Rank-one ECS manifolds of dilational type

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Abstract. We study ECS manifolds, that is, pseudo-Riemannian manifolds with parallel Weyl tensor which are neither conformally flat nor locally symmetric. Every ECS manifold has rank 1 or 2, the rank being the dimension of a distinguished null parallel distribution discovered by Olszak, and a rank-one ECS manifold may be called translational or dilational, depending on whether the holonomy group of a natural flat connection in the Olszak distribution is finite or infinite. Some such manifolds are in a natural sense generic, which refers to the algebraic structure of the Weyl tensor. Known examples of compact ECS manifolds, in every dimension greater than 4, are all of rank 1 and translational, some of them generic, none of them locally homogeneous. As we show, generic compact rank-one ECS manifolds must be translational or locally homogeneous, provided that they arise as isometric quotients of a specific class of explicitly constructed "model" manifolds. This result is relevant since the clause starting with "provided that" may be dropped: according to a theorem which we prove in another paper, the models just mentioned include the isometry types of the pseudo-Riemannian universal coverings of all generic compact rank-one ECS manifolds.

Introduction

By *ECS manifolds* [**3**] one means those pseudo-Riemannian manifolds of dimensions $n \geq 4$ which have parallel Weyl tensor, but not for one of two obvious reasons: conformal flatness or local symmetry. Both their existence, for every $n \geq 4$, and indefiniteness of their metrics, are results of Roter [**12**, Corollary 3], [**2**, Theorem 2], and their local structure has been completely described [**4**].

The acronym 'ECS' stands for *essentially conformally symmetric.*

On every ECS manifold (M, g) there exists a naturally distinguished null parallel distribution $\mathcal{D}$, known as the *Olszak distribution* [**11**], [**4**, p. 119]. Its dimension, necessarily equal to 1 or 2, is referred to as the *rank* of (M, g). Any rank-one ECS

2020 *Mathematics Subject Classification.* Primary 53C50.

The first author's research was supported in part by a FAPESP-OSU 2015 Regular Research Award (FAPESP grant: 2015/50265-6).

manifold is *translational* or *dilational*, in the sense of finiteness or infiniteness of the holonomy group of the flat connection in $\mathcal{D}$, induced by the Levi-Civita connection.

Examples of *compact* rank-one ECS manifolds have been found [**5, 6**] in all dimensions $n \geq 5$. They are all geodesically complete, translational, and none of them is locally homogeneous. It remains an open question whether a compact ECS manifold may be dilational, or incomplete, or locally homogeneous, or have rank two, or be of dimension four.

The present paper provides a partial answer to the first part of the above question: our Theorem A, combined with a result of [**8**] mentioned below, implies that a rank-one compact ECS manifold cannot be dilational except, possibly, in very special circumstances – when it is nongeneric or locally homogeneous.

Here are some details. In Section 4 we describe specific rank-one ECS *model manifolds* [**12**, p. 93], representing all dimensions $n \geq 4$ and all indefinite metric signatures. Some of them are generic in the sense of Remark 4.4.

Since the Olszak distribution $\mathcal{D}$ is a *real line bundle* over the rank-one ECS manifold in question, the holonomy group K of the flat connection in $\mathcal{D}$ induced by the Levi-Civita connection is a multiplicative subgroup of $\mathrm{I\!R} \smallsetminus \{0\}$ (see Section 1), and we will repeatedly refer to

(0.1) the positive holonomy group $K_+ = K \cap (0, \infty)$ of the flat connection in $\mathcal{D}$.

Our main result can be stated as follows.

Theorem A. *Every generic compact isometric quotient of a rank-one ECS model manifold is either translational or locally homogeneous.*

In the locally-homogeneous case the group (0.1) *is dense in* $(0, \infty)$.

According to the final clause of Theorem 3.2, compact locally homogeneous rank-one ECS model manifolds, if they exist, are necessarily dilational. Theorem A thus has the following consequence.

Corollary B. *For a generic compact rank-one ECS manifold arising as an isometric quotient of a model manifold, the property of being dilational is equivalent to local homogeneity.*

Both Theorem A and Corollary B do not really require assuming that the manifold is an isometric quotient of a model. Namely, as we show in [**8**, Corollary D], *the pseudo-Riemannian universal covering of any generic compact rank-one ECS manifold is necessarily isometric to one of the model manifolds.*

Furthermore, another result (Theorem E) of the same paper [**8**] excludes the option of local homogeneity in Theorem A, thus also rendering moot its final clause. However, Theorem A as stated here is a crucial step in the arguments of [**8**].

Instead of establishing Theorem A directly, we derive it from

Theorem C. *In a generic compact isometric quotient of a rank-one ECS model manifold, the group* K_+ *in* (0.1) *is not infinite cyclic.*

Theorem A trivially follows from Theorem C (established in Section 8) combined with the next fact, which we prove at the very end of Section 3. Note that, unlike the previous three, this result holds in a more abstract setting, with no reference to either genericity or model manifolds.

Theorem D. *Given a compact rank-one ECS manifold* (M,g)*, with* K_+ *in* (0.1) *not infinite cyclic,* K_+ *may be trivial, which makes* (M,g) *translational, or else* K_+ *is dense in* $(0,\infty)$*, and then* (M,g) *must be locally homogeneous.*

The paper is organized as follows. After Section 2, presenting a combinatorial argument (Theorem 2.1) needed to establish Theorem C, there are two sections dealing with rank-one ECS manifolds, followed by some material from linear algebra and algebraic number theory (genericity of nilpotent self-adjoint linear endomorphisms of pseudo-Euclidean spaces, and the cyclic root-group condition for $\mathrm{GL}(\mathbb{Z})$-polynomials), in Sections 5 and 7. Those two are separated by a section devoted to subspaces of certain spaces $\mathcal{E}$ of vector-valued functions on $(0,\infty)$, invariant under an operator $CT:\mathcal{E}\to\mathcal{E}$ which is relevant to the existence question for generic compact isometric quotients of rank-one ECS model manifolds.

In the final Section 8 we prove Theorem C by contradiction, assuming that its hypotheses hold and yet K_+ in (0.1) is infinite cyclic. Lemma 8.2 provides the first important consequence of this assumption: the existence of a CT-invariant vector subspace, of the type discussed in Section 6, with the additional properties (8.5). Such a subspace necessarily satisfies further conditions, listed in Lemma 8.4, and leading – for reasons stated at the very end of Section 8 – to a combinatorial structure, the existence of which contradicts Theorem 2.1.

1. Preliminaries

Unless stated otherwise, manifolds and mappings are smooth, the former connected. The group $\mathrm{Aff}(\mathbb{R})$ of affine transformations $t\mapsto qt+p$ of $\mathbb{R}$, with real p and $q\neq 0$, has the index-two subgroup $\mathrm{Aff}^+(\mathbb{R})=\{(q,p)\in\mathrm{Aff}(\mathbb{R}):q>0\}$, and

$$\text{(1.1)}\qquad \text{nontrivial finite subgroups of } \mathrm{Aff}(\mathbb{R}) \text{ have the form } \{(1,0),(-1,2c)\}$$

with any center $c\in\mathbb{R}$ of the reflection $(-1,2c)$. In fact, the square of any (q,p) in such a subgroup Ξ lies in the intersection $\Xi\cap\mathrm{Aff}^+(\mathbb{R})$, which due to its finiteness must consist of translations, and hence be trivial.

Every $(q,p)\in\mathrm{Aff}^+(\mathbb{R})\smallsetminus\{(1,0)\}$ is either a translation ($q=1$), or has a unique fixed point c (and then we call it a *dilation* with center c, since by choosing

c as the new origin we turn c into 0 and (q,p) into $(q,0)$). Now,

$$\text{any Abelian subgroup of } \mathrm{Aff}^+(\mathrm{I\!R}) \text{ consists of translations, or of dilations with a single center,} \tag{1.2}$$

as two commuting self-mappings of a set preserve each other's fixed-point sets, and so in $\mathrm{Aff}^+(\mathrm{I\!R}) \smallsetminus \{(1,0)\}$ two dilations with different centers cannot commute with each other or with a translation.

Lemma 1.1. *Let $(\cdot,\cdot)$ be a symmetric bilinear form in a real vector space. If a coset S of a $(\cdot,\cdot)$-null one-dimensional subspace Q is not contained in the $(\cdot,\cdot)$-orthogonal complement of Q, then S contains a unique $(\cdot,\cdot)$-null vector.*

In fact, S is parametrized by $t \mapsto x = v+tu$, where u spans Q and $(v,u) \neq 0$, so that $(x,x) = (v,v) + 2t(v,u)$ vanishes for a unique $t \in \mathrm{I\!R}$.

Let a group Γ act on a manifold $\widehat{M}$ freely by diffeomorphisms. One calls the action of Γ *properly discontinuous* if there exists a locally diffeomorphic surjective mapping $\pi : \widehat{M} \to M$ onto some manifold M such that the π-preimages of points of M coincide with the orbits of the Γ action. One then refers to M as the *quotient* of $\widehat{M}$ under the action of Γ and writes $M = \widehat{M}/\Gamma$.

For $\pi, \widehat{M}, M, \Gamma$ as above and a flat linear connection ∇ in a vector bundle $\mathcal{Z}$ over M, let $\hat{\mathcal{Z}}$ and $\hat{\nabla}$ be the π-pullbacks of $\mathcal{Z}, \nabla$ to $\widehat{M}$. If $\widehat{M}$ is also simply connected, the vector space $\mathcal{F}$ of all $\hat{\nabla}$-parallel sections of $\hat{\mathcal{Z}}$ trivializes $\hat{\mathcal{Z}}$, and a homomorphism $\Gamma \to \mathrm{GL}(\mathcal{F})$, known as the *holonomy representation* of ∇, assigns to $\gamma \in \Gamma$ the composite isomorphism

$$\mathcal{F} \to \hat{\mathcal{Z}}_y \to \mathcal{Z}_x \to \hat{\mathcal{Z}}_{\gamma(y)} \to \mathcal{F}, \tag{1.3}$$

described with the aid of any given $y \in \widehat{M}$ and $x = \pi(y)$, where the two middle arrows denote the identity automorphism of $\hat{\mathcal{Z}}_y = \mathcal{Z}_x = \hat{\mathcal{Z}}_{\gamma(y)}$, and the first/last one is the evaluation operator or its inverse. Note that (1.3) does not depend on the choice of $y \in \widehat{M}$, being locally (and hence globally) constant as a function of y. To see this, we choose connected neighborhoods $\widehat{U}$ of y in $\widehat{M}$ and U of $x = \pi(y)$ in M such that $\mathcal{Z}$ restricted to U is trivialized by the space $\mathcal{F}_U$ of its ∇-parallel sections and π maps $\widehat{U}$ diffeomorphically onto U. The isomorphism $\mathcal{F} \to \mathcal{F}_U$ arising as the restriction to $\widehat{U}$ followed by the "identity" identification via π then allows us to apply (1.3) to a fixed section from $\mathcal{F}$, using all $y \in \widehat{U}$ at once.

When $\mathcal{Z}$ is a real line bundle, with the multiplicative group $\mathrm{GL}(\mathcal{F}) = \mathrm{I\!R} \smallsetminus \{0\}$,

$$\text{for any } x \in M, \text{ the image of the holonomy representation } \Gamma \to \mathrm{I\!R} \smallsetminus \{0\} \text{ coincides with the holonomy group of } \nabla \text{ at } x, \tag{1.4}$$

the latter meaning the group of the ∇-parallel transports $\mathcal{Z}_x \to \mathcal{Z}_x$ along all the loops at x. In fact, if (1.3) assigns to $\gamma \in \Gamma$ the multiplication by $q \in \mathrm{I\!R} \smallsetminus \{0\}$ and $y \in \pi^{-1}(x)$ is fixed, the ∇-parallel transport Θ along the π-image of any curve joining y to $\gamma(y)$ in $\widehat{M}$ is $\mathcal{F} \leftarrow \hat{\mathcal{Z}}_y \leftarrow \mathcal{Z}_x$ followed by $\mathrm{Id}_{\mathcal{F}}$ followed by

$\mathcal{Z}_x \leftarrow \hat{\mathcal{Z}}_{\gamma(y)} \leftarrow \mathcal{F}$, the reversed arrows representing the inverses of those in (1.3). Writing $\mathrm{Id}_{\mathcal{F}}$ as q^{-1} times (1.3), we get Θ equal to q^{-1} times the identity of $\mathcal{Z}_x$.

