UNFLAT CONNECTIONS IN 3 SPHERE BUNDLES OVER $S^4$

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Abstract. The paper concerns connections in 3-sphere bundles over 4-manifolds having the property of unflatness, which is a necessary condition in order that a natural construction give a Riemannian metric of positive sectional curvature in the total space. It is shown that, as conjectured by A. Weinstein, the only 3-sphere bundle over $S^4$ with an unflat connection is the Hopf bundle.

1. Introduction. Suppose $H$ is a closed subgroup of a compact connected Lie group $G$ and let $\omega$ be a connection in a principal $G$-bundle $P \to M$. Following A. Weinstein [13] one can call $\omega$ $H$-unflat if, for any $p \in P$, the curvature form $\Omega_p$ restricted to the horizontal space at $p$ has the property that its composite with any nonzero functional on the Lie algebra of $G$, annihilating the Lie algebra of $H$, is nondegenerate (cf. also [2]). (The term "fat" is used instead of "unflat" in [13].)

The motivation for this concept comes from a natural idea of constructing a metric on the total space $E = P \times_G G/H = P/H$ of the bundle with fibre $G/H$, associated to $P$, by means of a connection $\omega$ in $P$ and a Riemannian metric $h$ on the base manifold $M$. The construction consists in declaring the fibres orthogonal to the horizontal spaces, the former being isometric to $G/H$ with a fixed normal homogeneous metric, while the latter carry the inner products pulled back from $h$. The $H$-unflatness of $\omega$ is then equivalent to the positivity of the sectional curvatures of all planes spanned by one horizontal and one vertical vector in $E$.

However, the condition of positivity of all sectional curvatures in $E$ is much stronger. In fact, since the fibres of $E$ are totally geodesic, the normal homogeneous space $G/H$ must then have positive sectional curvature (unless $\dim G/H = 1$). The same must hold for $(M, h)$ in view of the O'Neill formulae [10], as $E \to M$ is a Riemannian submersion. This imposes natural restrictions on the bases and fibres of bundles for which the construction of unflat connections could be a tool for finding new compact manifolds with positive curvature.

There are also examples of total spaces of principal $G$-bundles with $G$-invariant positively curved metrics, for which the corresponding connections are not $\{1\}$-unflat, in particular, the exceptional positively curved normal homogeneous space $Sp(2)/Sp(1)$ of Berger [3] fibers principally over $S^4$ with structure group $S^3$ acting
by isometries. In this case, the fibers are not totally geodesic and the connection that occurs is not unflat, which follows, e.g., from Corollary 1 below.

The results of this paper show that, in the case of 3-sphere bundles over four-manifolds, unflat connections appear to be scarce. Our argument is based on constructing a conformal structure in the base manifold, intrinsically associated to the given unflat connection. Namely, we prove (Theorem 2) that the characteristic numbers of a bundle with such a connection satisfy a certain inequality, which implies (Theorem 3) that the only principal \( SO(4) \)-bundle with an \( SO(3) \)-unflat connection over \( S^4 \) is the one that has the Hopf fibration \( S^7 \to S^4 \) as the associated 3-sphere bundle. This proves a conjecture of Weinstein [13], implying at the same time that the above procedure does not yield positively curved Riemannian metrics on exotic 7-spheres. However, it does, in some cases, yield Riemannian metrics of nonnegative sectional curvature ([11]; see also [4]).

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2. Unflat forms. Let \( T \) and \( V \) be real vector spaces of positive dimensions, \( \Omega \) a \( V \)-valued exterior 2-form on \( T \). The form \( \Omega \) is called unflat if, for any nonzero functional \( f \in V^* \), the composite \( f \circ \Omega: T \wedge T \to \mathbb{R} \) is nondegenerate, that is, \( \text{rank}(f \circ \Omega) = \dim T \). This is clearly equivalent to saying that the map \( T \ni v \mapsto \Omega(u, v) \in V \) is surjective for any nonzero \( u \in T \) (cf. [13]).

