Changes to the third printing May 18, 2013

Measure, Topology, and Fractal Geometry **by Gerald A. Edgar**

Page 34 Line 9. After summer program. **add** Exercise 1.6.3 and 1.6.4 are stated as "either/or". It is possible that *both* alternatives occur: there is some number with more than one expansion, while there is another number with no expansion. For example, modify the Eisenstein number system (Exercise 1.6.10) by using base $b = -2$, but using only 3 of the digits, say $D = \{0, 1, \omega\}$. (Thanks to Peter Hinow for this example.)

Page 43 Line 6. Replace open set U **by** open set T

Page 43 Line 7. Replace $U = \bigcup_{A \in \mathcal{A}} A$ **by** $T = \bigcup_{U \in \mathcal{A}} U$

Page 84 Line 7. After 3.1.1. **add** Let W and F be as on p. 83.

Page 96 Line 5. Replace no cover of $\mathbb{R}^2$ by bounded open sets with order 1.

by no cover with order 1 of $\mathbb{R}^2$ by open sets with bounded diameter.

Page 97 Line 9. Replace $F \cup \bigcup_{i=2}^{n} B_i$ **by** $F_1 \cup \bigcup_{i=2}^{n+2} B_i$

Page 97 Line 10. Replace $\bigcup_{i=2}^{n+2} B_i$ **by** $\bigcup_{i=3}^{n+2} B_i$

Page 97 Line 15. Replace $F_2 \subseteq B_1$ **by** $F_1 \subseteq B_1$

Page 97 Line -14. Replace Theorem 3.4.2 **by** Theorem 3.4.3

Page 100 Line 12. Replace Exercise by

By Theorems (3.4.4) and (3.4.5), we may conclude that $\operatorname{Cov} \mathbb{R} = 1$ since $\operatorname{ind} \mathbb{R} = 1$. Instead, compute $\operatorname{Cov} \mathbb{R}$ directly from the definition of covering dimension.

Page 114 Line 10. After Mauldin–Williams graph

add For technical reasons we assume each vertex has at least one edge leaving it.

Page 115 Line -8. Replace complete metric **by** nonempty complete metric

Page 129 Line -18. Replace two paragraphs beginning Now we take **by**

Now we take the case $\mathcal{L}(A) = \infty$. All of the sets $A \cap [-n, n]$ are measurable, so there exist open sets $U_n \supseteq A \cap [-n, n]$ and compact sets $F_n \subseteq A \cap [-n, n]$ with $\mathcal{L}(U_n \setminus F_n) < \varepsilon/2^{n+2}$. Define $U_n' = U_n \cap \big((-\infty, -n+1+\varepsilon/2^{n+2}) \cup (n-1-\varepsilon/2^{n+2}, \infty)\big)$ and $F_n' = F_n \cap \big([-n, -n+1] \cup [n-1, n]\big)$, so that U_n' is open, F_n' is compact, $U_n' \supseteq A \cap \big([-n, -n+1] \cup [n-1, n]\big) \supseteq F_n'$ and $\mathcal{L}(U_n' \setminus F_n') < 3\varepsilon/2^{n+2} < \varepsilon/2^n$. Now $U = \bigcup U_n'$ is open, and $F = \bigcup F_n'$ is closed (Exercise 2.1.42). We have $U \supseteq A \supseteq F$, and $U \setminus F \subseteq \bigcup_{n \in \mathbb{N}} (U_n' \setminus F_n')$, so that $\mathcal{L}(U \setminus F) \le \sum \mathcal{L}(U_n' \setminus F_n') < \varepsilon$.

Conversely, suppose sets U and F exist. By Theorem (5.1.12) they are measurable. First assume $\overline{\mathcal{L}}(A) < \infty$. Then $\mathcal{L}(F) < \infty$, and $\mathcal{L}(U) \le \mathcal{L}(U \setminus F) + \mathcal{L}(F) < \varepsilon + \mathcal{L}(F) < \infty$. Now $\overline{\mathcal{L}}(A) \le \overline{\mathcal{L}}(U) = \mathcal{L}(U) < \mathcal{L}(F) + \varepsilon = \underline{\mathcal{L}}(F) + \varepsilon \le \underline{\mathcal{L}}(A) + \varepsilon$. This is true for any $\varepsilon > 0$, so $\overline{\mathcal{L}}(A) = \underline{\mathcal{L}}(A)$, so A is measurable.

Page 134 Line 10. Add (Recall that $\inf \varnothing = \infty$, so if there is no countable cover at all of B by sets of $\mathcal{A}$, then $\overline{\mathcal{M}}(B) = \infty$.)