Lemma 1.2. *Suppose that* $q \in \mathrm{I\!R} \smallsetminus \{1, -1\}$ *and a diffeomorphism* $\gamma \in \mathrm{Diff}\,\widehat{M}$ *of a manifold* $\widehat{M}$ *pushes a complete nontrivial vector field* w *forward onto* qw*. If* $\mathrm{I\!R} \ni t \mapsto \phi(t, \cdot\,) \in \mathrm{Diff}\,\widehat{M}$ *denotes the flow of* w*, while a subgroup* $\Gamma \subseteq \mathrm{Diff}\,\widehat{M}$ *contains* γ *and* $\phi(t, \cdot\,)$ *for some* $t \neq 0$*, then the action of* Γ *on* $\widehat{M}$ *cannot be properly discontinuous.*

Proof. The kth iteration γ^k of γ, for $k \in \mathbb{Z}$, pushes w forward onto $q^k w$, giving $\gamma^k \circ \phi(t, \cdot\,) = \phi(q^k t, \cdot\,) \circ \gamma^k$ for all t, so that $\phi(q^k t, \cdot\,) \in \Gamma$ with our fixed t. Choosing $x \in \widehat{M}$ such that $w_x \neq 0$, and setting $\eta = \mathrm{sgn}(1 - |q|)$, we thus get a sequence $\phi(q^{\eta k} t, x)$ with mutually distinct terms when k is large, tending to x as $k \to \infty$, which obviously precludes proper discontinuity. $\square$

The conclusion of Lemma 1.2 remains valid when, instead of $\phi(qt, \cdot\,) \in \Gamma$ for some t, one assumes periodicity of the flow of w, and replaces the condition $\gamma, \phi(t, \cdot\,) \in \Gamma$ with just $\gamma \in \Gamma$ (and then uses t equal to the period of the flow).

Remark 1.3. A submersion from a compact manifold into a connected manifold is a bundle projection, which is the compact case of Ehresmann's fibration theorem [**9**, Corollary 8.5.13].

2. The combinatorial argument

The result established here provides the final step needed to prove Theorem C.

Any $m, k \in \mathbb{Z}$ with $m \geq 2$ give rise to functions $E, \Phi : \mathbb{Z} \to \mathbb{Z}$ and integers a_0, a_1 such that, for any $a, b \in \mathbb{Z}$,

$$\begin{array}{rl} \text{i)} & E(a) = m - (-1)^a k - a, \qquad \text{ii)}\ \Phi(a) = 2m - 2(-1)^a k - a, \\ \text{iii)} & E \text{ is bijective and } \Phi \text{ is an involution}, \\ \text{iv)} & E^{-1}(b) = m - (-1)^{m+k+b} k - b, \qquad \text{v)}\ \Phi(a) = E^{-1}(-E(a)), \\ \text{vi)} & a_1 = E^{-1}(1) = m + (-1)^{m+k} k - 1, \\ \text{vii)} & a_0 = E^{-1}(0) = m - (-1)^{m+k} k, \qquad \text{viii)}\ a_0 + a_1 = 2m - 1. \end{array} \tag{2.1}$$

Let integers $m \geq 2$ and k be fixed, $\mathcal{V} = \{1, \ldots, 2m\}$, and $|\;|$ denote cardinality.

Theorem 2.1. *There is no set* $\mathcal{S} \subseteq \mathcal{V}$ *with the following properties.*

(a) $a_1 \in \mathcal{S}$ *and* $\Phi(a_1) \notin \mathcal{S}$*.*
(b) $a_0 \in \mathcal{S}$ *if and only if* m *is even.*
(c) *If* $a, b \in \mathcal{V}$ *and* $a + b = 2m + 1$*, then exactly one of* a, b *lies in* $\mathcal{S}$*.*
(d) *For every* $a \in \mathcal{S} \smallsetminus \{a_1\}$ *there exists* $b \in \mathcal{S}$ *with* $E(b) = -E(a)$*.*
(e) $|\mathcal{S} \cap \{1, 2, \ldots, 2j\}| \leq j$ *whenever* $j \in \{1, \ldots, m\}$*.*

Proof. Equivalently, (c) states that $\mathcal{S}$ is a selector for the m-element family $\{\{a,b\}\subseteq\mathcal{V}: a+b=2m+1\}$. Hence $|\mathcal{S}|=m$. In addition,

$$\text{i) } |\mathcal{S}|=m\geq 3,\quad \text{ii) } |k|\leq m-1,\quad \text{iii) } \varPhi(\mathcal{S}\smallsetminus\{a_1\})=\mathcal{S}\smallsetminus\{a_1\}. \tag{2.2}$$

In fact, as $a_1\neq a_0$ and $a_0+a_1=2m-1$ by (2.1-viii), having $m=2$ in (2.2-i) would, by (a) – (b), give $\mathcal{S}=\{a_0,a_1\}\subseteq\{1,2,3,4\}$ and $a_0+a_1=3$, implying that $\mathcal{S}=\{1,2\}$, contrary to (e) for $j=1$. Next, (d) and (2.1-v) give $\varPhi(\mathcal{S}\smallsetminus\{a_1\})\subseteq\mathcal{S}\smallsetminus\{a_1\}$, cf. (2.1-iii), with the image not containing a_1, as otherwise, by (2.1-iii), $\varPhi(a_1)$ would lie in $\mathcal{S}$, which contradicts (2.1-i); and (2.1-iii) makes the inclusion an equality, proving (2.2-iii). Finally, using (2.2-i), we may fix $a\in\mathcal{S}\smallsetminus\{a_1,2m\}$. Thus, by (2.2-iii) and (2.1-ii), $1\leq\varPhi(a)=2m-2(-1)^a k-a\leq 2m$. When a is even (odd) this becomes $2\leq 2m-2k-a\leq 2m$ (or, $1\leq 2m+2k-a\leq 2m-1$), yielding $1-m\leq k\leq m-2$ (or $1-m\leq k\leq m-1$), and (2.2-ii) follows.

Let us now define $c_\pm\in\mathbb{Z}$ by

$$c_\pm=m\mp k,\ \text{ so that } 1\leq c_\pm\leq 2m-1 \text{ due to (2.2-ii)}, \tag{2.3}$$

denote by $\mathcal{V}_\pm$ (or, $\mathcal{S}_\pm$) the set of all $a\in\mathcal{V}$ (or, $a\in\mathcal{S}\smallsetminus\{a_1\}$) having $(-1)^a=\pm1$ and, finally, given $a,b\in\mathcal{V}_\pm$ with $a\leq b$, set $[a,b]_\pm=[a,b]\cap\mathcal{V}_\pm$, referring to any such $[a,b]_\pm$ as an *even/odd subinterval* of $\mathcal{V}$. Finally, we let $\mathcal{R}_\pm$ stand for the maximal even/odd subinterval of $\mathcal{V}$ which is symmetric about $c_\pm$. Then

$$\begin{aligned}
&\text{i)} && \mathcal{S}=\mathcal{S}_+\cup\mathcal{S}_-\cup\{a_1\},\qquad \varPhi(\mathcal{S}_\pm)=\mathcal{S}_\pm,\qquad \mathcal{S}_\pm\subseteq\mathcal{R}_\pm,\\
&\text{ii)} && \mathcal{R}_+=[2,2m-2k-2]_+,\quad \mathcal{R}_-=[2k+1,2m-1]_- \ \text{ if } k\geq 0,\\
&\text{iii)} && \mathcal{R}_+=[-2k,2m]_+,\quad \mathcal{R}_-=[1,2m+2k-1]_- \ \text{ if } k<0,\\
&\text{iv)} && \varPhi \text{ restricted to even/odd integers is the reflection about } c_\pm.
\end{aligned} \tag{2.4}$$

In fact, the first relation in (2.4-i) is obvious, the second immediate from (2.2-iii) since, by (2.1-ii), $\varPhi:\mathbb{Z}\to\mathbb{Z}$ preserves parity, Also, (2.1-ii) yields (2.4-iv), which in turns shows that $\mathcal{S}_\pm=\varPhi(\mathcal{S}_\pm)$ is a (possibly empty) union of sets $\{a,b\}$ having $c_\pm$ as the midpoint, and so $\mathcal{S}_\pm\subseteq\mathcal{R}_\pm$. Finally, depending on whether $c_\pm=m\mp k$ is less (or, greater) than the midpoint $m+1/2$ of $\mathcal{V}$, one endpoint of $\mathcal{R}_\pm$ must lie in $\{1,2\}$ (or, in $\{2m-1,2m\}$), and the other endpoint added to this one must yield $2c_\pm$, which proves (2.4-ii) – (2.4-iii).

Note that, as an obvious consequence of (2.4),

$$\begin{aligned}
&\mathcal{S}\smallsetminus\{a_1\} \text{ fails to include specific integers from } \mathcal{V}, \text{ which are:}\\
&\text{the lowest } k \text{ odd and highest } k+1 \text{ even ones when } k>0,\\
&\text{the highest } |k| \text{ odd and lowest } |k|-1 \text{ even ones for } k<0,\\
&\text{the integer } 2m \text{ if } k=0.
\end{aligned} \tag{2.5}$$

Furthermore, one necessarily has

$$k\in\{0,-1\}. \tag{2.6}$$

To see this, we begin by excluding the possibility that $k \ge 2$ (or, $k \le -3$). Namely, if this was the case, (2.5) would give $1, 3, 2m-2, 2m \notin \mathcal{S} \smallsetminus \{a_1\}$ (when $k \ge 2$), or $2, 4, 2m-3, 2m-1 \notin \mathcal{S} \smallsetminus \{a_1\}$ (for $k \le -3$). From the two pairs $\{1, 2m\}$, $\{3, 2m-2\}$ (or, $\{2, 2m-1\}$, $\{4, 2m-3\}$) we would choose one, $\{a, b\}$, having $a_1 \notin \{a, b\}$ and $a + b = 2m + 1$, as well as $a, b \notin \mathcal{S}$, which contradicts (c).

The next two cases that need to be excluded are $k = 1$ and $k = -2$. If one of them occurred, (2.5) would give $1, 2m \notin \mathcal{S} \smallsetminus \{a_1\}$ (if $k = 1$), or $2, 2m-1 \notin \mathcal{S} \smallsetminus \{a_1\}$ (for $k = -2$), which would again contradict (c), unless $a_1 \in \{1, 2m\}$ and $k = 1$, or $a_1 \in \{2, 2m-1\}$ and $k = -2$. However, each of the resulting four possible values $(1, 1), (2m, 1), (2, -2), (2m-1, -2)$ for the ordered pair (a_1, k) leads, via (2.1-vi), to the immediate conclusion that $m \le 1$, contrary to (2.2-i), and so (2.6) follows.

As the next step, we write $m = 2j$ (m even) or $m = 2j + 1$ (m odd), so that $j \ge 1$ by (2.2-i), and proceed to establish the inclusion

$$\mathcal{S}' \cup \{a_*\} \subseteq \mathcal{S} \cap \{1, 2, \ldots, 2j\}, \text{ with } |\mathcal{S}' \cup \{a_*\}| = j + 1, \tag{2.7}$$

which will contradict (e), thus completing the proof of the theorem. Here $\mathcal{S}'$ is the j-element set consisting of all integers from $\{1, 2, \ldots, 2j\}$ with a specific parity (even if $k = -1$, odd for $k = 0$), and $a_* = a_0$ (m even) or $a_* = a_1$ (m odd).

To derive (2.7), we list various conclusions in two separate columns (one for either possible value of k):

(A)	$k = 0$	$k = -1$
(B)	$\mathcal{S}' = \{1, 3, \ldots, 2j-1\}$	$\mathcal{S}' = \{2, 4, \ldots, 2j\}$
(C)	$a_* = 2j \in \mathcal{S}$	$a_* = 2j - 1 \in \mathcal{S}$
(D)	$a_1 = m - 1$	$a_1 = m - 1 + (-1)^m$
(E)	$a_0 = m$	$a_0 = m - (-1)^m$
(F)	$2m \notin \mathcal{S}$	$2m - 1 \notin \mathcal{S}$
(G)	$1 \in \mathcal{S}$	$2 \in \mathcal{S}$.

In fact, (B) is the definition of $\mathcal{S}'$, (E), (D), (C) follow from (2.1-vii) – (2.1-viii), with $a_* \in \mathcal{S}$ due to (a) – (b), while (F) is immediate from (2.5) for $k \in \{-1, 0\}$, and (G) from (F) and (c). What still remains to be shown, for (2.7), is the inclusion

$$\mathcal{S}' \subseteq \mathcal{S}, \tag{2.8}$$

as (2.8) combined with (B) – (C) obviously yields (2.7).

