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UNFLAT CONNECTIONS IN 3-SPHERE BUNDLES OVER S^4

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ABSTRACT. The paper concerns connections in 3-sphere bundles over 4-manifolds having the property of unflatness, which is a necessary condition in order that a natural construction give a Riemannian metric of positive sectional curvature in the total space. It is shown that, as conjectured by A. Weinstein, the only 3-sphere bundle over S^4 with an unflat connection is the Hopf bundle.

1. Introduction. Suppose H is a closed subgroup of a compact connected Lie group G and let ω be a connection in a principal G-bundle $P \to M$. Following A. Weinstein [**13**] one can call ω *H-unflat* if, for any $p \in P$, the curvature form Ω_p restricted to the horizontal space at p has the property that its composite with any nonzero functional on the Lie algebra of G, annihilating the Lie algebra of H, is nondegenerate (cf. also [**2**]). (The term "fat" is used instead of "unflat" in [**13**].)

The motivation for this concept comes from a natural idea of constructing a metric on the total space $E = P \times_G G/H = P/H$ of the bundle with fibre G/H, associated to P, by means of a connection ω in P and a Riemannian metric h on the base manifold M. The construction consists in declaring the fibres orthogonal to the horizontal spaces, the former being isometric to G/H with a fixed normal homogeneous metric, while the latter carry the inner products pulled back from h. The H-unflatness of ω is then equivalent to the positivity of the sectional curvatures of all planes spanned by one horizontal and one vertical vector in E. However, the condition of positivity of *all* sectional curvatures in E is much stronger. In fact, since the fibres of E are totally geodesic, the normal homogeneous space G/H must then have positive sectional curvature (unless $\dim G/H = 1$). The same must hold for (M, h) in view of the O'Neill formulae [**10**], as $E \to M$ is a Riemannian submersion. This imposes natural restrictions on the bases and fibres of bundles for which the construction of unflat connections could be a tool for finding new compact manifolds with positive curvature.

There are also examples of total spaces of principal G-bundles with G-invariant positively curved metrics, for which the corresponding connections are not $\{1\}$-unflat, in particular, the exceptional positively curved normal homogeneous space $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ of Berger [**3**] fibers principally over S^4 with structure group S^3 acting

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by isometries. In this case, the fibers are not totally geodesic and the connection that occurs is not unflat, which follows, e.g., from Corollary 1 below.

The results of this paper show that, in the case of 3-sphere bundles over four-manifolds, unflat connections appear to be scarce. Our argument is based on constructing a conformal structure in the base manifold, intrinsically associated to the given unflat connection. Namely, we prove (Theorem 2) that the characteristic numbers of a bundle with such a connection satisfy a certain inequality, which implies (Theorem 3) that the only principal $SO(4)$-bundle with an $SO(3)$-unflat connection over S^4 is the one that has the Hopf fibration $S^7 \to S^4$ as the associated 3-sphere bundle. This proves a conjecture of Weinstein **[13]**, implying at the same time that the above procedure does not yield positively curved Riemannian metrics on exotic 7-spheres. However, it does, in some cases, yield Riemannian metrics of nonnegative sectional curvature (**[11]**; see also **[4]**).

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2. Unflat forms. Let T and V be real vector spaces of positive dimensions, Ω a V-valued exterior 2-form on T. The form Ω is called *unflat* if, for any nonzero functional $f \in V^*$, the composite $f \circ \Omega$: $T \wedge T \to \mathbf{R}$ is nondegenerate, that is, $\operatorname{rank}(f \circ \Omega) = \dim T$. This is clearly equivalent to saying that the map $T \ni v \mapsto \Omega(u, v) \in V$ is surjective for any nonzero $u \in T$ (cf. **[13]**).

As observed by Weinstein **[13]**, the property of unflatness is very restrictive: Given an unflat form Ω: $T \wedge T \to V$, choose an inner product in T and let S be the unit sphere. Then the maps $\Omega(u, \cdot)$: $T_uS \to V$, $u \in S$, define a vector bundle epimorphism of TS onto $S \times V$, i.e., TS contains a trivial subbundle of dimension $\dim V$. In particular, if $\dim V > 1$, then $\dim T \equiv 0 \pmod 4$.

