FRACTAL DIMENSION OF SELF-AFFINE SETS: SOME EXAMPLES

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One of the most common mathematical ways to construct a fractal is as a "self-similar" set. A **similarity** in $\mathbb{R}^d$ is a function $f\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\|f(x) - f(y)\| = r\,\|x - y\|$$

for some constant r. We call r the **ratio** of the map f. If $f_1, f_2, \cdots, f_n$ is a finite list of similarities, then the **invariant set** or **attractor** of the iterated function system is the compact nonempty set K satisfying

$$K = f_1[K] \cup f_2[K] \cup \cdots \cup f_n[K].$$

The set K obtained in this way is said to be **self-similar**. If f_i has ratio $r_i < 1$, then there is a unique attractor K. The **similarity dimension** of the attractor K is the solution s of the equation

$$\sum_{i=1}^{n} r_i^s = 1. \tag{1}$$

This theory is due to Hausdorff [13], Moran [16], and Hutchinson [14]. The similarity dimension defined by (1) is the Hausdorff dimension of K, provided there is not "too much" overlap, as specified by Moran's open set condition. See [14], [6], [10].

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I will be interested here in a generalization of self-similar sets, called **self-affine** sets. In particular, I will be interested in the computation of the Hausdorff dimension of such sets.

If points $x \in \mathbb{R}^d$ are identified with $d \times 1$ column vectors, then an **affine** transformation $f : \mathbb{R}^d \to \mathbb{R}^d$ is a map of the form

$$f(x) = Ax + b,$$

where A is a $d \times d$ matrix and $b \in \mathbb{R}^d$. Often we will assume that $\|A\| < 1$. Here $\|A\|$ is the operator norm of A [the square-root of the largest eigenvalue of $A^T A$].

Let

$$f_i(x) = A_i x + b_i \qquad i = 1, 2, \cdots, n$$

be a finite set of affine transformations of $\mathbb{R}^d$. The **invariant set** or **attractor** of this iterated function system is the compact nonempty set K satisfying

$$K = f_1[K] \cup f_2[K] \cup \cdots \cup f_n[K].$$

The set K obtained in this way is said to be **self-affine**. Self-affine sets have been studied recently by Falconer [8], Urbanski [19], McMullen [15], Bedford [2], Fickel [11], and others.

Falconer's Theorem

The **singular values** of a real $d \times d$ matrix A are the square-roots of the eigenvalues of the matrix $A^T A$. They are nonnegative, so they may be arranged

$$s_1(A) \geq s_2(A) \geq \cdots \geq s_d(A) \geq 0.$$

For each positive number $s \leq d$, define a function ϕ^s on the $d \times d$ real matrices by:

$$\phi^s(A) = s_1(A)\, s_2(A) \cdots s_k(A)\, s_{k+1}(A)^{s-k},$$

where $k = [s]$ is the greatest integer in s.

Now suppose f_i, $i = 1, 2, \cdots, n$, is an iterated function system of affine maps, as above. We consider strings (or sequences) chosen from the alphabet $\{1, 2, \cdots, n\}$. If $\alpha = e_1 e_2 \cdots e_k$ is such a string of length k, we will write $|\alpha| = k$. A matrix will be associated with each such string by

$$A_\alpha = A_{e_1} A_{e_2} \cdots A_{e_k}.$$

The "nominal dimension" associated with the iterated function system is the critical value s_0 such that

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \phi^s(A_\alpha) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0 \end{cases} \tag{2}$$

These definitions come from Falconer [8]. Falconer's theorem shows how the Hausdorff dimension is related to the nominal dimension:

Theorem. *Let $d \times d$ matrices $A_1, A_2, \cdots, A_n$ be given. Suppose $\|A_i\| < 1/3$ for all i. Then for almost all choices of $b_1, b_2, \cdots, b_n \in \mathbb{R}^d$, the attractor of the iterated function system*

$$f_i(x) = A_i x + b_i \qquad i = 1, 2, \cdots, n$$

has Hausdorff dimension equal to the nominal dimension. For all choices of b_i, the Hausdorff dimension is $\leq$ the nominal dimension.

I will next provide a few examples illustrating this theorem of Falconer.