As observed by Weinstein [13], the property of unflatness is very restrictive: Given an unflat form \( \Omega: T \wedge T \to V \), choose an inner product in \( T \) and let \( S \) be the unit sphere. Then the maps \( \Omega(u, \cdot): T \to V, u \in S \), define a vector bundle epimorphism of \( TS \) onto \( S \times V \), i.e., \( TS \) contains a trivial subbundle of dimension \( \dim V \). In particular, if \( \dim V > 1 \), then \( \dim T \equiv 0 \pmod 4 \).

EXAMPLES. (1) Unflat forms \( \Omega: T \wedge T \to \mathbb{R} \) are just symplectic structures in \( T \).

(2) Let \( V = \text{Im} \mathbf{H} \), the 3-dimensional space of pure quaternions. For \( T = \mathbf{H}^k \), the space of \( k \)-tuples of quaternions, an unflat form \( \Omega: T \wedge T \to \mathbb{R} \) can be defined by

\[
\Omega((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \text{Im}(x_1y_1 + \cdots + x_ky_k),
\]

where \( \text{Im} \) denotes the pure quaternion part.

(3) If \( \Omega: T \wedge T \to V_1 \) is unflat and \( F: V_1 \to V_2 \) is an epimorphism, then \( F \circ \Omega: T \wedge T \to V_2 \) is unflat.

From an obvious dimension argument, we obtain

**Lemma 1.** For real vector spaces, \( T, V \) with \( \dim T = \dim V + 1 \), a form \( \Omega: T \wedge T \to V \) is unflat if and only if \( \Omega(u, v) \neq 0 \) for every pair of independent vectors \( u, v \in T \).

For the remainder of this section, we assume that \( T \) and \( V \) are real vector spaces with \( \dim T = 4 \), \( \dim V = 3 \).

The formula

\[
\det[a_{ij}] = (a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23})^2,
\]

valid for any skew-symmetric \( 4 \times 4 \)-matrix (cf. [9, p. 309]), together with the definition of unflatness, yields
Lemma 2. If $\Omega: T \wedge T \to V$ is unflat, then the assignment

$$V^* \ni f \mapsto [\det(f \circ \Omega)]^{1/2} \in \mathbb{R}$$

is a positive definite quadratic form in $V^*$, determined by $\Omega$ up to a scaling factor (since $\det$ is not well defined).

Thus, (2) defines a conformal structure in $V^*$, and hence also in $V$. Having in mind the latter, we shall call a basis of $V$ $\Omega$-conformal if it is orthonormal for some inner product within the conformal structure.

For later convenience, let us introduce some notations. Given a vector space $W$, denote by $LW$ the set of all bases of $W$. For $e = (e_1, \ldots, e_n) \in LW$ and $0 = t \in \mathbb{R}$, set $te = (te_1, \ldots, te_n)$. Finally, for vector spaces $T, V$ with $\dim T = 4, \dim V = 3$, a form $\Omega: T \wedge T \to V$ and bases $e \in LT, x \in LV$, we shall write, by abuse of notation,

$$\Omega(e) = x$$

instead of

$$\Omega(e_1, e_2) = \Omega(e_3, e_4) = x_1,
\Omega(e_1, e_3) = \Omega(e_4, e_2) = x_2,
\Omega(e_1, e_4) = \Omega(e_2, e_3) = x_3.$$

Lemma 3. If $\Omega: T \wedge T \to V$ is unflat and $X \in LV$ is $\Omega$-conformal, then for any nonzero vector $e_i \in T$ there exist unique vectors $e_2, e_3, e_4 \in T$ and a unique real number $\lambda \neq 0$ such that $e = (e_1, e_2, e_3, e_4) \in LT$ and $\Omega(e) = \lambda X$. Moreover, the orientation of $V$ determined by $\lambda X$ depends only on $\Omega$, i.e., $\Omega$ distinguishes an orientation in $V$.