To this end, consider $\Psi : \mathbb{Z} \to \mathbb{Z}$ given by $\Psi(a) = 2m + 1 - a$, so that (c) amounts to $|\mathcal{S} \cap \{a, \Psi(a)\}| = 1$ for all $a \in \mathcal{V}$ or, equivalently, $\Psi(\mathcal{S}) = \mathcal{V} \smallsetminus \mathcal{S}$ and $\Psi(\mathcal{V} \smallsetminus \mathcal{S}) = \mathcal{S}$. Now, in our case, given an integer i,

$$\text{if } 1 \le i < m - 2 \text{ and } i \in \mathcal{S}, \text{ then } i + 2 \in \mathcal{S}. \tag{2.9}$$

Namely, for the sign $\pm$ such that $(-1)^i = \pm 1$, (2.2-iii) and (2.4-iv) yield

$$\underset{\text{in}}{i} \xrightarrow[\Phi]{} \underset{\text{in}}{2m \mp 2k - i} \xrightarrow[\Psi]{} \underset{\text{out}}{i \pm 2k + 1} \xrightarrow[\Phi]{} \underset{\text{out}}{2m - i - 1} \xrightarrow[\Psi]{} \underset{\text{in}}{i + 2},$$

'in' or 'out' meaning lying in $\mathcal{S}$ or in $\mathcal{V}\smallsetminus\mathcal{S}$. In fact, the four sums of pairs of adjacent integers in the above displayed line are $2(m\mp k)=2c_{\pm}$, $2m+1$, $2(m\pm k)=2c_{\mp}$, $2m+1$, as required in the definitions of the reflections $\varPsi$ and $\varPhi$, the latter restricted to even/odd integers. On the other hand, the inequality $i<m-2$ implies, via (D), that $i\neq a_1\neq 2m-i-1$ (and so $2m-i-1\notin\mathcal{S}$, for otherwise $i\pm 2k+1=\varPhi(2m-i-1)$ would lie in $\mathcal{S}$).

Now (2.9) combined with (G) and (B) proves (2.8) by induction on i. Specifically, the highest value of odd (or, even) i such that this yields $i\in\mathcal{S}$ is the one with $i-2<m-2\leq i$, which is the required value $2j-1$ (or, $2j$) except for even m and $k=-1$. In the latter case, although we get $2j-2$ instead of $2j=m$, we have $2j=m=a_1\in\mathcal{S}$ nevertheless, due to (D) and (a). □

3. Compact rank-one ECS manifolds

Throughout this section $(\widehat{M},\widehat{\mathrm{g}})$ is the pseudo-Riemannian universal covering space of a compact rank-one ECS manifold (M,g) of dimension $n\geq 4$, defined as in the Introduction, $\mathcal{D}$ stands for the (one-dimensional, null, parallel) Olszak distribution on (M,g), and $\mathcal{D}^\perp$ for its orthogonal complement, while $\widehat{\mathcal{D}}$, $\widehat{\mathcal{D}}^\perp$ are the analogous distributions on $(\widehat{M},\widehat{\mathrm{g}})$. Thus, $M=\widehat{M}/\Gamma$ for a subgroup Γ of the full isometry group $\mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})$ isomorphic to the fundamental group of M, and acting on $\widehat{M}$ freely and properly discontinuously via deck transformations. The connection in $\widehat{\mathcal{D}}$ induced by the Levi-Civita connection $\widehat{\nabla}$ of $(\widehat{M},\widehat{\mathrm{g}})$ is always flat [**7**, Sect. 9]. Thus, due to simple connectivity of $\widehat{M}$,

$$\begin{array}{l}\widehat{\mathcal{D}}\text{ is spanned by the parallel gradient}\\ \widehat{\nabla}t\text{ of a surjective function }t:\widehat{M}\to I\end{array}\tag{3.1}$$

onto an open interval $I\subseteq\mathrm{I\!R}$ (which is the case even without assuming the existence of a compact quotient). The Olszak distribution being a local geometric invariant of the ECS metric in question [**4**, Sect. 2], (3.1) determines $\widehat{\nabla}t$ and t uniquely up to multiplication by nonzero constants and, respectively, affine substitutions, meaning replacements of t with $qt+p$, where $(q,p)\in\mathrm{Aff}(\mathrm{I\!R})$ (for $\mathrm{Aff}(\mathrm{I\!R})$ as in Section 1: $q,p\in\mathrm{I\!R}$ and $q\neq 0$). Consequently, we have group homomorphisms

$$\text{a) }\mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})\ni\gamma\mapsto(q,p)\in\mathrm{Aff}(\mathrm{I\!R}),\qquad\text{b) }\mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})\ni\gamma\mapsto q\in\mathrm{I\!R}\smallsetminus\{0\},\tag{3.2}$$

characterized, for any $\gamma\in\mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})$, by $t\circ\gamma=qt+p$ and $\gamma^*dt=q\,dt$, that is,

$$(d\gamma)\widehat{\nabla}t\;=\;q^{-1}\widehat{\nabla}t.\tag{3.3}$$

According to [**7**, formula (5.4) and the end of Sect. 11],

$$\begin{array}{l}\widehat{\mathcal{D}}^\perp=\;\mathrm{Ker}\,dt,\text{ the levels of }\;t:\widehat{M}\to I\;\text{ are all}\\ \text{connected and coincide with the leaves of }\;\widehat{\mathcal{D}}^\perp.\end{array}\tag{3.4}$$

On the other hand, as a consequence of (1.1) and (1.4),

(3.5) the image of Γ under (3.2-a) is infinite, while its (3.2-b)-image coincides with the holonomy group of the flat connection in $\mathcal{D}$.

In fact, the first image, if finite, would, by (1.1), lie within some $\{(1,0),(-1,2c)\}$, making $(t-c)^2$ descend to a nonconstant function with at most one critical value on the compact manifold M. The second claim follows from (1.4): by (3.1) and (3.3), the action (1.3) of any $\gamma\in\Gamma$ on the parallel section $\widehat{\nabla}t$ spanning $\widehat{\mathcal{D}}$ equals the multiplication by the corresponding q^{-1}. Namely, the two middle arrows in (1.3) now are restrictions of $d\pi_y$ and $[d\pi_{\gamma(y)}]^{-1}$, so that their composite $\hat{\mathcal{Z}}_y\to\mathcal{Z}_x\to\hat{\mathcal{Z}}_{\gamma(y)}$ equals $d\gamma_y$. (From $\pi\circ\gamma=\pi$ we get $d\pi_{\gamma(y)}\circ d\gamma_y=d\pi_y$.) Thus, (1.3) takes $w=\widehat{\nabla}t$ first to w_y, then (two successive arrows) to $d\gamma_y w_y$ which – by (3.3) – equals $q^{-1}w_{\gamma(y)}$, the evaluation at $\gamma(y)$ of $q^{-1}w$.

The translational/dilational dichotomy of the Introduction, meaning finiteness/infiniteness of the holonomy group of the flat connection in $\mathcal{D}$ induced by the Levi-Civita connection of g, can now be summarized in terms of the homomorphism (3.2-b) restricted to Γ. Specifically, due to (3.5), the two cases are

(3.6)
a) translational: $|q|=1$ for each $\gamma\in\Gamma$,
b) dilational: $|q|\neq 1$ for some $\gamma\in\Gamma$.

Lemma 3.1. *With the assumptions and notations as above,*

(a) *the parallel vector field* $\widehat{\nabla}t$ *on* $\widehat{M}$*, spanning* $\widehat{\mathcal{D}}$*, is complete,*

(b) *in case* (3.6-b)*,* $\phi(t,\,\cdot\,)\notin\Gamma$ *for all* $t\in\mathrm{I\!R}\smallsetminus\{0\}$*,*

$\mathrm{I\!R}\ni t\mapsto\phi(t,\,\cdot\,)\in\mathrm{Diff}\,\widehat{M}$ *being the flow of* $\widehat{\nabla}t$*.*

In fact, (a) appears in [**7**, the second conclusion in Sect. 11], while (b) follows from (a) and Lemma 1.2 combined with (3.3).

The remainder of this section uses the assumptions preceding (3.1) along with

(3.7) transversal orientability of $\mathcal{D}^\perp$ which, by (3.4), reads $\Gamma\subseteq\mathrm{Iso}^+(\widehat{M},\widehat{\mathsf{g}})$,

for the normal subgroup $\mathrm{Iso}^+(\widehat{M},\widehat{\mathsf{g}})$ forming the (3.2-b)-preimage of $(0,\infty)$. This can always be achieved by replacing (M,g) (or, Γ) with a two-fold isometric covering (or, an index-two subgroup).

Theorem 3.2. *In the dilational case* (3.6-b)*, with* (3.7)*, the image of* Γ *under* (3.2-a) *consists of dilations with a single center. The replacement of* t *in* (3.1) *by a suitable affine function of* t *then makes the center appear as* $t=0$*, the interval* I *as* $(0,\infty)$*, and all* (q,p) *in the* (3.2-a)*-image of* Γ *as having* $p=0$*.*

Then the image of Γ *under* (3.2-a)*, always an infinite multiplicative subgroup of* $(0,\infty)$*, must be infinite cyclic unless* $(\widehat{M},\widehat{\mathsf{g}})$ *is locally homogeneous.*

On the other hand, (3.6-b) *follows if one assumes local homogeneity of* $(\widehat{M},\widehat{\mathsf{g}})$*.*

Proof. As shown in [**7**, the beginning of Sect. 11], (3.7) implies the existence of a C^∞ function $\psi : \widehat{M} \to (0,\infty)$ such that the 1-form $\psi\, dt$ is π-projectable onto M (in other words, Γ-invariant), and closed. According to (3.4), the t-levels in $\widehat{M}$ are all connected, and so closedness of $\psi\, dt$ makes t *globally* a function of t, with $\psi = \chi \circ t$ for some C^∞ function $\chi : I \to (0,\infty)$. A fixed antiderivative ϕ of χ thus constitutes a strictly increasing C^∞ diffeomorphism $\phi : I \to J$ onto some open interval $J \subseteq \mathrm{I\!R}$, while Γ-invariance of $d(\phi \circ t) = \psi\, dt$ means that Γ acts on $\phi \circ t$ by translations: $\phi \circ t \circ \gamma = \phi \circ t + a$ with constants $a \in \mathrm{I\!R}$ depending on $\gamma \in \Gamma$. The mappings $t : \widehat{M} \to I$ and $\phi \circ t : \widehat{M} \to J$ are Γ-equivariant relative to Γ acting on I and J via the homomorphisms (3.2-a), restricted to Γ, and $\gamma \mapsto a$. As the diffeomorphism $\phi : I \to J$ makes the two mappings equivariantly equivalent, the two homomorphisms have the same kernel $\Sigma \subseteq \Gamma$, leading to an isomorphism $(q,p) \mapsto \gamma\Sigma \mapsto a$ between the images of the two homomorphisms. The former image must thus be Abelian (as that of $\gamma \mapsto a$ is a group of translations) and so, due to (3.6-b) and (1.2), it consists of dilations with a single center. An affine substitution of t turns this center into 0, and elements of the (3.2-a)-image of Γ into pairs (q,p) with $q > 0$ and $p = 0$. As a result, for our open interval I,

(i) 0 lies in the closure of I, but not in I itself.

The first claim of (i) is obvious: by (3.6-b) – (3.7), for some $q \in (0,\infty) \smallsetminus \{1\}$,

(ii) I is closed under multiplications by powers of q with integer exponents.

To verify the second one, note that, as shown in [**7**, formulae (5.6) – (5.8)], some nonconstant C^∞ function $f : \widehat{M} \to \mathrm{I\!R}$ has

(iii) $f \circ \gamma = q^{-2} f$ for all $\gamma \in \Gamma$ and $(q,p) \in \mathrm{Aff}^+(\mathrm{I\!R})$ with $t \circ \gamma = qt + p$.

This f is also globally a function of t [**7**, the end of Sect. 11]. Treating f, informally, as a function $I \to \mathrm{I\!R}$, and noting that all (q,p) in the (3.2-a)-image of Γ now have $q > 0$ and $p = 0$, we get $f(t) = q^2 f(qt)$ for such q, while these q, due to (3.5), form an infinite subgroup of $(0,\infty)$. Thus, $0 \notin I$, or else, fixing any t in the equality $f(t) = q^2 f(qt)$ and letting $q \to 0$, we would get $f(t) = 0$, even though f is nonconstant.

