction to $\widehat{U}$ followed by the "identity" identification via π then allows us to apply (2.3) to a fixed section from $\mathcal{F}$, using all $y \in \widehat{U}$ at once.

When $\mathcal{Z}$ is a real line bundle, with the multiplicative group $\mathrm{GL}(\mathcal{F}) = \mathbb{R} \smallsetminus \{0\}$,

$$\begin{aligned}&\text{for any } x \in M, \text{ the image of the holonomy representation}\\&\Gamma \to \mathbb{R} \smallsetminus \{0\} \text{ coincides with the holonomy group of } \nabla \text{ at } x,\end{aligned} \tag{2.4}$$

the latter meaning the group of the ∇-parallel transports $\mathcal{Z}_x \to \mathcal{Z}_x$ along all the loops at x. In fact, if (2.3) assigns to $\gamma \in \Gamma$ the multiplication by $q \in \mathbb{R} \smallsetminus \{0\}$ and $y \in \pi^{-1}(x)$ is fixed, the ∇-parallel transport Θ along the π-image of any curve joining y to $\gamma(y)$ in $\widehat{M}$ is $\mathcal{F} \leftarrow \widehat{\mathcal{Z}}_y \leftarrow \mathcal{Z}_x$ followed by $\mathrm{Id}_{\mathcal{F}}$ followed by $\mathcal{Z}_x \leftarrow \widehat{\mathcal{Z}}_{\gamma(y)} \leftarrow \mathcal{F}$, the reversed arrows representing the inverses of those in (2.3). Writing $\mathrm{Id}_{\mathcal{F}}$ as q^{-1} times (2.3), we get Θ equal to q^{-1} times the identity of $\mathcal{Z}_x$.

Lemma 2.2. *Suppose that $q \in \mathbb{R} \smallsetminus \{1, -1\}$ and a diffeomorphism $\gamma \in \mathrm{Diff}\,\widehat{M}$ of a manifold $\widehat{M}$ pushes a complete nontrivial vector field w forward onto qw. If $\mathbb{R} \ni t \mapsto \phi(t, \cdot) \in \mathrm{Diff}\,\widehat{M}$ denotes the flow of w, while a subgroup $\Gamma \subseteq \mathrm{Diff}\,\widehat{M}$ contains γ and $\phi(t, \cdot)$ for some $t \neq 0$, then the action of Γ on $\widehat{M}$ cannot be properly discontinuous.*

Proof. The kth iteration γ^k of γ, for $k \in \mathbb{Z}$, pushes w forward onto $q^k w$, giving $\gamma^k \circ \phi(t, \cdot) = \phi(q^k t, \cdot) \circ \gamma^k$ for all t and all $k \in \mathbb{Z}$, so that $\phi(q^k t, \cdot) \in \Gamma$ with our fixed t. Choosing $x \in \widehat{M}$ such that $w_x \neq 0$, and setting $\eta = \mathrm{sgn}(1 - |q|)$, we thus get a sequence $\phi(q^{\eta k} t, x)$ with mutually distinct terms when k is large, tending to x as $k \to \infty$, which obviously precludes proper discontinuity. ■

The conclusion of Lemma 2.2 remains valid when, instead of $\phi(qt, \cdot) \in \Gamma$ for some t, one assumes periodicity of the flow of w, while replacing the condition $\gamma, \phi(t, \cdot) \in \Gamma$ with just $\gamma \in \Gamma$ (and then using t equal to the period of the flow).

Remark 2.3. A submersion from a compact manifold into a connected manifold is a bundle projection, which is the compact case of Ehresmann's fibration theorem [10, Corollary 8.5.13].

3. Compact rank-one ECS manifolds

Throughout this section, $(\widehat{M}, \widehat{g})$ is the pseudo-Riemannian universal covering space of a compact rank-one ECS manifold (M, g) of dimension $n \geq 4$, defined as in the Introduction, $\mathcal{D}$ stands for the (one-dimensional, null, parallel) Olszak distribution on (M, g), and $\mathcal{D}^\perp$ for its orthogonal complement, while $\widehat{\mathcal{D}}$, $\widehat{\mathcal{D}}^\perp$ are the analogous distributions on $(\widehat{M}, \widehat{g})$. Thus, $M = \widehat{M}/\Gamma$ for a subgroup Γ of the full isometry group $\mathrm{Iso}(\widehat{M}, \widehat{g})$ isomorphic to the fundamental group of M, and acting on $\widehat{M}$ freely and properly discontinuously via deck transformations. The connection in $\widehat{\mathcal{D}}$ induced by the Levi-Civita connection $\widehat{\nabla}$ of $(\widehat{M}, \widehat{g})$ is always flat [7, Sect. 10]. Thus, due to simple connectivity of $\widehat{M}$,

$$\begin{aligned}&\widehat{\mathcal{D}} \text{ is spanned by the parallel gradient}\\ &\widehat{\nabla} t \text{ of a surjective function } t : \widehat{M} \to I\end{aligned} \tag{3.1}$$

onto an open interval $I \subseteq \mathbb{R}$ (which is the case even without assuming the existence of a compact quotient). The Olszak distribution being a local geometric invariant of the ECS metric in question [4, Sect. 2], (3.1) determines $\widehat{\nabla} t$ and t uniquely up to multiplication by nonzero constants and, respectively, affine substitutions, meaning replacements of t with $qt + p$, where $(q, p) \in \mathrm{Aff}(\mathbb{R})$ (for $\mathrm{Aff}(\mathbb{R})$ as in Section 2: $q, p \in \mathbb{R}$ and $q \neq 0$). Consequently, we have group homomorphisms

$$\mathrm{Iso}(\widehat{M}, \widehat{g}) \ni \gamma \mapsto (q, p) \in \mathrm{Aff}(\mathbb{R}), \tag{3.2a}$$

$$\mathrm{Iso}(\widehat{M}, \widehat{g}) \ni \gamma \mapsto q \in \mathbb{R} \smallsetminus \{0\}, \tag{3.2b}$$

characterized, for any $\gamma \in \mathrm{Iso}(\widehat{M}, \widehat{g})$, by $t \circ \gamma = qt + p$ and $\gamma^* \, dt = q \, dt$, that is,

$$(d\gamma)\widehat{\nabla} t = q^{-1} \widehat{\nabla} t. \tag{3.3}$$

According to [7, formula (6.4) and the end of Sect. 12],

$$\begin{aligned}&\widehat{\mathcal{D}}^\perp = \mathrm{Ker}\, dt, \text{ the levels of } t : \widehat{M} \to I \text{ are all}\\ &\text{connected and coincide with the leaves of } \widehat{\mathcal{D}}^\perp.\end{aligned} \tag{3.4}$$

Lemma 3.1. *The above hypotheses imply that the image of* Γ *under* (3.2a) *is infinite, while its image under* (3.2b) *coincides with the holonomy group of the flat connection in* $\mathcal{D}$.

Proof. The first image, if finite, would lie within some $\{(1, 0), (-1, 2c)\}$, cf. (2.1), causing $(t - c)^2$ to descend to a nonconstant function with at most one critical value on the compact manifold M. The second claim follows from (2.4): by (3.1) and (3.3), the action (2.3) of any $\gamma \in \Gamma$ on the parallel section $\widehat{\nabla} t$ spanning $\widehat{\mathcal{D}}$ equals the multiplication by the corresponding q^{-1}. Namely, the two middle arrows in (2.3) now are restrictions of $d\pi_y$ and $[d\pi_{\gamma(y)}]^{-1}$, so that their composite $\widehat{\mathcal{Z}}_y \to \mathcal{Z}_x \to \widehat{\mathcal{Z}}_{\gamma(y)}$ equals $d\gamma_y$. (From $\pi \circ \gamma = \pi$ we get $d\pi_{\gamma(y)} \circ d\gamma_y = d\pi_y$.) Thus, (2.3) takes $w = \widehat{\nabla} t$ first to w_y, then (two successive arrows) to $d\gamma_y w_y$ which – by (3.3) – equals $q^{-1} w_{\gamma(y)}$, the evaluation at $\gamma(y)$ of $q^{-1} w$. ∎

The translational/dilational dichotomy of the Introduction, meaning finiteness/infiniteness of the holonomy group of the flat connection in $\mathcal{D}$ induced by the Levi-Civita connection of g, can now be summarized in terms of the homomorphism (3.2b) restricted to Γ. Specifically, by Lemma 3.1, the two cases are

$$\text{translational: } |q| = 1 \text{ for each } \gamma \in \Gamma, \tag{3.5a}$$

$$\text{dilational: } |q| \neq 1 \text{ for some } \gamma \in \Gamma. \tag{3.5b}$$

Lemma 3.2. *With the assumptions and notations as above,*

(a) *the parallel vector field* $\widehat{\nabla} t$ *on* $\widehat{M}$*, spanning* $\widehat{\mathcal{D}}$*, is complete,*

(b) *in case* (3.5b)*,* $\phi(t,\cdot) \notin \Gamma$ *for all* $t \in \mathbb{R} \smallsetminus \{0\}$*,*

$\mathbb{R} \ni t \mapsto \phi(t,\cdot) \in \mathrm{Diff}\,\widehat{M}$ *being the flow of* $\widehat{\nabla} t$*.*

In fact, (a) appears in [7, the second italicized conclusion in Sect. 12], while (b) follows from (a) and Lemma 2.2 combined with (3.3).

The remainder of this section uses the assumptions preceding (3.1) along with

$$\text{transversal orientability of } \mathcal{D}^\perp \text{ which, by (3.4), reads } \Gamma \subseteq \mathrm{Iso}^+(\widehat{M},\hat{g}), \tag{3.6}$$

for the normal subgroup $\mathrm{Iso}^+(\widehat{M},\hat{g})$ forming the (3.2b)-preimage of $(0,\infty)$. This can always be achieved by replacing (M, g) (or, Γ) with a two-fold isometric covering (or, an index-two subgroup), and has an obvious consequence: the translational case then means precisely that the holonomy group is trivial.

Theorem 3.3. *In the dilational case* (3.5b)*, with* (3.6)*, the image of* Γ *under* (3.2a) *consists of dilations with a single center. The replacement of* t *in* (3.1) *by a suitable affine function of* t *then makes this center appear as* $t = 0$*, the interval* I *as* $(0,\infty)$*, and all* (q, p) *in the* (3.2a)*-image of* Γ *as having* $p = 0$*.*

Then the image of Γ *under* (3.2a)*, always an infinite multiplicative subgroup of* $(0,\infty)$*, must be infinite cyclic unless* $(\widehat{M},\hat{g})$ *is locally homogeneous. On the other hand,* (3.5b) *follows if one assumes local homogeneity of* $(\widehat{M},\hat{g})$*.*

Proof. As shown in [7, the beginning of Sect. 12], (3.6) implies the existence of a C^∞ function $\psi : \widehat{M} \to (0,\infty)$ such that the 1-form $\psi\, dt$ is π-projectable onto M (in other words, Γ-invariant), and closed. According to (3.4), the t-levels in $\widehat{M}$ are all connected, and so closedness of $\psi\, dt$ makes t *globally* a function of t, with $\psi = \chi \circ t$ for some C^∞ function $\chi : I \to (0,\infty)$. A fixed antiderivative ϕ of χ thus constitutes a strictly increasing C^∞ diffeomorphism $\phi : I \to J$ onto some open interval $J \subseteq \mathbb{R}$, while Γ-invariance of $d(\phi \circ t) = \psi\, dt$ means that Γ acts on $\phi \circ t$ by translations: $\phi \circ t \circ \gamma = \phi \circ t + a$ with constants $a \in \mathbb{R}$ depending on $\gamma \in \Gamma$. The mappings $t : \widehat{M} \to I$ and $\phi \circ t : \widehat{M} \to J$ are Γ-equivariant relative to Γ acting on I and J

via the homomorphisms (3.2a), restricted to Γ, and $\gamma \mapsto a$. As the diffeomorphism $\phi : I \to J$ makes the two mappings equivariantly equivalent, the two homomorphisms have the same kernel $\Sigma \subseteq \Gamma$, leading to an isomorphism $(q, p) \mapsto \gamma\Sigma \mapsto a$ between the images of the two homomorphisms. The former image must thus be Abelian (as that of $\gamma \mapsto a$ is a group of translations) and so, due to (3.5b) and (2.2), it consists of dilations with a single center. An affine substitution of t turns this center into 0, and elements of the (3.2a)-image of Γ into pairs (q, p) with $q > 0$ and $p = 0$. As a result, for our open interval I,

(i) 0 lies in the closure of I, but not in I itself.

