WEAK SEPARATION IN SELF-SIMILAR FRACTALS

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ABSTRACT. We consider the “weak separation property” for graph-directed iterated function systems of similitudes in Euclidean space. The formulations that are known to be equivalent in the case of strongly connected graph are, in general, no longer equivalent when the graph is not strongly connected.

1. Introduction

We consider graph-directed self-similar fractals. The figures illustrate an example. (This is Example 4.6, discussed in a more technical way below.) Figure 1A is a directed multigraph.

Figure 1A. A directed multigraph

Figure 1B illustrates an iterated function system (IFS) directed by that graph. There are four sets, and maps sending some...
of the sets inside others. For example, node 1 in the graph has two edges leaving it, one to node 3 and one to node 4. Correspondingly, set 1 contains one image of set 3 and one image of set 4. Similarly for all the other nodes and edges of the graph. We are dealing specifically with “self-similar” sets, so the maps are all similitudes.

Here is a more detailed description of how the similitudes are illustrated in the figures. There are four nodes in the graph in Figure 1A. This means we will be constructing four sets in the plane. But for clarity they are shown in four separate pictures of the plane. In Figures 1B and 1C, the four pictures are labeled 1, 2, 3, and 4. There is an arrow from node 1 to node 3 in Figure 1A. This is supposed to correspond to a similitude of the plane. It maps picture 3 to picture 1 in Figure 1B. In picture 3 there is a large irregular pentagon surrounding everything else. The image of that pentagon is the pentagon shown in picture 1. (Because the pentagon has no symmetries, this uniquely determines the similitude of the plane.) There is an arrow from node 1 to node 4 in Figure 1A. This is supposed to correspond to a similitude of the plane. It maps picture 4 to picture 1 in Figure 1B. In picture 4 there is a large irregular hexagon surrounding everything else. The image of that hexagon is the hexagon shown in picture 1. Departing from node 2 of the graph is only a single arrow, going back to node 2. This indicates a similitude, mapping picture 2 into itself. The image of the large square is the small square. (In fact, because of the symmetries of the square, this does not uniquely determine the similitude. For the pictures I chose the one that rotates clockwise by less than 45 degrees.) Departing from node 3 there are three arrows, two going back to 3 itself, and one going to 2. The corresponding similitudes...
are represented in Figure 1B. The large pentagon in 3 is mapped to each of the smaller pentagons by the first two similitudes, and the large square in picture 2 maps to the square in picture 3. Departing from node 4 there are three arrows, two going back to 4 itself, and one going to 2. These are represented in the same way as before by the two small hexagons and the square in picture 4.

Figure 1C illustrates the “attractors” or “invariant sets” defined by this IFS. (Set 2 is a single point at the origin.) Each of these sets is made up of images of the others as directed by the graph. For example, set 1 is the union of the image of set 3 and the image of set 4 under the similitudes described.

![Figure 1C. The four attractors](image)

When we study such self-similar sets, and try to determine a fractal dimension of the attractors (such as Hausdorff dimension or box dimension), the most familiar situation involves the open set condition (OSC). This example was chosen so that the OSC fails. There is no way to choose four nonempty open sets, so that the images as specified by the IFS are inside the appropriate sets, and disjoint. (For example, the open sets 3 and 4 should have images inside open set 1, but disjoint from each other.) This is described more technically in Section 2, below.

When (as in this case) an IFS fails the OSC, that means there is overlap of the images. Computation of the fractal dimension may be difficult. Two papers that study this problem in one dimension are: Keane, Simon & Solomyak [7] and Sidorov [13]. The weak separation property (WSP) has been proposed by Lau & Ngai [8] to single out cases where the dimension computation is simplified by the combinatorial nature of the overlaps. We studied this in our previous paper [4] under the assumption that the graph directing
the IFS is strongly connected. This work was in turn motivated by Zerner [14].

A special subclass of the IFS with WSP, are those that are of \textbf{finite type}. These were studied in Das & Ngai [5] and Ngai & Wang [10]. An IFS of finite type gives rise to a new graph-directed IFS. Oftentimes, even though the original IFS may be determined by a strongly connected graph, the new IFS may be determined by a graph that is not strongly connected. This raises the question as to which separation condition to use, and what it might even mean. It is worth noting that Bandt & Rao [3] gives an analysis of the relationship between the open set condition and finite type for connected self-similar sets in the plane. The relationship between finite-type and WSP was first proved in Nguyen [11].

So what should the definition be when the graph is not strongly connected? One natural possibility is simply to require WSP separately for each connected component of the graph. In fact, that is probably the best choice. But our concern in this paper is what can be said when the definition of WSP is taken for the entire IFS.

Note that we are adapting only the first part of [4]. Adaptation to non-strongly-connected graphs of the later parts of that paper (such as: similarity and growth dimensions, finite type) is left for the future.

The paper is organized as follows: Section 2 describes the setting. The weak separation property is discussed in Section 3. Finally, in Section 4, we provide several examples, both to illustrate the results and to provide counterexamples.

\section{The setting}

It may be noted that our setting can be thought of as a generalization of \textit{finitely generated semigroups of contracting homotheties}, which have been studied earlier in the literature. But in that study, there is no mention of the self-similar sets that are generated, whereas for us these sets are the main item of interest.

\subsection{Directed multigraph}

The \textit{weak separation property} for iterated function systems with overlap was defined and studied for the one-node setting by Lau & Ngai [8] and Zerner [14], following work by Band & Graf [2] and Schief [12]. In [4], the authors generalized this to the graph-directed setting, but only for strongly

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\section{References}


connected graphs. In the present paper we consider the definition for general graphs, not necessarily strongly connected.

The setting is the same as in [4], but the definitions are repeated here. Begin with a directed multigraph $G = (V, E)$. So $V$ is a finite set (of "vertices"), $E$ is a finite set (of "edges"), for each $u, v \in V$, $E_{uv} \subseteq E$ (the set of edges "from $u$ to $v$ "). For convenience we assume that $E$ is the disjoint union of the sets $E_{uv}$. If $e \in E_{uv}$ then $e$ has initial vertex $u$ and final vertex $v$. Again for convenience we assume that every node $u$ is the initial vertex for at least one edge. Write $E^{(k)}_{uv}$ for paths of length $k$, say $\sigma = e_1 e_2 \cdots e_k$ where $e_1$ has initial vertex $u$, the final vertex of each $e_i$ matches the initial vertex of the next one $e_{i+1}$, and the final vertex of $e_k$ is $v$. Then $E^{(\ast)}_{uv} = \bigcup_{k=0}^{\infty} E^{(k)}_{uv}$ is the forest of all paths in $G$.