By (i) and (ii), I equals $(0,\infty)$ or $(-\infty,0)$ and, replacing t with $-t$ if necessary, we get $I = (0,\infty)$, proving the first assertion of the theorem.

To establish the second one, suppose that the (3.2-a)-image of Γ, infinite as a consequence of (3.5), is not cyclic. This makes the image dense in $(0,\infty)$, so that, from continuity of f, our equation $f(t) = q^2 f(qt)$ holds for *all* $t, q \in (0,\infty)$. Setting $t = 1$, we get $f(q) = f(1)/q^2$. The resulting linearity of the function $|f|^{-1/2}$ amounts – see [**7**, Theorem 6.3] – to local homogeneity of $(\widehat{M}, \widehat{\mathrm{g}})$.

Finally, suppose that $(\widehat{M}, \widehat{\mathrm{g}})$ is locally homogeneous. The preceding lines now yield linearity of $|f|^{-1/2}$, that is, $f(t) = f(1)/t^2$ for all $t \in (0,\infty)$, and so f

is unbounded on $(0,\infty)$. This gives (3.6-b), since (3.6-a) would, by (iii), imply Γ-invariance of f, leading to its boundedness, as $M=\widehat{M}/\Gamma$ is compact. ☐

Proof of Theorem D. Due to (3.5) we may, without loss of generality, assume (3.6-b) and (3.7). Our claim now follows from Theorem 3.2. ☐

4. The rank-one ECS model manifolds

In this section we fix the data $f, I, n, V, \langle\cdot,\cdot\rangle, A$ consisting of

(4.1) an integer $n\geq 4$, a real vector space V of dimension $n-2$, a pseudo-Euclidean inner product $\langle\cdot,\cdot\rangle$ on V, a nonzero, traceless, $\langle\cdot,\cdot\rangle$-self-adjoint linear endomorphism A of V, and a nonconstant C^∞ function $f:I\to\mathrm{I\!R}$ on an open interval $I\subseteq\mathrm{I\!R}$.

Treating $\langle\cdot,\cdot\rangle$ as a flat (constant) metric on V, and following [**12**], we define the simply connected n-dimensional pseudo-Riemannian manifold

$$(\widehat{M},\widehat{\mathrm{g}}) \,=\, (I\times\mathrm{I\!R}\times V,\, \kappa\, dt^2 +\, dt\, ds\, +\, \langle\cdot,\cdot\rangle), \tag{4.2}$$

where t,s are the Cartesian coordinates on $I\times\mathrm{I\!R}$, we identify dt, ds and $\langle\cdot,\cdot\rangle$ with their pullbacks to $\widehat{M}$, and the function $\kappa:\widehat{M}\to\mathrm{I\!R}$ is defined by $\kappa(t,s,v)=f(t)\langle v,v\rangle+\langle Av,v\rangle$.

It is well known [**4**, Theorem 4.1] that (4.2) is a rank-one ECS manifold. To describe its isometry group, we need two ingredients. The first is

(4.3) the subgroup S of $\mathrm{Aff}(\mathrm{I\!R})\times\mathrm{O}(V)$ formed by triples (q,p,C) such that $CAC^{-1}=q^2A$, while $qt+p\in I$ and $f(t)=q^2f(qt+p)$ for all $t\in I$,

$\mathrm{O}(V)$ being the group of linear $\langle\cdot,\cdot\rangle$-isometries $C:V\to V$.

The second ingredient is the $2(n-2)$-dimensional real

(4.4) vector space $\mathcal{E}$ of all solutions $u:I\to V$ to the second-order ordinary differential equation $\ddot{u}=fu+Au$, carrying the symplectic form $\varOmega:\mathcal{E}\times\mathcal{E}\to\mathrm{I\!R}$ given by $\varOmega(u^+,u^-)\,=\,\langle\dot{u}^+,u^-\rangle\,-\,\langle u^+,\dot{u}^-\rangle$.

Note that $q,(q,p),C$ all depend homomorphically on the triple $\sigma=(q,p,C)$, and S acts from the left on $C^\infty(I,V)$ via

$$[\sigma u](t) \,=\, Cu((t-p)/q), \tag{4.5}$$

while the operator $u\mapsto\sigma u$ leaves the solution space $\mathcal{E}$ invariant.

Theorem 4.1. *For* $(\widehat{M},\widehat{\mathrm{g}})$ *and* S *as in* (4.1) – (4.3)*, the full isometry group* $\mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})$ *is isomorphic to the set* $\mathrm{G}=\mathrm{S}\times\mathrm{I\!R}\times\mathcal{E}\subseteq\mathrm{Aff}(\mathrm{I\!R})\times\mathrm{O}(V)\times\mathrm{I\!R}\times\mathcal{E}$

endowed with the group operation

$$\begin{aligned}&(q,p,C,r,u)(\hat q,\hat p,\hat C,\hat r,\hat u)\\ &\qquad=(q\hat q,\,q\hat p+p,\,C\hat C,\,-\varOmega(u,(q,p,C)\hat u)+r+\hat r/q,\,(q,p,C)\hat u+u)\\ &\text{or, in the notation of (4.4) – (4.5), with } \sigma=(q,p,C),\\ &(\sigma,r,u)(\hat\sigma,\hat r,\hat u)=(\sigma\hat\sigma,\,\varOmega(\sigma\hat u,u)+r+\hat r/q,\,\sigma\hat u+u).\end{aligned}\tag{4.6}$$

The required isomorphism amounts to the following left action on $\widehat M$ *by the group* G *with the operation* (4.6):

$$\begin{aligned}&(q,p,C,r,u)(t,s,v)\\ &=(qt+p,\,-\langle\dot u(qt+p),2Cv+u(qt+p)\rangle+r+s/q,\,Cv+u(qt+p)).\end{aligned}\tag{4.7}$$

Proof. This is precisely [**1**, Theorem 2], plus [**1**, p. 24, formula (22)] describing the group operation, except for the fact that [**1**] assumes real-analyticity of f along with $I=\mathrm{I\!R}$, and it is these assumptions that force $|q|$, for any $(q,p,C)\in\mathrm{S}$ – see (4.3) – to equal 1. If one ignores the last conclusion and the assumptions that led to it, the proof in [**1**] repeated almost verbatim in our case yields our assertion. However, the resulting right-hand side in (4.7) is not ours, but instead reads

$$(qt+p,\,-\langle\dot u(t),2Cv+u(t)\rangle+r+s/q,\,Cv+u(t))$$

due to the fact that u, instead of $\mathcal{E}$, now lies in the solution space $\mathcal{E}_q$ of the q-dependent equation $\ddot u=fu+q^2Au$. We reconcile both versions by observing that the replacement of u with $t\mapsto u(qt+p)$ defines an isomorphism $\mathcal{E}_q\to\mathcal{E}$.

The notation of [**1**] differs from ours: our $q,p,C,r,u,t,s,v,V,f,\kappa,A,\langle\cdot,\cdot\rangle,\varOmega$ correspond to $\varepsilon,T,H^\lambda_\mu,r,C^\lambda,x^1,2x^n,\mathrm{I\!R}^{n-2},A,\varphi,a_{\lambda\mu},k_{\lambda\mu},2\omega$ in [**1**]. ☐

By (4.6), $\mathrm{G}\ni\gamma=(\sigma,r,u)\mapsto\sigma\in\mathrm{S}$ is a group homomorphism, leading to

$$\text{the normal subgroup } \mathrm{H}=\{(1,0,\mathrm{Id})\}\times\mathrm{I\!R}\times\mathcal{E} \text{ of } \mathrm{G}.\tag{4.8}$$

The group operation (4.6) restricted to H becomes

(a) $(1,0,\mathrm{Id},\hat r,\hat u)(1,0,\mathrm{Id},r,u)=(1,\,0,\,\mathrm{Id},\,\varOmega(u,\hat u)+\hat r+r,\,\hat u+u)$,

and the action (4.7) of H on $\widehat M$ is explicitly given by

(b) $(1,0,\mathrm{Id},r,u)(t,s,v)=(t,\,-\langle\dot u(t),2v+u(t)\rangle+r+s,\,v+u(t))$.

Treating the vector space $\mathcal{E}$ as an Abelian group we get, from (a), an obvious

(c) group homomorphism $\mathrm{H}\ni(1,0,\mathrm{Id},r,u)\mapsto u\in\mathcal{E}$.

Also, as stated in [**7**, formula (5.5)], with a suitable affine substitution,

(d) t in (4.2) can always be made equal to t chosen as in (3.1),

so that, in view of (4.6) – (4.7),

(e) the homomorphism $\mathrm{G}\ni(q,p,C,r,u)\mapsto(q,p)$ coincides with (3.2-a).

Furthermore, according to [**6**, the lines following formula (3.6)], $\widehat\nabla t$ in (3.1) equals twice the coordinate vector field in the s coordinate direction, and so

(f) the flow of $\widehat\nabla t$ on $\widehat M$ is given by $\mathrm{I\!R}\ni r\mapsto(1,0,\mathrm{Id},2r,0)\in\mathrm{H}\subseteq\mathrm{G}$.

In other words, cf. (b), the flow acts on $\widehat{M}$ via $(\tau,(t,s,v))\mapsto(t,s+2\tau,v)$. Also,

(g) $\sigma^*\varOmega=q^{-1}\varOmega$, as an obvious consequence of (4.4) – (4.5).

The subgroup H (canonically identified with $\mathrm{I\!R}\times\mathcal{E}$) acts both on the product $I\times\mathrm{I\!R}\times\mathcal{E}$, by left H-translations of the $\mathrm{H}\approx\mathrm{I\!R}\times\mathcal{E}$ factor, and on $\widehat{M}$, via (b). The following mapping is H-equivariant for these two actions:

$$(4.9)\quad I\times\mathrm{I\!R}\times\mathcal{E}\ni(t,z,u)\,\mapsto(t,s,v)\,=\,(t,z-\langle\dot u(t),u(t)\rangle,\,u(t))\in\widehat{M}=I\times\mathrm{I\!R}\times V.$$

as one easily verifies using (a), (b) and the definition of $\varOmega$ in (4.4).

Remark 4.2. It is useful to note that $(\sigma,r,u)^{-1}=(\sigma^{-1},-qr,-\sigma^{-1}u)$ in G, which yields, for $(\sigma,\hat r,\hat u)=(q,p,C,\hat r,\hat u)\in\mathrm{G}$ and $(1,0,\mathrm{Id},r,u)\in\mathrm{H}$, the equality

$$(\sigma,\hat r,\hat u)(1,0,\mathrm{Id},r,u)(\sigma,\hat r,\hat u)^{-1}=(1,0,\mathrm{Id},\,2\varOmega(\sigma u,\hat u)+r/q,\,\sigma u).$$

Remark 4.3. Nondegeneracy of $\varOmega$ gives $\dim\mathcal{L}'=\dim\mathcal{E}-\dim\mathcal{L}$ for any vector subspace $\mathcal{L}\subset\mathcal{E}$ and its $\varOmega$-orthogonal complement $\mathcal{L}'$. Thus, $2\dim\mathcal{L}\le\dim\mathcal{E}$ whenever $\mathcal{L}$ is isotropic in the sense that $\varOmega(u,u')=0$ for all $u,u'\in\mathcal{L}$.

Remark 4.4. We refer to a rank-one ECS model manifold (4.2) as *generic* when so is A in (4.1), by which we mean that A commutes with only finitely many linear $\langle\cdot,\cdot\rangle$-isometries of V. Genericity of A in (4.1) is an intrinsic property of the metric $\widehat{\mathrm{g}}$, rather than just a condition imposed on the construction (4.1) – (4.2): as stated in [**7**, the paragraph following formula (6.3)], the algebraic type of the pair $\langle\cdot,\cdot\rangle,A$, up to rescaling of A, can be explicitly defined in terms of $\widehat{\mathrm{g}}$ and its Weyl tensor.

5. Generic self-adjoint nilpotent endomorphisms

Throughout this section V denotes a real vector space of dimension $m\ge 2$.