The first claim of (i) is obvious: by (3.5b)–(3.6), for some $q \in (0, \infty) \smallsetminus \{1\}$,

(ii) I is closed under multiplications by powers of q with integer exponents.

To verify the second one, note that, as shown in [7, formulae (6.6)–(6.8)], some non-constant C^∞ function $f : \widehat{M} \to \mathbb{R}$ has

(iii) $f \circ \gamma = q^{-2} f$ for all $\gamma \in \Gamma$ and $(q, p) \in \mathrm{Aff}^+(\mathbb{R})$ with $t \circ \gamma = qt + p$.

This f is also globally a function of t [7, the end of Sect. 12]. Treating f, informally, as a function $I \to \mathbb{R}$, and noting that all (q, p) in the (3.2a)-image of Γ now have $q > 0$ and $p = 0$, we get $f(t) = q^2 f(qt)$ for such q, while these q, due to Lemma 3.1, form an infinite subgroup of $(0, \infty)$. Thus, $0 \notin I$, or else, fixing any t in the equality $f(t) = q^2 f(qt)$ and letting $q \to 0$, we would get $f(t) = 0$, even though f is nonconstant.

By (i) and (ii), I equals $(0, \infty)$ or $(-\infty, 0)$ and, replacing t with $-t$ if necessary, we get $I = (0, \infty)$, proving the first assertion of the theorem.

To establish the second one, suppose that the (3.2a)-image of Γ, infinite as a consequence of Lemma 3.1, is not cyclic. This makes the image dense in $(0, \infty)$, so that, from continuity of f, our equation $f(t) = q^2 f(qt)$ holds for *all* $t, q \in (0, \infty)$. Setting $t = 1$, we get $f(q) = f(1)/q^2$. The resulting linearity of the function $|f|^{-1/2}$ amounts – see [7, Theorem 7.3] – to local homogeneity of $(\widehat{M}, \widehat{g})$.

Finally, suppose that $(\widehat{M}, \widehat{g})$ is locally homogeneous. The preceding lines now yield linearity of $|f|^{-1/2}$, that is, $f(t) = f(1)/t^2$ for all $t \in (0, \infty)$, and so f is unbounded on $(0, \infty)$. This gives (3.5b), since (3.5a) would, by (iii), imply Γ-invariance of f, leading to its boundedness, as $M = \widehat{M}/\Gamma$ is compact. ∎

Proof of Theorem B. Due to Lemma 3.1 we may, without loss of generality, assume (3.5b) and (3.6). Our claim now follows from Theorem 3.3. ∎

4. The rank-one ECS model manifolds

In this section, we fix the data $f, I, n, V, \langle\cdot,\cdot\rangle, A$ consisting of

$$\begin{aligned}&\text{an integer } n \geq 4, \text{ a real vector space } V \text{ of dimension } n-2,\\ &\text{a pseudo-Euclidean inner product } \langle\cdot,\cdot\rangle \text{ on } V, \text{ a nonzero, traceless,}\\ &\langle\cdot,\cdot\rangle\text{-self-adjoint linear endomorphism } A \text{ of } V, \text{ and a nonconstant}\\ &C^\infty \text{ function } f : I \to \mathbb{R} \text{ on an open interval } I \subseteq \mathbb{R}.\end{aligned} \tag{4.1}$$

Treating $\langle\cdot,\cdot\rangle$ as a flat (constant) metric on V, and following [13], we define the simply connected n-dimensional pseudo-Riemannian manifold

$$(\widehat{M}, \hat{g}) = (I \times \mathbb{R} \times V, \kappa\, dt^2 + dt\, ds + \langle\cdot,\cdot\rangle), \tag{4.2}$$

where t, s are the Cartesian coordinates on $I \times \mathbb{R}$, we identify dt, ds and $\langle\cdot,\cdot\rangle$ with their pullbacks to $\widehat{M}$, and the function $\kappa : \widehat{M} \to \mathbb{R}$ is defined by

$$\kappa(t, s, v) = f(t)\langle v, v\rangle + \langle Av, v\rangle.$$

Thus, translations in the s direction are isometries of $(\widehat{M}, \hat{g})$.

It is well known [4, Theorem 4.1] that (4.2) is a rank-one ECS manifold. To describe its isometry group, we need two ingredients. The first is

$$\begin{aligned}&\text{the subgroup S of } \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V) \text{ formed by}\\ &\text{triples } (q, p, C) \text{ such that } CAC^{-1} = q^2 A, \text{ while}\\ &qt + p \in I \text{ and } f(t) = q^2 f(qt + p) \text{ for all } t \in I,\end{aligned} \tag{4.3}$$

$\mathrm{O}(V)$ being the group of linear $\langle\cdot,\cdot\rangle$-isometries $C : V \to V$.

The second ingredient is the $2(n-2)$-dimensional real

$$\begin{aligned}&\text{vector space } \mathcal{E} \text{ of all solutions } u : I \to V \text{ to the second-order ordinary}\\ &\text{differential equation } \ddot{u} = fu + Au, \text{ carrying the symplectic}\\ &\text{form } \Omega : \mathcal{E} \times \mathcal{E} \to \mathbb{R} \text{ given by } \Omega(u^+, u^-) = \langle \dot{u}^+, u^-\rangle - \langle u^+, \dot{u}^-\rangle.\end{aligned} \tag{4.4}$$

Note that $q, (q, p), C$ all depend homomorphically on the triple $\sigma = (q, p, C)$, and S acts from the left on $C^\infty(I, V)$ via

$$[\sigma u](t) = Cu((t - p)/q), \tag{4.5}$$

while the operator $u \mapsto \sigma u$ leaves the solution space $\mathcal{E}$ invariant.

Theorem 4.1. *For* $(\widehat{M}, \hat{g})$ *and* S *as in* (4.1)–(4.3)*, the full isometry group* $\mathrm{Iso}(\widehat{M}, \hat{g})$ *is isomorphic to the set* $\mathrm{G} = \mathrm{S} \times \mathbb{R} \times \mathcal{E} \subseteq \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V) \times \mathbb{R} \times \mathcal{E}$ *endowed with the group operation*

$$\begin{aligned}(q, p, C, r, u)&(\hat{q}, \hat{p}, \hat{C}, \hat{r}, \hat{u})\\ &= (q\hat{q}, q\hat{p} + p, C\hat{C}, -\Omega(u, (q, p, C)\hat{u}) + r + \hat{r}/q, (q, p, C)\hat{u} + u) \quad (4.6)\end{aligned}$$

or, in the notation of (4.4)–(4.5)*, with* $\sigma = (q, p, C)$*,*

$$(\sigma, r, u)(\hat{\sigma}, \hat{r}, \hat{u}) = (\sigma\hat{\sigma}, \Omega(\sigma\hat{u}, u) + r + \hat{r}/q, \sigma\hat{u} + u).$$

The required isomorphism amounts to the following left action on $\widehat{M}$ *by the group* G *with the operation* (4.6)*:*

$$\begin{aligned}(q, p, C, r, u)&(t, s, v)\\ &= (qt + p, -\langle\dot{u}(qt + p), 2Cv + u(qt + p)\rangle + r + s/q, Cv + u(qt + p)).\end{aligned} \quad (4.7)$$

Proof. This is precisely [1, Theorem 2], plus [1, p. 24, formula (22)] describing the group operation, except for the fact that [1] assumes real-analyticity of f along with $I = \mathbb{R}$, and it is because of these assumptions that $|q| = 1$ whenever $(q, p, C) \in \mathrm{S}$, cf. (4.3). If one ignores the last conclusion and the assumptions that led to it, the proof in [1] repeated almost verbatim in our case yields our assertion. However, the resulting right-hand side in (4.7) is not ours, but instead reads

$$(qt + p, -\langle\dot{u}(t), 2Cv + u(t)\rangle + r + s/q, Cv + u(t))$$

due to the fact that u, instead of $\mathcal{E}$, now lies in the solution space $\mathcal{E}_q$ of the q-dependent equation $\ddot{u} = fu + q^2Au$. We reconcile both versions by observing that the replacement of u with $t \mapsto u(qt + p)$ defines an isomorphism $\mathcal{E}_q \to \mathcal{E}$.

The notation of [1] differs from ours: our $q, p, C, r, u, t, s, v, V, f, \kappa, A, \langle\cdot,\cdot\rangle, \Omega$ correspond to $\varepsilon, T, H^{\lambda}_{\mu}, r, C^{\lambda}, x^1, 2x^n, \mathbb{R}^{n-2}, A, \varphi, a_{\lambda\mu}, k_{\lambda\mu}, 2\omega$ in [1]. ■

By (4.6), $\mathrm{G} \ni \gamma = (\sigma, r, u) \mapsto \sigma \in \mathrm{S}$ is a group homomorphism, leading to

$$\text{the normal subgroup } \mathrm{H} = \{(1, 0, \mathrm{Id})\} \times \mathbb{R} \times \mathcal{E} \text{ of G.} \quad (4.8)$$

The group operation (4.6) restricted to H becomes

(a) $(1, 0, \mathrm{Id}, \hat{r}, \hat{u})(1, 0, \mathrm{Id}, r, u) = (1, 0, \mathrm{Id}, \Omega(u, \hat{u}) + \hat{r} + r, \hat{u} + u)$,

and the action (4.7) of H on $\widehat{M}$ is explicitly given by

(b) $(1, 0, \mathrm{Id}, r, u)(t, s, v) = (t, -\langle\dot{u}(t), 2v + u(t)\rangle + r + s, v + u(t))$.

Treating the vector space $\mathcal{E}$ as an Abelian group we get, from (a), an obvious

(c) group homomorphism $\mathrm{H} \ni (1, 0, \mathrm{Id}, r, u) \mapsto u \in \mathcal{E}$.

Also, as stated in [7, formula (6.5)], with a suitable affine substitution,

(d) t in (4.2) can always be made equal to t chosen as in (3.1),

so that, in view of (4.6)–(4.7),

(e) the homomorphism $\mathrm{G} \ni (q, p, C, r, u) \mapsto (q, p)$ coincides with (3.2a).

Furthermore, according to [6, the lines following formula (3.6)], $\widehat{\nabla} t$ in (3.1) equals twice the coordinate vector field in the s coordinate direction, and so

(f) the flow of $\widehat{\nabla} t$ on $\widehat{M}$ is given by $\mathbb{R} \ni r \mapsto (1, 0, \mathrm{Id}, 2r, 0) \in \mathrm{H} \subseteq \mathrm{G}$.