Proof. By unflatness, there exist $u_2, u_3, u_4 \in T$ such that $\Omega(e_1, u_{i+1}) = X_i, i = 1, 2, 3$, and they complete $e_1$ to a basis of $T$. Let $\Omega = \Sigma_i \Omega_i X_i$. Setting $v_2 = u_2 - \Omega_2(u_2, u_3)e_1, v_3 = u_3 + \Omega_1(u_2, u_3)e_1, v_4 = u_4 + \Omega_1(u_2, u_4)e_1$, we obtain

$$\Omega(e_1, v_{i+1}) = X_i, \quad i = 1, 2, 3, \quad \Omega(v_2, v_3) = \delta X_3,$$

for some $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$. On the other hand, since $X$ is $\Omega$-conformal, (1) and (5) yield

$$\mu(x^2 + y^2 + z^2)^2 = \det(x\Omega_1 + y\Omega_2 + z\Omega_3)$$

for some $\mu > 0$ and arbitrary real $x, y, z$, the determinant being calculated in the basis $e_1, v_2, v_3, v_4$. Therefore $\alpha = -\beta = \delta \neq 0$ and $\gamma = \zeta = \epsilon = 0$. Our assertion is now satisfied by $e_{i+1} = \alpha^{-1}v_{i+1}, i = 1, 2, 3$, and $\lambda = \alpha^{-1}$. To prove the uniqueness statement, assume $\Omega(e) = \lambda X, \Omega(e') = \lambda' X$ and $e'_1 = e_1$. Since $\Omega$ is unflat, the kernel of $T \ni u \mapsto \Omega(e_1, u) \in V$ is spanned by $e_1$, which implies (cf. (4)) that $e'_{i+1} = \lambda\lambda^{-1}e_{i+1} + t_i e_i$ for some real $t_i, i = 1, 2, 3$. Evaluating now $\Omega(e'_i, e'_j), 2 < i < j < 4$, we obtain $\lambda' = \lambda$ and $t_i = 0, i = 1, 2, 3$, as desired. Finally, to show that the
orientation of \( \lambda X \) depends only on \( \Omega \), it is sufficient to observe that the map 
\((e_1, X) \mapsto \lambda X \) is continuous and sends \((e_1, -X) \) to \((-\lambda(-X) = \lambda X \). This completes the proof.

For an unflat form \( \Omega: T \wedge T \to V \), let us denote by \( C_\Omega V \) the set of all \( \Omega \)-conformal bases \( X \in LV \), compatible with the orientation determined by \( \Omega \), i.e., satisfying (3) for some \( e \in LT \), and by \( C_\Omega T \) the set of all bases \( e \in LT \) such that

\[
\Omega(e, e) = \Omega(e_\kappa, e_\eta)
\]  

(6)

for any even permutation \((i, j, k, \ell)\) of \((1, 2, 3, 4)\), i.e., satisfying (3) for some \( X \in LV \).

**Remark 1.** The matrix group \( SO(4) \) acts on \( LT \) from the left in the obvious way, which gives rise to an action of its universal covering \( Spin(4) = S^3 \times S^3 \). The latter can be described as follows: \( S^3 \times S^3 \times LT \ni (q_1, q_2, e) \mapsto q_1e_\ell q_2 \in LT \), the left (resp. right) action of the unit quaternions on \( LT \) being given, for \( e = (e_1, e_2, e_3, e_4) \in LT \), by \( ie = (-e_2, e_1, -e_4, e_3) \), \( je = (-e_3, e_4, e_1, -e_2) \), \( ke = (-e_4, -e_3, e_2, e_1) \) (resp. by \( ei = (-e_2, e_1, e_4, -e_3) \), \( ej = (-e_3, -e_4, e_1, e_2) \), \( ek = (-e_4, e_3, -e_2, e_1) \)) and then extended linearly. In terms of the obvious left action of \( SO(3) \) on \( LV \) and the covering homomorphism \( \varphi: S^3 \to SO(3) \), it is easy to verify that, given an unflat form \( \Omega: T \wedge T \to V \), a real number \( t \neq 0 \), \( q \in S^3 \) and \( e \in LT \), \( X \in LV \) such that \( \Omega(e) = X \), we have

\[
\Omega(te) = t^2X, \quad \Omega(eq) = X, \quad \Omega(qe) = \varphi(q)X.
\]  

(7)

**Lemma 4.** Let \( \Omega: T \wedge T \to V \) be unflat. Then, for any fixed \( X \in C_\Omega V \), the set of all \( e \in LT \) satisfying (3) forms precisely one orbit of the right action of \( S^3 \) on \( LT \).