Given a pseudo-Euclidean inner product $\langle\cdot,\cdot\rangle$ on V, we refer to $\langle\cdot,\cdot\rangle$ as *semi-neutral* if its positive and negative indices differ by at most one, and – following the terminology of Remark 4.4 – call a $\langle\cdot,\cdot\rangle$-self-adjoint endomorphism of V *generic* when it commutes with only a finite number of linear $\langle\cdot,\cdot\rangle$-isometries of V. As we show below (Remark 5.4), for $\langle\cdot,\cdot\rangle$-self-adjoint endomorphisms A of V which are nilpotent, genericity is equivalent to having $A^{m-1}\ne 0$ (while $A^m=0$).

Generally, for any endomorphism A of our vector space V and any integer $j\ge 1$, the inclusions $\operatorname{Ker}A^{j-1}\subseteq\operatorname{Ker}A^j$ allow us to define the quotient spaces $\operatorname{Ker}A^j/\operatorname{Ker}A^{j-1}$, and then A obviously descends to *injective linear operators*

$$(5.1)\qquad A:\operatorname{Ker}A^{j+1}/\operatorname{Ker}A^j\to\operatorname{Ker}A^j/\operatorname{Ker}A^{j-1},\quad j=1,\dots,m-1.$$

Setting $d_j=\dim[\operatorname{Ker}A^j/\operatorname{Ker}A^{j-1}]$ we thus have $d_j\ge d_{j+1}$ and, if A is nilpotent,

$$(5.2)\qquad d_1\ge\ldots\ge d_m\ge 0\ \text{ and }\ \dim V=\,d_1+\ldots+d_m,$$

while, whenever $j = 0, \ldots, m$,

$$\dim \operatorname{Ker} A^j = d_1 + \ldots + d_j, \quad \operatorname{rank} A^j = d_{j+1} + \ldots + d_m. \tag{5.3}$$

Thus, $d_m \geq 1$ in the case where A is nilpotent and $A^{m-1} \neq 0$, and then, by (5.2),

$$d_1 = \ldots = d_m = 1 \text{ and (5.1) is an isomorphism for } j = 1, \ldots, m-1. \tag{5.4}$$

Theorem 5.1. *Let a $\langle\cdot,\cdot\rangle$-self-adjoint nilpotent endomorphism A of an m-dimensional pseudo-Euclidean vector space V have $A^{m-1} \neq 0$. Then the inner product $\langle\cdot,\cdot\rangle$ is semi-neutral and there exist exactly two bases $e_1, \ldots, e_m$ of V, differing by an overall sign change, as well as a unique sign factor $\varepsilon = \pm 1$, such that $Ae_j = e_{j-1}$ and $\langle e_i, e_k\rangle = \varepsilon\delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$, where $e_0 = 0$ and $k = m + 1 - j$. Equivalently, the matrix representing A or, respectively, $\langle\cdot,\cdot\rangle$ in our basis has zero entries except those immediately above main diagonal, all equal to 1 or, respectively, except those on the main antidiagonal, all equal to ε.*

Conversely, if A and $\langle\cdot,\cdot\rangle$ are of the above form in some basis $e_1, \ldots, e_m$ of V, then A is $\langle\cdot,\cdot\rangle$-self-adjoint, nilpotent and $A^{m-1} \neq 0$.

Proof. For $j = 0, \ldots, m$, the symmetric bilinear form $(v, v') \mapsto \langle A^j v, v'\rangle$ on V, briefly denoted by $\langle A^j\cdot,\cdot\rangle$, and the subspaces $V_j = A^j(V) \subseteq V$,

(a) $\dim V_j = m - j$ and $V_j \subseteq V_{j-1}$ if $j \geq 1$,

(b) $\langle A^{m-j}\cdot,\cdot\rangle$ descends to the j-dimensional quotient space V/V_j,

(a) being obvious from (5.3) – (5.4), and (b) from

(c) self-adjointness of A along with the relation $A^m = 0$.

As $A^{m-1} \neq 0$, the form resulting from (b) on the line V/V_1 is nonzero, and hence positive or negative definite, which proves the existence and uniqueness of a sign factor $\varepsilon \in \{1, -1\}$ such that $\langle A^{m-1}v, v\rangle = \varepsilon$ for some $v \in V$. More precisely, ε is the *semidefiniteness sign* of $\langle A^{m-1}\cdot,\cdot\rangle$, and

(d) vectors with $\langle A^{m-1}v, v\rangle = \varepsilon$ form a pair of opposite cosets of V_1 in V.

We now prove, by induction on $j = 1, \ldots, m$, the existence of an ordered j-tuple $(S_1, \ldots, S_j) \in V/V_1 \times \ldots \times V/V_j$ of cosets such that $S_j \subseteq \ldots \subseteq S_1$ while, for ε in (d) and every $v \in S_j$,

$$\langle A^{m-1}v, v\rangle = \varepsilon, \ \ \langle A^{m-2}v, v\rangle = \ldots = \langle A^{m-j}v, v\rangle = 0, \tag{5.5}$$

along with uniqueness of $(S_1, \ldots, S_j)$ up to its replacement by $(-S_1, \ldots, -S_j)$. As (d) yields our claim for $j = 1$, suppose that it holds for some $j - 1 \geq 1$. Since $V_j \subseteq V_{j-1} \subseteq \ldots \subseteq V_1$, cf. (a),

(e) the spaces $V_{j-1}, \ldots, V_1$ project onto subspaces $Q_1, \ldots, Q_{j-1}$ of dimensions $1, \ldots, j-1$ in the j-dimensional quotient $Q_j = V/V_j$,

and $Q_1 \subseteq \ldots \subseteq Q_{j-1}$, while the cosets $S_{j-1}, \ldots, S_1$ of $V_{j-1}, \ldots, V_1$ in V, assumed to exist (and be unique up to an overall sign), project onto an ascending chain of cosets of $Q_1, \ldots, Q_{j-1}$ in Q_j. Let us fix a vector $v \in S_{j-1}$, denote by $\hat{R}_1, \ldots, \hat{R}_{j-1}$

the latter cosets (of dimensions $1,\dots,j-1$), and by $(\cdot,\cdot)$ the symmetric bilinear form on Q_j induced by $\langle A^{m-j}\cdot,\cdot\rangle$ via (b). Since (5.5) is assumed to hold for our v, with j replaced by $j-1$, if we set $v_i = A^{j-i}v$, $i=1,\dots,j$, then, for all $i,k\in\{1,\dots,j\}$, due to (c) and the first equality in this version of (5.5), $(v_i,v_k)=0$ if $i+k\le j$ and $(v_i,v_k)=\varepsilon$ when $i+k=j+1$. The $j\times j$ matrix of these $(\cdot,\cdot)$-inner products thus has the entries all equal to ε on the main antidiagonal, and all zero above it. Due to the resulting nondegeneracy of the matrix and the presence of the zero entries, $v_1,\dots,v_j$ project onto a basis $\hat v_1,\dots,\hat v_j$ of Q_j, with $\hat v_i\in Q_i$, $i=1,\dots,j$, and $(\cdot,\cdot)$ is a semi-neutral pseudo-Euclidean inner product in Q_j. Thus, $\hat v_1\in Q_1$ is $(\cdot,\cdot)$-orthogonal to the basis $\hat v_1,\dots,\hat v_{j-1}$ of Q_{j-1}, which makes Q_{j-1} the $(\cdot,\cdot)$-orthogonal complement of the $(\cdot,\cdot)$-null line Q_1. At the same time, the coset $\hat R_1$ of Q_1 is not contained in the $(\cdot,\cdot)$-orthogonal complement Q_{j-1} of Q_1, since $(v_1,v)=(A^{j-1}v,v)=\langle A^{m-1}v,v\rangle=\varepsilon\ne 0$ in the $j-1$ version of (5.5), and so the vector $v=v_j\in S_{j-1}$, projecting onto $\hat v_j\in\hat R_1$, is not $(\cdot,\cdot)$-orthogonal to $\hat v_1$ spanning the line Q_1. By Lemma 1.1, $\hat R_1$ intersects the $(\cdot,\cdot)$-null cone at exactly one point, and so does $-\hat R_1$. This "point" in the j-dimensional quotient $Q_j=V/V_j$ is actually a coset S_j of V_j in V, contained in S_{j-1}, and its lying in the $(\cdot,\cdot)$-null cone amounts to $\langle A^{m-j}v,v\rangle=0$ for all $v\in S_j$, which establishes the inductive step and thus proves the existence and uniqueness claim about (5.5).

This last claim, for $j=m$, yields a unique (up to a sign) coset S_m of $V_m=\{0\}$, that is, a unique pair $\{v,-v\}$ of opposite vectors in V, with

$$\text{(5.6)}\qquad \langle A^{m-1}v,v\rangle=\varepsilon \quad\text{and}\quad \langle A^i v,v\rangle=0 \quad\text{whenever}\quad i\ge 0 \text{ and } i\ne m-1,$$

the case of $i<m-1$ being due to (5.5) for $j=m$, that of $i\ge m$ immediate from (c). Note that S_m uniquely determines the other cosets S_j as $S_m\subseteq\ldots\subseteq S_1$. Setting $e_i=A^{m-i}v$, $i=1,\dots,m$, we obtain an m-tuple of vectors leading to matrices for A and $\langle\cdot,\cdot\rangle$ described in the statement of the theorem, cf. (c) and (5.6). Nondegeneracy of the latter matrix, along with the abundance of zero entries in it, establishes both linear independence of $e_1,\dots,e_m$ and the semi-neutral signature of $\langle\cdot,\cdot\rangle$. Uniqueness of $\{v,-v\}$ clearly implies uniqueness of $e_1,\dots,e_m$ up to their replacement by $-e_1,\dots,-e_m$.

For the converse statement it suffices to note that the basis $e_1,\dots,e_m$ has the form $A^{m-1}v, A^{m-2}v,\dots,Av,v$, and so self-adjointness of A amounts to requiring that the matrix of $\langle\cdot,\cdot\rangle$ have a single value of the entries in each antidiagonal. □

Corollary 5.2. *The only linear isometries of a pseudo-Euclidean space of dimension m commuting with a given generic nilpotent self-adjoint endomorphism A such that $A^{m-1}\ne 0$ are* Id *and* −Id.

In fact, due the up-to-a-sign uniqueness of the basis in Theorem 5.1, such a linear isometry must transform this basis into itself or its opposite.

Corollary 5.3. *Let a nilpotent self-adjoint endomorphism* A *of a pseudo-Euclidean space* V *have* $A^{m-1} \neq 0$, *where* $m = \dim V$. *Then, for every* $q \in (0,\infty)$, *there exists a unique pair* $\{C, -C\}$ *of mutually opposite linear isometries of* V *with* $CAC^{-1} = q^2A$.

Such C *is diagonalized by a basis* $e_1, \dots, e_m$ *chosen as in Theorem* 5.1, *with the respective eigenvalues* $q^{m-1}, q^{m-3}, \dots, q^{1-m}$, *or their opposites, so that* $Ce_j = \pm q^{m+1-2j} e_j$ *for* $j = 1, \dots, m$ *and some fixed sign* $\pm$.

Proof. Uniqueness is immediate from Corollary 5.2 since two such linear isometries differ, composition-wise, by one commuting with A. Existence: defining the linear automorphism C by $Ce_j = \widetilde{e}_j$, for $\widetilde{e}_j = q^{m+1-2j} e_j$, we get the inner products $\langle \widetilde{e}_i, \widetilde{e}_k \rangle = \varepsilon \delta_{ij}$, and $q^2 A \widetilde{e}_j = \widetilde{e}_{j-1}$, for all $i, j \in \{1, \dots, m\}$, with $k = m + 1 - j$ and $\widetilde{e}_0 = 0$, as required. $\square$

Remark 5.4. For a nilpotent self-adjoint endomorphism A of an m-dimensional pseudo-Euclidean space V, five conditions are mutually equivalent:

(i) $A^{m-1} \neq 0$.

(ii) $\operatorname{rank} A = m - 1$ (in other words, $\dim \operatorname{Ker} A = 1$).

(iii) $\pm\mathrm{Id}$ are the only linear self-isometries of V commuting with A.

(iv) A is generic (commutes with only finitely many linear isometries).

(v) 0 is the only skew-adjoint endomorphism of V commuting with A.

In fact, (i) yields (ii) due to (5.3) – (5.4), and the converse is immediate as (ii) and (5.2) – (5.3) force all d_j to equal 1. The implications (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) are obvious from Corollary 5.2. Finally, (v) implies (ii) as any two vectors $v, v' \in \operatorname{Ker} A$ are linearly dependent: the skew-adjoint endomorphism $v \wedge v' = \langle v, \cdot\,\rangle v' - \langle v', \cdot\,\rangle v$, where $\langle \cdot,\cdot \rangle$ is the inner product, commutes with A.