2.2. The IFS. For each $u \in V$ we associate a complete metric space $X_u$. For now we will assume each $X_u \subseteq \mathbb{R}^d$ for a certain $d$. For each $e \in E_{uv}$ we associate a similitude $S_e : X_v \to X_u$, with contraction ratio $\rho_e$:

$$|S_e(x) - S_e(y)| = \rho_e |x - y|.$$ 

(A similitude defined on a subset of Euclidean space can be extended to a similitude on the whole space, so when convenient we will assume $S_e$ is defined on all of $\mathbb{R}^d$. Also recall that a similitude $R$ on $\mathbb{R}^d$ is affine so that it has the form $R(x) = Ax + b$ for some matrix $A$ and vector $b$. When considered defined on all of $\mathbb{R}^d$, the similitude $S_e$ has an inverse from $\mathbb{R}^d$ to itself. If $\sigma \in E^{(\ast)}_{uv}$ and $\dim X_u = \dim X_v$, then since $S_{\sigma}$ maps $X_v$ into $X_u$ it will be bijective and the inverse maps $X_u$ to $X_v$. But if $\dim X_u > \dim X_v$, of course the inverse will not map $X_u$ into $X_v$.) Assume $0 < \rho_e < 1$. Write $\rho_{\min} = \min \{ \rho_e : e \in E \}$, $\rho_{\max} = \max \{ \rho_e : e \in E \}$. For $\sigma = e_1 e_2 \cdots e_k$ write $S_{\sigma} = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$ and $\rho(\sigma) = \rho_{e_1} \cdots \rho_{e_k}$. This formulation is found in [9, 6].

The original version of an IFS, where no graph is specified, can be fit into this scheme by using a graph $G = (V, E)$ where $V$ has exactly one element. Then all edges are loops from that node to itself. To emphasize this case, we will sometimes call it the one-node case.
There is then a unique family \( \{ K_u : u \in V \} \) of nonempty compact sets such that
\[
K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(K_v)
\]
for all \( u \in V \). These are the **attractors or invariant sets** defined by the IFS.

Sometimes it will be convenient to allow \( \rho_e \geq 1 \) for some edges \( e \). But as long as \( \rho(\sigma) < 1 \) for all cycles \( \sigma \), the situation reduces to the previous case by multiplying the metric in each \( X_u \) by an appropriate constant [6, Exercise (4.3.9)].

If \( R \) is a similitude, write \( \rho(R) \) for its contraction ratio. So in our setting, \( \rho(S_\sigma) = \rho(\sigma) \).

We will assume that \( X_u \) is the affine span of the attractor \( K_u \). This means that if \( R_1, R_2 : X_u \to \mathbb{R}^d \) are affine, and \( R_1(x) = R_2(x) \) for all \( x \in K_u \), then \( R_1(x) = R_2(x) \) for all \( x \in X_u \). If \( E^{(\ast)}_{uv} \neq \emptyset \), so that there is a path from \( u \) to \( v \) in the graph \( G \), we say \( u \) **precedes** \( v \). So if \( u \) precedes \( v \), then the attractor \( K_u \) contains a similar image of \( K_v \), and therefore the affine dimensions satisfy \( \dim X_u \geq \dim X_v \). Within a component of \( G \) all dimensions \( \dim X_u \) are the same, but if \( G \) is not strongly connected, then \( \dim X_u \) may differ from component to component. If \( \dim X_u = \dim X_v \) for all \( u, v \), then we will say that \( (S_e) \) has **uniform affine dimension**.

And when this is true, there is no harm in assuming \( X_u = \mathbb{R}^d \) for all \( u \).

### 2.3. Definitions.

We give next a few definitions formulated in terms of a graph-directed iterated function system. Note that many of these are computed in the examples below, see especially Example 4.3. Reading the examples may help in understanding the definitions.

Let \( u, v \in V \), \( 0 < a < b \), \( I \subseteq \mathbb{R} \) an interval, \( 0 < r < 1 \), \( U \subseteq X_u \) bounded, \( M \subseteq X_v \) nonempty. Define
\[
\mathcal{R}_{uv} = \{ R : R \text{ is a similitude from } X_v \text{ to } X_u \}
\]
\[
\mathcal{R}_{uv}(I) = \{ R \in \mathcal{R}_{uv} : \rho(R) \in I \}
\]
\[
E^{(\ast)}_{uv}([a, b]) = \left\{ \sigma \in E^{(\ast)}_{uv} : S_\sigma \in \mathcal{R}_{uv}([a, b]) \right\}
\]
\[
F_{uv}([a, b]) = \left\{ S_\sigma : \sigma \in E^{(\ast)}_{uv}([a, b]) \right\}
\]
\[ \mathcal{F}_{uv}(a, b) = \{ T^{-1} \circ S : T, S \in F_{uv}(a, b) \} \]

(These map \( X_u \) into \( \mathbb{R}^d \).)

\[ \mathcal{F}_{uv}(r) = \bigcup_{b > 0} \mathcal{F}_{uv}(\lfloor rb, b \rfloor) \]

\[ \mathcal{F}_{uv} = \bigcup_{0 < a < b} \mathcal{F}_{uv}([a, b]) = \bigcup_{0 < r < 1} \mathcal{F}_{uv}(r) \]

\[ = \left\{ S^{-1}_r \circ S_{\sigma} : \tau, \sigma \in E^{(s)} \right\} \]

\[ F_{uv}([a, b], U, M) = \{ T \in F_{uv}([a \ \text{diam} U, b \ \text{diam} U]) : T(M) \cap U \neq \emptyset \} \]

\[ \gamma_{uv}([a, b], M) = \sup \{ \#F_{uv}([a, b], U, M) : U \subseteq X_u \ \text{bounded} \} \]

**Proposition 2.1.** \( \mathcal{F}_{uv}(r) = \mathcal{F}_{uv} \cap \mathcal{R}_{uv}([r, r^{-1}]) \)

**Proof.** Let \( R \in \mathcal{F}_{uv}(r) \). Then there is \( b \) so that \( R = T^{-1} \circ S \) with \( T, S \in F_{uv}(\lfloor rb, b \rfloor) \). So \( \rho(R) = \rho(T)^{-1} \rho(S) < \lfloor rb \rfloor^{-1} b = r^{-1} \) and \( \rho(R) = \rho(T)^{-1} \rho(S) > b^{-1}(rb) = r \). So \( R \in \mathcal{F}_{uv} \cap \mathcal{R}_{uv}([r, r^{-1}]) \).