EXAMPLES. (1) Unflat forms Ω: $T \wedge T \to \mathbf{R}$ are just symplectic structures in T.

(2) Let $V = \operatorname{Im} \mathbf{H}$, the 3-dimensional space of pure quaternions. For $T = \mathbf{H}^k$, the space of k-tuples of quaternions, an unflat form Ω: $T \wedge T \to V$ can be defined by

$$\Omega((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \operatorname{Im}(x_1\bar{y}_1 + \cdots + x_k\bar{y}_k),$$

where Im denotes the pure quaternion part.

(3) If Ω: $T \wedge T \to V_1$ is unflat and F: $V_1 \to V_2$ is an epimorphism, then $F \circ \Omega$: $T \wedge T \to V_2$ is unflat.

From an obvious dimension argument, we obtain

LEMMA 1. *For real vector spaces, T, V with* $\dim T = \dim V + 1$, *a form* Ω: $T \wedge T \to V$ *is unflat if and only if* $\Omega(u, v) \neq 0$ *for every pair of independent vectors* $u, v \in T$.

For the remainder of this section, we assume that T and V are real vector spaces with $\dim T = 4$, $\dim V = 3$.

The formula

$$\det[a_{ij}] = (a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23})^2, \tag{1}$$

valid for any skew-symmetric 4×4-matrix (cf. **[9**, p. 309**]**), together with the definition of unflatness, yields

LEMMA 2. *If* Ω: $T \wedge T \to V$ *is unflat, then the assignment*

$$V^* \ni f \mapsto [\det(f \circ \Omega)]^{1/2} \in \mathbf{R} \tag{2}$$

is a positive definite quadratic form in V^*, *determined by* Ω *up to a scaling factor* (*since* det *is not well defined*).

Thus, (2) defines a *conformal structure* in V^*, and hence also in V. Having in mind the latter, we shall call a basis of V Ω-*conformal* if it is orthonormal for some inner product within the conformal structure.

For later convenience, let us introduce some notations. Given a vector space W, denote by LW the set of all bases of W. For $e = (e_1, \dots, e_n) \in LW$ and $0 \neq t \in \mathbf{R}$, set $te = (te_1, \dots, te_n)$. Finally, for vector spaces T, V with $\dim T = 4$, $\dim V = 3$, a form Ω: $T \wedge T \to V$ and bases $e \in LT$, $X \in LV$, we shall write, by abuse of notation,

$$\Omega(e) = X \tag{3}$$

instead of

$$\begin{aligned} \Omega(e_1, e_2) &= \Omega(e_3, e_4) = X_1, \\ \Omega(e_1, e_3) &= \Omega(e_4, e_2) = X_2, \\ \Omega(e_1, e_4) &= \Omega(e_2, e_3) = X_3. \end{aligned} \tag{4}$$

LEMMA 3. *If* Ω: $T \wedge T \to V$ *is unflat and* $X \in LV$ *is* Ω-*conformal, then for any nonzero vector* $e_1 \in T$ *there exist unique vectors* $e_2, e_3, e_4 \in T$ *and a unique real number* $\lambda \neq 0$ *such that* $e = (e_1, e_2, e_3, e_4) \in LT$ *and* $\Omega(e) = \lambda X$. *Moreover, the orientation of* V *determined by* λX *depends only on* Ω, *i.e.*, Ω *distinguishes an orientation in* V.