In other words, cf. (b), the flow acts on $\widehat{M}$ via $(\tau, (t, s, v)) \mapsto (t, s + 2\tau, v)$. Also,

(g) $\sigma^*\Omega = q^{-1}\Omega$, as an obvious consequence of (4.4)–(4.5).

The subgroup H (canonically identified with $\mathbb{R} \times \mathcal{E}$) acts both on the product $I \times \mathbb{R} \times \mathcal{E}$, by left H-translations of the $\mathrm{H} \approx \mathbb{R} \times \mathcal{E}$ factor, and on $\widehat{M}$, via (b). The following mapping is H-equivariant for these two actions:

$$I \times \mathbb{R} \times \mathcal{E} \ni (t, z, u) \mapsto (t, s, v) = (t, z - \langle \dot{u}(t), u(t)\rangle, u(t)) \in \widehat{M} = I \times \mathbb{R} \times V \quad (4.9)$$

as one easily verifies using (a), (b) and the definition of Ω in (4.4).

Remark 4.2. It is useful to note that $(\sigma, r, u)^{-1} = (\sigma^{-1}, -qr, -\sigma^{-1}u)$ in G, which yields, for $(\sigma, \hat{r}, \hat{u}) = (q, p, C, \hat{r}, \hat{u}) \in \mathrm{G}$ and $(1, 0, \mathrm{Id}, r, u) \in \mathrm{H}$, the equality

$$(\sigma, \hat{r}, \hat{u})(1, 0, \mathrm{Id}, r, u)(\sigma, \hat{r}, \hat{u})^{-1} = (1, 0, \mathrm{Id}, 2\Omega(\sigma u, \hat{u}) + r/q, \sigma u).$$

Remark 4.3. Nondegeneracy of Ω gives $\dim \mathcal{L}' = \dim \mathcal{E} - \dim \mathcal{L}$ for any vector subspace $\mathcal{L} \subset \mathcal{E}$ and its Ω-orthogonal complement $\mathcal{L}'$. Thus, $2 \dim \mathcal{L} \leq \dim \mathcal{E}$ whenever $\mathcal{L}$ is isotropic in the sense that $\Omega(u, u') = 0$ for all $u, u' \in \mathcal{L}$.

Remark 4.4. We refer to a rank-one ECS model manifold (4.2) as *generic* when so is A in (4.1), by which we mean that A commutes with only finitely many linear $\langle\cdot,\cdot\rangle$-isometries of V. Genericity of A in (4.1) is an intrinsic property of the metric $\hat{g}$, rather than just a condition imposed on the construction (4.1)–(4.2): as stated in [7, the paragraph following formula (7.3)], the algebraic type of the pair $\langle\cdot,\cdot\rangle$, A, up to rescaling of A, can be explicitly defined in terms of $\hat{g}$ and its Weyl tensor.

Remark 4.5. The relation $CAC^{-1} = q^2 A$ in (4.3) with $|q| \neq 1$ implies nilpotency of A, as all complex characteristic roots of A then obviously equal 0.

5. Generic self-adjoint nilpotent endomorphisms

Throughout this section, V denotes a real vector space of dimension $m \geq 2$.

Given a pseudo-Euclidean inner product $\langle\cdot,\cdot\rangle$ on V, we refer to $\langle\cdot,\cdot\rangle$ as *semi-neutral* if its positive and negative indices differ by at most one, and – following the terminology of Remark 4.4 – call a $\langle\cdot,\cdot\rangle$-self-adjoint endomorphism of V *generic* when it commutes with only a finite number of linear $\langle\cdot,\cdot\rangle$-isometries of V. As we show below (Remark 5.4), for $\langle\cdot,\cdot\rangle$-self-adjoint endomorphisms A of V which are nilpotent, genericity is equivalent to having $A^{m-1} \neq 0$ (while $A^m = 0$).

Nilpotent endomorphisms are relevant to our discussion due to Remark 4.5. Generally, for any endomorphism A of our vector space V and any integer $j \geq 1$, the inclusions $\operatorname{Ker} A^{j-1} \subseteq \operatorname{Ker} A^j$ lead to the quotient spaces $\operatorname{Ker} A^j / \operatorname{Ker} A^{j-1}$, and then A obviously descends to *injective linear operators*

$$A : \operatorname{Ker} A^{j+1} / \operatorname{Ker} A^j \to \operatorname{Ker} A^j / \operatorname{Ker} A^{j-1}, \quad j = 1, \ldots, m-1. \tag{5.1}$$

Setting $d_j = \dim[\operatorname{Ker} A^j / \operatorname{Ker} A^{j-1}]$ we thus have $d_j \geq d_{j+1}$ and, if A is nilpotent,

$$d_1 \geq \cdots \geq d_m \geq 0 \quad \text{and} \quad \dim V = d_1 + \cdots + d_m, \tag{5.2}$$

while, whenever $j = 0, \ldots, m$,

$$\dim \operatorname{Ker} A^j = d_1 + \cdots + d_j, \quad \operatorname{rank} A^j = d_{j+1} + \cdots + d_m. \tag{5.3}$$

Thus, $d_m \geq 1$ in the case where A is nilpotent and $A^{m-1} \neq 0$, and then, by (5.2),

$$d_1 = \cdots = d_m = 1 \text{ and (5.1) is an isomorphism for } j = 1, \ldots, m-1. \tag{5.4}$$

Theorem 5.1. *Let a $\langle\cdot,\cdot\rangle$-self-adjoint nilpotent endomorphism A of an m-dimensional pseudo-Euclidean vector space V have $A^{m-1} \neq 0$. Then the inner product $\langle\cdot,\cdot\rangle$ is semi-neutral and there exist exactly two bases $e_1, \ldots, e_m$ of V, differing by an overall sign change, as well as a unique sign factor $\varepsilon = \pm 1$, such that $Ae_j = e_{j-1}$ and $\langle e_i, e_k \rangle = \varepsilon\delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$, where $e_0 = 0$ and $k = m + 1 - j$. Equivalently, the matrix representing A or, respectively, $\langle\cdot,\cdot\rangle$ in our basis has zero entries except those immediately above the main diagonal, all equal to 1 or, respectively, except those on the main antidiagonal, all equal to ε.*

Conversely, if A and $\langle\cdot,\cdot\rangle$ are of the above form in some basis $e_1, \ldots, e_m$ of V, then A is $\langle\cdot,\cdot\rangle$-self-adjoint, nilpotent and $A^{m-1} \neq 0$.

Proof. For $j = 0, \ldots, m$, the symmetric bilinear form $(v, v') \mapsto \langle A^j v, v' \rangle$ on V, briefly denoted by $\langle A^j \cdot, \cdot \rangle$, and the subspaces $V_j = A^j(V) \subseteq V$, we have

(a) $\dim V_j = m - j$ and $V_j \subseteq V_{j-1}$ if $j \geq 1$,

(b) $\langle A^{m-j} \cdot, \cdot \rangle$ descends to the j-dimensional quotient space V/V_j,

with (a) being obvious from (5.3)–(5.4), and (b) from

(c) self-adjointness of A along with the relation $A^m = 0$.

As $A^{m-1} \neq 0$, the form resulting from (b) on the line V/V_1 is nonzero, and hence positive or negative definite, which proves the existence and uniqueness of a sign factor $\varepsilon \in \{1,-1\}$ such that $\langle A^{m-1}v, v\rangle = \varepsilon$ for some $v \in V$. More precisely, ε is the *semidefiniteness sign* of $\langle A^{m-1}\cdot,\cdot\rangle$, and

(d) vectors with $\langle A^{m-1}v, v\rangle = \varepsilon$ form a pair of opposite cosets of V_1 in V.

We now prove, by induction on $j = 1,\dots,m$, the existence of an ordered j-tuple $(S_1,\dots,S_j) \in V/V_1 \times\cdots\times V/V_j$ of cosets such that $S_j \subseteq\cdots\subseteq S_1$ while, for ε in (d) and every $v \in S_j$,

$$\langle A^{m-1}v, v\rangle = \varepsilon, \quad \langle A^{m-2}v, v\rangle = \cdots = \langle A^{m-j}v, v\rangle = 0, \tag{5.5}$$

along with uniqueness of $(S_1,\dots,S_j)$ up to its replacement by $(-S_1,\dots,-S_j)$. As (d) yields our claim for $j = 1$, suppose that it holds for some $j-1 \geq 1$. Since $V_j \subseteq V_{j-1} \subseteq\cdots\subseteq V_1$, cf. (a),

(e) the spaces $V_{j-1},\dots,V_1$ project onto subspaces $Q_1,\dots,Q_{j-1}$ of dimensions $1,\dots,j-1$ in the j-dimensional quotient $Q_j = V/V_j$,

and $Q_1 \subseteq\cdots\subseteq Q_{j-1}$, while the cosets $S_{j-1},\dots,S_1$ of $V_{j-1},\dots,V_1$ in V, assumed to exist (and be unique up to an overall sign), project onto an ascending chain of cosets of $Q_1,\dots,Q_{j-1}$ in Q_j. Let us fix a vector $v \in S_{j-1}$, denote by $\widehat{R}_1,\dots,\widehat{R}_{j-1}$ the latter cosets (of dimensions $1,\dots,j-1$), and by $(\cdot,\cdot)$ the symmetric bilinear form on Q_j induced by $\langle A^{m-j}\cdot,\cdot\rangle$ via (b). Since (5.5) is assumed to hold for our v, with j replaced by $j-1$, if we set $v_i = A^{j-i}v$, $i = 1,\dots,j$, then, for all $i,k \in \{1,\dots,j\}$, due to (c) and the first equality in this version of (5.5), $(v_i, v_k) = 0$ if $i+k \leq j$ and $(v_i, v_k) = \varepsilon$ when $i+k = j+1$. The $j\times j$ matrix of these $(\cdot,\cdot)$-inner products thus has the entries all equal to ε on the main antidiagonal, and all zero above it. Due to the resulting nondegeneracy of the matrix and the presence of the zero entries, $v_1,\dots,v_j$ project onto a basis $\widehat{v}_1,\dots,\widehat{v}_j$ of Q_j, with $\widehat{v}_i \in Q_i$, $i = 1,\dots,j$, and $(\cdot,\cdot)$ is a semi-neutral pseudo-Euclidean inner product in Q_j. Thus, $\widehat{v}_1 \in Q_1$ is $(\cdot,\cdot)$-orthogonal to the basis $\widehat{v}_1,\dots,\widehat{v}_{j-1}$ of Q_{j-1}, which makes Q_{j-1} the $(\cdot,\cdot)$-orthogonal complement of the $(\cdot,\cdot)$-null line Q_1. At the same time, the coset $\widehat{R}_1$ of Q_1 is not contained in the $(\cdot,\cdot)$-orthogonal complement Q_{j-1} of Q_1, since $(v_1, v) = (A^{j-1}v, v) = \langle A^{m-1}v, v\rangle = \varepsilon \neq 0$ in the $j-1$ version of (5.5), and so the vector $v = v_j \in S_{j-1}$, projecting onto $\widehat{v}_j \in \widehat{R}_1$, is not $(\cdot,\cdot)$-orthogonal to $\widehat{v}_1$ spanning the line Q_1. By Lemma 2.1, $\widehat{R}_1$ intersects the $(\cdot,\cdot)$-null cone at exactly one point, and so does $-\widehat{R}_1$. This "point" in the j-dimensional quotient $Q_j = V/V_j$ is actually a coset S_j of V_j in V, contained in S_{j-1}, and its lying in the $(\cdot,\cdot)$-null cone amounts to $\langle A^{m-j}v, v\rangle = 0$ for all $v \in S_j$, which establishes the inductive step and thus proves the existence and uniqueness claim about (5.5).