Conversely, let \( R \in \mathcal{F}_{uv} \cap \mathcal{R}_{uv}([r, r^{-1}]) \). Say \( R = T^{-1} \circ S \). First take the case \( \rho(T) \leq \rho(S) \). Let \( b = \rho(S) \) so that \( T, S \in F_{uv}(\lfloor rb, b \rfloor) \) and \( R \in \mathcal{F}_{uv}(r) \). For the other case \( \rho(T) > \rho(S) \), let \( b = \rho(T) \) and again \( R \in \mathcal{F}_{uv}(r) \). \( \square \)

3. **The Weak Separation Property**

For each \( u \in V \) let \( X_u \subseteq \mathbb{R}^d \) be the affine span of \( K_u \). For \( e \in E_{uv} \) let \( S_e : \mathbb{R}^d \to \mathbb{R}^d \) be a similitude with ratio \( \rho_e \) that maps \( X_v \) into \( X_u \). Let \( r \in [0, \rho_{\text{min}}) \). We will consider the conditions given below. The length of the list is justified by the fact that it exists in the literature: In [14] all ten of these conditions are proved equivalent in the one-node setting. For the graph-directed setting, in the case of strongly connected graph \( G \), it was proved in [4] that all ten conditions are equivalent.

- **(1a)** For all \( v \in V \), there exist \( x \in K_v \) and \( \varepsilon > 0 \) such that for all \( u \) preceding \( v \) and all \( R \in \mathcal{F}_{uv}(r) \), either \( R \) is the identity on \( X_v \) or \( |R(x) - x| \geq \varepsilon \).
- **(1b)** For all \( v \in V \) there exist \( x \in X_v \) and \( \varepsilon > 0 \) such that for all \( u \) preceding \( v \) and all \( R \in \mathcal{F}_{uv}(r) \), either \( R(x) = x \) or \( |R(x) - x| \geq \varepsilon \).
• (2a) For all \( v \in V \) there are \( \varepsilon > 0 \) and \( \{x_0, \ldots, x_m\} \subseteq X_v \) with affine span \( X_v \) such that for all \( u \) preceding \( v \), all \( R \in F_{uv}(r) \), and all \( j \), either \( R(x_j) = x_j \) or \( |R(x_j) - x_j| \geq \varepsilon \).

• (2b) For all \( v \in V \) there are \( \varepsilon > 0 \) and \( \{x_0, \ldots, x_m\} \subseteq X_v \) with affine span \( X_v \) such that for all \( u \) preceding \( v \) and all \( R \in F_{uv}(r) \), either \( R \) is the identity on \( X_v \) or \( |R(x_j) - x_j| \geq \varepsilon \) for some \( j \).

• (3a) For all \( u, v \in V \) with \( u \) preceding \( v \), the identity is an isolated point of \( F_{uv} \).

• (3b) For all \( u, v \in V \) with \( u \) preceding \( v \), the identity is an isolated point of \( F_{uv}(r) \).

• (4a) For all \( u, v \in V \) with \( u \) preceding \( v \), all bounded \( M \subseteq X_v \), and all \( b > 0 \), we have \( \gamma_{uv}(\lfloor rb, b \rfloor, M) < \infty \).

• (4b) For all \( u, v \in V \) with \( u \) preceding \( v \), there exist nonempty \( M \subseteq X_v \) and \( b > 0 \) such that \( \gamma_{uv}(\lfloor rb, b \rfloor, M) < \infty \).

• (5a) For all \( u, v, w \in V \) such that \( u \) precedes \( v \) precedes \( w \), and all \( z \in X_w \), there exists \( l \in \mathbb{N} \) such that for any \( \tau \in E_{uv}^{(s)} \) and any \( b > 0 \), every ball in \( X_u \) with radius \( b \) contains at most \( l \) elements of \( \{ S_{\sigma \tau}(z) : \sigma \in E_{uv}^{(s)}(\lfloor rb, b \rfloor) \} \).

• (5b) For all \( u, v, w \in V \) such that \( u \) precedes \( v \) precedes \( w \), there exist \( z \in X_w \) and \( l \in \mathbb{N} \) such that for any \( \tau \in E_{uv}^{(s)} \) and any \( b > 0 \), every ball in \( X_u \) with radius \( b \) contains at most \( l \) elements of \( \{ S_{\sigma \tau}(z) : \sigma \in E_{uv}^{(s)}(\lfloor rb, b \rfloor) \} \).

We will prove:

\[
\begin{align*}
(1a) & \quad \Downarrow \\
(1b) & \quad \iff (2a) \\
& \quad \Downarrow \\
(2b) & \quad \iff (3a) \iff (3b) \\
& \quad \Downarrow \\
(4a) & \quad \iff (4b) \iff (5a) \iff (5b)
\end{align*}
\]

and provide examples showing the vertical arrows cannot be reversed.

Theorem 3.1 proves the implications that do not use the assumption of uniform affine dimension, and Theorem 3.3 proves the
implications that do use that assumption. Most of the proofs are taken from [4], where they are carried out in the case of strongly connected graph.

It may also be noted that Theorem 3.1 does not use the system of attractors, so it may be thought of as a result for sets of affine maps of Euclidean space equipped with the compact-open topology which (depending on the situation) are either contracting homotheties or which are homotheties whose dilations are uniformly bounded from above and below. In particular, this point of view easily explains the equivalence of (2b), (3a), (3b).

**Theorem 3.1.** Let $G = (V, E)$ be a directed multigraph. Let the iterated function system $(S_e)_{e \in E}$ with invariant list $(K_u)_{u \in V}$ in $\mathbb{R}^d$ be directed by $G$. Then $(1a) \implies (1b) \iff (2a) \implies (2b) \iff (3a) \iff (3b) \iff (4a) \implies (4b) \iff (5a) \implies (5b)$.

**Proof.** $(1a) \implies (1b)$, $(2a) \implies (2b)$, $(3a) \implies (3b)$, $(4a) \implies (4b)$, $(5a) \implies (5b)$ are trivial.

$(2a) \implies (1b)$ is trivial: let $x = x_0$.

$(2b) \implies (3a)$: Assume $(2b)$. Let $u, v \in V$ be given with $u$ preceding $v$. Note that $R_{uv}([r, r^{-1}])$ is an open neighborhood of the identity in $R_{uv}$ and $F_{uv}(r) = F_{uv} \cap R_{uv}([r, r^{-1}])$. Let $\{x_0, \ldots, x_d\}$ and $\varepsilon$ be as in $(2b)$. The set

$$\{ R \in F_{uv}(r) : |R(x_j) - x_j| < \varepsilon \text{ for all } j \} = \{\text{id}\}$$

is an open neighborhood of the identity in $F_{uv}(r)$. So the identity is an isolated point of $F_{uv}$.