PROOF. By unflatness, there exist $u_2, u_3, u_4 \in T$ such that $\Omega(e_1, u_{i+1}) = X_i$, $i = 1, 2, 3$, and they complete e_1 to a basis of T. Let $\Omega = \sum_i \Omega_i X_i$. Setting $v_2 = u_2 - \Omega_2(u_2, u_3)e_1$, $v_3 = u_3 + \Omega_1(u_2, u_3)e_1$, $v_4 = u_4 + \Omega_1(u_2, u_4)e_1$, we obtain

$$\begin{aligned} \Omega(e_1, v_{i+1}) &= X_i, \quad i = 1, 2, 3, \quad \Omega(v_2, v_3) = \delta X_3, \\ \Omega(v_2, v_4) &= \beta X_2 + \varepsilon X_3, \quad \Omega(v_3, v_4) = \alpha X_1 + \gamma X_2 + \zeta X_3 \end{aligned} \tag{5}$$

for some α, β, γ, δ, ε, ζ. On the other hand, since X is Ω-conformal, (1) and (5) yield

$$\begin{aligned} \mu(x^2 + y^2 + z^2)^2 &= \det(x\Omega_1 + y\Omega_2 + z\Omega_3) \\ &= (\alpha x^2 - \beta y^2 + \delta z^2 + \gamma xy + \zeta xz - \varepsilon yz)^2 \end{aligned}$$

for some $\mu > 0$ and arbitrary real x, y, z, the determinant being calculated in the basis e_1, v_2, v_3, v_4. Therefore $\alpha = -\beta = \delta \neq 0$ and $\gamma = \zeta = \varepsilon = 0$. Our assertion is now satisfied by $e_{i+1} = \alpha^{-1}v_{i+1}$, $i = 1, 2, 3$, and $\lambda = \alpha^{-1}$. To prove the uniqueness statement, assume $\Omega(e) = \lambda X$, $\Omega(e') = \lambda' X$ and $e_1' = e_1$. Since Ω is unflat, the kernel of $T \ni u \mapsto \Omega(e_1, u) \in V$ is spanned by e_1, which implies (cf. (4)) that $e_{i+1}' = \lambda'\lambda^{-1}e_{i+1} + t_i e_1$ for some real t_i, $i = 1, 2, 3$. Evaluating now $\Omega(e_i', e_j')$, $2 \leqslant i < j \leqslant 4$, we obtain $\lambda' = \lambda$ and $t_i = 0$, $i = 1, 2, 3$, as desired. Finally, to show that the orientation of λX depends only on Ω, it is sufficient to observe that the map $(e_1, X) \mapsto \lambda X$ is continuous and sends $(e_1, -X)$ to $(-\lambda)(-X) = \lambda X$. This completes the proof.

For an unflat form Ω: $T \wedge T \to V$, let us denote by $C_\Omega V$ the set of all Ω-conformal bases $X \in LV$, compatible with the orientation determined by Ω, i.e., satisfying (3) for some $e \in LT$, and by $C_\Omega T$ the set of all bases $e \in LT$ such that

$$\Omega(e_i, e_j) = \Omega(e_k, e_l) \tag{6}$$

for any even permutation (i, j, k, l) of $(1, 2, 3, 4)$, i.e., satisfying (3) for some $X \in LV$.

Remark 1. The matrix group $SO(4)$ acts on LT from the left in the obvious way, which gives rise to an action of its universal covering $\mathrm{Spin}(4) = S^3 \times S^3$. The latter can be described as follows: $S^3 \times S^3 \times LT \ni (q_1, q_2, e) \mapsto q_1 e \bar{q}_2 \in LT$, the left (resp. right) action of the unit quaternions on LT being given, for $e = (e_1, e_2, e_3, e_4) \in LT$, by $ie = (-e_2, e_1, -e_4, e_3)$, $je = (-e_3, e_4, e_1, -e_2)$, $ke = (-e_4, -e_3, e_2, e_1)$ (resp. by $ei = (-e_2, e_1, e_4, -e_3)$, $ej = (-e_3, -e_4, e_1, e_2)$, $ek = (-e_4, e_3, -e_2, e_1)$) and then extended linearly. In terms of the obvious left action of $SO(3)$ on LV and the covering homomorphism φ: $S^3 \to SO(3)$, it is easy to verify that, given an unflat form Ω: $T \wedge T \to V$, a real number $t \neq 0$, $q \in S^3$ and $e \in LT$, $X \in LV$ such that $\Omega(e) = X$, we have

$$\Omega(te) = t^2 X, \qquad \Omega(eq) = X, \qquad \Omega(qe) = \varphi(q)X. \tag{7}$$