This last claim, for $j = m$, yields a unique (up to a sign) coset S_m of $V_m = \{0\}$, that is, a unique pair $\{v,-v\}$ of opposite vectors in V, with

$$\langle A^{m-1}v, v\rangle = \varepsilon \quad\text{and}\quad \langle A^i v, v\rangle = 0 \quad \text{whenever } i \geq 0 \text{ and } i \neq m-1, \tag{5.6}$$

the case of $i < m-1$ being due to (5.5) for $j = m$, that of $i \geq m$ immediate from (c). Note that S_m uniquely determines the other cosets S_j as $S_m \subseteq \cdots \subseteq S_1$. Setting $e_i = A^{m-i}v$, $i = 1, \ldots, m$, we obtain an m-tuple of vectors leading to matrices for A and $\langle\cdot,\cdot\rangle$ described in the statement of the theorem, cf. (c) and (5.6). Nondegeneracy of the latter matrix, along with the abundance of zero entries in it, establishes both linear independence of $e_1, \ldots, e_m$ and the semi-neutral signature of $\langle\cdot,\cdot\rangle$. Uniqueness of $\{v, -v\}$ clearly implies uniqueness of $e_1, \ldots, e_m$ up to their replacement by $-e_1, \ldots, -e_m$.

For the converse statement it suffices to note that the basis $e_1, \ldots, e_m$ has the form $A^{m-1}v, A^{m-2}v, \ldots, Av, v$, and so self-adjointness of A amounts to requiring that the matrix of $\langle\cdot,\cdot\rangle$ has a single value of the entries in each antidiagonal. ■

Corollary 5.2. *The only linear isometries of a pseudo-Euclidean space of dimension m commuting with a given generic nilpotent self-adjoint endomorphism A such that $A^{m-1} \neq 0$ are* Id *and* $-$Id.

In fact, due to the up-to-a-sign uniqueness of the basis in Theorem 5.1, such a linear isometry must transform this basis into itself or its opposite.

Corollary 5.3. *Let a nilpotent self-adjoint endomorphism A of a pseudo-Euclidean space V have $A^{m-1} \neq 0$, where $m = \dim V$. Then, for every $q \in (0, \infty)$, there exists a unique pair $\{C, -C\}$ of mutually opposite linear isometries of V with $CAC^{-1} = q^2 A$.*

Such C is diagonalized by a basis $e_1, \ldots, e_m$ chosen as in Theorem 5.1, with the respective eigenvalues $q^{m-1}, q^{m-3}, \ldots, q^{1-m}$, or their opposites, so that $Ce_j = \pm q^{m+1-2j}e_j$ for $j = 1, \ldots, m$ and some fixed sign $\pm$.

Proof. Uniqueness is immediate from Corollary 5.2 since two such linear isometries differ, composition-wise, by one commuting with A. Existence: defining the linear automorphism C by $Ce_j = \tilde{e}_j$, for $\tilde{e}_j = q^{m+1-2j}e_j$, we get the inner products $\langle\tilde{e}_i, \tilde{e}_k\rangle = \varepsilon\delta_{ij}$, and $q^2 A\tilde{e}_j = \tilde{e}_{j-1}$, for all $i, j \in \{1, \ldots, m\}$, with $k = m+1-j$ and $\tilde{e}_0 = 0$, as required. ■

Remark 5.4. For a nilpotent self-adjoint endomorphism A of an m-dimensional pseudo-Euclidean space V, five conditions are mutually equivalent:

(i) $A^{m-1} \neq 0$.

(ii) $\operatorname{rank} A = m-1$ (in other words, $\dim \operatorname{Ker} A = 1$).

(iii) $\pm$Id are the only linear self-isometries of V commuting with A.

(iv) A is generic (commutes with only finitely many linear isometries).

(v) 0 is the only skew-adjoint endomorphism of V commuting with A.

In fact, (i) yields (ii) due to (5.3)–(5.4), and the converse is immediate as (ii) and (5.2)–(5.3) force all d_j to equal 1. The implications (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) are obvious from Corollary 5.2. Finally, (v) implies (ii) as any two vectors $v, v' \in \operatorname{Ker} A$ are linearly dependent: the skew-adjoint endomorphism $v \wedge v' = \langle v, \cdot \rangle v' - \langle v', \cdot \rangle v$, where $\langle \cdot, \cdot \rangle$ is the inner product, commutes with A.

6. Invariant subspaces

This section uses the following assumptions and notations.

First, we fix $q \in (0, \infty) \smallsetminus \{1\}$, an integer $m \geq 2$, a generic self-adjoint nilpotent endomorphism A of an m-dimensional pseudo-Euclidean space V with the inner product $\langle \cdot, \cdot \rangle$, and a linear $\langle \cdot, \cdot \rangle$-isometry C of V having positive eigenvalues and satisfying the condition $CAC^{-1} = q^2 A$.

According to Remark 5.4, Theorem 5.1 and Corollary 5.3, the algebraic type of the above quadruple $(V, \langle \cdot, \cdot \rangle, A, C)$ is uniquely determined by m, q and a sign parameter $\varepsilon = \pm 1$. More precisely, we may choose a basis $e_1, \ldots, e_m$ of V such that, for some $\varepsilon \in \{1, -1\}$ and all $i, j \in \{1, \ldots, m\}$, with $e_0 = 0$ and $k = m + 1 - j$,

$$Ae_j = e_{j-1}, \quad \langle e_i, e_k \rangle = \varepsilon \delta_{ij}, \quad Ce_j = q^{m+1-2j} e_j. \tag{6.1}$$

Let the operator T act on functions $(0, \infty) \ni t \mapsto u(t)$, valued anywhere, by

$$[Tu](t) = u(t/q). \tag{6.2}$$

We also fix a C^∞ function

$$f : (0, \infty) \to \mathbb{R} \quad \text{with } q^2 f(qt) = f(t) \text{ whenever } t \in (0, \infty), \tag{6.3}$$

and define $\mathcal{W}, \mathcal{E}$ to be the vector spaces of dimensions 2 and $2m$ formed by all real-valued (or, V-valued) functions y (or, u) on $(0, \infty)$ such that

$$\ddot{y} = fy \tag{6.4-i}$$

or, respectively,

$$\ddot{u} = fu + q^2 Au, \tag{6.4-ii}$$

with $(\,)^\cdot = d/dt$. The operator T obviously preserves $\mathcal{W}$, and so we may select a basis y^+, y^- of the space of complex-valued solutions to (6.4-i) having

$$Ty^+ = \mu^+ y^+ \text{ and } Ty^- \text{ equal to } \mu^- y^- \text{ plus a multiple of } y^+, \tag{6.5}$$

for some eigenvalues $\mu^\pm \in \mathbb{C}$, the multiple being zero unless $\mu^+ = \mu^- \in \mathbb{R}$. Since the formula $\alpha(y^+, y^-) = \dot{y}^+ y^- - y^+ \dot{y}^-$ (a constant!) defines an area form on $\mathcal{W}$ such that $T^*\alpha = q^{-1}\alpha$, we have $\det T = q^{-1}$ in $\mathcal{W}$. Consequently,

$$\mu^+ \mu^- = q^{-1}. \tag{6.6}$$

In general, $\mathcal{E}$ is not preserved by either T or by C applied valuewise via $u \mapsto Cu$. Their composition $CT = TC$ however, does leave $\mathcal{E}$ invariant,

$$CT : \mathcal{E} \to \mathcal{E}, \tag{6.7}$$

as it coincides with the operator $u \mapsto \sigma u$ in (4.5). The solution space $\mathcal{E}$ of (6.4-ii) has an ascending m-tuple of CT-invariant vector subspaces

$$\mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_m = \mathcal{E} \quad \text{with } \dim \mathcal{E}_j = 2j, \tag{6.8}$$

each $\mathcal{E}_j$ consisting of solutions taking values in the space $\operatorname{Ker} A^j$. (Note that, as a consequence of (5.3)–(5.4), $\dim \operatorname{Ker} A^j = j$.)

Theorem 6.1. *Given $q, m, V, \langle\cdot,\cdot\rangle, A, C, e_1, \dots, e_m, T, f, \mathcal{W}, \mathcal{E}, y^\pm, \mu^\pm$ introduced earlier in this section, let $\mathcal{L}$ be any CT-invariant subspace of $\mathcal{E}$. Then in some basis $u_1^+, u_1^-, \dots, u_m^+, u_m^-$ of the complexification $\mathcal{E}^{\mathbb{C}}$ of $\mathcal{E}$, containing a basis of $\mathcal{L}^{\mathbb{C}}$, the matrix of CT is upper triangular with the diagonal $(\lambda_1^+, \lambda_1^-, \dots, \lambda_m^+, \lambda_m^-)$ where, for some combination coefficients $(0, \infty) \to \mathbb{C}$,*

$$\lambda_j^\pm = q^{m+1-2j}\mu^\pm \text{ and } u_j^\pm \text{ equals } y^\pm e_j \text{ plus a combination of } e_1, \dots, e_{j-1}, \tag{6.9}$$

$j = 1, \dots, m$. If $\mu^+, \mu^- \in \mathbb{R}$, we may replace "complex-valued" by "real-valued" and the complexifications $\mathbb{C}, \mathcal{E}^{\mathbb{C}}, \mathcal{E}_j^{\mathbb{C}}$ by the original real forms $\mathbb{R}, \mathcal{E}, \mathcal{E}_j$.

Proof. The equation $\ddot{u} = fu + Au$ imposed on $u = y_1 e_1 + \cdots + y_j e_j$, with $1 \le j \le m$ and complex-valued functions $y_1, \dots, y_j$, reads

$$\ddot{y}_j = f y_j \quad \text{and} \quad \ddot{y}_i = f y_i + y_{i+1} \quad \text{for } i < j. \tag{6.10}$$

Since, by (6.1), $e_1, \dots, e_j$ span $\operatorname{Ker} A^j$, such u lies in $\mathcal{E}_j^{\mathbb{C}}$, for $\mathcal{E}_j$ appearing in (6.8), and we can now define $u_j^\pm$ by (6.9), declaring y_j in (6.10) to be $y^\pm$ and then solving the equations $\ddot{y}_i = f y_i + y_{i+1}$ in the descending order $i = j-1, \dots, 1$, with a $2(j-1)$-dimensional freedom of choosing the functions y_i. As $u_j^\pm \notin \mathcal{E}_i^{\mathbb{C}}$ for $i < j$, the $2m$ solutions $u_j^\pm$ are linearly independent, and hence constitute a basis $u_1^+, u_1^-, \dots, u_m^+, u_m^-$ of $\mathcal{E}^{\mathbb{C}}$ which makes CT upper triangular with the required diagonal. More precisely, by (6.1)–(6.5), CTu_j^+ (or, CTu_j^-) equals $q^{m+1-2j}\mu^+ u_j^+$ (or, $q^{m+1-2j}\mu^- u_j^-$ plus a multiple of u_j^+), plus a linear combination of $u_i^\pm$ with $i < j$, the multiple being 0 unless $\mu^+ = \mu^- \in \mathbb{R}$.