Example 1: Exceptional Cases

There is (at most) a set of measure zero (in $\mathbb{R}^{nd}$) that gives rise to exceptional attractors with Hausdorff dimension strictly smaller than the nominal dimension. In the case of self-similar sets, these exceptions are included among those with too much overlap (as specified by Moran's open set condition). But in the case of self-affine sets, overlap is not the only way to obtain such an exceptional attractor. The open set condition is not sufficient to ensure equality of the dimensions.

Consider this example of an iterated function system in $\mathbb{R}^2$:

$$f_1 \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$
$$f_2 \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The unit square $[0, 1] \times [0, 1]$ is mapped into two nonoverlapping parallelograms by these two maps (as shown in Figure 1). So Moran's open set condition is satisfied. The attractor

Figure 1. An iterated function system.

consists of the origin only. So the Hausdorff dimension is 0, while the nominal dimension is larger than 1.

EXAMPLE 2: NORM > 1/3

Falconer's theorem has hypothesis $\|A_i\| < 1/3$. It would seem that the natural hypothesis is $\|A_i\| < 1$. The next example shows that it is not possible to make this change in Falconer's theorem. This observation comes from Przytycki and Urbanski [17].

Let $0 < \lambda < 1$. Consider the matrices

$$A_1 = A_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & \lambda \end{bmatrix}. \tag{3}$$

If the translation vectors b_1 and b_2 do not lie on the same horizontal or vertical line, then the attractor of the iterated function system

$$f_1(x) = A_1 x + b_1 \qquad f_2(x) = A_2 x + b_2$$

is an affine image of the case when $b_1 = 0$ and $b_2 = (1, \lambda/(1-\lambda))$. In this case, the attractor is the set K_λ consisting of all points (x, y) of the form

$$x = \sum_{i=1}^{\infty} a_i (1/2)^i$$
$$y = \sum_{i=1}^{\infty} a_i \lambda^i,$$

Figure 2. Another iterated function system.

where (a_i) runs through the infinite sequences of 0s and 1s. (The sequence (a_i) corresponding to the point (x, y) of K_λ is called the **address** of (x, y).)

Figure 2 illustrates this iterated function system. The large rectangle is transformed into two smaller rectangles by the two affine transformations. The image rectangles have horizontal dimension shrunk by factor 1/2 and vertical dimension shrunk by factor λ. (Except for the case $\lambda = 1/2$, the attractor K_λ is topologically a Cantor set, so it has topological dimension 0.) Figure 3 shows the attractor $K_{2/3}$.

Figure 3. The attractor $K_{2/3}$.

What is the nominal dimension for this iterated function system? If α is a string of 1s and 2s of length k, then

$$A_\alpha = \begin{bmatrix} (1/2)^k & 0 \\ 0 & \lambda^k \end{bmatrix}.$$

Consider first the case $\lambda \le 1/2$. Then the singular values of A_α are, in order, $(1/2)^k$ and λ^k. So

$$\phi^s(A_\alpha) = \begin{cases} (1/2)^{ks} & \text{for } 0 \le s \le 1 \\ (1/2)^k\, \lambda^{k(s-1)} & \text{for } 1 \le s \le 2. \end{cases}$$

There are 2^k strings of length k, so

$$\sum_{|\alpha|=k} \phi^s(A_\alpha) = \begin{cases} 2^{k(1-s)} & \text{for } 0 \le s \le 1 \\ \lambda^{k(s-1)} & \text{for } 1 \le s \le 2. \end{cases}$$

Thus the critical value given by (2) is $s_0 = 1$. The nominal dimension for this iterated function system is 1. Therefore the Hausdorff dimension for the attractor K_λ is ≤ 1. The projection of the attractor K_λ onto the x-axis is the entire interval $[0,1]$, so K_λ has Hausdorff dimension at least equal to the Hausdorff dimension 1 of $[0,1]$. So in this case, the nominal dimension is achieved by the Hausdorff dimension.