**Proof.** Suppose \( \Omega(e) = X \). If \( e' = eq \), then \( \Omega(e') = X \) by (7). Conversely, if \( \Omega(e') = X \), then we can clearly find \( q \in S^3 \) such that \( e' = \mu(eq)_1 \) for some \( \mu > 0 \). Since, by (7), \( \Omega(e') = X \) and \( \Omega(\mu eq) = \mu^2X \), the uniqueness statement of Lemma 3 implies \( \mu^2 = 1 \) and \( e' = \mu eq \), i.e., \( e' = eq \). This completes the proof.

The group \( GL(V) \) of all automorphisms of \( V \) acts from the left on the set of all unflat forms \( T \wedge T \to V \) by \((A\Omega)(u, v) = A(\Omega(u, v)) \), \( A \in GL(V) \), \( u, v \in T \).

Given real vector spaces \( T, W \) with dim \( T = 4 \), suppose that \( T \) is endowed with an **oriented conformal structure**, i.e., an orientation together with a homothety class of inner products. The Hodge star operator acting on \( \bigwedge^2T = T \wedge T \) is then defined by \* \((e_1 \wedge e_2) = e_3 \wedge e_4, e_1, \ldots, e_4 \) being an arbitrary oriented conformal basis of \( T \). A 2-form \( \Omega: T \wedge T \to W \) is called **self-dual** if \( \Omega \circ * = \Omega \).

**Lemma 5.** (i) For any unflat form \( \Omega: T \wedge T \to V \), the set \( C_\Omega T \) of all bases \( e \in LT \) satisfying (6) is an orbit of the natural action of \( \text{Conf}^+(4) = \mathbb{R}_+ \times SO(4) \) on \( LT \). In other words, \( \Omega \) defines an oriented conformal structure in \( T \).

(ii) Every unflat form \( \Omega: T \wedge T \to V \) is self-dual with respect to the oriented conformal structure which it determines in \( T \).

(iii) The oriented conformal structure in \( T \), determined as above by any unflat form \( T \wedge T \to V \), is invariant under the natural action of \( GL(V) \) on unflat forms.
Proof. We use the conventions introduced in Remark 1.

(i) If \( e \in C_T \), say \( \Omega(e) = X \), and \( e' = tq_1 e q_2 \), \( t \in \mathbb{R}^+ \), \( q_1, q_2 \in S^3 \), then, by (7), \( \Omega(e') = t^2 \varphi(q_1)X \), hence \( e' \in C_T \). Conversely, if \( e, e' \in C_T \), say, \( \Omega(e) = X \), \( \Omega(e') = X' \), then \( X, X' \in C_V \). Therefore \( X' = t^2 \varphi(q_1)X \) for some \( t > 0 \) and \( q_1 \in S^3 \), which implies \( \Omega(tq_1 e) = X' = \Omega(e') \) in view of (7). Lemma 4 now yields \( e' = tq_1 e q_2 \) for some \( q_2 \in S^3 \), as required.

(ii) Our assertion is immediate from the fact that the bases \( e \in LT \) compatible with the oriented conformal structure determined by the unflat form \( \Omega \) are characterized by (6).

(iii) If \( e \in C_T \) and \( A \in \text{GL}(V) \), say, \( \Omega(e) = X \), then \( (A\Omega)(e) = AX = (AX_1, AX_2, AX_3) \), so that \( e \in C_{AT} \). This completes the proof.

3. Relative unflatness. Let \( T \) and \( W \) be real vector spaces of positive dimensions. Given a subspace \( V \) of \( W \) and a 2-form \( \Omega: T \wedge T \to W \), one says that \( \Omega \) is \( V \)-unflat (cf. [13]) if the composite \( T \wedge T \to W \to W/ V \) of \( \Omega \) with the natural projection is unflat in the sense of §2.