The freedom of choosing y_i will now ensure that some $u_1^+, u_1^-, \ldots, u_m^+, u_m^-$ as above also contains a basis of $\mathcal{L}^{\mathbb{C}}$. Namely, for $\mathcal{L}_j = \mathcal{L} \cap \mathcal{E}_j$, we get inclusion-induced, obviously injective operators $\mathcal{L}_j/\mathcal{L}_{j-1} \to \mathcal{E}_j/\mathcal{E}_{j-1}$, where $1 \le j \le m$ and $\mathcal{L}_0 = \mathcal{E}_0 = \{0\}$, so that, by (6.8), $\delta_j \in \{0, 1, 2\}$, with $\delta_j = \dim(\mathcal{L}_j/\mathcal{L}_{j-1})$. Our $u_j^\pm$ may now be left completely arbitrary, as before, when $\delta_j = 0$. If j is fixed and $\delta_j = 2$, our operator $\mathcal{L}_j/\mathcal{L}_{j-1} \to \mathcal{E}_j/\mathcal{E}_{j-1}$ is an isomorphism, and so the cosets of $u_j^\pm$, forming a basis of $[\mathcal{E}_j/\mathcal{E}_{j-1}]^{\mathbb{C}}$, are also realized as $\mathcal{L}_{j-1}^{\mathbb{C}}$ cosets of solutions in $\mathcal{L}_j^{\mathbb{C}}$, which we select as the required modified versions of $u_j^\pm$. Finally, in the case $\delta_j = 1$, the embedded line $[\mathcal{L}_j/\mathcal{L}_{j-1}]^{\mathbb{C}}$ in $[\mathcal{E}_j/\mathcal{E}_{j-1}]^{\mathbb{C}}$, due to its CT-invariance, must be one of the two eigenvector cosets represented by $u_j^\pm$, and the latter can thus be modified (within our $2(j-1)$-dimensional freedom) so as to lie in $\mathcal{L}_j^{\mathbb{C}}$. Since $\delta_j = \dim(\mathcal{L}_j/\mathcal{L}_{j-1})$, the total number of modified solutions, $\delta_1 + \cdots + \delta_m$, equals $\dim \mathcal{L}$. Therefore, they form a basis of $\mathcal{L}^{\mathbb{C}}$. ∎

7. GL($\mathbb{Z}$)-polynomials

By a *root of unity*, or a GL($\mathbb{Z}$)*-polynomial* we mean here any complex number z such that $z^k = 1$ for some integer $k \ge 1$ or, respectively, any polynomial of degree $d \ge 1$ with integer coefficients, the leading coefficient $(-1)^d$, and the constant term 1 or -1. It is well known, cf. [5, p. 75], that

$$\begin{gathered}\mathrm{GL}(\mathbb{Z})\text{-polynomials of degree } d \text{ are precisely the}\\ \text{characteristic polynomials of matrices in } \mathrm{GL}(d, \mathbb{Z}).\end{gathered} \tag{7.1}$$

Every complex root a of a GL($\mathbb{Z}$)-polynomial P is an invertible algebraic integer and P, if also assumed irreducible, is the minimal monic polynomial of a. Then, due to minimality, a is not a root of the derivative of P, showing that

$$\text{the complex roots of an irreducible } \mathrm{GL}(\mathbb{Z})\text{-polynomial are all distinct.} \tag{7.2}$$

Irreducibility is always meant here to be over $\mathbb{Z}$ or, equivalently, over $\mathbb{Q}$.

We say that a GL($\mathbb{Z}$)-polynomial has a *cyclic root group* if its (obviously nonzero) complex roots generate a cyclic multiplicative group of nonzero complex numbers. The goal of this section is to show that

$$\begin{gathered}\text{the only irreducible } \mathrm{GL}(\mathbb{Z})\text{-polynomials with a cyclic}\\ \text{root group are the cyclotomic and quadratic ones.}\end{gathered} \tag{7.3}$$

We call an irreducible GL($\mathbb{Z}$)-polynomial *cyclotomic* if all of its roots are roots of unity which, up to a sign, agrees with the standard terminology [11]. The cyclic root-group condition clearly does hold for all cyclotomic polynomials and all quadratic GL($\mathbb{Z}$)-polynomials.

First, if an irreducible $\mathrm{GL}(\mathbb{Z})$-polynomial P has among its roots a and a^k, for some $a \in \mathbb{C} \smallsetminus \{1,-1\}$ and an integer $k \notin \{0,1,-1\}$, then

$$\text{every complex root of } P \text{ is a root of unity.} \tag{7.4}$$

In fact, if $k \geq 2$, then, for such P, a, all $\lambda \in \mathbb{C}$, all integers $r \geq 1$, and some $\mathrm{GL}(\mathbb{Z})$-polynomial Q,

$$P(\lambda^{k^r}) = Q(\lambda)Q(\lambda^k)\cdots Q(\lambda^{k^{r-1}})P(\lambda), \tag{7.5}$$

as one sees using induction on r, the case $r = 1$ being obvious as $\lambda \mapsto P(\lambda^k)$ has a as a root, which makes it divisible by the minimal polynomial P of a, and the induction step amounts to replacing λ in (7.5) by λ^k. Now (7.4) follows, or else P would have infinitely many roots. The extension of (7.4) to negative integers k is in turn immediate if one notes that $(PQ)^* = P^*Q^*$ and $P^{**} = P$ for the *inversion* P^* of a degree d polynomial P, defined by $P^*(\lambda) = \lambda^d P(1/\lambda)$ or, equivalently, $P^*(\lambda) = a_0\lambda^d + \cdots + a_{d-1}\lambda + a_d$ whenever $P(\lambda) = a_0 + a_1\lambda + \cdots + a_d\lambda^d$. More precisely, we then replace (7.5) with $P(\lambda^{k^r}) = Q^*(\lambda)Q(\lambda^k)\cdots Q^{[r]}(\lambda^{k^{r-1}})P^{[r]}(\lambda)$, where $P^{[r]}$ equals P or P^* depending on whether r is even or odd.

Remark 7.1. If a $\mathrm{GL}(\mathbb{Z})$-polynomial has the complex roots $c_1,\ldots,c_d$, and k is an integer, then $c_1^k,\ldots,c_d^k$ are the roots of a $\mathrm{GL}(\mathbb{Z})$-polynomial. (By (7.1), we may choose the latter polynomial to be the characteristic polynomial of the kth power of a matrix in $\mathrm{GL}(d,\mathbb{Z})$ with the characteristic roots $c_1,\ldots,c_d$.)

Lemma 7.2. *Let an irreducible* $\mathrm{GL}(\mathbb{Z})$*-polynomial* P *of degree* d *have a root* a^k *for some* $a \in \mathbb{C} \smallsetminus \{1,-1\}$ *and an integer* $k \neq 0$*. Then*

$$a \text{ is an invertible algebraic integer,} \tag{7.6}$$

having some $\mathrm{GL}(\mathbb{Z})$*-polynomial* S *as its minimal polynomial, and the complex roots* $c_1,\ldots,c_r$ *of* S *can be rearranged so that, with* $d \leq r$*,*

$$P(\lambda) = (c_1^k - \lambda)\cdots(c_d^k - \lambda) \quad \text{and} \quad \{c_1^k,\ldots,c_d^k\} = \{c_1^k,\ldots,c_r^k\}, \tag{7.7}$$

Proof. If $k > 0$, the polynomial $\lambda \mapsto P(\lambda^k)$ has the root a, which yields (7.6) and the equality $P(\lambda^k) = Q(\lambda)S(\lambda)$ for all $\lambda \in \mathbb{C}$ and some $\mathrm{GL}(\mathbb{Z})$-polynomial Q. Thus, the kth powers of all the roots $c_1,\ldots,c_r$ of S are roots of P. The polynomial R with the roots $c_1^k,\ldots,c_r^k$ is a $\mathrm{GL}(\mathbb{Z})$-polynomial (Remark 7.1), while each factor in its unique irreducible factorization has simple roots by (7.2), which are also roots of P, and irreducibility of P thus implies that the factor must equal P. In other words, R is a power of P, and (7.7) follows. When $k < 0$, the preceding assumptions (and conclusions) hold with k, P replaced by $|k|$, P^* (and a, S unchanged), so that $P^*(\lambda) = (c_1^{|k|} - \lambda)\cdots(c_d^{|k|} - \lambda)$, as required in (7.7). ■

Lemma 7.3. *If an irreducible* GL($\mathbb{Z}$)*-polynomial* P *has two roots of the form* a^k *and* a^ℓ *for* $a \in \mathbb{C} \smallsetminus \{1, 0, -1\}$ *and two distinct nonzero integers* $k, \ell \geq 2$*, then all roots of* P *have modulus* 1.

Proof. Let $k > \ell$. The two versions of (7.7), one for k and one for ℓ, involve the same roots $c_1, \dots, c_r$ of the same polynomial S, so that

$$\{|c_1|^k, \dots, |c_r|^k\} = \{|c_1|^\ell, \dots, |c_r|^\ell\}. \tag{7.8}$$

If the greatest (or, least) of the moduli $|c_1|, \dots, |c_r|$ were greater (or, less) than 1, its kth (or, ℓth) power would lie on the left-hand (or, right-hand) side of (7.8) and be greater than any number on the opposite side, contrary to the equality in (7.8). Thus, $|c_1| = \cdots = |c_r| = 1$. ■

Lemma 7.4. *If all roots of an irreducible* GL($\mathbb{Z}$)*-polynomial* P *have modulus* 1*, then they are roots of unity, that is,* P *is cyclotomic.*

Proof. A matrix in $\mathrm{GL}(d, \mathbb{Z})$ with the characteristic polynomial P, cf. (7.1), treated as an automorphism of $\mathbb{C}^d$ is, in view of (7.2), diagonalized by a suitable basis, with unit diagonal entries, so that its powers form a bounded sequence, with a convergent subsequence. As these powers preserve the real form $\mathbb{R}^d \subseteq \mathbb{C}^d$, the convergence takes place in $\mathrm{GL}(d, \mathbb{R})$ and discreteness of the subset $\mathrm{GL}(d, \mathbb{Z})$ makes the subsequence ultimately constant. ■

Proof of (7.3). Consider an irreducible GL($\mathbb{Z}$)-polynomial with a cyclic root group generated by $a \in \mathbb{C}$. By (7.2), we may assume that $a \notin \{1, 0, -1\}$. If a is (or is not) a root, our claim follows from (7.4) (or, Lemmas 7.3–7.4). ■

8. The combinatorial argument

The main result of this section, Theorem 8.1, will serve as the final step needed to prove Theorem A in Section 9.

Any $m, k \in \mathbb{Z}$ with $m \geq 2$ give rise to functions $E, \Phi : \mathbb{Z} \to \mathbb{Z}$ and integers a_0, a_1 such that, for all $a, b \in \mathbb{Z}$,

$$E(a) = m - (-1)^a k - a, \tag{8.1-i}$$

$$\Phi(a) = 2m - 2(-1)^a k - a, \tag{8.1-ii}$$

$$E \text{ is bijective and } \Phi \text{ is an involution,} \tag{8.1-iii}$$

$$E^{-1}(b) = m - (-1)^{m+k+b} k - b, \tag{8.1-iv}$$

$$\Phi(a) = E^{-1}(-E(a)), \tag{8.1-v}$$

$$a_1 = E^{-1}(1) = m + (-1)^{m+k} k - 1, \tag{8.1-vi}$$

$$a_0 = E^{-1}(0) = m - (-1)^{m+k}k, \tag{8.1-vii}$$

$$a_0 + a_1 = 2m - 1. \tag{8.1-viii}$$

Let integers $m \geq 2$ and k be fixed, $\mathcal{V} = \{1, \ldots, 2m\}$, and $|\ |$ denote cardinality.