Next consider the case $\lambda > 1/2$. Then the singular values of A_α are, in order, λ^k and $(1/2)^k$. So

$$\phi^s(A_\alpha) = \begin{cases} \lambda^{ks} & \text{for } 0 \le s \le 1 \\ \lambda^k\, (1/2)^{k(s-1)} & \text{for } 1 \le s \le 2. \end{cases}$$

Again there are 2^k strings of length k, so the critical value given by (2) is

$$s_0 = 2 - \frac{\log(1/\lambda)}{\log 2}. \tag{4}$$

The Hausdorff dimension of this attractor K_λ has been studied by Przytycki and Urbanski [17]. Lebesgue measure on $[0,1]$ in the x-axis is the projection of a unique measure on the attractor K_λ. The projection of this measure onto the y-axis yields a measure μ on the line. Przytycki and Urbanski showed that the Hausdorff dimension of K_λ agrees with the nominal dimension (4) if and only if the measure μ is absolutely continuous. This is a question studied in the literature. (See the Problem Section of this volume.) In particular, Erdös [7] showed that the measure μ is singular when λ has certain values, such as the reciprocal golden section $(\sqrt{5}-1)/2$. So at least for these values of λ the Hausdorff dimension is strictly smaller than the nominal dimension.

For almost all choices of translation vectors b_1 and b_2 (namely all cases where b_1 and b_2 are not on the same vertical or horizontal line) the iterated function system

$$f_1(x) = A_1 x + b_1 \qquad f_2(x) = A_2 x + b_2$$

produces an attractor that is an affine image of the set K_λ. So almost all such attractors have the same Hausdorff dimension as the set K_λ. Thus Falconer's theorem fails for the matrices (3) with $\lambda = (\sqrt{5}-1)/2 \approx 0.618$. Here, $\|A_i\| = \lambda$.

Example 3: Non-Commuting Matrices

In Example 2, the matrices A_i commute, so computation of A_α in closed form is easy. For the next example, we will consider non-commuting matrices. The set of labels for the iterated function system will be $\{\mathsf{L}, \mathsf{R}\}$. Fix a constant r with $0 < r < 1$. Let

$$A_{\mathsf{L}} = \begin{bmatrix} r & r \\ 0 & r \end{bmatrix}, \qquad A_{\mathsf{R}} = \begin{bmatrix} r & 0 \\ r & r \end{bmatrix}.$$

Now a closed form for the product matrices A_α is more difficult.

One helpful way to view the product matrices involves the **continuant** polynomials $K_n(x_1, x_2, \cdots, x_n)$. Polynomial K_n is a polynomial in n variables; the definition is recursive:

$$\begin{aligned} K_0() &= 1, \\ K_1(x_1) &= x_1, \\ K_{n+2}(x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2}) &= K_{n+1}(x_1, x_2, \cdots, x_n, x_{n+1})\, x_{n+2} \\ &\qquad + K_n(x_1, x_2, \cdots, x_n). \end{aligned}$$

The term "continuant" refers to the relation with continued fractions:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}} = \frac{K_{n+1}(a_0, a_1, \cdots, a_n)}{K_n(a_1, \cdots, a_n)}.$$

If α is a finite string of Ls and Rs, then the product matrix A_α may be written in terms of continuant polynomials. There are four cases, depending on whether the first letter is an L

or an R, and whether the last letter is an L or an R. For example, if $\alpha = \mathsf{R}^{a_0}\mathsf{L}^{a_1}\mathsf{R}^{a_2}\cdots\mathsf{L}^{a_n}$, then

$$A_\alpha = r^{a_0+\cdots+a_n}\begin{bmatrix} K_{n-1}(a_1,\cdots,a_{n-1}) & K_n(a_1,\cdots,a_n) \\ K_n(a_0,\cdots,a_{n-1}) & K_{n+1}(a_0,\cdots,a_n)\end{bmatrix}.$$

(Reference: [12, p. 288*ff*].)

This form of the product matrices A_α can be used to obtain information about the iterated function systems using the two matrices A_{L} and A_{R}. For example: $K_n(a_1,\cdots,a_n) \le F_{a_1+\cdots+a_n+1}$, where a_i are positive integers, and F_k denotes a Fibonacci number. (Equality holds when all $a_i = 1$.) This can be used to show that an iterated function system

$$f_{\mathsf{L}}(x) = A_{\mathsf{L}}x + b_{\mathsf{L}}, \qquad f_{\mathsf{R}}(x) = A_{\mathsf{R}}x + b_{\mathsf{R}}$$

has a nonempty compact attractor, provided $r < (\sqrt{5}-1)/2 \approx 0.618$.

Figure 4. A non-commuting iterated function system, $r = 0.6$.