Suppose now that \( \dim T = 4 \), \( \dim V = 3 \), \( W = V + V \) and that an inner product has been chosen in \( V \). Let \( SO(V) \) be the group of orientation preserving linear isometries of \( V \). We are interested in \( D \)-unflat forms \( T \wedge T \to V + V \), where \( D = \{(X, X): X \in V \} \subset V + V \) is the diagonal. The group \( SO(V) \times SO(V) \) acts then on 2-forms \( \Omega: T \wedge T \to V + V \) by \( ((A, B)\Omega)(u, v) = (A\Omega_1(u, v), B\Omega_{-1}(u, v)) \), \( \Omega_1 \) and \( \Omega_{-1} \) being the components of \( \Omega \) in \( V + V \). This action does not, in general, preserve \( D \)-unflatness, since \( D \) is not invariant under \( SO(V) \times SO(V) \).

Lemma 6. Let \( \dim T = 4 \), \( \dim V = 3 \). Suppose \( V \) is endowed with an inner product. For a 2-form \( \Omega = (\Omega_1, \Omega_{-1}): T \wedge T \to V + V \), the following three conditions are equivalent:

(i) Every 2-form in the \( SO(V) \times SO(V) \)-orbit of \( \Omega \) is \( D \)-unflat, \( D \subset V + V \) being the diagonal.

(ii) For every pair of independent vectors \( u, v \in T \), \( |\Omega_1(u, v)| \neq |\Omega_{-1}(u, v)| \).

(iii) For some \( \varepsilon \in \{1, -1\} \), \( \Omega_{-\varepsilon} \) is unflat (as a \( V \)-valued form) and we have

\[
|\Omega_\varepsilon(u, v)| < |\Omega_{-\varepsilon}(u, v)|
\]

whenever \( u, v \in T \) are independent.

Proof. By Lemma 1, \( D \)-unflatness of \( \Omega = (\Omega_1, \Omega_{-1}) \) is equivalent to \( \Omega_1(u, v) \neq \Omega_{-1}(u, v) \) for arbitrary independent \( u, v \in T \). Thus, the orbit of \( \Omega \) consists of \( D \)-unflat forms if and only if \( A\Omega_1(u, v) \neq B\Omega_{-1}(u, v) \) for all \( A, B \in SO(V) \) and arbitrary independent \( u, v \in T \), which is obviously equivalent to (ii). Assume now (ii). From a connectivity argument we obtain (8) for some \( \varepsilon \in \{1, -1\} \) and any independent \( u, v \in T \). By Lemma 1, \( \Omega_{-\varepsilon} \) is unflat, which completes the proof.

4. Unflatness in principal bundles. Let \( P \to M \) be a differentiable principal \( G \)-bundle, \( G \) being a Lie group with Lie algebra \( g \). By a horizontal tensorial 2-form on \( P \) we shall mean a \( g \)-valued 2-form \( \Omega \) on \( P \) such that \( \Omega(u, \cdot) = 0 \) for any vertical tangent vector \( u \) and \( \Omega(ua, va) = ad a^{-1} \cdot \Omega(u, v) \) for \( a \in G \) and \( u, v \in TP \), \( ad \) being the adjoint representation. For example, these conditions are satisfied by the
curvature form of any connection in $P$. A horizontal tensorial 2-form $\Omega$ in $P$ is called *unflat* if, for any $p \in P$, $\Omega_p$ restricted to a complement of the vertical subspace at $p$ is unflat.

For a four-manifold $M$ endowed with an oriented conformal structure, the Hodge star operator is a vector bundle endomorphism $*: \bigwedge^2M \to \bigwedge^2M$ with $*^2 = 1$. Therefore $\bigwedge^2M$ splits as the direct sum $\bigwedge^+_M + \bigwedge^-_M$ of the 3-dimensional subbundles of *self-dual* and *anti-self-dual* forms, constituted by the $\pm 1$-eigenspaces of $. Clearly, the bundles $\bigwedge^\pm_M$ do not essentially depend on the conformal structure, i.e., they are determined up to equivalence by the *oriented* 4-manifold $M$ alone. The principal $SO(3)$-bundle associated to $\bigwedge^\pm_M$ will be denoted by $P \bigwedge^\pm_M$. More generally, replacing the tangent bundle $TM$ by any oriented 4-plane bundle $\xi$ over any paracompact space $N$, one can similarly define the oriented 3-plane bundles $\bigwedge^\pm\xi$ over $N$ by means of an arbitrary fibre metric (or conformal structure) in $\xi$. We have the following formulae for characteristic classes, the coefficient field being, respectively, $\mathbb{R}$ or $\mathbb{Z}_2$:

$$p_1(\bigwedge^+\xi) = p_1(\xi) + 2e(\xi), \quad (9)$$

$$w_2(\bigwedge^+\xi) = w_2(\xi). \quad (10)$$

In fact, (9) follows immediately from the curvature description of characteristic classes [9, pp. 308, 311], while (10) can be easily obtained with the aid of a splitting map for $\xi$ [6, p. 235].

**Remark 2.** Suppose we are given an unflat horizontal tensorial 2-form $\Omega$ in a principal $G$-bundle $P \to M$, where $\dim M = 4$ and $G = SO(3)$ or $G = S^3$. For any $x \in M$ and any $p \in \pi^{-1}(x)$, the horizontal form $\Omega_p$ projects onto a $\mathfrak{so}(3)$-valued unflat form on $T_xM$ and, since $\Omega$ is tensorial, the oriented conformal structure in $T_xM$ defined by $\Omega_p$ is independent of $p \in \pi^{-1}(x)$ (cf. Lemma 5). Thus, $\Omega$ defines an oriented conformal structure on $M$. With respect to this structure, $\Omega_p$ is self-dual when viewed as a form on $T_{\pi(p)}M$, i.e., $f \circ \Omega_p \in \bigwedge^+_M(\pi(p))$ for any $f \in \mathfrak{so}(3)^*$. In other words, considering $\Omega$ as a 2-form on $M$ valued in the adjoint bundle $ad P = P \times_{ad} \mathfrak{so}(3)$ of Lie algebras, we have $\Omega \circ * = \Omega$.

The following theorem can be viewed as a special case of Weinstein’s Theorem 7.2 of [13] (cf. Remark 3).

**Theorem 1.** Let $\pi: P \to M$ be a principal $SO(3)$-bundle over a four-manifold $M$. If $P$ admits an unflat horizontal tensorial 2-form $\Omega$, then $M$ admits an orientation such that $P$ is isomorphic to $P \bigwedge^+_M$.

**Proof.** Fix a basis $X_1, X_2, X_3$ of $\mathfrak{so}(3)$ and set $\Omega = \Sigma_i \omega_i X_i$. Viewing the forms $\omega_i$ and $\omega_i(p)$ as defined in $T_{\pi(p)}M$, it is clear from Remark 2 that $\Psi(p) = (\omega_1(p), \omega_2(p), \omega_3(p))$ is a basis of $\bigwedge^+_M(\pi(p))$ for any $p \in P$, the oriented conformal structure involved being the one determined by $\Omega$. It is now immediate that the map $\Psi: P \to B$ is $SO(3)$-equivariant, $B$ being the principal $GL(3)$-bundle of $\bigwedge^+_M$. This completes the proof.

**Remark 3.** In [13] A. Weinstein proved the following (Theorem 7.2): “Let $P$ be a principal $SO(4)$-bundle over a compact orientable 4-manifold $M$. Denote by $\xi$ the
four-plane bundle associated to $P$. If $P$ admits an $SO(3)$-unflat connection, then, for properly chosen orientations in $\xi$ and $M$, $\wedge_+ \xi$ is isomorphic to $\wedge_+ M$". Note that, for a fixed base manifold, the functors \{oriented 4-plane bundles\} $\ni \xi \mapsto \wedge_+ \xi \in \{oriented 3-plane bundles\}$ correspond, on the principal bundle level, to \{principal $SO(4)$-bundles\} $\ni P \mapsto P/S^3 \in \{principal SO(3)$-bundles\}, $\epsilon = \pm 1$. Here $S^3$ denotes the unique connected proper normal subgroups of $SO(4)$, both isomorphic to $S^3$. In terms of the unique connected proper normal subgroups of $SO(4)$, both isomorphic to $S^3$. In terms of the covering homomorphism $F: \text{Spin}(4) = S^3 \times S^3 \to SO(4)$, which assigns to $(q_1, q_2) \in S^3 \times S^3$ the isometry $x \mapsto q_1 x q_2$ of $H = \mathbb{R}^4$, we have $S^3_1 = F(\{1\} \times S^3)$ and $S^3_{-1} = F(S^3 \times \{1\})$. The assertion of Weinstein's theorem says precisely that, for some $\epsilon \in \{1, -1\}$ and a suitable orientation of $M$, $P \wedge_+ M$ is isomorphic to $P_\epsilon$.