Theorem 8.1. *There is no set $\mathcal{S} \subseteq \mathcal{V}$ with the following properties:*

(a) $a_1 \in \mathcal{S}$ *and* $\Phi(a_1) \notin \mathcal{S}$.

(b) $a_0 \in \mathcal{S}$ *if and only if m is even.*

(c) *If $a, b \in \mathcal{V}$ and $a + b = 2m + 1$, then exactly one of a, b lies in $\mathcal{S}$.*

(d) *For every $a \in \mathcal{S} \smallsetminus \{a_1\}$ there exists $b \in \mathcal{S}$ with $E(b) = -E(a)$.*

(e) $|\mathcal{S} \cap \{1, 2, \ldots, 2j\}| \leq j$ *whenever* $j \in \{1, \ldots, m\}$.

Proof. Equivalently, (c) states that $\mathcal{S}$ is a selector for the m-element family $\{\{a, b\} \subseteq \mathcal{V} : a + b = 2m + 1\}$. Hence $|\mathcal{S}| = m$. In addition,

$$|\mathcal{S}| = m \geq 3, \tag{8.2-i}$$

$$|k| \leq m - 1, \tag{8.2-ii}$$

$$\Phi(\mathcal{S} \smallsetminus \{a_1\}) = \mathcal{S} \smallsetminus \{a_1\}. \tag{8.2-iii}$$

In fact, as $a_1 \neq a_0$ and $a_0 + a_1 = 2m - 1$ by (8.1-viii), having $m = 2$ in (8.2-i) would, by (a)–(b), give $\mathcal{S} = \{a_0, a_1\} \subseteq \{1, 2, 3, 4\}$ and $a_0 + a_1 = 3$, implying that $\mathcal{S} = \{1, 2\}$, contrary to (e) for $j = 1$. Next, (d) and (8.1-v) give $\Phi(\mathcal{S} \smallsetminus \{a_1\}) \subseteq \mathcal{S} \smallsetminus \{a_1\}$, cf. (8.1-iii), with the image not containing a_1, as otherwise, by (8.1-iii), $\Phi(a_1)$ would lie in $\mathcal{S}$, which contradicts (8.1-i); and (8.1-iii) makes the inclusion an equality, proving (8.2-iii). Finally, using (8.2-i), we may fix $a \in \mathcal{S} \smallsetminus \{a_1, 2m\}$. Thus, by (8.2-iii) and (8.1-ii), $1 \leq \Phi(a) = 2m - 2(-1)^a k - a \leq 2m$. When a is even (odd) this becomes $2 \leq 2m - 2k - a \leq 2m$ (or, $1 \leq 2m + 2k - a \leq 2m - 1$), yielding $1 - m \leq k \leq m - 2$ (or, $1 - m \leq k \leq m - 1$), and (8.2-ii) follows.

Let us now define $c_\pm \in \mathbb{Z}$ by

$$c_\pm = m \mp k, \quad \text{so that } 1 \leq c_\pm \leq 2m - 1 \text{ due to (8.2-ii)}, \tag{8.3}$$

denote by $\mathcal{V}_\pm$ (or, $\mathcal{S}_\pm$) the set of all $a \in \mathcal{V}$ (or, $a \in \mathcal{S} \smallsetminus \{a_1\}$) having $(-1)^a = \pm 1$ and, finally, given $a, b \in \mathcal{V}_\pm$ with $a \leq b$, set $[a, b]_\pm = [a, b] \cap \mathcal{V}_\pm$, referring to any such $[a, b]_\pm$ as an *even/odd subinterval* of $\mathcal{V}$. Finally, we let $\mathcal{R}_\pm$ stand for the maximal even/odd subinterval of $\mathcal{V}$ which is symmetric about $c_\pm$. Then

$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \{a_1\}, \quad \Phi(\mathcal{S}_\pm) = \mathcal{S}_\pm, \quad \mathcal{S}_\pm \subseteq \mathcal{R}_\pm, \tag{8.4-i}$$

$$\mathcal{R}_+ = [2, 2m - 2k - 2]_+, \quad \mathcal{R}_- = [2k + 1, 2m - 1]_- \quad \text{if } k \geq 0, \tag{8.4-ii}$$

$$\mathcal{R}_+ = [-2k, 2m]_+, \quad \mathcal{R}_- = [1, 2m + 2k - 1]_- \quad \text{if } k < 0, \tag{8.4-iii}$$

$$\Phi \text{ restricted to even/odd integers is the reflection about } c_\pm. \tag{8.4-iv}$$

In fact, the first relation in (8.4-i) is obvious, the second immediate from (8.2-iii) since, by (8.1-ii), $\Phi : \mathbb{Z} \to \mathbb{Z}$ preserves parity. Also, (8.1-ii) yields (8.4-iv), which in turn shows that $\mathcal{S}_\pm = \Phi(\mathcal{S}_\pm)$ is a (possibly empty) union of sets $\{a, b\}$ having $c_\pm$ as the midpoint, and so $\mathcal{S}_\pm \subseteq \mathcal{R}_\pm$. Finally, depending on whether $c_\pm = m \mp k$ is less (or, greater) than the midpoint $m + 1/2$ of $\mathcal{V}$, one endpoint of $\mathcal{R}_\pm$ must lie in $\{1, 2\}$ (or, in $\{2m - 1, 2m\}$), and the other endpoint added to this one must yield $2c_\pm$, which proves (8.4-ii)–(8.4-iii).

Note that, as an obvious consequence of (8.4),

$$\begin{aligned}&\mathcal{S} \smallsetminus \{a_1\} \text{ fails to include specific integers from } \mathcal{V}, \text{ which are:}\\ &\text{the lowest } k \text{ odd and highest } k+1 \text{ even ones when } k > 0,\\ &\text{the highest } |k| \text{ odd and lowest } |k|-1 \text{ even ones for } k < 0,\\ &\text{the integer } 2m \text{ if } k = 0.\end{aligned} \tag{8.5}$$

Furthermore, one necessarily has

$$k \in \{0, -1\}. \tag{8.6}$$

To see this, we begin by excluding the possibility that $k \geq 2$ (or, $k \leq -3$). Namely, if this was the case, (8.5) would give $1, 3, 2m - 2, 2m \notin \mathcal{S} \smallsetminus \{a_1\}$ (when $k \geq 2$), or $2, 4, 2m-3, 2m-1 \notin \mathcal{S} \smallsetminus \{a_1\}$ (for $k \leq -3$). From the two pairs $\{1, 2m\}, \{3, 2m-2\}$ (or, $\{2, 2m - 1\}, \{4, 2m - 3\}$) we would choose one, $\{a, b\}$, having $a_1 \notin \{a, b\}$ and $a + b = 2m + 1$, as well as $a, b \notin \mathcal{S}$, which contradicts (c).

The next two cases that need to be excluded are $k = 1$ and $k = -2$. If one of them occurred, (8.5) would give $1, 2m \notin \mathcal{S} \smallsetminus \{a_1\}$ (if $k = 1$), or $2, 2m - 1 \notin \mathcal{S} \smallsetminus \{a_1\}$ (for $k = -2$), which would again contradict (c), unless $a_1 \in \{1, 2m\}$ and $k = 1$, or $a_1 \in \{2, 2m - 1\}$ and $k = -2$. However, each of the resulting four possible values $(1, 1), (2m, 1), (2, -2), (2m - 1, -2)$ for the ordered pair (a_1, k) leads, via (8.1-vi), to the immediate conclusion that $m \leq 1$, contrary to (8.2-i), and so (8.6) follows.

As the next step, we write $m = 2j$ (m even) or $m = 2j + 1$ (m odd), so that $j \geq 1$ by (8.2-i), and proceed to establish the inclusion

$$\mathcal{S}' \cup \{a_*\} \subseteq \mathcal{S} \cap \{1, 2, \ldots, 2j\}, \quad \text{with } |\mathcal{S}' \cup \{a_*\}| = j + 1, \tag{8.7}$$

which will contradict (e), thus completing the proof of the theorem. Here $\mathcal{S}'$ is the j-element set consisting of all integers from $\{1, 2, \ldots, 2j\}$ with a specific parity (even if $k = -1$, odd for $k = 0$), and $a_* = a_0$ (m even) or $a_* = a_1$ (m odd).

To derive (8.7), we list various conclusions in two separate columns (one for either possible value of k),

$$\begin{aligned}&k = 0, && k = -1, && \text{(A)}\\ &\mathcal{S}' = \{1, 3, \ldots, 2j - 1\}, && \mathcal{S}' = \{2, 4, \ldots, 2j\}, && \text{(B)}\\ &a_* = 2j \in \mathcal{S}, && a_* = 2j - 1 \in \mathcal{S}, && \text{(C)}\end{aligned}$$

$$a_1 = m - 1, \qquad a_1 = m - 1 + (-1)^m, \tag{D}$$
$$a_0 = m, \qquad a_0 = m - (-1)^m, \tag{E}$$
$$2m \notin \mathcal{S}, \qquad 2m - 1 \notin \mathcal{S}, \tag{F}$$
$$1 \in \mathcal{S}, \qquad 2 \in \mathcal{S}. \tag{G}$$

In fact, (B) is the definition of $\mathcal{S}'$, (E), (D), (C) follow from (8.1-vii)–(8.1-viii), with $a_* \in \mathcal{S}$ due to (a)–(b), while (F) is immediate from (8.5) for $k \in \{-1, 0\}$, and (G) from (F) and (c). What still remains to be shown, for (8.7), is the inclusion

$$\mathcal{S}' \subseteq \mathcal{S}, \tag{8.8}$$

as (8.8) combined with (B)–(C) obviously yields (8.7).

To this end, consider $\Psi : \mathbb{Z} \to \mathbb{Z}$ given by $\Psi(a) = 2m + 1 - a$, so that (c) amounts to $|\mathcal{S} \cap \{a, \Psi(a)\}| = 1$ for all $a \in \mathcal{V}$ or, equivalently, $\Psi(\mathcal{S}) = \mathcal{V} \smallsetminus \mathcal{S}$ and $\Psi(\mathcal{V} \smallsetminus \mathcal{S}) = \mathcal{S}$. Now, in our case, given an integer i,

$$\text{if } 1 \le i < m - 2 \text{ and } i \in \mathcal{S}, \text{ then } i + 2 \in \mathcal{S}. \tag{8.9}$$

Namely, for the sign $\pm$ such that $(-1)^i = \pm 1$, (8.2-iii) and (8.4-iv) yield

$$\underset{\text{in}}{i} \xrightarrow[\Phi]{} \underset{\text{in}}{2m \mp 2k - i} \xrightarrow[\Psi]{} \underset{\text{out}}{i \pm 2k + 1} \xrightarrow[\Phi]{} \underset{\text{out}}{2m - i - 1} \xrightarrow[\Psi]{} \underset{\text{in}}{i + 2}$$

"in" or "out" meaning lying in $\mathcal{S}$ or in $\mathcal{V} \smallsetminus \mathcal{S}$. In fact, the four sums of pairs of adjacent integers in the above displayed line are $2(m \mp k) = 2c_\pm$, $2m + 1$, $2(m \pm k) = 2c_\mp$, $2m + 1$, as required in the definitions of the reflections Ψ and Φ, the latter restricted to even/odd integers. On the other hand, the inequality $i < m - 2$ implies, via (D), that $i \ne a_1 \ne 2m - i - 1$ (and so $2m - i - 1 \notin \mathcal{S}$, for otherwise $i \pm 2k + 1 = \Phi(2m - i - 1)$ would lie in $\mathcal{S}$).