Page 154 Line 16. Replace $h(a)$ **by** $h(f(a))$ **and replace** $h(b)$ **by** $h(f(b))$

Page 160 Line 18. After system **add** on a complete metric space

Page 161 Line 16. After (6.3.11) add Let f_i and K be as in Theorem (4.1.3)

Page 190 Line -10. Replace let **by** Let

Page 208 Line 7. Replace Let **by** For $\sigma \in E^{(\omega)}$, let

Page 213 Line 20. Replace $-\log a / \log 5$ where $a \approx 0.3992$ is a solution of $a^4 - a^3 - a^2 + 3a - 1 = 0$. The dimension is approximately 0.57058;

by $\log\big((\sqrt{5}+3)/2\big)/\log 5 \approx 0.59799$;

Changes: from second printing to third printing February 7, 1995
Measure, Topology, and Fractal Geometry by Gerald A. Edgar

Page 68 Line 10. Replace The points of F **by** The points of F_1
Page 83 Line 8. After $\{U_1, U_2, \cdots\}$.
add (If $\mathcal{B}$ is finite, repeat the basic sets over and over.)
Page 109 Line 4. Replace complex number **by** complex number, $|b| > 1$
Page 145, after line 3. Add

An example of a set in the line not measurable for Lebesgue measure may be found in many texts. For example: [6, pp. 36–37], [9, Theorem 1.4.7], [29, (10.28)], or [48, Chapter 3, Section 4].

Page 187 Line -14. Replace Theorem 6.5.2 **by** Theorem 6.2.5

Changes: from first printing to second printing March 28, 1992
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Page ix Line 14. Delete and
Page ix Line 15. Add and Kenneth Dreyhaupt
Page 4 Line 23. Replace 1.6.7 **by** 1.6.2
Page 17 Line 8. Replace 24 **by** 2.4
Page 29 last display.

$$\textbf{Replace}\quad (2\pi/3) \quad \textbf{by}\quad \frac{2\pi}{3}$$

Page 29–30. Replace a **and** b **by** u **and** v
Page 31 Figure 1.6.8. Missing label L_4**.**
Page 32 Line -9. Replace is has come **by** it has come
Page 33 Line 2.
Replace called by Mandelbrot “Sierpiński's carpet”
by called Sierpiński's carpet (*dywan Sierpinskiego*)
Page 49 Line -10.
Replace $\left\{ y \in \mathbb{R}^d : \rho(x,y) = r \right\}$ **by** $\{ y \in S : \rho(x,y) = r \}$
Page 57 Line 4. Replace $\rightarrow z_1$ **by** $= z_1$
Page 57 Line 7. Replace $\rightarrow z_2$ **by** $= z_2$
Page 57 Line 9. Replace $\rightarrow z_j$ **by** $= z_j$
Page 65 Line 5. Replace 2.3.15 **by** 2.3.16
Page 66 Line 19. Replace first paragraph of proof by

Proof. First, clearly $D(A,B) \geq 0$ and $D(A,B) = D(B,A)$. Since A and B are compact, they are bounded, so $D(A,B) < \infty$.

Page 66 Line -3. Add and let A be a nonempty compact subset of S

Page 67–68. Replace (2.4.4) by

(2.4.4) Theorem. *Suppose S is a complete metric space. Then the space $\mathcal{K}(S)$ is complete.*

Proof. Suppose (A_n) is a Cauchy sequence in $\mathcal{K}(S)$. I must show that A_n converges. Let

$$A = \{\, x : \text{there is a sequence } (x_k) \text{ with } x_k \in A_k \text{ and } x_k \to x \,\}.$$

I must show that $D(A_n, A) \to 0$ and A is nonempty and compact.

Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}$ so that $n, m \geq N$ implies $D(A_n, A_m) < \varepsilon/2$. Let $n \geq N$. I claim that $D(A_n, A) \leq \varepsilon$.

If $x \in A$, then there is a sequence (x_k) with $x_k \in A_k$ and $x_k \to x$. So, for large enough k, we have $\rho(x_k, x) < \varepsilon/2$. Thus, if $k \geq N$, then (since $D(A_k, A_n) < \varepsilon/2$) there is $y \in A_n$ with $\rho(x_k, y) < \varepsilon/2$, and we have $\rho(y, x) \leq \rho(y, x_k) + \rho(x_k, x) < \varepsilon$. This shows that $A \subseteq N_\varepsilon(A_n)$.