6. Invariant subspaces

This section uses the following assumptions and notations.

First, we fix $q \in (0,\infty) \smallsetminus \{1\}$, an integer $m \geq 2$, a generic self-adjoint nilpotent endomorphism A of an m-dimensional pseudo-Euclidean space V with the inner product $\langle \cdot,\cdot \rangle$, and a linear $\langle \cdot,\cdot \rangle$-isometry C of V having positive eigenvalues and satisfying the condition $CAC^{-1} = q^2 A$.

According to Remark 5.4, Theorem 5.1 and Corollary 5.3, the algebraic type of the above quadruple $(V, \langle \cdot,\cdot \rangle, A, C)$ is uniquely determined by m, q and a sign parameter $\varepsilon = \pm 1$. More precisely, we may choose a basis $e_1, \dots, e_m$ of V such that, for some $\varepsilon \in \{1, -1\}$ and all $i, j \in \{1, \dots, m\}$, with $e_0 = 0$ and $k = m+1-j$,

$$(6.1) \qquad Ae_j = e_{j-1}, \qquad \langle e_i, e_k \rangle = \varepsilon \delta_{ij}, \qquad Ce_j = q^{m+1-2j} e_j.$$

Let the operator T act on functions $(0,\infty) \ni t \mapsto u(t)$, valued anywhere, by

$$(6.2) \qquad [Tu](t) = u(t/q).$$

We also fix a C^∞ function

$$f : (0,\infty) \to \mathrm{I\!R} \quad \text{with} \quad q^2 f(qt) \,=\, f(t) \ \text{ whenever } \ t \in (0,\infty), \tag{6.3}$$

and define $\mathcal{W}, \mathcal{E}$ to be the vector spaces of dimensions 2 and $2m$ formed by all real-valued (or, V-valued) functions y (or, u) on $(0,\infty)$ such that

$$\text{i) } \ \ddot y \,=\, fy \quad \text{or, respectively,} \quad \text{ii) } \ \ddot u = fu + q^2 Au, \tag{6.4}$$

with $(\,)^\cdot = d/dt$. The operator T obviously preserves $\mathcal{W}$, and so we may select a basis y^+, y^- of the space of complex-valued solutions to (6.4-i) having

$$Ty^+ = \mu^+ y^+ \text{ and } Ty^- \text{ equal to } \mu^- y^- \text{ plus a multiple of } y^+, \tag{6.5}$$

for some eigenvalues $\mu^\pm \in \mathbb{C}$, the multiple being zero unless $\mu^+ = \mu^- \in \mathrm{I\!R}$. Since the formula $\alpha(y^+, y^-) = \dot y^+ y^- - y^+ \dot y^-$ (a constant!) defines an area form on $\mathcal{W}$ such that $T^*\alpha = q^{-1}\alpha$, we have $\det T = q^{-1}$ in $\mathcal{W}$. Consequently,

$$\mu^+ \mu^- \,=\, q^{-1}. \tag{6.6}$$

In general, $\mathcal{E}$ is not preserved by either T or by C applied valuewise via $u \mapsto Cu$. Their composition $CT = TC$ however, does leave $\mathcal{E}$ invariant:

$$CT : \mathcal{E} \to \mathcal{E}, \tag{6.7}$$

as it coincides with the operator $u \mapsto \sigma u$ in (4.5). The solution space $\mathcal{E}$ of (6.4-ii) has an ascending m-tuple of CT-invariant vector subspaces:

$$\mathcal{E}_1 \,\subseteq\, \ldots \,\subseteq\, \mathcal{E}_m \,=\, \mathcal{E} \ \text{ with } \ \dim \mathcal{E}_j \,=\, 2j, \tag{6.8}$$

each $\mathcal{E}_j$ consisting of solutions taking values in the space $\operatorname{Ker} A^j$. (Note that, as a consequence of (5.3) – (5.4), $\dim \operatorname{Ker} A^j = j$.)

Theorem 6.1. *Given* $q, m, V, \langle\cdot,\cdot\rangle, A, C, e_1, \ldots, e_m, T, f, \mathcal{W}, \mathcal{E}, y^\pm, \mu^\pm$ *introduced earlier in this section, let* $\mathcal{L}$ *be any* CT*-invariant subspace of* $\mathcal{E}$.

Then in some basis $u_1^+, u_1^-, \ldots, u_m^+, u_m^-$ *of the complexification* $\mathcal{E}^{\mathbb{C}}$ *of* $\mathcal{E}$, *containing a basis of* $\mathcal{L}^{\mathbb{C}}$, *the matrix of* CT *is upper triangular with the diagonal* $(\lambda_1^+, \lambda_1^-, \ldots, \lambda_m^+, \lambda_m^-)$ *where, for some combination coefficients* $(0,\infty) \to \mathbb{C}$,

$$\lambda_j^\pm = q^{m+1-2j}\mu^\pm \text{ and } u_j^\pm \text{ equals } y^\pm e_j \text{ plus a combination of } e_1, \ldots, e_{j-1}, \tag{6.9}$$

$j = 1, \ldots, m$. *If* $\mu^+, \mu^- \in \mathrm{I\!R}$, *we may replace 'complex-valued' by 'real-valued' and the complexifications* $\mathbb{C}, \mathcal{E}^{\mathbb{C}}, \mathcal{E}_j^{\mathbb{C}}$ *by the original real forms* $\mathrm{I\!R}, \mathcal{E}, \mathcal{E}_j$.

Proof. The equation $\ddot u = fu + Au$ imposed on $u = y_1 e_1 + \ldots + y_j e_j$, with $1 \le j \le m$ and complex-valued functions $y_1, \ldots, y_j$, reads

$$\ddot y_j \,=\, f y_j \quad \text{and} \quad \ddot y_i = f y_i + y_{i+1} \text{ for } i < j. \tag{6.10}$$

Since, by (6.1), $e_1, \ldots, e_j$ span $\operatorname{Ker} A^j$, such u lie in $\mathcal{E}_j^{\mathbb{C}}$, for $\mathcal{E}_j$ appearing in (6.8), and we can now define $u_j^\pm$ by (6.9), declaring y_j in (6.10) to be $y^\pm$ and then solving the equations $\ddot y_i \,=\, f y_i + y_{i+1}$ in the descending order $i \,=\, j-1, \ldots, 1$, with a $2(j-1)$-dimensional freedom of choosing the functions y_i. As $u_j^\pm \notin \mathcal{E}_i^{\mathbb{C}}$ for $i < j$, the $2m$ solutions $u_j^\pm$ are linearly independent, and hence constitute a basis $u_1^+, u_1^-, \dots, u_m^+, u_m^-$ of $\mathcal{E}^{\mathbb{C}}$ which makes CT upper triangular with the required diagonal. More precisely, by (6.1) – (6.5), CTu_j^+ (or, CTu_j^-) equals $q^{m+1-2j}\mu^+ u_j^+$ (or, $q^{m+1-2j}\mu^- u_j^-$ plus a multiple of u_j^+), plus a linear combination of $u_i^\pm$ with $i < j$, the multiple being 0 unless $\mu^+ = \mu^- \in \mathrm{I\!R}$.

The freedom of choosing y_i will now ensure that some $u_1^+, u_1^-, \dots, u_m^+, u_m^-$ as above also contains a basis of $\mathcal{L}^{\mathbb{C}}$. Namely, for $\mathcal{L}_j = \mathcal{L} \cap \mathcal{E}_j$, we get inclusion-induced, obviously injective operators $\mathcal{L}_j/\mathcal{L}_{j-1} \to \mathcal{E}_j/\mathcal{E}_{j-1}$, where $1 \le j \le m$ and $\mathcal{L}_0 = \mathcal{E}_0 = \{0\}$, so that, by (6.8), $\delta_j \in \{0,1,2\}$, with $\delta_j = \dim(\mathcal{L}_j/\mathcal{L}_{j-1})$. Our $u_j^\pm$ may now be left completely arbitrary, as before, when $\delta_j = 0$. If j is fixed and $\delta_j = 2$, our operator $\mathcal{L}_j/\mathcal{L}_{j-1} \to \mathcal{E}_j/\mathcal{E}_{j-1}$ is an isomorphism, and so the cosets of $u_j^\pm$, forming a basis of $[\mathcal{E}_j/\mathcal{E}_{j-1}]^{\mathbb{C}}$, are also realized as $\mathcal{L}_{j-1}^{\mathbb{C}}$ cosets of solutions in $\mathcal{L}_j^{\mathbb{C}}$, which we select as the required modified versions of $u_j^\pm$. Finally, in the case $\delta_j = 1$, the embedded line $[\mathcal{L}_j/\mathcal{L}_{j-1}]^{\mathbb{C}}$ in $[\mathcal{E}_j/\mathcal{E}_{j-1}]^{\mathbb{C}}$, due to its CT-invariance, must be one of the two eigenvector cosets represented by $u_j^\pm$, and the latter can thus be modified (within our $2(j-1)$-dimensional freedom) so as to lie in $\mathcal{L}_j^{\mathbb{C}}$. Since $\delta_j = \dim(\mathcal{L}_j/\mathcal{L}_{j-1})$, the total number of modified solutions, $\delta_1 + \ldots + \delta_m$, equals $\dim \mathcal{L}$. Therefore, they form a basis of $\mathcal{L}^{\mathbb{C}}$. $\square$

7. GL($\mathbb{Z}$)-polynomials

By a *root of unity*, or a GL($\mathbb{Z}$)*-polynomial* we mean here any complex number z such that $z^k = 1$ for some integer $k \ge 1$ or, respectively, any polynomial of degree $d \ge 1$ with integer coefficients, the leading coefficient $(-1)^d$, and the constant term 1 or -1. It is well known, cf. [**5**, p. 75], that

(7.1) GL($\mathbb{Z}$)-polynomials of degree d are precisely the characteristic polynomials of matrices in $\mathrm{GL}(d,\mathbb{Z})$.

Every complex root a of a GL($\mathbb{Z}$)-polynomial P is an invertible algebraic integer and P, if also assumed irreducible, is the minimal monic polynomial of a. Then, due to minimality, a is not a root of the derivative of P, showing that

(7.2) the complex roots of an irreducible GL($\mathbb{Z}$)-polynomial are all distinct.

Irreducibility is always meant here to be over $\mathbb{Z}$ or, equivalently, over $\mathbb{Q}$.

We say that a GL($\mathbb{Z}$)-polynomial has a *cyclic root group* if its (obviously nonzero) complex roots generate a cyclic multiplicative group of nonzero complex numbers. The goal of this section is to show that

(7.3) the only irreducible GL($\mathbb{Z}$)-polynomials with a cyclic root group are the cyclotomic and quadratic ones.

We call an irreducible GL($\mathbb{Z}$)-polynomial *cyclotomic* if all of its roots are roots of unity which, up to a sign, agrees with the standard terminology [**10**]. The cyclic root-group condition clearly does hold for all cyclotomic polynomials and all quadratic $\mathrm{GL}(\mathbb{Z})$-polynomials.

First, if an irreducible $\mathrm{GL}(\mathbb{Z})$-polynomial P has among its roots a and a^k, for some $a \in \mathbb{C} \smallsetminus \{1, -1\}$ and an integer $k \notin \{0, 1, -1\}$, then

$$\text{every complex root of } P \text{ is a root of unity.} \tag{7.4}$$

In fact, if $k \geq 2$, then, for such P, a, all $\lambda \in \mathbb{C}$, all integers $r \geq 1$, and some $\mathrm{GL}(\mathbb{Z})$-polynomial Q,

$$P(\lambda^{k^r}) \,=\, Q(\lambda)Q(\lambda^k)\ldots Q(\lambda^{k^{r-1}})P(\lambda), \tag{7.5}$$

as one sees using induction on r, the case $r = 1$ being obvious as $\lambda \mapsto P(\lambda^k)$ has a as a root, which makes it divisible by the minimal polynomial P of a, and the induction step amounts to replacing λ in (7.5) by λ^k. Now (7.4) follows, or else P would have infinitely many roots. The extension of (7.4) to negative integers k is in turn immediate if one notes that $(PQ)^* = P^*Q^*$ and $P^{**} = P$ for the *inversion* P^* of a degree d polynomial P, defined by $P^*(\lambda) = \lambda^d P(1/\lambda)$ or, equivalently, $P^*(\lambda) = a_0\lambda^d + \ldots + a_{d-1}\lambda + a_d$ whenever $P(\lambda) = a_0 + a_1\lambda + \ldots + a_d\lambda^d$. More precisely, we then replace (7.5) with $P(\lambda^{k^r}) \,=\, Q^*(\lambda)Q(\lambda^k)\ldots Q^{[r]}(\lambda^{k^{r-1}})P^{[r]}(\lambda)$, where $P^{[r]}$ equals P or P^* depending on whether r is even or odd.