Now (8.9) combined with (G) and (B) proves (8.8) by induction on i. Specifically, the highest value of odd (or, even) i such that this yields $i \in \mathcal{S}$ is the one with $i - 2 < m - 2 \le i$, which is the required value $2j - 1$ (or, $2j$) except for even m and $k = -1$. In the latter case, although we get $2j - 2$ instead of $2j = m$, we have $2j = m = a_1 \in \mathcal{S}$ nevertheless, due to (D) and (a). ■

9. Proof of Theorem A

We argue by contradiction. Suppose that, for some rank-one ECS model manifold $(\widehat{M}, \widehat{g})$ defined by (4.2), with (4.1), and for G as in Theorem 4.1, there exists

$$\begin{aligned} &\text{a subgroup } \Gamma \subseteq \mathrm{G} \text{ acting on } \widehat{M} \text{ freely and properly discontinuously} \\ &\text{with a generic compact quotient manifold } M = \widehat{M}/\Gamma, \end{aligned} \tag{9.1}$$

yet K_+ in (1.2) is infinite cyclic. As $K_+ = K \cap (0, \infty)$, by Lemma 3.1, for the image K of the homomorphism $\Gamma \ni (q, p, C, r, u) \mapsto q$, we get (3.5b). Theorem 3.3 now allows us to set $I = (0, \infty)$ in (4.1), and all $(q, p, C, r, u) \in \Gamma$ have $p = 0$. We fix

$$\hat{\gamma} = (q, 0, C, \hat{r}, \hat{u}) \in \Gamma \text{ such that } q \text{ is a generator of } K_+. \tag{9.2}$$

From (4.3) and Theorem 4.1, we have (6.3) and $CAC^{-1} = q^2 A$, for f, A in (4.1). Using the notations of (6.2)–(6.7), with $m = n - 2$, we replace Γ, without loss of generality, by a finite-index subgroup Γ_+, which allows us to assume that

$$q \in (0, \infty) \smallsetminus \{1\}, C \text{ has positive eigenvalues, and } \mu^\pm \in \mathbb{C} \smallsetminus (-\infty, 0]. \tag{9.3}$$

Namely, each of these additional requirements amounts to passing from Γ to a subgroup of index at most 2 (or, equivalently, from M to the corresponding finite isometric covering). Specifically, we successively intersect Γ with the kernels of the homomorphisms $\Gamma \to \{1, -1\}$ sending $(q, 0, C, r, u)$ to $\operatorname{sgn} q$ and $\operatorname{sgn} C$, the latter sign accounting for positivity or negativity of the eigenvalues of C. (According to Corollary 5.3, one of these cases must take place, and all C occurring in G form an Abelian group.) The last condition (positivity of $\mu^\pm$ when they are real) is achieved by replacing $\hat{\gamma}, q, C, \mu^\pm$ with their squares and Γ with the corresponding homomorphic preimage of the index-two subgroup of K_+ generated by q^2, which is to be done only if $\mu^\pm$ are real and negative, cf. (6.6). Finally, we define a linear operator $\Pi : \mathbb{R} \times \mathcal{E} \to \mathbb{R} \times \mathcal{E}$ by

$$\Pi(r, u) = (2\Omega(CTu, \hat{u}) + r/q, CTu). \tag{9.4}$$

From the assumption that K_+ is infinite cyclic we will derive, in Lemma 9.2, the existence of a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ having the following properties:

$$\dim \mathcal{L} = m, \text{ where } m = n - 2; \tag{9.5-A}$$

$$CT \text{ leaves } \mathcal{L} \text{ invariant}; \tag{9.5-B}$$

$$\Pi(\Sigma') = \Sigma' \text{ for some lattice } \Sigma' \text{ in } \mathbb{R} \times \mathcal{L}; \tag{9.5-C}$$

$$\Omega(u, u') = 0 \text{ whenever } u, u' \in \mathcal{L}; \tag{9.5-D}$$

$$u \mapsto u(t) \text{ is an isomorphism } \mathcal{L} \to V \text{ for every } t \in (0, \infty). \tag{9.5-E}$$

Remark 9.1. For any rank-one ECS model manifold (4.1)–(4.2), with H and the solution space $\mathcal{E}$ defined in (4.8) and (4.4), if a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ satisfies (9.5-E), with any I instead of $I = (0, \infty)$, then, restricting (4.9) to $(0, \infty) \times \mathbb{R} \times \mathcal{L}$ we clearly obtain an H-equivariant diffeomorphism

$$I \times \mathbb{R} \times \mathcal{L} \to \widehat{M} = I \times \mathbb{R} \times V,$$

its bijectivity being due to (9.5-E), and smoothness of its inverse – to the smooth dependence of the isomorphism $\mathcal{L} \ni u \mapsto u(t) \in V$ on t along with real-analyticity of the isomorphism-inversion operation.

Lemma 9.2. *A vector subspace $\mathcal{L} \subseteq \mathcal{E}$ with* (9.5) *exists if the conditions preceding* (9.4) *are all satisfied.*

Proof. The surjective submersion $\widehat{M} \ni (t, s, v) \mapsto (\log t)/(\log q) \in \mathbb{R}$, being clearly equivariant relative to the homomorphism

$$\Gamma_+ \ni \gamma' = (q', 0, C', r', u') \mapsto (\log q')/(\log q) \in \mathbb{Z} \tag{9.6}$$

along with the obvious actions of Γ on $\widehat{M}$, via (4.7) with $p = 0$, and $\mathbb{Z}$ on $\mathbb{R}$ by translations, descends to a surjective submersion $M \to S^1$ which is

$$\text{a bundle projection } \widehat{M}/\Gamma_+ \to \mathbb{R}/\mathbb{Z} = S^1, \tag{9.7}$$

according to Remark 2.3. The kernel Σ of (9.6) equals $\Sigma = \{(1, 0, \mathrm{Id})\} \times \Sigma'$ for some set $\Sigma' \subseteq \mathbb{R} \times \mathcal{E}$, since C' in (9.6), due to its positivity, (4.3) and Corollary 5.3, is uniquely determined by q'. Thus, $\Sigma \subseteq \mathrm{H}$, for H given by (4.8). As a consequence of Lemma 3.2 (b) and assertion (f) in Section 4, the restriction to Σ of the homomorphism (c) in Section 4 is injective, making Σ Abelian. Now (a) in Section 4 implies that the image of Σ' under the projection $(r, u) \mapsto u$ spans a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ satisfying condition (9.5-D), and so Remark 4.3 gives $\dim \mathcal{L} \le n - 2$. Due to (9.5-D) and (a) in Section 4, $\mathrm{H}' = \{(1, 0, \mathrm{Id})\} \times \mathbb{R} \times \mathcal{L}$ *is an Abelian subgroup of* H, *containing* Σ, *and the group operation in* H' *identified with* $\mathbb{R} \times \mathcal{L}$ *coincides with the addition in the vector space* $\mathbb{R} \times \mathcal{L}$.

At the same time, the (necessarily compact) fiber of the bundle (9.7) over the $\mathbb{Z}$-coset of $(\log t)/(\log q)$ is obviously the quotient $M_t = [\{t\} \times \mathbb{R} \times V]/\Sigma$. Compactness of M_t implies surjectivity of the linear operator $\mathcal{L} \ni u \mapsto u(t) \in V$ for every $t \in (0, \infty)$, since otherwise a nonzero linear functional vanishing on its image, composed with the projection $\{t\} \times \mathbb{R} \times V \to V$, would descend – according to (b) in Section 4 – to an unbounded function $M_t \to \mathbb{R}$. Thus, $\dim \mathcal{L} \ge n - 2 = \dim V$ which, due to the opposite inequality in the last paragraph, gives both (9.5-A) and (9.5-E). Remark 9.1 with $I = (0, \infty)$ and the italicized conclusion of the preceding paragraph, combined with compactness of each of the quotients M_t (and the obvious proper discontinuity of the action of Σ on $\{t\} \times \mathbb{R} \times V$) show that Σ' is a lattice in $\mathbb{R} \times \mathcal{L}$.

Finally, according to Remark 4.2, the right-hand side of (9.4) describes the conjugation by our $\hat{\gamma}$ in (9.2) applied to $(1, 0, \mathrm{Id}, r, u) \in \Sigma$, which we identify here with (r, u). As this conjugation obviously sends the kernel Σ onto itself, we get (9.5-C), and so $\Pi(\mathbb{R} \times \mathcal{L}) = \mathbb{R} \times \mathcal{L}$ (since Σ' is a lattice in $\mathbb{R} \times \mathcal{L}$). Now (9.4) yields (9.5-B), which completes the proof. ∎

Lemma 9.3. *Under the hypotheses preceding* (9.4)*, let a vector subspace* $\mathcal{L} \subseteq \mathcal{E}$ *satisfy* (9.5-A)–(9.5-C)*, a basis* $u_1^+, u_1^-, \ldots, u_m^+, u_m^-$ *of* $\mathcal{E}^{\mathbb{C}}$ *containing a basis* $u_1, \ldots, u_m$ *of* $\mathcal{L}^{\mathbb{C}}$ *be chosen as in Theorem* 6.1*, and* $\lambda_1, \ldots, \lambda_m$ *be the corresponding complex characteristic roots of* $CT : \mathcal{E} \to \mathcal{E}$ *selected from* $\lambda_1^+, \lambda_1^-, \ldots, \lambda_m^+, \lambda_m^-$ *given by* (6.9)*. Then*

(i) $\lambda_0 = q^{-1}$ *and* $\lambda_1, \ldots, \lambda_m$ *form a* $\mathrm{GL}(\mathbb{Z})$*-spectrum,*

in the sense that they are the complex roots of some $\mathrm{GL}(\mathbb{Z})$*-polynomial of degree* $m+1$*, defined as in Section* 7*, and*

(ii) *the product* $\lambda_1 \cdots \lambda_m$ *equals* q *or* $-q$*.*

Furthermore, assuming in addition that

(iii) *one of* $\mu^{\pm}$ *is a power of* q *with a rational exponent,*

we have the following conclusions:

(iv) *Both* $\mu^{\pm}$ *are powers of* q *with integer exponents.*

(v) $\lambda_1^+, \lambda_1^-, \ldots, \lambda_m^+, \lambda_m^-$ *are all distinct, real and positive.*

(vi) *Exactly one of* $\lambda_1, \ldots, \lambda_m$ *equals* q*.*

(vii) *Just one, or none of* $\lambda_1, \ldots, \lambda_m$ *equals* 1 *if* n *is even, or odd.*

(viii) *Those* $\lambda_1, \ldots, \lambda_m$ *not equal to* q *or* 1 *form pairs of mutual inverses.*

(ix) $\Omega(u_i^{\pm}, u_j^{\pm}) = 0$ *for all* $i, j \in \{1, \ldots, m\}$ *and both signs* $\pm$*.*

(x) $\Omega(u_i^{\pm}, u_j^{\mp}) \neq 0$ *if and only if* $i + j = m + 1$*.*

Proof. Assertion (i) is immediate from (9.4) and (9.5-C) along with (7.1), and (ii) from (i). Assuming (iii), we see – using (6.9), (6.6) and (7.3) – that, for the $\mathrm{GL}(\mathbb{Z})$-polynomial P with the roots $\lambda_0, \ldots, \lambda_m$,

(xi) the irreducible factors of P must all be linear or quadratic,

higher degree cyclotomic polynomials being excluded since the roots are all real. Thus, one of $\lambda_1, \ldots, \lambda_m$ equals q, to match $\lambda_0 = q^{-1}$, and (6.9) combined with (6.6) yields (iv). Since $|\lambda_j^{\pm}|$ is, for either sign $\pm$, a strictly monotone function of j, to prove (v) it suffices to consider the case $q^{m+1-2j}\mu^{\pm} = q^{m+1-2i}\mu^{\mp}$, that is, $\mu^{\pm}/\mu^{\mp} = q^{2(j-i)}$. Multiplied by $\mu^{\pm}\mu^{\mp} = q^{-1}$, cf. (6.6), this makes $(\mu^{\pm})^2$ a power of q with an *odd* integer exponent, contrary to (iv), so that (v) follows. From (iii) and (xi) we now get (viii).

