REALCOMPACTNESS AND MEASURE-COMPACTNESS OF THE UNIT BALL IN A BANACH SPACE*

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Abstract. It is proved that the unit ball (with its weak topology) is not real-compact in the Banach spaces $\ell_\infty/c_0$ and $J(\omega_1)$. It is stated, but not proved, that the unit ball is not measure-compact in the Banach space $\ell_\infty$.

1. Let $X$ be a Banach space. Topological properties of the weak topology $\sigma(X, X^*)$ have been of interest recently (for example [4][9]). The unit ball $B_X = \{x \in X : ||x|| \leq 1\}$ in the relative weak topology can also be considered. Since $(B_X, \text{weak})$ is a closed subset of $(X, \text{weak})$, we see that if $(X, \text{weak})$ is real-compact (measure-compact), so is $(B_X, \text{weak})$. The question I will be concerned with in this paper is whether the converse is true.

I do not have an answer to the question in general. In this paper, some concrete Banach spaces $X$ are considered that are known not to be realcompact (or measure-compact), and it is proved that $B_X$ is also not realcompact (or measure-compact). In some cases this is more difficult for $B_X$ than for $X$. Reasons for the extra difficulty are hard to pin down. Corson's criterion for realcompactness in $X$ [1, p. 10] is false when applied to $B_X$ (see Theorem 5.3, below). The $\sigma$-algebra of Baire sets for $X$ is generated by $X^*$ [4, Theorem 2.3] but this is not necessarily true for $B_X$ (see Section 3).

Topological words and phrases will always refer to the weak topology $\sigma(X, X^*)$ unless the contrary is specified. If $T$ is a topological space, we write $C(T)$ for the set of all continuous, real-valued functions on $T$.

General background on realcompactness can be found in [8]; on measure-compactness can be found in [9].

2. In this preliminary section, we will recast some topological conditions in terms of nets. Doubtless this could be avoided in the sequel, but I find it helpful.

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2.1 Definition. A $\sigma$-directed set is a directed set such that every countable subset has an upper bound. A $\sigma$-net is a net whose domain is a $\sigma$-directed set.

The proofs of the following observations are omitted.

A topological space $T$ is Lindelöf if and only if every $\sigma$-net in $T$ has a cluster point.

A $\sigma$-net that converges in $\mathbb{R}$ is eventually constant. A $\sigma$-net in $\mathbb{R}$ that does not converge has at least two finite cluster points.

If a $\sigma$-net is in a countable union $\bigcup_{n=1}^{\infty} A_n$, then it is frequently in $A_n$ (for some $n$).

Let $I$ be a set whose cardinal is not 2-valued measurable [that is, the discrete space $I$ is realcompact]. If $(x_\xi)$ is a $\sigma$-net in a union $\bigcup_{i \in I} A_i$ that is not eventually in any $A_i$, then there exist disjoint $I_1, I_2 \subseteq I$ such that $(x_\xi)$ is frequently in each of the sets $\bigcup_{i \in I_1} A_i, \bigcup_{i \in I_2} A_i$.

Let $T$ be a topological space. Then $T$ is realcompact if and only if each $\sigma$-net $(x_\xi)$ such that $h(x_\xi)$ converges for all continuous $h : T \to \mathbb{R}$ is convergent. (In general, the limits of such nets are the points of the Hewitt real compactification $uX$.)

3. I include here an example where Baire $(B_X, \text{weak}) \not= \text{Baire}(X, \text{weak}) \cap B_X$. Some of the later examples have the same property, but the verification is simpler in this case.

Let $X = \ell_1(R)$, where $\text{card } R > 2^0$. Define

$$G = \{f : ||f|| \leq 1, f(\gamma) > \frac{3}{4} \text{ for some } \gamma \in R\}.$$

Then (1) $G$ is a cozero set in $B_X$; and (2) there is no Baire set $D$ in $X$ with $D \cap B_X = G$.