Figure 4 shows such an iterated function system for $r = 0.6$. The fixed points for f_{L} and f_{R} were chosen as $(0,1)$ and $(1,0)$, respectively. (The two vertices at the upper right.) So the iterated function system consists of:

$$f_{\mathsf{L}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0 & 0.6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -0.6 \\ 0.4 \end{bmatrix},$$
$$f_{\mathsf{R}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0.6 & 0.6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.4 \\ -0.6 \end{bmatrix}.$$

The large hexagon is mapped to two smaller non-overlapping hexagons by the two affine transformations. The attractor in this case is shown in Figure 5. This attractor is topologically a Cantor set; the diameters of the images of the original hexagon converge to 0. But those hexagons become very distorted, the length much larger than the width. This

Figure 5. The attractor, $r = 0.6$.

reflects the fact that the two singular values of the matrix A_α are unequal; the smaller singular value approaches 0 much more rapidly than the larger singular value.

Computation of the exact nominal dimension by (2) seems difficult in general. It can be done in the case $r = 1/3$. (See below.) For this case, the nominal dimension is 1.

I do not know whether the Hausdorff dimension agrees with the nominal dimension for this particular case. Now the choice of fixed points (or translation vectors b_i) makes a difference in the appearance of the attractor. In Figure 6, the attractor has been illustrated for many choices of fixed points. (The pictures are labeled by the angle with the x-axis made by the vector connecting the two fixed points.) This figure shows some frames of an animation illustrating the way that the attractor depends continuously on the parameters of the iterated function system. This is colorfully known as "a tree blowing in the wind" in Barnsley [1]. Each choice of vectors b_i (except $b_1 = b_2$) will produce an attractor that is an affine image of one of these. So, according to Falconer's theorem, almost all of these sets have Hausdorff dimension 1. (But, strictly speaking, the norms are larger than 1/3, so Falconer's theorem might not apply.)

Two particular cases deserve special note. Angle 135 has fixed points at $(1, 0)$ and $(0, 1)$ for example. This is in Figure 7. This is the case where the image sets line up beside each other as much as possible, so this is the case where dimension strictly smaller than 1 is most likely. (But I do not know whether the Hausdorff dimension is actually < 1.) Angle 45 has fixed points at $(0, 0)$ and $(1, 1)$ for example. This is in Figure 8. The iterated function system maps the triangle shown into two smaller trangles. Because of the intersection here, the attractor itself is a connected curve. Certainly it has Hausdorff dimension 1, equal to the nominal dimension. This attractor is an example of a curve that can be obtained by "corner cutting". For example, this curve is called C_1 in de Rham [18].

Figure 6. Dependence on a parameter.

(Thanks to N. Fickel and R. Gardner for pointing out this reference.) This attractor is a differentiable curve, of course. But it has curvature 0 almost everywhere [18].

Figure 7. The attractor, $r = 1/3$, $\theta = 135$.

Figure 8. The corner cutting iterated function system, and its attractor.

The String Model

We will study here a "string model" (in the sense of [6]) for the iterated function systems constructed from the two matrices

$$A_{\mathsf{L}} = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{bmatrix}, \qquad A_{\mathsf{R}} = \begin{bmatrix} 1/3 & 0 \\ 1/3 & 1/3 \end{bmatrix}.$$

One result of this will be the computation of the nominal dimension 1. This nominal dimension is the Hausdorff dimension of the string model $E^{(\omega)}$. This will be the case since only the dominant singular value is used in (2) when $s \leq 1$. (The idea is that the details of Euclidean space are eliminated, and computations are done in an idealized setting. The parts of $E^{(\omega)}$ are far away from each other, so they do not interfere with each other.)

The notation of [6] is used for this construction. The string model consists of the set $E^{(\omega)}$ of infinite strings made up from the two-letter alphabet $E = \{\mathsf{L}, \mathsf{R}\}$. We will define a metric ρ on $E^{(\omega)}$ that induces the usual product topology, but is compatible with the product matrices A_α. We will also construct a metric outer measure $\mathcal{M}$ on $E^{(\omega)}$ for use in the estimates involved in the Hausdorff dimension.