Remark 7.1. If a $\mathrm{GL}(\mathbb{Z})$-polynomial has the complex roots $c_1, \ldots, c_d$, and k is an integer, then $c_1^k, \ldots, c_d^k$ are the roots of a $\mathrm{GL}(\mathbb{Z})$-polynomial. (By (7.1), we may choose the latter polynomial to be the characteristic polynomial of the kth power of a matrix in $\mathrm{GL}(d, \mathbb{Z})$ with the characteristic roots $c_1, \ldots, c_d$.)

Lemma 7.2. *Let an irreducible* $\mathrm{GL}(\mathbb{Z})$*-polynomial* P *of degree* d *have a root* a^k *for some* $a \in \mathbb{C} \smallsetminus \{1, -1\}$ *and an integer* $k \neq 0$*. Then*

$$a \text{ is an invertible algebraic integer} \tag{7.6}$$

having some $\mathrm{GL}(\mathbb{Z})$*-polynomial* S *as its minimal polynomial, and the complex roots* $c_1, \ldots, c_r$ *of* S *can be rearranged so that, with* $d \leq r$*,*

$$P(\lambda) \,=\, (c_1^k - \lambda)\ldots(c_d^k - \lambda) \quad\text{and}\quad \{c_1^k, \ldots, c_d^k\} = \{c_1^k, \ldots, c_r^k\}, \tag{7.7}$$

Proof. If $k > 0$, the polynomial $\lambda \mapsto P(\lambda^k)$ has the root a, which yields (7.6) and the equality $P(\lambda^k) = Q(\lambda)S(\lambda)$ for all $\lambda \in \mathbb{C}$ and some $\mathrm{GL}(\mathbb{Z})$-polynomial Q. Thus, the kth powers of all the roots $c_1, \ldots, c_r$ of S are roots of P. The polynomial R with the roots $c_1^k, \ldots, c_r^k$ is a $\mathrm{GL}(\mathbb{Z})$-polynomial (Lemma 7.1), while each factor in its unique irreducible factorization has simple roots by (7.2), which are also roots of P, and irreducibility of P thus implies that the factor must equal P. In other words, R is a power of P, and (7.7) follows. When $k < 0$, the preceding assumptions (and conclusions) hold with k, P replaced by $|k|, P^*$ (and a, S unchanged), so that $P^*(\lambda) = (c_1^{|k|} - \lambda)\ldots(c_d^{|k|} - \lambda)$, as required in (7.7). $\square$

Lemma 7.3. *If an irreducible* $\mathrm{GL}(\mathbb{Z})$*-polynomial* P *has two roots of the form* a^k *and* a^ℓ *for* $a \in \mathbb{C} \smallsetminus \{1, 0, -1\}$ *and two distinct nonzero integers* $k, \ell \geq 2$*, then all roots of* P *have modulus* 1.

Proof. Let $k > \ell$. The two versions of (7.7), one for k and one for ℓ, involve the same roots $c_1, \dots, c_r$ of the same polynomial S, so that

$$\{|c_1|^k, \dots, |c_r|^k\} = \{|c_1|^\ell, \dots, |c_r|^\ell\}. \tag{7.8}$$

If the greatest (or, least) of the moduli $|c_1|, \dots, |c_r|$ were greater (or, less) than 1, its kth (or, ℓth) power would lie on the left-hand (or, right-hand) side of (7.8) and be greater than any number on the opposite side, contrary to the equality in (7.8). Thus, $|c_1| = \ldots = |c_r| = 1$. ☐

Lemma 7.4. *If all roots of an irreducible* $\mathrm{GL}(\mathbb{Z})$*-polynomial* P *have modulus* 1*, then they are roots of unity, that is,* P *is cyclotomic.*

Proof. A matrix in $\mathrm{GL}(d, \mathbb{Z})$ with the characteristic polynomial P, cf. (7.1), treated as an automorphism of $\mathbb{C}^d$ is, in view of (7.2), diagonalized by a suitable basis, with unit diagonal entries, so that its powers form a bounded sequence, with a convergent subsequence. As these powers preserve the real form $\mathbb{R}^d \subseteq \mathbb{C}^d$, the convergence takes place in $\mathrm{GL}(d, \mathbb{R})$ and discreteness of the subset $\mathrm{GL}(d, \mathbb{Z})$ makes the subsequence ultimately constant. ☐

Proof of (7.3). Consider an irreducible $\mathrm{GL}(\mathbb{Z})$-polynomial with a cyclic root group generated by $a \in \mathbb{C}$. By (7.2), we may assume that $a \notin \{1, 0, -1\}$. If a is (or is not) a root, our claim follows from (7.4) (or, Lemmas 7.3 – 7.4). ☐

8. Proof of Theorem C

We argue by contradiction. Suppose that, for some rank-one ECS model manifold $(\widehat{M}, \widehat{\mathrm{g}})$ defined by (4.2), with (4.1), and for G as in Theorem 4.1, there exists

(8.1) a subgroup $\Gamma \subseteq \mathrm{G}$ acting on $\widehat{M}$ freely and properly discontinuously with a generic compact quotient manifold $M = \widehat{M}/\Gamma$,

while K_+ in (0.1) is infinite cyclic. Since, by (3.5), $K_+ = K \cap (0, \infty)$ for the image K of the homomorphism $\Gamma \ni (q, p, C, r, u) \mapsto q$, we get (3.6-b). Theorem 3.2 now allows us to set $I = (0, \infty)$ in (4.1), and all $(q, p, C, r, u) \in \Gamma$ have $p = 0$. We fix

$$\widehat{\gamma} = (q, 0, C, \hat{r}, \hat{u}) \in \Gamma \text{ such that } q \text{ is a generator of } K_+. \tag{8.2}$$

By (4.3) and Theorem 4.1, we have (6.3) and $CAC^{-1} = q^2 A$, for f, A in (4.1). Using the notations of (6.2) – (6.7), with $m = n - 2$, we replace Γ, without loss of generality, by a finite-index subgroup Γ_+, which allows us to assume that

$$q \in (0, \infty) \smallsetminus \{1\}, \quad C \text{ has positive eigenvalues, and } \mu^\pm \in \mathbb{C} \smallsetminus (-\infty, 0]. \tag{8.3}$$

Namely, each of these additional requirements amounts to passing from Γ to a subgroup of index at most 2 (or, equivalently, from M to the corresponding finite isometric covering). Specifically, we successively intersect Γ with the kernels of the homomorphisms $\Gamma \to \{1,-1\}$ sending $(q,0,C,r,u)$ to $\operatorname{sgn} q$ and $\operatorname{sgn} C$, the latter sign accounting for positivity of negativity of the eigenvalues of C. (According to Corollary 5.3, one of these cases must take place, and all C occurring in G form an Abelian group.) The last condition (positivity of $\mu^{\pm}$ when they are real) is achieved by replacing $\widehat{\gamma}, q, C, \mu^{\pm}$ with their squares and Γ with the corresponding homomorphic preimage of the index-two subgroup of K_{+} generated by q^2, which is to be done only if $\mu^{\pm}$ are real and negative, cf. (6.6). Finally, we define a linear operator $\varPi : \mathrm{I\!R} \times \mathcal{E} \to \mathrm{I\!R} \times \mathcal{E}$ by

$$\varPi(r,u) \,=\, (2\varOmega(CTu,\hat{u}) \,+\, r/q,\, CTu). \tag{8.4}$$

From the assumption that K_{+} is infinite cyclic we will derive, in Lemma 8.2, the existence of a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ having the following properties.

$$\begin{array}{ll} \text{(A)} & \dim \mathcal{L} = m, \text{ where } m = n-2. \\ \text{(B)} & CT \text{ leaves } \mathcal{L} \text{ invariant.} \\ \text{(C)} & \varPi(\Sigma') = \Sigma' \text{ for some lattice } \Sigma' \text{ in } \mathrm{I\!R} \times \mathcal{L}. \\ \text{(D)} & \varOmega(u,u') = 0 \text{ whenever } u, u' \in \mathcal{L}. \\ \text{(E)} & u \mapsto u(t) \text{ is an isomorphism } \mathcal{L} \to V \text{ for every } t \in (0,\infty). \end{array} \tag{8.5}$$

Remark 8.1. For any rank-one ECS model manifold (4.1) – (4.2), with H and the solution space $\mathcal{E}$ defined in (4.8) and (4.4), if a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ satisfies (8.5-E), with any I instead of $I = (0,\infty)$, then, restricting (4.9) to $(0,\infty) \times \mathrm{I\!R} \times \mathcal{L}$ we clearly obtain an H-equivariant diffeomorphism

$$I \times \mathrm{I\!R} \times \mathcal{L} \,\to\, \widehat{M} \,=\, I \times \mathrm{I\!R} \times V,$$

its bijectivity being due to (8.5-E), and smoothness of its inverse – to the smooth dependence of the isomorphism $\mathcal{L} \ni u \mapsto u(t) \in V$ on t along with real-analyticity of the isomorphism-inversion operation.

Lemma 8.2. *A vector subspace* $\mathcal{L} \subseteq \mathcal{E}$ *with* (8.5) *exists if the conditions preceding* (8.4) *are all satisfied.*

Proof. The surjective submersion $\widehat{M} \ni (t,s,v) \mapsto (\log t)/(\log q) \in \mathrm{I\!R}$, being clearly equivariant relative to the homomorphism

$$\Gamma_{+} \ni \gamma' = (q',0,C',r',u') \mapsto (\log q')/(\log q) \in \mathbb{Z} \tag{8.6}$$

along with the obvious actions of Γ on $\widehat{M}$, via (4.7) with $p = 0$, and $\mathbb{Z}$ on $\mathrm{I\!R}$ by translations, descends to a surjective submersion $M \to S^1$ which is

$$\text{a bundle projection } \widehat{M}/\Gamma_{+} \to \mathrm{I\!R}/\mathbb{Z} = S^1. \tag{8.7}$$

according to Remark 1.3. The kernel Σ of (8.6) equals $\Sigma = \{(1,0,\mathrm{Id})\} \times \Sigma'$ for some set $\Sigma' \subseteq \mathrm{I\!R} \times \mathcal{E}$, since C' in (8.6), due to its positivity, (4.3) and Corollary 5.3, is uniquely determined by q'. Thus, $\Sigma \subseteq \mathrm{H}$, for H given by (4.8). As a consequence of Lemma 3.1(b) and (f) in Section 4, the restriction to Σ of the homomorphism (c) in Section 4 is injective, making Σ Abelian. Now (a) in Section 4 implies that the image of Σ' under the projection $(r,u) \mapsto u$ spans a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ satisfying condition (8.5-D), and so Remark 4.3 gives $\dim \mathcal{L} \le n-2$. Due to (8.5-D) and (a) in Section 4, $\mathrm{H}' = \{(1,0,\mathrm{Id})\} \times \mathrm{I\!R} \times \mathcal{L}$ *is an Abelian subgroup of* H*, containing* Σ*, and the group operation in* H' *identified with* $\mathrm{I\!R} \times \mathcal{L}$ *coincides with the addition in the vector space* $\mathrm{I\!R} \times \mathcal{L}$.

At the same time, the (necessarily compact) fibre of the bundle (8.7) over the $\mathbb{Z}$-coset of $(\log t)/(\log q)$ is obviously the quotient $M_t = [\{t\} \times \mathrm{I\!R} \times V]/\Sigma$. Compactness of M_t implies surjectivity of the linear operator $\mathcal{L} \ni u \mapsto u(t) \in V$ for every $t \in (0,\infty)$, since otherwise a nonzero linear functional vanishing on its image, composed with the projection $\{t\} \times \mathrm{I\!R} \times V \to V$, would descend – according to (b) in Section 4 – to an unbounded function $M_t \to \mathrm{I\!R}$. Thus, $\dim \mathcal{L} \ge n-2 = \dim V$ which, due to the opposite inequality in the last paragraph, gives both (8.5-A) and (8.5-E). Remark 8.1 with $I = (0,\infty)$ and the italicized conclusion of the preceding paragraph, combined with compactness of each of the quotients M_t (and the obvious proper discontinuity of the action of Σ on $\{t\} \times \mathrm{I\!R} \times V$) show that Σ' is a lattice in $\mathrm{I\!R} \times \mathcal{L}$.