For our basis $u_j^{\pm}$ of $\mathcal{E}$, diagonalizing CT with the eigenvalues $\lambda_j^{\pm} = q^{m+1-2j}\mu^{\pm}$, (g) in Section 4 gives

$$q^{-1}\Omega(u_i^{\pm}, u_j^{\pm}) = q^{2m+2-2i-2j}(\mu^{\pm})^2\Omega(u_i^{\pm}, u_j^{\pm}),$$
$$q^{-1}\Omega(u_i^{\pm}, u_j^{\mp}) = q^{2m+2-2i-2j}\mu^{+}\mu^{-}\Omega(u_i^{\pm}, u_j^{\mp}).$$

Thus, the inequality $\Omega(u_i^\pm, u_j^\pm) \neq 0$ would, again, make $(\mu^\pm)^2$ a power of q with an odd integer exponent, contradicting (iv), which yields (ix). Similarly, assuming that $\Omega(u_i^\pm, u_j^\mp) \neq 0$, we now get, from (6.6), $i + j = m + 1$. The converse implication needed in (x) follows, via (ix), from nondegeneracy of Ω. ■

Lemma 9.4. *With the assumptions and notations of Lemma 9.3, let $\mathcal{L}$ this time satisfy all of (9.5). Then conditions (i)–(x) in Lemma 9.3 all hold, so that $\mu^\pm$ and $\lambda_j^\pm$ are all real, while*

(i) *the number of pluses is different from that of minuses*

among the $\pm$ superscripts of those $\lambda_1^+, \lambda_1^-, \dots, \lambda_m^+, \lambda_m^-$ which form the characteristic roots $\lambda_1, \dots, \lambda_m$ of $CT : \mathcal{L} \to \mathcal{L}$. Finally, for the basis $\mathcal{B} = \{u_1, \dots, u_m\}$ of $\mathcal{L}$ contained in the basis $\{u_1^+, u_1^-, \dots, u_m^+, u_m^-\}$ of $\mathcal{E}$, with $|\;|$ denoting cardinality,

(ii) $|\mathcal{B} \cap \{u_1^+, u_1^-, \dots, u_j^+, u_j^-\}| \le j$ *whenever* $j = 1, \dots, m$,

(iii) $|\mathcal{B} \cap \{u_i^+, u_j^-\}| = 1$ *if* $i, j \in \{1, \dots, m\}$ *and* $i + j = m + 1$.

Proof. If (ii) failed to hold, the evaluation operator in (9.5-E), complexified if necessary, would send $\{u_1, \dots, u_{j+1}\}$ into the span of the vectors $e_1, \dots, e_j$ appearing in (6.9), contrary to its injectivity. From (ii) we obtain

(iv) $k(j) \ge j$ for all $j = 1, \dots, m$,

$k(j) \in \{1, \dots, m\}$ being such that $u_j = u_{k(j)}^\pm$ with some sign $\pm$, since, otherwise, $\mathcal{B} \cap \{u_1^+, u_1^-, \dots, u_{k(j)}^+, u_{k(j)}^-\}$ would have at least $j > k(j)$ elements.

To prove (i), we now assume its negation, and evaluate the product of those $\lambda_j^\pm = q^{m+1-2j}\mu^\pm$ in (6.9) which constitute $\lambda_1, \dots, \lambda_m$. Both factors μ^+, μ^- appear in this product the same number of times, $m/2$, which makes m even, and by (6.6) their occurrences contribute to our product $\lambda_1 \cdots \lambda_m$ a total factor of $q^{-m/2}$. On the other hand, the set $\{q^{m+1-2j} : 1 \le j \le m\} = \{q^{m-1}, q^{m-3}, \dots, q^{1-m}\}$ is closed under taking inverses, so that $\prod_{j=1}^m q^{m+1-2j} = 1$. Writing $k(j) = j + \ell(j)$, with $\ell(j) \ge 0$ due to (iv), we now have

$$\lambda_j = \lambda_{k(j)}^\pm = q^{m+1-2k(j)}\mu^\pm = q^{m+1-2j}\mu^\pm q^{-2\ell(j)}, \tag{9.8}$$

making $\lambda_1 \cdots \lambda_m$ equal to 1 times $q^{-m/2}$ times $\prod_{j=1}^m q^{-2\ell(j)}$, that is, a power of q with a negative exponent, contrary to Lemma 9.3 (ii).

Next, (i) implies that $\mu^\pm$ and $\lambda_j^\pm$ are all real, for otherwise λ_j in (9.8), forming along with $\lambda_0 = q^{-1}$ the spectrum of a real matrix, would come in nonreal conjugate pairs, with the same number of positive real parts as negative ones. Thus, by (9.3), $\mu^\pm > 0$. Using (i) and reality of $\mu^\pm$ we now evaluate the product $\lambda_1 \cdots \lambda_m = \pm q$ in Lemma 9.3 (ii), observing that not all μ^+, μ^- undergo pairwise "cancellations" (forming the product q^{-1}), but instead Lemma 9.3 (ii) equates some power of μ^+

or μ^-, with a positive integer exponent, to a power of q, and so positivity of $\mu^\pm$ yields condition (iii) in Lemma 9.3, which in turn implies (iv)–(x).

Finally, the m-element family $\mathcal{P} = \{\{u_i^+, u_j^-\} : i + j = m + 1\}$ forms a partition of $\{u_1^+, u_1^-, \ldots, u_m^+, u_m^-\}$ into disjoint two-element subsets, while the mapping $F : \mathcal{B} \to \mathcal{P}$ given by $u \in F(u)$ is injective: $|\mathcal{B} \cap \{u_i^+, u_j^-\}| \leq 1$ if $i + j = m + 1$, or else Lemma 9.3 (x) would contradict (9.5-D). As $|\mathcal{B}| = m$, surjectivity of F thus follows, proving (iii). ∎

We now complete the proof of Theorem A by observing that a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ with (9.5) gives rise to a subset $\mathcal{S}$ of $\mathcal{V} = \{1, \ldots, 2m\}$, for $m = n - 2$, satisfying conditions (a)–(e) in Theorem 8.1, which – according to Theorem 8.1 – cannot exist. Namely, using Lemma 9.3 (iv) we define $k \in \mathbb{Z}$ by $\mu^+ = q^k$, so that, by (6.6), $\mu^- = q^{-k-1}$. Next, the obvious order-preserving bijection

$$\mathcal{V} = \{1, \ldots, 2m\} \to \{u_1^+, u_1^-, \ldots, u_m^+, u_m^-\} \tag{9.9}$$

(notation of Lemma 9.3) which, explicitly, sends $a \in \mathcal{V}$ to u_j^- when $a = 2j$ is even, or to u_j^+ for odd $a = 2j - 1$, is used from now on to identify the two sets, and we declare $\mathcal{S}$ to be the subset of $\mathcal{V}$ corresponding under (9.9) to the basis $\mathcal{B} = \{u_1, \ldots, u_m\}$ of $\mathcal{L}$. The function assigning to each $u_j^\pm$ the corresponding eigenvalue $\lambda_j^\pm = q^{m+1-2j}\mu^\pm$ treated, via (9.9), as defined on $\mathcal{V}$, is now easily seen to be given by $\mathcal{V} \ni a \mapsto q^{E(a)}$, with (8.1-i). Referring to (a)–(e) in Theorem 8.1 simply as (a)–(e), we observe that assertions (ii) and (iii) of Lemma 9.4 yield (e) and (c), while (b), the first claim in (a), and (d) trivially follow from Lemma 9.3 (vi)–(viii) (the latter guaranteed to hold by Lemma 9.4). Finally, the relation $\Phi(a_1) \notin \mathcal{S}$ in (a) which, in view of (8.1-iii) and (8.1-v), amounts to $q^{-1} \notin \{\lambda_1, \ldots, \lambda_m\}$, is thus immediate since otherwise, due to Lemma 9.3 (viii), the inverse q of q^{-1} would occur on the list $\lambda_1, \ldots, \lambda_m$ *twice*, contradicting Lemma 9.3 (v).

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References

[1] A. Derdziński, On conformally symmetric Ricci-recurrent manifolds with Abelian fundamental groups. *Tensor (N.S.)* **34** (1980), no. 1, 21–29 Zbl 0438.53046 MR 570559

[2] A. Derdziński and W. Roter, On conformally symmetric manifolds with metrics of indices 0 and 1. *Tensor (N.S.)* **31** (1977), no. 3, 255–259 Zbl 0379.53027 MR 467596

[3] A. Derdzinski and W. Roter, Global properties of indefinite metrics with parallel Weyl tensor. In *Pure and applied differential geometry—PADGE 2007*, pp. 63–72, Ber. Math., Shaker Verlag, Aachen, 2007 Zbl 1140.53034 MR 2497674

[4] A. Derdzinski and W. Roter, The local structure of conformally symmetric manifolds. *Bull. Belg. Math. Soc. Simon Stevin* **16** (2009), no. 1, 117–128 Zbl 1165.53011 MR 2498963

[5] A. Derdzinski and W. Roter, Compact pseudo-Riemannian manifolds with parallel Weyl tensor. *Ann. Global Anal. Geom.* **37** (2010), no. 1, 73–90 Zbl 1193.53147 MR 2575471

[6] A. Derdzinski and I. Terek, New examples of compact Weyl-parallel manifolds. 2022, arXiv:2210.03660, to appear in *Monatsh. Math.*

[7] A. Derdzinski and I. Terek, The topology of compact rank-one ECS manifolds. *Proc. Edinb. Math. Soc.* **66** (2023), no. 3, 789–809

[8] A. Derdzinski and I. Terek, Compact locally homogeneous manifolds with parallel Weyl tensor. 2023, arXiv:2306.01600

[9] A. Derdzinski and I. Terek, The metric structure of compact rank-one ECS manifolds. 2023, arXiv:2304.10388, to appear in *Ann. Global Anal. Geom.*

[10] B. I. Dundas, *A short course in differential topology*. Cambridge Math. Textbooks, Cambridge University Press, Cambridge, 2018 Zbl 1397.57001 MR 3793640

[11] H. Maier, Anatomy of integers and cyclotomic polynomials. In *Anatomy of integers*, pp. 89–95, CRM Proc. Lecture Notes 46, American Mathematical Society, Providence, RI, 2008 Zbl 1186.11010 MR 2437967

[12] Z. Olszak, On conformally recurrent manifolds, I: Special distributions. *Zesz. Nauk. Politech. Śl., Mat.-Fiz.* **68** (1993), 213–225 Zbl 0841.53033

[13] W. Roter, On conformally symmetric Ricci-recurrent spaces. *Colloq. Math.* **31** (1974), 87–96 Zbl 0292.53014 MR 372768

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