To see that (1) is true, consider the function.

$$f \mapsto \frac{3}{4} \max_{\gamma \in \gamma} f(\gamma)$$

on $(B_X, \text{weak})$. It is continuous since the closure of any set $A_\gamma = \{f : f(\gamma) > \frac{3}{4}\}$ is disjoint from the closure of the union of all the rest.
For (2), suppose $D$ is a Baire set in $(X, \text{weak})$ with $D \cap B_X = G$. Then [4, Theorem 2.3] $D$ is determined by countably many linear functionals $(g_1, g_2, \ldots) \subseteq \ell_1(r)^\star$. Let $e_\gamma$ be the canonical unit vectors in $\ell_1(r)$. Since $\text{card } r > 2^{\aleph_0}$, there is an uncountable $r_0 \subseteq r$ with $g_i(e_\gamma) = g_i(e_\gamma')$ for all $\gamma, \gamma' \in r_0$ and all $i = 1, 2, \ldots$. Now $e_\gamma \in G \subseteq D$, so $\frac{1}{2} (e_\gamma + e_\gamma') \in D$ when $\gamma, \gamma' \in r_0$, but not in $G$. So $D \cap B_X \neq G$.

4. The next example is the space $X = \ell_\infty/c_0$, which Corson showed is not realcompact [1, p. 12]. The proof that $B_X$ is not realcompact is similar to Corson's proof, but greater care must be taken, since Corson's criterion for realcompactness of $X$ may fail for $B_X$.

We consider $\ell_\infty/c_0 = C(\beta\mathbb{N}\setminus\mathbb{N})$. For countable ordinals $\alpha$, there exist clopen sets $T_\alpha$ in $\beta\mathbb{N}\setminus\mathbb{N}$ such that if $\alpha < \beta$ then $T_\alpha \subseteq T_\beta$ [1, p. 13]. Let $x_\alpha = x_{\alpha T_\alpha} \in C(\beta\mathbb{N}\setminus\mathbb{N}) = X$, and $F = x_{\cup T_\alpha} \in X^\star$. Corson showed $F \notin X$ but $x_\alpha + F$ in $\nu_X$. In fact, $||x_\alpha|| = 1$, $||F|| = 1$, so I must show that $h(x_\alpha)$ converges for any $h \in C(B_X)$. Suppose not. Then there exist $a < b$ such that $h(x_\alpha) > b$ frequently and $h(x_\alpha) < a$ frequently.

Note that if $H \subseteq \beta\mathbb{N}\setminus\mathbb{N}$ is the support of a measure, then (by countable additivity) there exists $\beta < \omega_1$ such that $H \cap (\cup_{\alpha} T_\alpha) = H \cap T_\beta$. So for each $\alpha$ such that $h(x_\alpha) > b$ [respectively, $h(x_\alpha) < a$], choose a basic neighborhood of $x_\alpha$ so that $h(x) > b$ [respectively, $h(x) < a$] on it. By considering finitely many supports of measures, it follows that there exists $\alpha < \omega_1$ so that $x|T_\alpha = x|T_\alpha$ then $h(x) > b$ [resp., $h(x) < a$]. So, we can choose ordinals $\alpha_1 < \alpha_2 < \ldots$ such that $h(x_{\alpha_k}) > b$ for $k$ odd, $h(x_{\alpha_k}) < a$ for $k$ even, $\alpha_{k+1} > \alpha_k$, $\alpha_{k+1} > \alpha_k$. Choose $\beta = \sup_k \alpha_k$. Let $y_k = x_{\alpha_k} - x_{\alpha_{k+1}} + x_\beta$. Then $y_k|T_{\alpha_k} = x_{\alpha_k}|T_{\alpha_k}$, so $h(y_k)$ does not converge. But $||y_k|| = 1$ so $y_k \in B_X$ and $y_k + x_\beta$ (pointwise on $\beta\mathbb{N}\setminus\mathbb{N}$ and hence weakly in $C(\beta\mathbb{N}\setminus\mathbb{N})$ by the dominated convergence theorem). So $h$ is not continuous on $C(B_X)$.

5. The next example is the long James space $X = J(\omega_1)$. Notation will be the same as in [6], which I assume is familiar to the reader. Write $B = B_X$.

5.1 THEOREM. If $\mathcal{U}$ is a discrete family of nonempty open sets in $3B$, then
{U ∈ ℰ : U ∩ B ≠ ∅} is countable.

Proof. Begin with the following observation: if α < ω₁, and ℰ is an uncountable family of nonempty open sets in B, then (since J(α) is separable) there exists f ∈ B such that

\[ \{U ∈ ℰ : \text{there exists } g ∈ U, g\big|_{[0, α]} = f\big|_{[0, α]} \} \]

is uncountable.