Let $E^{(*)}$ be the set of all finite strings from the same alphabet E. (This includes the empty string Λ.) Then $E^{(*)}$ has the structure of an infinite binary tree: the root is Λ, and each node $\alpha \in E^{(*)}$ has children $\alpha\mathsf{L}$ and $\alpha\mathsf{R}$. (Figure 9.) For each $\alpha \in E^{(*)}$ let $[\alpha]$ denote the set of all infinite strings that begin with the string α (a "cylinder"). Now, for each $\alpha \in E^{(*)}$, the diameter of the set $[\alpha] \subseteq E^{(\omega)}$ should be the largest singular value of A_α (that is, $s_1(A_\alpha) = \|A_\alpha\|$, the operator norm of A_α): If $\sigma \neq \tau \in E^{(\omega)}$, then $\rho(\sigma,\tau) = s_1(A_\alpha)$, where α is the longest common prefix of σ and τ. There is such an ultrametric ρ on $E^{(\omega)}$ since $\|A_\alpha\|$ decreases to 0 as more letters are added to the right of α (as in [6, p. 71]).

Figure 9. Binary tree.

Now suppose vectors b_{L} and b_{R} are given. Then the iterated function system

$$f_{\mathsf{L}}(x) = A_{\mathsf{L}}x + b_{\mathsf{L}}, \qquad f_{\mathsf{R}}(x) = A_{\mathsf{R}}x + b_{\mathsf{R}}$$

has a unique nonempty compact attractor K. There is a unique continuous function $h\colon E^{(\omega)} \to \mathbb{R}^2$ (the **model map**) such that $f_{\mathsf{L}}\big(h(\sigma)\big) = h(\mathsf{L}\sigma)$ and $f_{\mathsf{R}}\big(h(\sigma)\big) = h(\mathsf{R}\sigma)$. The range $h\big[E^{(\omega)}\big]$ is the attractor K. According to the metric just defined, the model map h satisfies the Lipschitz condition

$$|h(\sigma) - h(\tau)| \leq \rho(\sigma,\tau). \tag{5}$$

Next we will define a measure. The product matrices A_α are shown in Figure 10. For $\alpha \in E^{(*)}$, let

$$w_\alpha = [\,1/2 \quad 1/2\,]\; A_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6}$$

$$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$$

$$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right] \qquad \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$$

$$\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right] \quad \left[\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right] \quad \left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right] \quad \left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]$$

$$\left[\begin{smallmatrix}1&3\\0&1\end{smallmatrix}\right] \; \left[\begin{smallmatrix}3&2\\1&1\end{smallmatrix}\right] \; \left[\begin{smallmatrix}2&3\\1&2\end{smallmatrix}\right] \; \left[\begin{smallmatrix}3&1\\2&1\end{smallmatrix}\right] \; \left[\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right] \; \left[\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right] \; \left[\begin{smallmatrix}1&1\\2&3\end{smallmatrix}\right] \; \left[\begin{smallmatrix}1&0\\3&1\end{smallmatrix}\right]$$

$$\cdots$$

Figure 10. Matrices $3^{|\alpha|}\, A_\alpha$.

$$1$$

$$\frac{1}{2} \qquad \frac{1}{2}$$

$$\frac{2}{9} \quad \frac{5}{18} \quad \frac{5}{18} \quad \frac{2}{9}$$

$$\frac{5}{54} \; \frac{7}{54} \; \frac{4}{27} \; \frac{7}{54} \; \frac{7}{54} \; \frac{4}{27} \; \frac{7}{54} \; \frac{5}{54}$$

$$\cdots$$

Figure 11. Measures $\mathcal{M}\big([\alpha]\big)$.

The measure $\mathcal{M}$ should be defined on cylinders so that $\mathcal{M}\big([\alpha]\big) = w_\alpha$. This will define a measure since $w_\alpha = w_{\alpha\mathsf{L}} + w_{\alpha\mathsf{R}}$. See Figure 11. This defines a metric outer measure. So Borel sets are measurable, and cylinders $[\alpha]$ in particular are measurable.