$(3b) \implies (2b)$: Assume $(3b)$. Let $v$ be given. Let $\{x_0, \ldots, x_d\}$ be any set with affine span $X_v$. The map $R \mapsto (R(x_0), \ldots, R(x_d))$ is a homeomorphism since it is bijective and affine from one Euclidean space onto another [from the set of affine maps on $\mathbb{R}^d$ to the set $(X_u)^{d+1}$]. But if $u$ precedes $v$, then the identity is isolated in $F_{uv}(r)$, so there is $\varepsilon_u > 0$ so that for all $R \in F_{uv}(r)$ except the identity, $|R(x_j) - x_j| \geq \varepsilon_u$. Let $\varepsilon = \min\{\varepsilon_u : u \text{ precedes } v\}$.

$(2b) \implies (4a)$: Assume $(2b)$. Let $u$ precede $v$, let $M \subseteq X_v$ be bounded, and let $b > 0$. Then apply $(2b)$ with node $v$ to get $\{x_0, \ldots, x_d\}$ with affine span $X_v$ and $\varepsilon > 0$. Let $k = \text{diam} (M \cup \{x_0, \ldots, x_d\})$. We must show $\gamma_{uv}([rb, b], M) < \infty$. Let $U \subseteq X_u$ be bounded. Now if $T, S \in F_{uv}([rb, b], U, M)$, and $T \neq S$, then there exists $j = j(S, T) \in \{0, \ldots, d\}$ with $|T^{-1}(S(x_j)) - x_j| \geq \varepsilon$, and thus
\[|S(x_j) - T(x_j)| \geq r_b \varepsilon \text{diam } U.\] This choice of \(j(S, T)\) is a "colouring" of all pairs from \(F_{uv}([rb, b], U, M)\) in \(d+1\) colours. From Ramsey's Theorem it follows that if \(\sup_U \#F_{uv}([rb, b], U, M) = \infty\), then \(\sup_U \#F_{U'} = \infty\) as well, for some choice of \(F_{U'} \subseteq F_{uv}([rb, b], U, M)\) such that all pairs \(T, S \in F_{U'}\) have the same colour. But suppose all pairs in \(F_{U'}\) have colour \(j\). Then the balls \(B(T(x_j), (r_b \varepsilon /2) \text{diam } U)\), \(T \in F_{U'}\), are disjoint, and all their centers have distance at most \(bk_b\) from \(U\). So these balls are all contained in a ball of radius \((1 + bk + r_b \varepsilon /2) \text{diam } U\). Comparing volumes we see that \(\#F_{U'} \leq (1 + bk + r_b \varepsilon /2)^d / (r_b \varepsilon /2)^d\), where \(d\) is the affine dimension of \(X_u\), a bound independent of \(U\). So in fact \(\gamma_{uv}([rb, b], M) = \sup_U \#F_{uv}([rb, b], U, M) < \infty.\)

(4a) \(\Rightarrow\) (5a): Assume (4a). Let \(u, v, w \in V\) and \(z \in X_w\) be given. Then the set
\[
M = \left\{ S_\tau(z) : \tau \in E_{vw}^{(s)} \right\}
\]
is bounded. Let \(l = \gamma_{uv}([r/2, 1/2], M) < \infty\). Then for any \(\tau \in E_{vw}^{(s)}\), any \(b > 0\), and any ball \(U\) in \(X_u\) of diameter \(b\) (and diameter \(2b\)):
\[
\# \left\{ S_{\sigma \tau}(z) : \sigma \in E_{uv}^{(s)}([rb, b]) \right\} \cap U \leq \# \left\{ T \in F_{uv}([rb, b]) : T(S_\tau(z)) \in U \right\} \leq \# \left\{ T \in F_{uv}([rb, b]) : T(M) \cap U \neq \emptyset \right\} = \#F_{uv}([r/2, 1/2], U, M) \leq l.
\]
(4b) \(\Rightarrow\) (4a): Assume (4b). Let \(u, v \in V\). There exist \(M_0 \neq \emptyset\) and \(b_0 > 0\) with \(\gamma_{uv}([rb_0, b_0], M_0) < \infty\). Then since \(M_0 \neq \emptyset\), there is \(y_0 \in M_0\) with \(\gamma_{uv}([rb_0, b_0], \{y_0\}) < \infty.\)

We claim now that \(\gamma_{uv}([rb, b], \{y_0\}) < \infty\) for all \(b > 0\). Indeed, let \(c\) be the number of balls of diameter \(b\) required to cover a set in \(X_u\) of diameter \(b_0\). Given a bounded set \(U \subseteq X_u\), write \(k = \text{diam } U\), cover \(U\) by \(c\) balls \(V_i\) of diameter \(kb/b_0\). Then
\[
F_{uv}([rb, b], U, \{y_0\}) = \left\{ T \in F_{uv}([rbk, bk]) : T(y_0) \in U \right\} = \bigcup_i \left\{ T \in F_{uv} \left( \left[ r_{b_0} \frac{kb}{b_0}, b_0 \frac{kb}{b_0} \right] \right) : T(y_0) \in V_i \right\},
\]
so \( \#F_{uv}([rb, b], U, \{y_0\}) \leq c_{\gamma_{uv}}([rb_0, b_0], \{y_0\}) \). Taking supremum on \( U \), we conclude
\[
\gamma_{uv}([rb, b], \{y_0\}) \leq c_{\gamma_{uv}}([rb_0, b_0], \{y_0\})
\]

Now we are ready to prove (4a). Let \( M \subseteq X_v \) be bounded, and let \( b > 0 \). We claim there exists \( b' > 0 \) such that
\[
\gamma_{uv}([rb, b], M) \leq \gamma_{uv}([rb', b'], \{y_0\})
\]
To see this: let \( k = \text{diam} (M \cup \{y_0\}) \), and \( b' = b/(1 + 2bk) \). Let \( U \subseteq X_u \) be a bounded set. Define \( U' = B(U, bk \text{diam} U) \), the open set of all points within distance less than \( bk \text{diam} U \) of the set \( U \). So \( \text{diam} U' = \text{diam} U + 2bk \text{diam} U = (1 + 2bk) \text{diam} U \). We claim that
\[
F_{uv}([rb, b], U, M) \subseteq F_{uv}([rb', b'], U', \{y_0\})
\]
Indeed, let \( T \in F_{uv}([rb, b], U, M) \). So \( \rho(T) \in [rb \text{diam} U, bk \text{diam} U] \) and \( T(M) \cap U \neq \emptyset \). So there exists \( y \in M \) with \( T(y) \in U \). Now \( |y - y_0| \leq k \), and \( |T(y) - T(y_0)| \leq bk \text{diam} U \), so \( T(y_0) \in U' \). Also
\[
\rho(T) \in \left[ \frac{rb}{1 + 2bk} \text{diam} U', \frac{b}{1 + 2bk} \text{diam} U' \right].
\]
Thus \( T \in F_{uv}([rb', b'], U', \{y_0\}) \), as required. Now we have
\[
\#F_{uv}([rb, b], U, M) \leq \#F_{uv}([rb', b'], U', \{y_0\}) \leq \gamma_{uv}([rb', b'], \{y_0\})
\]
This is true for all \( U \), so
\[
\gamma_{uv}([rb, b], M) \leq \gamma_{uv}([rb', b'], \{y_0\}) \leq \infty.
\]
This completes the proof of (4a).