Suppose ℰ₀ = {U ∈ ℰ : U ∩ B ≠ ∅} is uncountable. Let α₀ = 1. Then there exists f₁ ∈ B such that

\[ \{U ∈ ℰ₀ : \text{there exists } g ∈ U, g\big|_{[0, α₀]} = f₁\big|_{[0, α₀]} \} \]

is uncountable. Choose U₁ ∈ ℰ₁. Then choose α₁ so that: α₁ > α₀, f₁ is constant on [α₁, ω₁], and if f = f₁ on [0, α₁] then f ∈ U₁. Continue recursively. If αₖ, fₖ, Uₖ have been chosen, there exists fₖ₊₁ ∈ B such that fₖ = fₖ₊₁ on [0, αₖ], and

\[ \{U ∈ ℰₖ : \text{there exists } g ∈ U, g\big|_{[0, αₖ]} = fₖ₊₁\big|_{[0, αₖ]} \} \]

is countable. Choose Uₖ₊₁ ∈ ℰₖ₊₁ different from U₁, ..., Uₖ. Then choose αₖ₊₁ so that: αₖ₊₁ > αₖ, fₖ₊₁ is constant on [αₖ₊₁, ω₁], and if f = fₖ₊₁ on [0, αₖ₊₁], then f ∈ Uₖ₊₁. This completes the recursive construction.

Now let β = sup αₖ. Define g : [0, ω₁] → ℝ by g(α) = limₖ fₖ(α). So in fact, g(α) = fₖ(α) if α ≤ αₖ₋₁, and g(α) = f₁(ω₁) for α ≥ β. Now ||g|| ≤ sup ||fₖ|| ≤ 1, so limₖ≤β g(α) exists, possibly not equal to g(β). Let g₁(α) = g(α) for α ≠ β, g₁(β) = limₖ≤β g(α). Then g₁ ∈ B. Note that g₁ = fₖ on [0, αₖ₋₁], g₁(ω₁) = fₖ(ω₁).

Now consider hₖ = g₁ + fₖ - fₖ₊₁. Then hₖ ∈ B₈. Also g₁ = fₖ₊₁ on [0, αₖ], so hₖ = fₖ on [0, αₖ]. Thus hₖ ∈ Uₖ. Also, limₖ hₖ(α) = g₁(α) for all α. This shows that every neighborhood if g₁ in B₈ meets infinitely many Uₖ's, so ℰ is not discrete on B₈. □
5.2 Corollary. There is an uncountable discrete family of open sets in $B$.
Therefore, there is no (weakly) continuous retraction of $3B$ onto $B$, and in particular, there is no retraction of $X$ onto $B$.

Proof. If $0 < \alpha < \omega_1$, let

$$V_\alpha = \{ f \in B : f(\alpha) < \frac{1}{10}, f(\alpha + 1) > \frac{9}{10} \}.$$ 

Then $\mathcal{U} = \{ V_\alpha : 0 < \alpha < \omega_1 \}$ is an uncountable discrete family of open sets in $B$. □

The problem of finding retractions onto the unit ball has been studied by Wheeler [10].

If $X = J(\omega_1)$ is the long James space, it is proved in [6] that $X$ is not realcompact. This is done as follows. Identifying $X^{**}$ with $J(\omega_1)$, we may define $F \in X^{**}$ by:

$$F(\alpha) = 0 \text{ for } \alpha < \omega_1, \quad F(\omega_1) = 1.$$ 

It is easily seen from Corson's criterion that $F \in uX$, but $F$ is not continuous at $\omega_1$, so $F \notin X$. Thus $X$ is not realcompact. Note that $\|F\| = 1$, so $F \in B^{***}$. But $F$ cannot be used to show that $B$ is not realcompact, as the following result shows. The wording is somewhat awkward because it is not clear that $uB$ can be identified with a subset of $X^{**}$; certainly the inclusion $B \to X$ extends to a canonical map $uB \to uX \subseteq X^{**}$.

5.3 Theorem. Let $X = J(\omega_1)$. There is no element of $uB$ whose image in $uX$ is $F$ defined in (1).

Proof. Let $(f_\xi)$ be a $\sigma$-net in $B$, suppose $f_\xi(\alpha) \to 0$ for $\alpha < \omega_1$ and $f_\xi(\omega_1) = 1$. I will show that there is $h \in C(B)$ such that $h(f_\xi)$ does not converge. This suffices to prove the result, as noted in Section 2.