Next we need to know that the metric and the measure are properly related. In fact, there are constants $p, q > 0$ so that

$$p \operatorname{diam}\big([\alpha]\big) \geq \mathcal{M}\big([\alpha]\big) \geq q \operatorname{diam}\big([\alpha]\big). \tag{7}$$

To see this, suppose

$$A_\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are positive rationals, and $ad - bc > 0$. Then $\mathcal{M}\big([\alpha]\big) = (a+b+c+d)/2$ and $\operatorname{diam}\big([\alpha]\big) = s_1(A_\alpha)$. So we have

$$\begin{aligned} s_1(A_\alpha)^2 \le s_1(A_\alpha)^2 + s_2(A_\alpha)^2 &= \operatorname{trace}\big(A_\alpha^T A_\alpha\big) \\ &= (a+c)^2 + (b+d)^2 \le (a+b+c+d)^2, \end{aligned}$$

so $s_1(A_\alpha) \le 2\mathcal{M}\big([\alpha]\big)$. Also

$$\begin{aligned} s_1(A_\alpha)^2 &\ge \frac{1}{2}\, \big(s_1(A_\alpha)^2 + s_2(A_\alpha)^2\big) \\ &= \frac{1}{2}\, \big((a+c)^2 + (b+d)^2\big) \ge \frac{1}{8}\, (a+b+c+d)^2\,, \end{aligned}$$

so $\sqrt{2}\, s_1(A_\alpha) \ge \mathcal{M}\big([\alpha]\big)$.

Now we are ready to compute the dimension. With $s = 1$ in (2), we can see by (7)

$$\sum_{|\alpha|=k} \phi^1(A_\alpha) = \sum_{|\alpha|=k} s_1(A_\alpha) \le \frac{1}{q} \sum_{|\alpha|=k} \mathcal{M}\big([\alpha]\big) = \frac{1}{q},$$

and similarly

$$\sum_{|\alpha|=k} \phi^1(A_\alpha) \ge \frac{1}{p}.$$

So clearly $s_0 = 1$ is the critical value in (2). Thus the nominal dimension for this iterated function system is 1.

This calculation can also prove that the Hausdorff dimension of $E^{(\omega)}$ is 1. Indeed, the covers $\{\, [\alpha] : |\alpha| = k \,\}$ show that $\mathcal{H}^1\big(E^{(\omega)}\big) \le 1/q < \infty$, so $\dim E^{(\omega)} \le 1$. On the other hand, if $\{U_i\}_{i=1}^\infty$ is a cover of $E^{(\omega)}$ by sets with diameter $\le \varepsilon$, then there exist cylinders $[\alpha_i]$ such that $U_i \subseteq [\alpha_i]$ and $\operatorname{diam} U_i = \operatorname{diam}\,[\alpha_i]$. (See [6, p. 72].) Now

$$\begin{aligned} \sum \operatorname{diam} U_i = \sum \operatorname{diam}\,[\alpha_i] &\ge \frac{1}{p} \sum \mathcal{M}\big([\alpha_i]\big) \\ &\ge \frac{1}{p} \mathcal{M}\big(E^{(\omega)}\big) = \frac{1}{p}. \end{aligned}$$

Therefore $\mathcal{H}^1_\varepsilon\big(E^{(\omega)}\big) \ge 1/p$. This is true for all $\varepsilon > 0$, so $\mathcal{H}^1\big(E^{(\omega)}\big) \ge 1/p > 0$. Thus $\dim E^{(\omega)} \ge 1$.

The model map h is Lipschitz (see (5)), so this proves that the Hausdorff dimension of the attractor K is ≤ 1, regardless of the choices of translations b_{L} and b_{R}.

It should be noted here that in the string space $E^{(\omega)}$, a ball $B_\varepsilon(\sigma)$ of radius $\varepsilon < 1$ always has diameter $\geq \varepsilon/6$. So the same measure $\mathcal{M}$ can be used to prove that the packing dimension of $E^{(\omega)}$ is also 1. And again the packing dimension of the attractor K is ≤ 1.

Entropy and Dimension

Now we will return to the case of general r in Example 3. Computation of the exact nominal dimension seems difficult. Even computation of the exact Hausdorff dimension of the string model $E^{(\omega)}$ seems difficult. Let us consider a related computation which is easier.

Instead of the Hausdorff dimension of a *set* it is sometimes useful to consider the Hausdorff dimension of a finite measure μ. By definition,

$$\mathcal{H}^s_\varepsilon(\mu) = \inf \sum_i (\operatorname{diam} U_i)^s,$$

where the infimum is over all countable families $\{U_i\}$ covering μ in the sense that the complement of the union has measure zero: $\mu\big((\bigcup U_i)^c\big) = 0$. Then

$$\mathcal{H}^s(\mu) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(\mu).$$

Finally, $\dim \mu$ is the critical value s_0 such that

$$\mathcal{H}^s(\mu) = \begin{cases} \infty, & \text{if } s < s_0 \\ 0, & \text{if } s > s_0. \end{cases}$$

If K is the closed support of μ, then clearly $\dim K \geq \dim \mu$. The reverse inequality is sometimes true, but sometimes false. (Of course, the computation above of the Hausdorff dimension of the space $E^{(\omega)}$ also computed $\dim \mathcal{M} = 1$.)