Lemma 4. *Let Ω: $T \wedge T \to V$ be unflat. Then, for any fixed $X \in C_\Omega V$, the set of all $e \in LT$ satisfying* (3) *forms precisely one orbit of the right action of S^3 on LT.*

Proof. Suppose $\Omega(e) = X$. If $e' = eq$, then $\Omega(e') = X$ by (7). Conversely, if $\Omega(e') = X$, then we can clearly find $q \in S^3$ such that $e'_1 = \mu(eq)_1$ for some $\mu > 0$. Since, by (7), $\Omega(e') = X$ and $\Omega(\mu eq) = \mu^2 X$, the uniqueness statement of Lemma 3 implies $\mu^2 = 1$ and $e' = \mu eq$, i.e., $e' = eq$. This completes the proof.

The group $\mathrm{GL}(V)$ of all automorphisms of V acts from the left on the set of all unflat forms $T \wedge T \to V$ by $(A\Omega)(u, v) = A(\Omega(u, v))$, $A \in \mathrm{GL}(V)$, $u, v \in T$.

Given real vector spaces T, W with $\dim T = 4$, suppose that T is endowed with an *oriented conformal structure*, i.e., an orientation together with a homothety class of inner products. The Hodge star operator acting on $\bigwedge^2 T = T \wedge T$ is then defined by $*(e_1 \wedge e_2) = e_3 \wedge e_4$, $e_1, \ldots, e_4$ being an arbitrary oriented conformal basis of T. A 2-form Ω: $T \wedge T \to W$ is called *self-dual* if $\Omega \circ * = \Omega$.

Lemma 5. (i) *For any unflat form Ω: $T \wedge T \to V$, the set $C_\Omega T$ of all bases $e \in LT$ satisfying* (6) *is an orbit of the natural action of* $\mathrm{Conf}^+(4) = \mathbf{R}_+ \times SO(4)$ *on LT. In other words, Ω defines an oriented conformal structure in T.*

(ii) *Every unflat form Ω: $T \wedge T \to V$ is self-dual with respect to the oriented conformal structure which it determines in T.*

(iii) *The oriented conformal structure in T, determined as above by any unflat form $T \wedge T \to V$, is invariant under the natural action of* $\mathrm{GL}(V)$ *on unflat forms.*

PROOF. We use the conventions introduced in Remark 1.

(i) If $e \in C_\Omega T$, say $\Omega(e) = X$, and $e' = tq_1e\bar{q}_2$, $t \in \mathbf{R}_+$, $q_1, q_2 \in S^3$, then, by (7), $\Omega(e') = t^2\varphi(q_1)X$, hence $e' \in C_\Omega T$. Conversely, if $e, e' \in C_\Omega T$, say, $\Omega(e) = X$, $\Omega(e') = X'$, then $X, X' \in C_\Omega V$. Therefore $X' = t^2\varphi(q_1)X$ for some $t > 0$ and $q_1 \in S^3$, which implies $\Omega(tq_1e) = X' = \Omega(e')$ in view of (7). Lemma 4 now yields $e' = tq_1e\bar{q}_2$ for some $q_2 \in S^3$, as required.

(ii) Our assertion is immediate from the fact that the bases $e \in LT$ compatible with the oriented conformal structure determined by the unflat form Ω are characterized by (6).

(iii) If $e \in C_\Omega T$ and $A \in \mathrm{GL}(V)$, say, $\Omega(e) = X$, then $(A\Omega)(e) = AX = (AX_1, AX_2, AX_3)$, so that $e \in C_{A\Omega}T$. This completes the proof.

3. Relative unflatness. Let T and W be real vector spaces of positive dimensions. Given a subspace V of W and a 2-form $\Omega: T \wedge T \to W$, one says that Ω is V-unflat (cf. [**13**]) if the composite $T \wedge T \to W \to W/V$ of Ω with the natural projection is unflat in the sense of §2.