**Remark 4.** Let $P \to M$ be a principal $SO(3)$-bundle over an oriented compact four-manifold $M$ with a fixed conformal structure. For any connection in $P$, the curvature form $\Omega$ can be viewed as a 2-form on $M$, valued in the adjoint bundle $\text{ad} P = P \times_{ad} so(3)$ of Lie algebras. Using any Riemannian metric on $M$, compatible with the conformal structure, and the bi-invariant metric on $SO(3)$, we have the Chern-Weil formula

$$c_4 \int_M \langle \Omega, \Omega \circ * \rangle = p_1(P)[M],$$

where $c_4$ is a universal constant and $p_1(P)$ denotes the first Pontryagin class of $\text{ad} P$. Moreover, the Schwartz inequality yields

$$|p_1(P)[M]| \leq c_4 \int_M |\Omega|^2$$

with equality if and only if $\Omega$ is self-dual or anti-self-dual, i.e., $\Omega \circ * = \pm \Omega$ (cf. [1]).

For a closed subgroup $H$ of a Lie group $G$ and a connection $\omega$ in a principal $G$-bundle $P \to M$, Weinstein's definition of unflatness was given in §1. If $G = SO(4)$ and $H = SO(3)$ is embedded in $SO(4)$ in the obvious way as the set of all orthogonal $4 \times 4$ matrices keeping the vector $(1, 0, 0, 0)$ fixed, then $\omega$ is $H$-unflat if and only if, for any $p \in P$, the curvature form $\Omega_p$, restricted to the horizontal space at $p$, is $D$-unflat, $D$ being the diagonal subspace of $so(4) = so(3) + so(3)$.

In the notations of Remark 3, we have

**Theorem 2.** Let $\pi: P \to M$ be a principal $SO(4)$-bundle with an $SO(3)$-unflat connection $\omega$ over a compact four-manifold $M$. Then there exist an orientation in $M$ and $\epsilon \in \{1, -1\}$ such that

(i) $P_\epsilon$ is isomorphic to $P \wedge_+ M$, and

(ii) $0 \leq |p_1(P)[M]| < 3\tau(M) + 2\chi(M)$, $\tau(M)$ being the signature and $\chi(M)$ the Euler characteristic of $M$.

**Proof.** As in Remark 3, we can form the quotient principal $SO(3)$-bundles $P_\epsilon$, $\epsilon = \pm 1$, with equivariant projections $\pi_\epsilon: P \to P_\epsilon$. Now $\omega$ projects onto connections $\hat{\omega}_\epsilon$ in $P_\epsilon$ with curvature forms $\Omega_\epsilon$ such that $\pi_\epsilon^* \Omega_\epsilon = \Omega_\epsilon$, $\epsilon = \pm 1$, where $\Omega_1$, $\Omega_{-1}$ are the components of $\Omega$ in $so(3) + so(3)$ [7, pp. 79–80]. On the other hand, for any $p \in P$, $\Omega_p$ may be viewed as a form in $T_{\pi(p)}M$ and it is easy to see that it satisfies
the hypotheses of Lemma 6. Therefore, for some $\epsilon \in \{1, -1\}$, $\Omega_{-\epsilon}$ is unflat and (8) holds for any independent vectors $u, v$ tangent to $M$. Thus, the curvature form $\Omega_{-\epsilon}$ in $P_{-\epsilon}$ is unflat and hence self-dual with respect to the oriented conformal structure that it induces in $M$ (cf. Remark 2), while $P_{-\epsilon}$ is isomorphic to $P \setminus _+ M$ by Theorem 1. In view of (9), (8), Remark 4 and Hirzebruch's signature theorem [9, p. 224], we have