Now suppose $y \in A_n$. Choose integers $k_1 < k_2 < \cdots$ so that $k_1 = n$ and $D(A_{k_j}, A_m) < 2^{-j}\varepsilon$ for all $m \geq k_j$. Then define a sequence (y_k) with $y_k \in A_k$ as follows: For $k < n$, choose $y_k \in A_k$ arbitrarily. Choose $y_n = y$. If y_{k_j} has been chosen, and $k_j < k \leq k_{j+1}$, choose $y_k \in A_k$ with $\rho(y_{k_j}, y_k) < 2^{-j}\varepsilon$. Then y_k is a Cauchy sequence, so it converges. Let x be its limit. So $x \in A$. We have $\rho(y, x) = \lim_k \rho(y, y_k) < \varepsilon$. So $y \in N_\varepsilon(A)$. This shows that $A_n \subseteq N_\varepsilon(A)$. Note that, taking $\varepsilon = 1$ in this argument, I have also proved that $A \neq \varnothing$.

So we have $D(A, A_n) \leq \varepsilon$. This concludes the proof that (A_n) converges to A.

Next I show that A is "totally bounded": that is, for every $\varepsilon > 0$, there is a finite ε-net in A. Choose n so that $D(A_n, A) < \varepsilon/3$. By (2.2.5), there is a finite $(\varepsilon/3)$-net for A_n, say $\{y_1, y_2, \cdots, y_m\}$. Now for each y_i, there is $x_i \in A$ with $\rho(x_i, y_i) < \varepsilon/3$. The finite set $\{x_1, x_2, \cdots, x_m\}$ is an ε-net for A.

Now I will show that A is a closed subset of S. Let x belong to the closure $\overline{A}$ of A. Then there exists a sequence (y_n) in A with $\rho(x, y_n) < 2^{-n}$. For each n there is a point $z_n \in A_n$ with $\rho(z_n, y_n) < D(A_n, A) + 2^{-n}$. Now

$$\rho(z_n, x) \leq \rho(z_n, y_n) + \rho(y_n, x) < D(A_n, A) + 2^{-n} + 2^{-n}.$$

This converges to 0, so $z_n \to x$. Thus $x \in A$. This shows that A is closed.

Finally, to show that A is compact, I will show that it is countably compact. Let F be an infinite subset of A. There is a finite $(1/2)$-net B for A, so each element of F is within distance $1/2$ of some element of B. Now F is infinite and B is finite, so there is an element of B within distance $1/2$ of infinitely many elements of F. Let $F_1 \subseteq F$ be that infinite subset. The points of F_1 are all within distance 1 of each other; that is, $\operatorname{diam} F_1 \leq 1$. In the same way, there is an infinite set $F_2 \subseteq F_1$ with $\operatorname{diam} F_2 \leq 1/2$; and so on. There are infinite sets F_j with $\operatorname{diam} F_j \leq 2^{-j}$ and $F_{j+1} \subseteq F_j$ for all j. Now if x_j is chosen from F_j, we have $\rho(x_j, x_k) \leq 2^{-j}$ if $j < k$, so (x_j) is a Cauchy sequence. Since S is complete, (x_j) converges, say $x_j \to x$. Since A is closed, $x \in A$. But then x is a cluster point of the set F. Therefore A is compact. ☺

Page 74 headline. Replace *25.12* **by** *2.5.12*
Page 76 Line -6.
Replace Then apply 2.2.18
by But A is closed, so any point with distance 0 from A belongs to A.
Page 86 Line -15. After empty set. **add**
Note that a space S has ind $S \leq k$ if and only if a point $\{x\}$ and a closed set B not containing x can be separated by a set L with ind $L \leq k-1$. Indeed, there is a basic open set U with $x \in U \subseteq S \setminus B$ and $L = \partial U$ separates $\{x\}$ and B.
Page 92 Line -1. Replace $(ay + bx)$ **by** $(ay - bx)$
Page 93 Line -10. Delete comma
Page 94 Line -12. Replace $g_1(x) \geq x_2 \geq -1$ **by** $g_1(x) \geq x_1 \geq -1$
Page 98 Line -3. Replace finite paths **by** of finite paths
Page 102 Line -4. Replace $(i = 1, 2, \cdots, m)$ **by** $(i = 1, 2, \cdots, n)$
Page 103 Line -16. Replace Exersise 3.1.7 **by** Exercise 3.1.8
Page 108 Line -7. Replace a cluster of **by** a cluster point of
Page 109 Line 5. Replace $\cdots d_k$ **by** $\cdots, d_k$
Page 113 Line 7. After similarities. **add** (See Plate 8.)
Page 113 Line -1. After made. **add**
The transformations map the large square and rectangle shown at the top into the same square and rectangle shown at the bottom.
Page 114 Line 9. Replace $r\colon V \to (0, \infty)$ **by** $r\colon E \to (0, \infty)$
Page 117 Line 5. Replace paragraph by
Under the new metrics, what happens to the maps f_e? If $e \in E_{uv}$, then

$$\begin{aligned} \rho'_u\big(f_e(x), f_e(y)\big) &= a_u \rho_u\big(f_e(x), f_e(y)\big) \\ &= a_u r(e) \rho_v(x, y) \\ &= \frac{a_u r(e)}{a_v}\, \rho'_v(x, y). \end{aligned}$$