By taking a cofinal subset of the directed set, we may assume $f_\xi(\omega_1) = 1$ for all $\xi$. Also, $f_\xi(0) = 0$ for all $\xi$ and $\|f_\xi\| < 1$, so $0 < f_\xi(\alpha) \leq 1$ for
all ξ and all α ∈ [0, ω₁]. Let

\[ P_{α,ε} = \{ f ∈ B : f = 0 \text{ on } [0, α], f(α + 1) > ε \}. \]

Then

\[ f_ξ ∈ \bigcup_{n=1}^{∞} \bigcup_{α<ω₁} P_{α,1/n} \]

for all ξ, so (again taking a cofinal subset) we may assume

\[ f_ξ ∈ \bigcup_{α<ω₁} P_{α,ε} \]

for some fixed ε > 0. That is, for every ξ there exists α_ξ < ω₁, such that

\[ f_ξ = 0 \text{ on } [0, α_ξ] \text{ and } f(α_ξ + 1) > ε. \]

Given this ε, choose δ > 0 so small that 3δ < ε and \((ε - 2δ)^2 + (1 - 2δ)^2 > 1\).

For each α < ω₁, define

\[ U_α = \{ f ∈ B : f(α) < δ, f(α + 1) > ε - δ, f(ω₁) > 1 - δ \}, \]

\[ \overline{U}_α = \{ f ∈ B : f(α) < δ, f(α + 1) > ε - δ, f(ω₁) ≥ 1 - δ \}, \]

so that \( f_ξ ∈ U_{α_ξ} \). The sets \( U_α \) are cozero sets in \( B \). I claim that the \( \overline{U}_α \) are disjoint, since δ is so small: indeed, suppose \( f ∈ \overline{U}_α \) are \( β > α + 2 \). Then

\[ ||f||^2 ≥ |f(β) - f(α + 1)|^2 + |f(ω₁) - f(β)|^2, \]

so that if \( f(β) < δ \), then \( ||f||^2 ≥ (ε - 2δ)^2 + (1 - 2δ)^2 > 1 \). Thus \( f(β) > δ \), so \( f \notin \overline{U}_β \). Also, \( f \notin \overline{U}_{α+1} \) since \( 3δ < ε \).

Next, I claim that any subcollection \( \{ U_α \}_{α ∈ A} \) of the \( \overline{U}_α \) has closed union. Let \( g \) be in the closure of \( U_α ∈ A \overline{U}_α \). Let \( α_0 < ω₁ \) be such that \( g \) is constant on \([α_0, ω₁]\). Then \( g \) is not close to any member of \( U_α ∈ A ω₂ A \overline{U}_α \), so \( g \) is in the closure of \( U_α ∈ A ω₂ A \overline{U}_α \). Let \( β_0 \) be the smallest ordinal such that \( g \) is in
the closure of \( U_{\alpha \leq \beta_0, \alpha \in A} \). If \( \beta_0 \) is a successor ordinal, then \( g \in \overline{U}_{\beta_0} \). If \( \beta_0 \) is a limit ordinal, and \( g \in \overline{U}_{\beta_0} \), then continuity of \( g \) yields \( \beta_1 < \beta_0 \) such that \( |g(\alpha) - g(\alpha + 1)| < \delta / 2 \) for \( \beta_1 < \alpha < \beta_0 \). Then \( g \) is not close to any member of \( \bigcup_{\beta_1 < \alpha < \beta, \alpha \in A} \overline{U}_\alpha \), so \( g \) is in the closure of \( \bigcup_{\alpha \leq \beta_1, \alpha \in A} \overline{U}_\alpha \), a contradiction.

Finally, note that \((f_\xi)\) is a \( \sigma \)-net in the disjoint union \( U_{\alpha \leq \omega_1} U_\alpha \) and \( \kappa_1 \) is not a 2-valued measurable cardinal, so there exist disjoint \( A_1, A_2 \subseteq [0, \omega_1) \) such that \( f_\xi \in U_{\alpha \in A_1} U_\alpha \) frequently and \( f_\xi \in U_{\alpha \in A_2} U_\alpha \) frequently. By the closedness of the unions above, there is a continuous function \( h \in C(\mathbb{B}) \) such that \( h = 0 \) on \( \overline{U}_\alpha \) for \( \alpha \in A_2 \), but

\[
h(f) = (\delta - f(\alpha))(f(\alpha + 1) - \epsilon + \delta)(f(\omega_1) - 1 + \delta)
\]
on \( \overline{U}_\alpha \) for \( \alpha \in A_1 \). Thus \( h(f_\xi) = 0 \) frequently and \( h(f_\xi) > \delta^3 \) frequently. So \( h(f_\xi) \) does not converge.