**Lemma 3.2.** Assume \((S_e)\) is an IFS with uniform affine dimension. Let \( u, v \in V \) such that \( u \) precedes \( v \). Assume (5b) holds. Then there exist \( w, z, l \) as in (5b), a constant \( C \), and \( \tau \in E_{uv}^{(w)} \) such that \( v \) precedes \( w \) and for all \( y \in X_u \), and all \( b > 0 \),
\[
\# \{ T \in F_{uv}([br, b]) : T(S_\tau(z)) = y \} \leq C.
\]
**Proof.** For each \( w \) preceded by \( v \) use (5b) with \( u, v, w \) to choose \( z_w \in X_w \) and \( l_w \in \mathbb{N} \). Let \( l = \max_w l_w \). Write
\[
A = \left\{ S_\tau(z_w) : v \text{ precedes } w, \tau \in E_{uv}^{(w)} \right\}.
\]
We claim that the affine span of \( A \) is \( X_v \). Indeed, suppose not. Then \( A \subseteq L \), where \( L \) is a proper affine subspace. But \( K_v \) spans
$X_v$, so there is $x \in K_v \setminus L$. There is $w_1$ preceded by $v$ and $\tau_1 \in E_{v,w_1}^{(s)}$ so that $x \in S_{\tau_1}(K_{w_1})$ and $S_{\tau_1}(K_{w_1}) \cap L = \emptyset$. Then if $d_0 = \max \{ \text{dist}(z_w, K_w) : u \text{ precedes } w \}$, then there is $w_2$ preceded by $w_1$ and $\tau_2 \in E_{v,w_2}^{(s)}$ as long as $d_0\rho(\tau_1\tau_2) < \text{dist}(S_{\tau_1}(K_{w_1}), L)$. So $S_{\tau_1\tau_2}(z_{w_2}) \notin L$.

Let $d$ be the dimension of $X_v$, and let $x_0, \ldots, x_d \in A$ have affine span $X_v$, with $x_j = S_{\tau_j}(z_{w_j})$. Let $t = \max \{ |x_j - x_0| : 0 \leq j \leq d \}$, let $c_t$ be the number of balls of radius $1$ required to cover a ball in $X_u$ of radius $t$, write $m = (d + 1)c_t l$ and $C = m(m - 1)(m - 2) \cdots (m - d + 1)$.

Now let $y \in X_u$ and $b > 0$ be given. The ball $\overline{B}(y, bt)$ is covered by $c_t$ balls of radius $b$, so for each $j \in \{0, \ldots, d\}$

$$\# \left\{ T(x_j) : T \in F_{uv}([br, b]), T(x_j) \in \overline{B}(y, bt) \right\}$$

$$= \# \left\{ S_{\sigma\tau_j}(z) : \sigma \in E_{uv}^{(s)}([rb, b]) \cap \overline{B}(y, bt) \right\} \leq c_t l.$$

If $T \in F_{uv}([br, b])$ and $T(x_0) = y$, then $|T(x_j) - y| = |T(x_j) - T(x_0)| \leq bt$ for all $j$. So

$$\# \left\{ T(x_j) : T \in F_{uv}([br, b]), j \in \{0, \ldots, d\}, T(x_0) = y \right\} \leq (d + 1)c_t l = m.$$

And a similitude is determined by its values on $\{x_0, \ldots, x_d\}$, so

$$\# \left\{ T \in F_{uv}([br, b]) : T(x_0) = y \right\} \leq m(m - 1)(m - 2) \cdots (m - d + 1) = C.$$

\[\square\]

**Theorem 3.3.** Let the IFS $(S_\varepsilon)$ be as in Lemma 3.2. Assume also that it has uniform affine dimension. Then (1b) $\implies$ (2a) and (5b) $\implies$ (4b).

**Proof.** (1b) $\implies$ (2a): Assume (1b). For each $w \in V$, apply (1b): there exist $y_w \in X_w$ and $\varepsilon_w > 0$ so that for all $u$ preceding $w$ and all $R \in F_{uw}(r)$ either $R(y_w) = y_w$ or $|R(y_w) - y_w| \geq \varepsilon_w$.

Let $v \in V$ be given. Then $X_v$ is the affine span of $K_v$, so the set

$$\{ S_{\tau}(y_w) : v \text{ precedes } w, \tau \in E_{uv}^{(s)} \}$$

also has $X_v$ as affine span. So there exist $w_0, w_1, \ldots, w_d$ preceded by $v$ and $\tau_j \in E_{v,w_{j}}^{(s)}$ such that $\{ S_{\tau_j}(y_{w_j}) : 0 \leq j \leq d \}$ spans $X_v$. 

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Write $x_j = S_{\tau_j}(y_{w_j})$. Define $\varepsilon = \min_j \varepsilon_{w_j} \rho(\tau_j)$. Let $u$ precede $v$, let $R \in \mathcal{F}_{uv}(r)$, and $j \in \{0, \ldots, d\}$. Then $R = T^{-1} \circ S$ with $T, S \in \mathcal{F}_{uv}([rb, b])$ for some $b$. So $T \circ S_{\tau_j}, S \circ S_{\tau_j} \in \mathcal{F}_{uw_j}([rbp(\tau_j), b(\rho(\tau_j))])$ and $(T \circ S_{\tau_j})^{-1} \circ (S \circ S_{\tau_j}) \in \mathcal{F}_{uw_j}(r)$, so

$$|T^{-1}(S(x_j)) - x_j| = \rho(\tau_j) |(T \circ S_{\tau_j})^{-1} \circ (S \circ S_{\tau_j})(y_{w_j}) - y_{w_j}| \geq \varepsilon$$

if it is not zero.