Thus, with the new metrics, the maps f_e realize a Mauldin-Williams graph (V, E, i, t, r'), where

$$r'(e) = \frac{a_u}{a_v}\, r(e) \qquad \text{for } e \in E_{uv}.$$

The Mauldin-Williams graph (V, E, i, t, r') is called a **rescaling** of the graph (V, E, i, t, r). A Mauldin-Williams graph (V, E, i, t, r) will be called **contracting** iff it is a rescaling of a strictly contracting graph.
Page 126 Line 5. Replace 1.5.2 **by** 5.1.2
Page 127 Line 11.
Replace and let $A \subseteq \bigcup_{j \in \mathbb{N}} [a_j, b_j)$ **by** and let $A \cup B \subseteq \bigcup_{j \in \mathbb{N}} [a_j, b_j)$

Page 129. Replace statement of Theorem (5.1.12) by

Compact subsets, closed subsets, and open subsets of $\mathbb{R}$ *are Lebesgue measurable.*

Page 129 Line 4. Replace the first paragraph of the proof by

PROOF. Let $K \subseteq \mathbb{R}$ be compact. It is bounded, so $K \subseteq [-n, n]$ for some n, and therefore $\overline{\mathcal{L}}(K) < \infty$. The compact set K is a subset of K, so $\underline{\mathcal{L}}(K) \geq \overline{\mathcal{L}}(K)$.

Let $F \subseteq \mathbb{R}$ be a closed set. Then for each $n \in \mathbb{N}$, the intersection $F \cap [-n, n]$ is compact, and therefore measurable. Thus F is measurable.

Page 131 Line 15. Replace $\overline{\mathcal{L}}(f[U] \setminus f[F]) < r\varepsilon$ **by** $\overline{\mathcal{L}}(f[U] \setminus f[F]) = \overline{\mathcal{L}}\big(f[U \setminus F]\big) < r\varepsilon$

Page 135 Line 12.

(wrong font) Replace $\overline{\mathcal{M}}$-measurable **by** $\overline{\mathcal{M}}$**-measurable**

Page 135 Line -3. Replace period by comma

Page 140 Line 13. Replace let $(f_1, f_2, \cdots f_n)$ is a **by** let $(f_1, f_2, \cdots, f_n)$ be a

Page 154 Line 2. Replace $\varepsilon/2$ **by** ε

Page 154 Line 10.

Replace $0 \leq y_1 < z_1 \leq y_2 < z_2 \leq \cdots \leq y_n < z_n \leq 1$

by $0 \leq y_1 \leq z_1 \leq y_2 \leq z_2 \leq \cdots \leq y_n \leq z_n \leq 1$

Page 154 Line 19.

Replace $\rho(u, v)$, so we have **by** $\rho(u, v)$, so h has bounded increase, and we have

Page 154 Line -6. Replace then the sets $f\big[[x_{i-1}, x_i)\big]$ are measurable and disjoint **by** then the set $f\big[[x_{i-1}, x_i)\big] = f\big[[x_{i-1}, x_i]\big] \setminus \big\{f(x_i)\big\}$ is the difference of two compact sets, hence measurable. The sets $f\big[[x_{i-1}, x_i)\big]$ are disjoint

Page 155 Line 3. On Hausdorff dimension vs. topological dimension add a footnote:

*Optional material. (It depends on Section 3.4.)

Page 156 Line 20. After system. **add** Of course, K is a measurable set, since it is compact.

Page 163 Line 9. Replace *conditon* **by** *condition*

Page 180. Refer to Tricot as well as Taylor.

Page 186 Line 4. Replace A. N. Besicovitch **by** A. S. Besicovitch

Page 191 Line -2.

Replace inside of the fudgeflake **by** filled-in fudgeflake

Page 192 Line 12. Replace Exercise 4.3.12 **by** Figure 4.3.12

Page 224 Line -18. Replace Kurt **by** Curt

Page 228. Replace Maltese cross xii **by** Maltese cross xiii

Page 228.

Replace number systems 28, 34, 59, 83, 34, 109, 131, 165, 178, 204

by number systems 28, 34, 59, 83, 109, 131, 165, 177, 204

Page 228. Space between overlap **and** packing dimension