Suppose now that $\dim T = 4$, $\dim V = 3$, $W = V + V$ and that an inner product has been chosen in V. Let $SO(V)$ be the group of orientation preserving linear isometries of V. We are interested in D-unflat forms $T \wedge T \to V + V$, where $D = \{(X, X): X \in V\} \subset V + V$ is the diagonal. The group $SO(V) \times SO(V)$ acts then on 2-forms $\Omega: T \wedge T \to V + V$ by $((A, B)\Omega)(u, v) = (A\Omega_1(u, v), B\Omega_{-1}(u, v))$, Ω_1 and Ω_{-1} being the components of Ω in $V + V$. This action does not, in general, preserve D-unflatness, since D is not invariant under $SO(V) \times SO(V)$.

LEMMA 6. *Let* $\dim T = 4$, $\dim V = 3$. *Suppose V is endowed with an inner product. For a 2-form $\Omega = (\Omega_1, \Omega_{-1})$: $T \wedge T \to V + V$, the following three conditions are equivalent*:

(i) *Every 2-form in the $SO(V) \times SO(V)$-orbit of Ω is D-unflat, $D \subset V + V$ being the diagonal.*

(ii) *For every pair of independent vectors $u, v \in T$, $|\Omega_1(u, v)| \neq |\Omega_{-1}(u, v)|$.*

(iii) *For some $\varepsilon \in \{1, -1\}$, $\Omega_{-\varepsilon}$ is unflat (as a V-valued form) and we have*

$$|\Omega_\varepsilon(u, v)| < |\Omega_{-\varepsilon}(u, v)| \tag{8}$$

whenever $u, v \in T$ are independent.

PROOF. By Lemma 1, D-unflatness of $\Omega = (\Omega_1, \Omega_{-1})$ is equivalent to $\Omega_1(u, v) \neq \Omega_{-1}(u, v)$ for arbitrary independent $u, v \in T$. Thus, the orbit of Ω consists of D-unflat forms if and only if $A\Omega_1(u, v) \neq B\Omega_{-1}(u, v)$ for all $A, B \in SO(V)$ and arbitrary independent $u, v \in T$, which is obviously equivalent to (ii). Assume now (ii). From a connectivity argument we obtain (8) for some $\varepsilon \in \{1, -1\}$ and any independent $u, v \in T$. By Lemma 1, $\Omega_{-\varepsilon}$ is unflat, which completes the proof.

4. Unflatness in principal bundles. Let $P \to M$ be a differentiable principal G-bundle, G being a Lie group with Lie algebra g. By a *horizontal tensorial 2-form* on P we shall mean a g-valued 2-form Ω on P such that $\Omega(u, \cdot) = 0$ for any vertical tangent vector u and $\Omega(ua, va) = \mathrm{ad}\, a^{-1} \cdot \Omega(u, v)$ for $a \in G$ and $u, v \in TP$, ad being the adjoint representation. For example, these conditions are satisfied by the curvature form of any connection in P. A horizontal tensorial 2-form Ω in P is called *unflat* if, for any $p \in P$, Ω_p restricted to a complement of the vertical subspace at p is unflat.

For a four-manifold M endowed with an oriented conformal structure, the Hodge star operator is a vector bundle endomorphism $*: \bigwedge^2 M \to \bigwedge^2 M$ with $*^2 = 1$. Therefore $\bigwedge^2 M$ splits as the direct sum $\bigwedge_+ M + \bigwedge_- M$ of the 3-dimensional subbundles of *self-dual* and *anti-self-dual* forms, constituted by the ± 1-eigenspaces of $*$. Clearly, the bundles $\bigwedge_\pm M$ do not essentially depend on the conformal structure, i.e., they are determined up to equivalence by the *oriented* 4-manifold M alone. The principal $SO(3)$-bundle associated to $\bigwedge_\pm M$ will be denoted by $P \bigwedge_\pm M$. More generally, replacing the tangent bundle TM by any oriented 4-plane bundle ξ over any paracompact space N, one can similarly define the oriented 3-plane bundles $\bigwedge_\pm \xi$ over N by means of an arbitrary fibre metric (or conformal structure) in ξ. We have the following formulae for characteristic classes, the coefficient field being, respectively, $\mathbf{R}$ or $\mathbf{Z}_2$:

$$p_1(\textstyle\bigwedge_+ \xi) = p_1(\xi) + 2e(\xi), \tag{9}$$

$$w_2(\textstyle\bigwedge_+ \xi) = w_2(\xi). \tag{10}$$

In fact, (9) follows immediately from the curvature description of characteristic classes [**9**, pp. 308, 311], while (10) can be easily obtained with the aid of a splitting map for ξ [**6**, p. 235].

REMARK 2. Suppose we are given an unflat horizontal tensorial 2-form Ω in a principal G-bundle $P \xrightarrow{\pi} M$, where $\dim M = 4$ and $G = SO(3)$ or $G = S^3$. For any $x \in M$ and any $p \in \pi^{-1}(x)$, the horizontal form Ω_p projects onto an $so(3)$-valued unflat form on $T_x M$ and, since Ω is tensorial, the oriented conformal structure in $T_x M$ defined by Ω_p is independent of $p \in \pi^{-1}(x)$ (cf. Lemma 5). Thus, Ω defines an oriented conformal structure on M. With respect to this structure, Ω_p is self-dual when viewed as a form on $T_{\pi(p)}M$, i.e., $f \circ \Omega_p \in \bigwedge_+ M_{\pi(p)}$ for any $f \in so(3)^*$. In other words, considering Ω as a 2-form on M valued in the adjoint bundle $\operatorname{ad} P = P \times_{\operatorname{ad}} so(3)$ of Lie algebras, we have $\Omega \circ * = \Omega$.

The following theorem can be viewed as a special case of Weinstein's Theorem 7.2 of [**13**] (cf. Remark 3).

THEOREM 1. *Let $\pi: P \to M$ be a principal $SO(3)$-bundle over a four-manifold M. If P admits an unflat horizontal tensorial 2-form Ω, then M admits an orientation such that P is isomorphic to $P \bigwedge_+ M$.*

PROOF. Fix a basis X_1, X_2, X_3 of $so(3)$ and set $\Omega = \sum_i \Omega_i X_i$. Viewing the forms Ω_p and $\Omega_i(p)$ as defined in $T_{\pi(p)}M$, it is clear from Remark 2 that $\Psi(p) = (\Omega_1(p), \Omega_2(p), \Omega_3(p))$ is a basis of $\bigwedge_+ M_{\pi(p)}$ for any $p \in P$, the oriented conformal structure involved being the one determined by Ω. It is now immediate that the map $\Psi: P \to B$ is $SO(3)$-equivariant, B being the principal $GL(3)$-bundle of $\bigwedge_+ M$. This completes the proof.

REMARK 3. In [**13**] A. Weinstein proved the following (Theorem 7.2): "*Let P be a principal $SO(4)$-bundle over a compact orientable 4-manifold M. Denote by ξ the* *four-plane bundle associated to P. If P admits an SO(3)-unflat connection, then, for properly chosen orientations in ξ and M, $\bigwedge_+\xi$ is isomorphic to $\bigwedge_+ M$*". Note that, for a fixed base manifold, the functors {oriented 4-plane bundles} $\ni \xi \mapsto \bigwedge_\pm \xi \in$ {oriented 3-plane bundles} correspond, on the principal bundle level, to {principal $SO(4)$-bundles} $\ni P \mapsto P_\varepsilon = P/S_\varepsilon^3 \in$ {principal $SO(3)$-bundles}, $\varepsilon = \pm 1$. Here S_ε^3 denotes the unique connected proper normal subgroups of $SO(4)$, both isomorphic to S^3. In terms of the covering homomorphism F: $\mathrm{Spin}(4) = S^3 \times S^3 \to SO(4)$, which assigns to $(q_1, q_2) \in S^3 \times S^3$ the isometry $x \mapsto q_1 x \bar{q}_2$ of $\mathbf{H} = \mathbf{R}^4$, we have $S_1^3 = F(\{1\} \times S^3)$ and $S_{-1}^3 = F(S^3 \times \{1\})$. The assertion of Weinstein's theorem says precisely that, for some $\varepsilon \in \{1, -1\}$ and a suitable orientation of M, $P \bigwedge_+ M$ is isomorphic to P_ε.