Now consider the string space $E^{(\omega)}$, where $E = \{\mathsf{L}, \mathsf{R}\}$. The matrices are

$$C_{\mathsf{L}} = \begin{bmatrix} r & r \\ 0 & r \end{bmatrix}, \qquad C_{\mathsf{R}} = \begin{bmatrix} r & 0 \\ r & r \end{bmatrix}.$$

So $C_e = (3r)\, A_e$ for $e \in E$, and therefore $C_\alpha = (3r)^{|\alpha|} A_\alpha$ for $\alpha \in E^{(*)}$. The metric will be changed: the diameter of $[\alpha]$ is $s_1(C_\alpha)$. (This is the old value multiplied by $(3r)^{|\alpha|}$.) The measure to be considered is the same measure $\mathcal{M}$ as used before, from (6). Thus with the new metric, we have

$$q\,(3r)^{-|\alpha|} \operatorname{diam}\big([\alpha]\big) \leq \mathcal{M}\big([\alpha]\big) \leq p\,(3r)^{-|\alpha|} \operatorname{diam}\big([\alpha]\big).$$

We are interested in evaluating the Hausdorff dimension of this measure. We will use the "entropy" of a shift on the string space $E^{(\omega)}$.

The numbers defined in (6) are additive ($w_\alpha = w_{\alpha\mathsf{L}} + w_{\alpha\mathsf{R}}$) since the column $[1 \quad 1]^T$ is a right eigenvector of the sum $A_\mathsf{L} + A_\mathsf{R}$ for the eigenvalue 1. We have also chosen the row $[1/2 \quad 1/2]$ to be a left eigenvector of $A_\mathsf{L} + A_\mathsf{R}$ for the eigenvalue 1. This means that $w_\alpha = w_{\mathsf{L}\alpha} + w_{\mathsf{R}\alpha}$. Or, in different language, the measure $\mathcal{M}$ is **invariant** for the left shift on the string space $E^{(\omega)}$. The **left shift** on $E^{(\omega)}$ is

$$\theta(e\sigma) = \sigma,$$

that is, drop the first letter in the string. The measure $\mathcal{M}$ is **invariant** for the shift θ in the sense that

$$\mathcal{M}\big(U\big) = \mathcal{M}\big(\theta^{-1}[U]\big)$$

for all measurable sets $U \subseteq E^{(\omega)}$. This can be seen by first proving it for cylinders, then approximating.

Now θ is ergodic on $\big(E^{(\omega)}, \mathcal{M}\big)$, and the partition $\big\{[\mathsf{L}], [\mathsf{R}]\big\}$ is a generator. (See [3].) Therefore the **entropy** of this system is

$$h = -\lim_k \frac{1}{k} \log \mathcal{M}\big([\sigma{\restriction}k]\big).$$

The limit exists for almost all $\sigma \in E^{(\omega)}$ by the Shannon-McMillan-Breiman theorem ([3]). The notation $\sigma{\restriction}k$ is used for the string consisting of the first k letters of σ. So, for a typical σ, we have

$$\mathcal{M}\big([\sigma{\restriction}k]\big) \approx e^{-hk}$$

for k large. The dimension computation can now be carried out in the same way as in [6, Theorem 7.4.6]. Heuristically, the dimension should be the exponent s so that $\mathcal{M}\big([\alpha]\big) \approx \big(\operatorname{diam}[\alpha]\big)^s$. But

$$\begin{aligned} &\mathcal{M}\big([\alpha]\big) \approx e^{-h|\alpha|} \\ &\operatorname{diam}[\alpha] \approx (3r)^{|\alpha|}\mathcal{M}\big([\alpha]\big) \approx \big(3r\, e^{-h}\big)^{|\alpha|}. \end{aligned}$$

The dimension is the solution s of the equation $(3re^{-h})^s = e^{-h}$, or

$$\dim \mathcal{M} = \frac{h}{h - \log(3r)}.$$

This shows how the computation of the entropy h is related to the computation of the dimension. See [3].

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