$$(5b) \implies (4b): \text{ Assume } (5b). \text{ Let } u, v \in V \text{ be given. Apply Lemma 3.2 to get } x_0 = S_{\tau}(z) \text{ and } C > 0, \text{ where } w, z, l \text{ are as in } (5b). \text{ Let } c \text{ be the number of balls of radius } 1 \text{ required to cover a set of diameter } 2 \text{ in } X_u. \text{ We claim that } \gamma_{uv}([r/2, 1/2], \{x_0\}) \leq cCl. \text{ Indeed, let } U \subseteq X_u \text{ be a bounded set. Write } b = \operatorname{diam} U. \text{ Now let } B \text{ be a ball in } X_u \text{ of radius } b/2. \text{ Write }$$

$$Q = \{ T(x_0) : T \in \mathcal{F}_{uv}([rb/2, b/2]) \} \cap B.$$  

Then $\# Q \leq l$, and

$$\# \{ T \in \mathcal{F}_{uv}([rb/2, b/2]) : T(x_0) \in B \} = \sum_{y \in Q} \# \{ T \in \mathcal{F}_{uv}([rb/2, b/2]) : T(x_0) = y \} \leq Cl.$$

Then since $U$ can be covered by at most $c$ balls of radius $b/2$,

$$\# \mathcal{F}_{uv}([r/2, 1/2], U, \{x_0\}) \leq cCl.$$

This is true for all $U$, so $\gamma_{uv}([r/2, 1/2], \{x_0\}) \leq cCl$.  \hfill \Box

4. Examples

We collect here some examples of iterated function systems $(S_e)$ to illustrate the alternate definitions for WSP. There is a lot of repetition in the descriptions, but we have kept it that way so that the examples can be read independently.

4.1. Example. Why we need span $K_u$ to be $X_u$.

For this example, let the graph have one node $V = \{2\}$, and two edges $E = \{d, e\}$. Let $X_2 = \mathbb{R}$, let $\alpha, \beta \in ]0, 1[ \ (to \ be \ specified \ later), \ and \ define \ the \ maps \ as:\n
$$S_d(x) = \alpha x, \quad S_e(x) = \beta x.$$  

So in this case we have $K_2 = \{0\}$ with affine span not equal to $X_2 = \mathbb{R}$.
Now for $\sigma \in E_2^{(n)}$ we get
\[ S_\sigma(x) = \alpha^k \beta^{n-k} x \]
for some $k$ with $0 \leq k \leq n$. And for $R \in F_2$ we get
\[ R(x) = \alpha^n \beta^m x \]
for any $n, m \in \mathbb{N}$. Now of course (1b) is satisfied in this case, for if we let $x_0 = 0$, then $R(x_0) = x_0$ for all $R$.

Now assume $\log \alpha$ and $\log \beta$ are not commensurable. This means that $n, m$ can be chosen so that $\alpha^n \beta^m$ is not 1 but as close to 1 as we like. So (2a) fails: for any $x_1 \neq 0$, any $r \in ]0, 1[$, and any $\varepsilon > 0$, we can choose $n, m$ so that $\alpha^n \beta^m \in ]r, r^{-1}[$ and $|R(x_1) - x_1| < \varepsilon$. Thus, in the proof that (1b) $\implies$ (2a), we really do need the assumption that the affine span of $K_u$ is $X_u$. Of course we can repair this case by declaring $X_1 = \{0\}$.

4.2. Example. Why we need uniform affine dimension.

Now consider a larger graph, in which we have the same node 2 as the previous example where $K_2$ has affine dimension 0, but also another node 1. Let the graph be as shown, $S_d, S_e$ as before, and
\[ S_a(x) = \alpha x, \quad S_b(x) = \alpha(x + 1), \quad S_c(x) = \alpha x. \]

If $\alpha < 1/2$, then the attractor $K_1$ is a Cantor set (together with countably many images of the point $K_2$, which are contained in that Cantor set). So $K_1$ has affine dimension 1. We assume $0 < \beta < \alpha < 1/2$ and $\log \alpha / \log \beta$ is irrational. We claim that (1b) holds but not (2a). The reason that (2a) fails is the same as the previous example. To see (1b), we must do some computations.

As before, maps $S \in F_2$ have the form
\[ S(x) = \alpha^k \beta^l x. \]
Maps $S \in F_1$ have the form
\[ S(x) = \alpha^n x + \sum_{i=1}^{n} \theta_i \alpha^i, \]
where $\theta_i \in \{0, 1\}$. Any choice $\theta_i$ of digits 0, 1 can be realized by choosing the appropriate string of $a, b$. And the attractor $K_1$ is
\[ \left\{ \sum_{i=1}^{\infty} \theta_i \alpha^i : \theta_i \in \{0, 1\} \right\}. \]
Maps \( S \in F_{12} \) have the form
\[
S(x) = \alpha^{n+k+1} \beta^l x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \quad \theta_i \in \{0, 1\}.
\]
We can always assume \( k = 0 \) by increasing \( n \) and adding extra digits \( \theta_i = 0 \). Now \( R \in F_{12} \) has the form \( R = S^{-1} \circ S \), where
\[
S(x) = \alpha^{n+1} \beta^l x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \quad S'(x) = \alpha^{n'+1} \beta^{l'} x + \sum_{i=1}^{n'} \theta'_i \alpha^{i+1},
\]
\( \theta_i, \theta'_i \in \{0, 1\} \), so that
\[
R(x) = \alpha^{n-n'} \beta^{l-l'} x + \sum_{i=1}^{n} \theta_i \alpha^{i+1-n'} \beta^{-l'} - \sum_{i=1}^{n'} \theta'_i \alpha^{i+1-n'} \beta^{-l'}.
\]
Now fix \( r \in ]0, \beta[ \). If \( R \in F_{12}(r) \), then \( \alpha^{n-n'} \beta^{l-l'} > r \), so \( \alpha^{-n'} \beta^{-l'} > r \alpha^{-n} \beta^{-l} \).