REMARK 4. Let $P \to M$ be a principal $SO(3)$-bundle over an oriented compact four-manifold M with a fixed conformal structure. For any connection in P, the curvature form Ω can be viewed as a 2-form on M, valued in the adjoint bundle $\mathrm{ad}\, P = P \times_{\mathrm{ad}} so(3)$ of Lie algebras. Using any Riemannian metric on M, compatible with the conformal structure, and the bi-invariant metric on $SO(3)$, we have the Chern-Weil formula

$$c_4 \int_M \langle \Omega, \Omega \circ * \rangle = p_1(P)[M],$$

where c_4 is a universal constant and $p_1(P)$ denotes the first Pontryagin class of ad P. Moreover, the Schwartz inequality yields

$$|p_1(P)[M]| \leqslant c_4 \int_M |\Omega|^2$$

with equality if and only if Ω is self-dual or anti-self-dual, i.e., $\Omega \circ * = \pm\Omega$ (cf. **[1]**).

For a closed subgroup H of a Lie group G and a connection ω in a principal G-bundle $P \to M$, Weinstein's definition of unflatness was given in §1. If $G = SO(4)$ and $H = SO(3)$ is embedded in $SO(4)$ in the obvious way as the set of all orthogonal 4×4-matrices keeping the vector $(1, 0, 0, 0)$ fixed, then ω is H-unflat if and only if, for any $p \in P$, the curvature form Ω_p, restricted to the horizontal space at p, is D-unflat, D being the diagonal subspace of $so(4) = so(3) + so(3)$.

In the notations of Remark 3, we have

THEOREM 2. *Let π: $P \to M$ be a principal $SO(4)$-bundle with an $SO(3)$-unflat connection ω over a compact four-manifold M. Then there exist an orientation in M and $\varepsilon \in \{1, -1\}$ such that*

(i) *$P_{-\varepsilon}$ is isomorphic to $P \bigwedge_+ M$, and*

(ii) $0 \leqslant |p_1(P_\varepsilon)[M]| < 3\tau(M) + 2\chi(M)$, *$\tau(M)$ being the signature and $\chi(M)$ the Euler characteristic of M.*

PROOF. As in Remark 3, we can form the quotient principal $SO(3)$-bundles P_ε, $\varepsilon = \pm 1$, with equivariant projections π_ε: $P \to P_\varepsilon$. Now ω projects onto connections $\bar{\omega}_\varepsilon$ in P_ε with curvature forms $\bar{\Omega}_\varepsilon$ such that $\pi_\varepsilon^* \bar{\Omega}_\varepsilon = \Omega_\varepsilon$, $\varepsilon = \pm 1$, where Ω_1, Ω_{-1} are the components of Ω in $so(3) + so(3)$ [7, pp. 79–80]. On the other hand, for any $p \in P$, Ω_p may be viewed as a form in $T_{\pi(p)}M$ and it is easy to see that it satisfies the hypotheses of Lemma 6. Therefore, for some $\varepsilon \in \{1, -1\}$, $\Omega_{-\varepsilon}$ is unflat and (8) holds for any independent vectors u, v tangent to M. Thus, the curvature form $\overline{\Omega}_{-\varepsilon}$ in $P_{-\varepsilon}$ is unflat and hence self-dual with respect to the oriented conformal structure that it induces in M (cf. Remark 2), while $P_{-\varepsilon}$ is isomorphic to $P \bigwedge_+ M$ by Theorem 1. In view of (9), (8), Remark 4 and Hirzebruch's signature theorem [**9**, p. 224], we have

$$0 \leqslant |p_1(P_\varepsilon)[M]| \leqslant c_4 \int_M |\overline{\Omega}_\varepsilon|^2 < c_4 \int_M |\overline{\Omega}_{-\varepsilon}|^2$$
$$= p_1(P_{-\varepsilon})[M] = p_1(\textstyle\bigwedge_+ M)[M] = 3\tau(M) + 2\chi(M),$$

which completes the proof.