$$0 < |p_1(P_{-\epsilon})[M]| < c_4 \int_M |\Omega_{-\epsilon}|^2 < c_4 \int_M |\Omega_{-\epsilon}|^2$$

$$= p_1(P_{-\epsilon})[M] = p_1(\setminus _+ M)[M] = 3\tau(M) + 2\chi(M),$$

which completes the proof.

**Remark 5.** By Theorem 2, the condition

$$3\tau(M) + 2\chi(M) > 0 \quad (11)$$

for an oriented compact 4-manifold $M$ is necessary in order that some principal $SO(4)$-bundle over $M$ admit an $SO(3)$-unflat connection (inducing the given orientation of $M$). However, (11) follows merely from the fact that $\setminus _+ M$ carries a nonflat self-dual connection (which leads to $p_1(\setminus _+ M)[M] > 0$, cf. [1]). For instance, (11) is satisfied by any oriented, compact, non-Ricci-flat Einstein 4-manifold: for such a manifold, the Riemannian connection of $\setminus _+ M$ is self-dual and nonflat (see [1]). Since (11) holds now for both orientations of $M$, we obtain the Thorpe-Hitchin inequality $|\tau(M)| < \frac{2}{3}\chi(M)$ (cf. [5]).

We can now use Theorem 2 to prove a conjecture of Weinstein [13].

**Theorem 3.** Let $P$ be a principal $SO(4)$-bundle over $S^4$ with an $SO(3)$-unflat connection. Then $P$ is isomorphic to the principal $SO(4)$-bundle, associated with the Hopf 3-sphere bundle $S^7 \to S^4 = HP^1$.

**Proof.** By (i) of Theorem 2, one of the quotient $SO(3)$-bundles of $P$, say, $P_{-\epsilon}$, is isomorphic to $P \setminus _+ S^4$. This is nothing but the principal Hopf bundle $RP^7 \to S^4$. In fact, the standard metric of $RP^7$ comes from the construction described in §1, so that the Hopf bundle carries a $\{1\}$-unflat connection and Theorem 1 works. On the other hand, since principal $SO(3)$-bundles over $S^4$ are pull-backs of the Hopf bundle, we have $p_1(P_{-\epsilon})[S^4] \equiv 0 \pmod{4}$ (cf. (9) and [12, p. 256]) and, by (ii) of Theorem 2, $P_{-\epsilon}$ is trivial. Our assertion follows now immediately from the classification of principal $SO(4)$-bundles over $S^4$.

**Remark 6.** A principal $SO(n)$-bundle $P \to M$ admits a spin structure, i.e., a double equivariant covering by a principal Spin($n$)-bundle over $M$ if and only if its second Stiefel-Whitney class $w_2 = 0$ [8, p. 199]. On the other hand, given a principal Spin($n$)-bundle $Q \to M$, one can use the normal subgroup $Z_2$ of Spin($n$) to form the quotient principal $SO(n)$-bundle $P = Q/Z_2 \to M$ with an equivariant projection $Q \to P$.

For $n = 3$, we can apply Theorem 1 to $P = Q/Z_2$ and use (10) to obtain
Corollary 1. Let $Q \to M$ be a principal $S^3$-bundle over a four-manifold $M$. If $Q$ admits an unflat horizontal tensorial 2-form, then

(i) $M$ is an orientable spin manifold, i.e., $w_1(M) = w_2(M) = 0$, and

(ii) $Q$ is a spin structure over $P \wedge_+ M$ for a suitable orientation of $M$.

Similarly, one can prove a statement analogous to Theorem 2 for principal Spin(4) (= $S^3 \times S^3$)-bundles with (diagonal $S^3$)-unflat connections over a compact four-manifold $M$. As in Corollary 1, we have in this case $w_2(M) = 0$.

References