**Proposition 4.2.1.** The IFS defined above satisfies (1b).

**Proof.** We will verify (1b) for node 2. (The argument for node 1 is similar but easier, since no \( \beta \) is involved.) Let \( x_0 = 0 \in K_2 \) and \( \varepsilon = r \alpha(1 - 2\alpha)/(1 - \alpha) \). Let \( R \in F_{12}(r) \), as computed above. Assume \( R(0) \neq 0 \). We claim that \( |R(0)| \geq \varepsilon \). Now
\[
R(0) = \sum_{i=1}^{n} \theta_i \alpha^{i+1-n'} \beta^{-l'} - \sum_{i=1}^{n'} \theta'_i \alpha^{i+1-n'} \beta^{-l'}.
\]
Since \( R(0) \neq 0 \), there exist \( i \) so that \( \theta_i \neq \theta'_i \) [where by convention nonexistent \( \theta \)s are 0]. Let \( i_0 \) be the least such \( i \). Then
\[
|R(0)| > \alpha^{i_0+1-n'} \beta^{-l'} \left( 1 - \sum_{j=1}^{\infty} \alpha^j \right) = \alpha^{i_0+1-n'} \beta^{-l'} \frac{1 - 2\alpha}{1 - \alpha}.
\]
If \( i_0 < n' \), then \( i_0 + 1 - n' \leq 0 \) so \( \alpha^{i_0 + 1 - n'} \geq 1 \); also \( \beta^{-l'} \geq 1 \), so
\[
|R(0)| > \frac{1 - 2\alpha}{1 - \alpha} > \varepsilon.
\]
On the other hand, if \( i_0 \geq n' \), then \( i_0 \leq n \) and \( \alpha^{i_0 + 1 - n'} \beta^{-l'} \geq r\alpha^{i_0 + 1 - n} \beta^{-l} \), \( \alpha^{i_0 + 1 - n} \geq \alpha \), \( \beta^{-l} \geq 1 \), and
\[
|R(0)| > \frac{r\alpha(1 - 2\alpha)}{1 - \alpha} = \varepsilon.
\]
This completes the proof of (1b).

Again this time we can try to get out of the difficulty by declaring \( X_2 = \{ 0 \} \). But then we do not know how to define \( F_{12} \) and related constructions. We must have \( X_1 = \mathbb{R} \) since the Cantor set \( K_1 \) is not a single point. If we allow \( \dim X_2 = 0 \neq 1 = \dim X_1 \), what should we do? In order to define maps like \( R = S_{\tau}^{-1} \circ S_{\sigma} \), we need \( S_c \) to have an inverse \( S_c^{-1} \) of some kind. And whatever that inverse is, if \( S_b(0) \neq 0 \), then we need \( S_c^{-1}(S_b(0)) \neq S_c^{-1}(0) \). Our solution in this paper is to require all similitudes be defined on \( \mathbb{R}^d \), and then maps \( R = S_{\tau}^{-1} \circ S_{\sigma} \) exist and have range in \( \mathbb{R}^d \), but need not map \( X_u \) into itself. Even in classical cases like the middle-thirds Cantor set in \( \mathbb{R} \), maps \( S_{\tau}^{-1} \circ S_{\sigma} \) do not map the attractor \( K_u \) into itself.

4.3. Example. An example with a graph that is not strongly connected. We will compute some of the conditions related to the WSP.

Let \( \alpha, \beta \in [0, 1/2] \) and \( \gamma > 0 \). The case \( \alpha = \beta = 1/3 \) is interesting, but for this example we will want \( \alpha \neq \beta \), and in fact \( \log \alpha, \log \beta \) incommensurable. The graph \( G = (V, E) \) is shown; \( V = \{ 1, 2 \} \), \( E = \{ a, b, c, d, e \} \). The two components in the graph are \( \{ 1 \} \) and \( \{ 2 \} \).

![Graph and IFS](image-url)

Figure 4.3A. The graph and the IFS
The IFS \((S_\sigma)\) is defined with: \(X_1 = X_2 = \mathbb{R}\),
\[
\begin{align*}
S_a(x) &= \alpha x, \\
S_b(x) &= \alpha x + \gamma, \\
S_c(x) &= \alpha x, \\
S_d(x) &= \beta x, \\
S_e(x) &= \beta x + (1 - \beta).
\end{align*}
\]

So \(E^{(\ast)}_{11} = \{ S_\sigma : \sigma \in \{a, b\}^{(\ast)} \}\) corresponds to a Cantor set \(C_1\) of dimension \(D_1 = \frac{\log 2}{\log(1/\alpha)}\). Component \(1\) by itself it satisfies OSC and the identity is isolated in \(F_{11}\). Similarly, \(E^{(\ast)}_{22} = \{ S_\sigma : \sigma \in \{d, e\}^{(\ast)} \}\) corresponds to a Cantor set \(C_2\) of dimension \(D_2 = \frac{\log 2}{\log(1/\beta)}\). Component \(2\) by itself it satisfies OSC and the identity is isolated in \(F_{22}\).

**Proposition 4.3.1.** Let \(\log \alpha, \log \beta\) be incommensurable. Then the identity is not isolated in \(F_{12}\).

**Proof.** Consider \(T = S_\tau, S = S_\sigma \in F_{12}\) defined by \(\tau = cd^n, \sigma = a^n c\). Then \(T^{-1} \circ S(x) = \beta^{-n} \alpha^m x\). Choose sequences \(n_k, m_k \to \infty\) with \(m_k \log \alpha - n_k \log \beta \to 0\). Then \(\beta^{-n_k} \alpha^{m_k} x \to x\). So the identity is not isolated. \(\square\)

Next we investigate \(\gamma_{uv}([a, b], M)\). Of course, by the result for the one-node case, we have \(\gamma_{11}([a, b], M) < \infty\) and \(\gamma_{22}([a, b], M) < \infty\) for all nonempty \(M\). And of course \(\gamma_{21}([a, b], M) = 0\). For \(\gamma_{12}\) we need a lemma.

**Lemma 4.3.2.** Let \(A, B > 0, A/B\) irrational. Then

\[
\limsup_{k \to \infty} \# \{ (n, m) \in \mathbb{N} \times \mathbb{N} : k - 1 < nA + mB < k \} = \infty.
\]

**Proof.** For \(k \in \mathbb{N}\) write

\[
S_k = \{ (n, m) \in \mathbb{N} \times \mathbb{N} : k - 1 < nA + mB < k \}.
\]

Since \(A/B\) is irrational, each line

\[
L_k = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : xA + yB = k \}
\]

can have at most one lattice point on it. The rectangle

\[
\left\{ (x, y) : 0 < x < \frac{k}{A}, 0 < y < \frac{k}{B} \right\}
\]
contains at least
\[
\left( \frac{k}{A} - 2 \right) \left( \frac{k}{B} - 2 \right)
\]
lattice points and is covered by the closures of \(2k\) strips \(S_1, \cdots, S_{2k}\).
At most \(2k + 1\) of the lattice lattice points lie on the lines \(L_k\), so there is at least one (open) strip with at least
\[
\frac{\left( \frac{k}{A} - 2 \right) \left( \frac{k}{B} - 2 \right) - (2k + 1)}{2k}
\]
lattice points in it. This goes to \(\infty\).

\[\square\]

**Proposition 4.3.3.** Assume \(\log \alpha, \log \beta\) are incommensurable and \(0 < a < b\). Then \(\gamma_{12}([a,b],\{0\}) = \infty\).