REMARK 5. By Theorem 2, the condition

$$3\tau(M) + 2\chi(M) > 0 \tag{11}$$

for an oriented compact 4-manifold M is necessary in order that some principal $SO(4)$-bundle over M admit an $SO(3)$-unflat connection (inducing the given orientation of M). However, (11) follows merely from the fact that $\bigwedge_+ M$ carries a nonflat self-dual connection (which leads to $p_1(\bigwedge_+ M)[M] > 0$, cf. [**1**]). For instance, (11) is satisfied by any oriented, compact, non-Ricci-flat Einstein 4-manifold: for such a manifold, the Riemannian connection of $\bigwedge_+ M$ is self-dual and nonflat (see [**1**]). Since (11) holds now for both orientations of M, we obtain the Thorpe-Hitchin inequality $|\tau(M)| < \frac{2}{3}\chi(M)$ (cf. [**5**]).

We can now use Theorem 2 to prove a conjecture of Weinstein [**13**].

THEOREM 3. *Let P be a principal $SO(4)$-bundle over S^4 with an $SO(3)$-unflat connection. Then P is isomorphic to the principal $SO(4)$-bundle, associated with the Hopf 3-sphere bundle $S^7 \to S^4 = \mathbf{H}P^1$.*

PROOF. By (i) of Theorem 2, one of the quotient $SO(3)$-bundles of P, say, $P_{-\varepsilon}$, is isomorphic to $P \bigwedge_+ S^4$. This is nothing but the principal Hopf bundle $\mathbf{R}P^7 \to S^4$. In fact, the standard metric of $\mathbf{R}P^7$ comes from the construction described in §1, so that the Hopf bundle carries a $\{1\}$-unflat connection and Theorem 1 works. On the other hand, since principal $SO(3)$-bundles over S^4 are pull-backs of the Hopf bundle, we have $p_1(P_\varepsilon)[S^4] \equiv 0 \pmod 4$ (cf. (9) and [**12**, p. 256]) and, by (ii) of Theorem 2, P_ε is trivial. Our assertion follows now immediately from the classification of principal $SO(4)$-bundles over S^4.

REMARK 6. A principal $SO(n)$-bundle $P \to M$ admits a *spin structure*, i.e., a double equivariant covering by a principal Spin(n)-bundle over M if and only if its second Stiefel-Whitney class $w_2 = 0$ [**8**, p. 199]. On the other hand, given a principal Spin(n)-bundle $Q \to M$, one can use the normal subgroup $\mathbf{Z}_2$ of Spin(n) to form the quotient principal $SO(n)$-bundle $P = Q/\mathbf{Z}_2 \to M$ with an equivariant projection $Q \to P$.

For $n = 3$, we can apply Theorem 1 to $P = Q/\mathbf{Z}_2$ and use (10) to obtain

COROLLARY 1. *Let $Q \to M$ be a principal S^3-bundle over a four-manifold M. If Q admits an unflat horizontal tensorial 2-form, then*

(i) *M is an orientable spin manifold, i.e., $w_1(M) = w_2(M) = 0$, and*

(ii) *Q is a spin structure over $P \bigwedge_+ M$ for a suitable orientation of M.*

Similarly, one can prove a statement analogous to Theorem 2 for principal Spin(4) ($= S^3 \times S^3$)-bundles with (diagonal S^3)-unflat connections over a compact four-manifold M. As in Corollary 1, we have in this case $w_2(M) = 0$.

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