It should be remarked that the above result shows that Corson's criterion for elements of \( uX \) fails to characterize \( u\mathbb{B} \).

Even though \( F \) cannot be used to prove it, the ball \( B \) of \( X = J(\omega_1) \) is not realcompact. We can use a small multiple of \( F \) for this.

5.4 THEOREM. There is an element of \( u\mathbb{B} \) whose image in \( uX \) is \((.1)F\).

Proof. For countable ordinal \( \alpha \), let \( f_\alpha = (.1)_{X(\alpha, \omega_1]} \). I will show that \( h(f_\alpha) \) converges for all \( h \in C(\mathbb{B}) \). This will mean that \((.1)F = \lim_\alpha f_\alpha \) is in (the image of) \( u\mathbb{B} \) but not in \( \mathbb{B} \), so \( \mathbb{B} \) is not realcompact.

Let \( h \in C(\mathbb{B}) \). Suppose (for purposes of contradiction) that \( h(f_\alpha) \) does not converge. Then (by uncountable confinality) there exist \( a < b \) such that \( h(f_\alpha) > b \) frequently and \( h(f_\alpha) < a \) frequently. Let \( A_1 = \{ \alpha : h(f_\alpha) > b \} \), \( A_2 = \{ \alpha : h(f_\alpha) < a \} \). Both are uncountable. For each \( \alpha \in A_1 \), choose an open neighborhood \( U_\alpha \) of \( f_\alpha \) determined by finitely many functionals: these functionals involve only countably many coordinates, say \( K_\alpha = [0, \overline{\alpha}] \cup \{ \omega_1 \} \), where \( \overline{\alpha} < \omega_1 \). Thus if \( f_{|K_\alpha} = f_\alpha_{|K_\alpha} \), then \( h(f) > b \).
Similarly, for $\alpha \in A_2$ we get $K_\alpha = [0, \omega_1] \cup \{\omega_1\}$, where $\omega_1 < \omega_1$, and if

$$f|_{K_\alpha} = f_\alpha|_{K_\alpha}$$

then $h(f) < a$.

Now define inductively $a_1 < a_2 < \ldots$ so that $a_{k+1} > a_k$, $a_k \in A_1$, for odd $k$, $a_k \in A_2$ for even $k$. Pick $\beta > \sup_k a_k$, $\beta < \omega_1$. Then let

$$g_k = (\cdot, 1)X(a_k, a_k] + (\cdot, 1)X(\beta, \omega_1]$$

then $g_k|_{K_\alpha k} = f_\alpha|_{K_\alpha k}$ so $h(g_k) > b$ for $k$ odd, $h(g_k) < a$ for $k$ even. But $g_k$ converges weakly to $(\cdot, 1)X(\beta, \omega_1]$, and this contradicts the continuity of $h$. \(\square\)

6. The next example is $X = \ell_\omega$. This space is realcompact, but not measure-compact. One way to see that $X$ is not measure-compact is based on an observation of Hagler (see [2, p. 43]). He exhibits a function $\phi : [0, 1] \to \ell_\omega$ which is scalarly measurable (and thus Baire measurable [4, Theorem 2.3]), but not scalarly equivalent to a Bochner measurable function, so that the image of Lebesgue measure under $\phi$ is not a $\tau$-smooth measure (see [4, Section 5]).

Now Hagler's function has range in $B_X$, so in order to show that $B_X$ is not measure-compact, it is enough to show that $\phi$ is Baire measurable into $B_X$. That is, if $h \in C(B_X)$, then $h \circ \phi$ is Lebesgue measurable. This can be done. But my proof is so long, and the result apparently so useless, that I will not include it here. Let me include only the following hints.

Suppose $h \circ \phi$ is not Lebesgue measurable. Then (restricting to a subset of positive measure) there exist $a < b$ so that on every set of positive measure, $h \circ \phi$ has values $> b$ and values $< a$. Something like the constructions in Theorems 4 and 5.4 can then be carried out (on branches of a binary tree) to find points $t_k \in [0, 1]$ and $t^* \in [0, 1]$ so that $y_k = \phi(t^*) + \phi(t_k) - \phi(t_{k+1})$ converges weakly to $\phi(t^*)$, but $h(y_k) > b$ for odd $k$, $h(y_k) < a$ for even $k$. This contradicts the continuity of $h$.

References