Finally, according to Remark 4.2, the right-hand side of (8.4) describes the conjugation by our $\widehat{\gamma}$ in (8.2) applied to $(1,0,\mathrm{Id},r,u) \in \Sigma$, which we identify here with (r,u). As this conjugation obviously sends the kernel Σ onto itself, we get (8.5-C), and so $\varPi(\mathrm{I\!R} \times \mathcal{L}) = \mathrm{I\!R} \times \mathcal{L}$ (since Σ' is a lattice in $\mathrm{I\!R} \times \mathcal{L}$). Now (8.4) yields (8.5-B), which completes the proof. □

Lemma 8.3. *Under the hypotheses preceding* (8.4)*, let a vector subspace* $\mathcal{L} \subseteq \mathcal{E}$ *satisfy* (8.5-A) – (8.5-C)*, a basis* $u_1^+, u_1^-, \dots, u_m^+, u_m^-$ *of* $\mathcal{E}^{\mathbb{C}}$ *containing a basis* $u_1, \dots, u_m$ *of* $\mathcal{L}^{\mathbb{C}}$ *be chosen as in Theorem* 6.1*, and* $\lambda_1, \dots, \lambda_m$ *be the corresponding complex characteristic roots of* $CT : \mathcal{E} \to \mathcal{E}$ *selected from* $\lambda_1^+, \lambda_1^-, \dots, \lambda_m^+, \lambda_m^-$ *given by* (6.9)*. Then*

(i) $\lambda_0 = q^{-1}$ *and* $\lambda_1, \dots, \lambda_m$ *form a* $\mathrm{GL}(\mathbb{Z})$*-spectrum,*

in the sense that they are the complex roots of some $\mathrm{GL}(\mathbb{Z})$*-polynomial of degree* $m+1$*, defined as in Section* 7*, and*

(ii) *the product* $\lambda_1 \dots \lambda_m$ *equals* q *or* $-q$.

Furthermore, assuming in addition that

(iii) *one of* $\mu^\pm$ *is a power of* q *with a rational exponent,*

we have the following conclusions.

(iv) *Both* $\mu^\pm$ *are powers of* q *with integer exponents.*

(v) $\lambda_1^+, \lambda_1^-, \dots, \lambda_m^+, \lambda_m^-$ *are all distinct, real and positive.*

(vi) *Exactly one of* $\lambda_1, \dots, \lambda_m$ *equals* q.

(vii) *Just one, or none of* $\lambda_1, \dots, \lambda_m$ *equals* 1 *if* n *is even, or odd.*

(viii) *Those* $\lambda_1, \dots, \lambda_m$ *not equal to* q *or* 1 *form pairs of mutual inverses.*

(ix) $\Omega(u_i^\pm, u_j^\pm) = 0$ *for all* $i, j \in \{1, \dots, m\}$ *and both signs* $\pm$.

(x) $\Omega(u_i^\pm, u_j^\mp) \neq 0$ *if and only if* $i + j = m + 1$,

Proof. Assertion (i) is immediate from (8.4) and (8.5-C) along with (7.1), and (ii) from (i). Assuming (iii), we see – using (6.9), (6.6) and (7.3) – that, for the $\mathrm{GL}(\mathbb{Z})$-polynomial P with the roots $\lambda_0, \dots, \lambda_m$,

(xi) the irreducible factors of P must all be linear or quadratic,

higher degree cyclotomic polynomials being excluded since the roots are all real. Thus, one of $\lambda_1, \dots, \lambda_m$ equals q, to match $\lambda_0 = q^{-1}$, and (6.9) combined with (6.6) yields (iv). Since $|\lambda_j^\pm|$ is, for either sign $\pm$, a strictly monotone function of j, to prove (v) it suffices to consider the case $q^{m+1-2j}\mu^\pm = q^{m+1-2i}\mu^\mp$, that is, $\mu^\pm/\mu^\mp = q^{2(j-i)}$. Multiplied by $\mu^\pm\mu^\mp = q^{-1}$, cf. (6.6), this makes $(\mu^\pm)^2$ a power of q with an *odd* integer exponent, contrary to (iv), so that (v) follows. From (iii) and (xi) we now get (viii).

For our basis $u_j^\pm$ of $\mathcal{E}$, diagonalizing CT with the eigenvalues $\lambda_j^\pm = q^{m+1-2j}\mu^\pm$, (g) in Section 4 gives

$$\begin{aligned} q^{-1}\Omega(u_i^\pm, u_j^\pm) &= q^{2m+2-2i-2j}(\mu^\pm)^2\Omega(u_i^\pm, u_j^\pm),\\ q^{-1}\Omega(u_i^\pm, u_j^\mp) &= q^{2m+2-2i-2j}\mu^+\mu^-\Omega(u_i^\pm, u_j^\mp). \end{aligned}$$

Thus, the inequality $\Omega(u_i^\pm, u_j^\pm) \neq 0$ would, again, make $(\mu^\pm)^2$ a power of q with an odd integer exponent, contradicting (iv), which yields (ix). Similarly, assuming that $\Omega(u_i^\pm, u_j^\mp) \neq 0$, we now get, from (6.6), $i+j = m+1$. The converse implication needed in (x) follows, via (ix), from nondegeneracy of Ω. □

Lemma 8.4. *With the assumptions and notations of Lemma* 8.3*, let* $\mathcal{L}$ *this time satisfy all of* (8.5)*. Then conditions* (i) – (x) *in Lemma* 8.3 *all hold, so that* $\mu^\pm$ *and* $\lambda_j^\pm$ *are all real, while*

(i) *the number of pluses is different from that of minuses*

among the $\pm$ *superscripts of those* $\lambda_1^+, \lambda_1^-, \dots, \lambda_m^+, \lambda_m^-$ *which form the characteristic roots* $\lambda_1, \dots, \lambda_m$ *of* $CT : \mathcal{L} \to \mathcal{L}$*. Finally, for the basis* $\mathcal{B} = \{u_1, \dots, u_m\}$ *of* $\mathcal{L}$ *contained in the basis* $\{u_1^+, u_1^-, \dots, u_m^+, u_m^-\}$ *of* $\mathcal{E}$*, with* $|\,|$ *denoting cardinality,*

(ii) $|\mathcal{B} \cap \{u_1^+, u_1^-, \dots, u_j^+, u_j^-\}| \leq j$ *whenever* $j = 1, \dots, m$,

(iii) $|\mathcal{B} \cap \{u_i^+, u_j^-\}| = 1$ *if* $i, j \in \{1, \dots, m\}$ *and* $i + j = m + 1$.

Proof. If (ii) failed to hold, the evaluation operator in (8.5-E), complexified if necessary, would send $\{u_1, \dots, u_{j+1}\}$ into the span of the vectors $e_1, \dots, e_j$ appearing in (6.9), contrary to its injectivity. From (ii) we obtain

(iv) $k(j) \geq j$ for all $j = 1, \dots, m$,

$k(j) \in \{1,\dots,m\}$ being such that $u_j = u^{\pm}_{k(j)}$ with some sign ±, since, otherwise, $\mathcal{B} \cap \{u^+_1, u^-_1, \dots, u^+_{k(j)}, u^-_{k(j)}\}$ would have at least $j > k(j)$ elements.

To prove (i), we now assume its negation, and evaluate the product of those $\lambda^{\pm}_j = q^{m+1-2j}\mu^{\pm}$ in (6.9) which constitute $\lambda_1,\dots,\lambda_m$. Both factors μ^+, μ^- appear in this product the same number of times, $m/2$, which makes m even, and by (6.6) their occurrences contribute to our product $\lambda_1 \dots \lambda_m$ a total factor of $q^{-m/2}$. On the other hand, the set $\{q^{m+1-2j} : 1 \le j \le m\} = \{q^{m-1}, q^{m-3}, \dots, q^{1-m}\}$ is closed under taking inverses, so that $\prod_{j=1}^m q^{m+1-2j} = 1$. Writing $k(j) = j + \ell(j)$, with $\ell(j) \ge 0$ due to (iv), we now have

$$\lambda_j = \lambda^{\pm}_{k(j)} = q^{m+1-2k(j)}\mu^{\pm} = q^{m+1-2j}\mu^{\pm}q^{-2\ell(j)}, \tag{8.8}$$

making $\lambda_1 \dots \lambda_m$ equal to 1 times $q^{-m/2}$ times $\prod_{j=1}^m q^{-2\ell(j)}$, that is, a power of q with a negative exponent, contrary to Lemma 8.3(ii).

Next, (i) implies that $\mu^{\pm}$ and $\lambda^{\pm}_j$ are all real, for otherwise λ_j in (8.8), forming along with $\lambda_0 = q^{-1}$ the spectrum of a real matrix, would come in nonreal conjugate pairs, with the same number of positive real parts as negative ones. Thus, by (8.3), $\mu^{\pm} > 0$. Using (i) and reality of $\mu^{\pm}$ we now evaluate the product $\lambda_1 \dots \lambda_m = \pm q$ in Lemma 8.3(ii), observing that not all μ^+, μ^- undergo pairwise "cancellations" (forming the product q^{-1}), but instead Lemma 8.3(ii) equates some power of μ^+ or μ^-, with a positive integer exponent, to a power of q, and so positivity of $\mu^{\pm}$ yields condition (iii) in Lemma 8.3, which in turn implies (iv) – (x).

Finally, the m-element family $\mathcal{P} = \{\{u^+_i, u^-_j\} : i + j = m + 1\}$ forms a partition of $\{u^+_1, u^-_1, \dots, u^+_m, u^-_m\}$ into disjoint two-element subsets, while the mapping $F : \mathcal{B} \to \mathcal{P}$ given by $u \in F(u)$ is injective: $|\mathcal{B} \cap \{u^+_i, u^-_j\}| \le 1$ if $i + j = m + 1$, or else Lemma 8.3(x) would contradict (8.5-D). As $|\mathcal{B}| = m$, surjectivity of F thus follows, proving (iii). □

We now complete the proof of Theorem C by observing that a vector subspace $\mathcal{L} \subseteq \mathcal{E}$ with (8.5) gives rise to a subset $\mathcal{S}$ of $\mathcal{V} = \{1,\dots,2m\}$, for $m = n - 2$, satisfying conditions (a) – (e) in Theorem 2.1, which – according to Theorem 2.1 – cannot exist. Namely, using Lemma 8.3(iv) we define $k \in \mathbb{Z}$ by $\mu^+ = q^k$, so that, by (6.6), $\mu^- = q^{-k-1}$. Next, the obvious order-preserving bijection

$$\mathcal{V} = \{1,\dots,2m\} \to \{u^+_1, u^-_1, \dots, u^+_m, u^-_m\} \tag{8.9}$$

(notation of Lemma 8.3) which, explicitly, sends $a \in \mathcal{V}$ to u^-_j when $a = 2j$ is even, or to u^+_j for odd $a = 2j - 1$, is used from now on to identify the two sets, and we declare $\mathcal{S}$ to be the subset of $\mathcal{V}$ corresponding under (8.9) to the basis $\mathcal{B} = \{u_1,\dots,u_m\}$ of $\mathcal{L}$. The function assigning to each $u^{\pm}_j$ the corresponding eigenvalue $\lambda^{\pm}_j = q^{m+1-2j}\mu^{\pm}$ treated, via (8.9), as defined on $\mathcal{V}$, is now easily seen to be given by $\mathcal{V} \ni a \mapsto q^{E(a)}$, with (2.1-i). Referring to (a) – (e) in Theorem 2.1 simply as (a) – (e), we observe that assertions (ii) and (iii) of Lemma 8.4 yield (e) and (c), while (b), the first claim in (a) and (d) trivially follow from Lemma 8.3(vi)-(viii) (the latter guaranteed to hold by Lemma 8.4). Finally, the relation $\varPhi(a_1) \notin \mathcal{S}$ in (a) which, in view of (2.1-iii) and (2.1-v), amounts to $q^{-1} \notin \{\lambda_1, \dots, \lambda_m\}$, is thus immediate since otherwise, due to Lemma 8.3(viii), the inverse q of q^{-1} would occur on the list $\lambda_1, \dots, \lambda_m$ *twice*, contradicting Lemma 8.3(v).

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