**Proof.** Write
\[
q = b \left( \frac{1}{\beta} - 1 \right), \quad p = q - 1, \quad a' = \max \left\{ a, \frac{p}{\beta - 1} \right\}.
\]
So \(0 < a' < b\), \(a \leq a'\), \(p \leq a'(\frac{1}{\beta} - 1)\), \(0 < a'/b < 1\). The two numbers
\[
A = \frac{\log \frac{1}{\alpha}}{\log \frac{b}{a'}} > 0 \quad \text{and} \quad B = \frac{\log \frac{1}{\beta}}{\log \frac{b}{a'}} > 0
\]
are incommensurable. Given \(C > 0\), there exists \(k \in \mathbb{N}\) so that
\[
\# \left\{ (m,n) : k - 1 < (m + 1) \log \frac{1}{\alpha} \log \frac{b}{a'} + (n + 1) \log \frac{1}{\beta} \log \frac{b}{a'} < k \right\} > C.
\]
So for all \((m,n)\) in the set,
\[
(k - 1) \log \frac{b}{a'} \leq (m + 1) \log \frac{1}{\alpha} + (n + 1) \log \frac{1}{\beta} \leq k \log \frac{b}{a'},
\]
\[
a'^{k-1} < \frac{1}{\alpha^{m+1} \beta^{n+1}} < \frac{b^k}{a'^k},
\]
\[
a'^{k-1} > \alpha^{m+1} \beta^{n+1} > \frac{b^k}{a'^k},
\]
\[
b^\varepsilon > \alpha^{m+1} \beta^{n+1} > a'^{\varepsilon},
\]
where \(\varepsilon = a'^{k-1}/b^k\) goes to \(0\) as \(k \to \infty\).
So let \( \tau = a^m c d^n e, T = S_\tau, M = \{0\}, U = ]p\varepsilon, q\varepsilon[, \) so \( \text{diam} U = \varepsilon, \rho(T) = \alpha^{m+1}\beta^{n+1}, T \in F_{12}([\text{diam} U, b\text{diam} U]) \),

\[
T(0) = \alpha^{m+1}\beta^n(1 - \beta) = \alpha^{m+1}\beta^{n+1}\left(\frac{1}{\beta} - 1\right),
\]

so \( T(0) < b\varepsilon(\frac{1}{\beta} - 1) = q\varepsilon, T(0) > a'\varepsilon(\frac{1}{\beta} - 1) \geq p\varepsilon. \) So \( T(0) \in U. \) Different \((m, n)\) yield different \( T. \) So \#\( F_{12}([a, b], U, \{0\}) > C, \) and therefore \( \gamma_{12}([a, b], \{0\}) = \infty. \)

**Proposition 4.3.4.** Assume \( \log \alpha, \log \beta \) are incommensurable and \( 0 < a < b \) and \( M \neq \emptyset. \) Then \( \gamma_{12}([a, b], M) = \infty. \)

**Proof.** Let \( y \in M. \) If \( y = 0, \) use the preceding. We do the case \( y > 0 \) here. The case \( y < 0 \) is similar.

Let \( p = by, q = p - 1, a' = \max\{a, q/y\}. \) Choose \((m, n)\) as before, so \( a'\varepsilon < \alpha^{m+1}\beta^{n+1} < b\varepsilon \) where \( \varepsilon = a^k/b^k \) goes to 0 as \( k \to \infty. \) Let \( U = ]q\varepsilon, p\varepsilon[, \text{diam} U = \varepsilon, \) let \( \tau = a^m c d^n e, T = S_\tau. \) Then \( \rho(T) = \alpha^{m+1}\beta^{n+1}, \)

\[
T(y) = \alpha^{m+1}\beta^{n+1}y < b\varepsilon y = p\varepsilon, \\
T(y) = \alpha^{m+1}\beta^{n+1}y > a'\varepsilon y \geq \frac{q}{y}\varepsilon y = q\varepsilon,
\]

so \( T(y) \in U \) for all these \( T. \) Therefore \#\( F_{12}([a, b], U, M) > C \) so \( \gamma_{12}([a, b], M) = \infty. \)

What of the attractors (invariant sets) for this IFS? The dimension is the maximum of \( D_1 \) and \( D_2. \) If \( \alpha < \beta, \) then the dimension is \( D_2 = \log 2/\log(1/\beta). \) And \( K_2 \) is a Cantor set of that dimension,

\[
0 < \mathcal{H}^{D_2}(K_2) < \infty.
\]

But \( K_1 \) is a Cantor set \( C_1 \) of dimension \( D_1 < D_2 \) plus countably many images of \( K_2. \) In fact, there are \( 2^n \) images of size \( \alpha^{n+1} \) for \( n = 0, 1, \ldots \). Then

\[
\mathcal{H}^{D_2}(K_1) \leq \mathcal{H}^{D_2}(C_1) + \sum_{n=0}^{\infty} 2^n (\alpha^{D_2})^{n+1} \mathcal{H}^{D_2}(K_2).
\]

The first term is 0, and since \( \alpha^{D_1} = 2, \beta^{D_2} = 2, \) we have \( 1/2 = \beta^{D_2} > \alpha^{D_2}, \) so \( 2\alpha^{D_2} < 1 \) and the series is a convergent geometric series. Therefore

\[
0 < \mathcal{H}^{D_2}(K_1) < \infty.
\]
What about the case $\alpha > \beta$, so the dimension is $D_1$? Again $K_2$ is a Cantor set of dimension $D_2 < D_1$, so $\mathcal{H}^{D_1}(K_2) = 0$. And $K_1$ is a Cantor set $C_1$ of dimension $D_1$ plus countably many images of $K_2$.

$$\mathcal{H}^{D_1}(K_1) \leq \mathcal{H}^{D_1}(C_1) + \sum_{n=0}^{\infty} 2^n (\alpha^{D_2})^{n+1} \mathcal{H}^{D_1}(K_2).$$

Now all the terms in the series are zero, so we get just $\mathcal{H}^{D_1}(C_1)$, which is positive and finite.

Finally consider the case $\alpha = \beta$. So $D_1 = D_2$. Now $K_2$ is a Cantor set of that dimension, $0 < \mathcal{H}^{D_1}(K_2) < \infty$. And $K_1$ consists of a Cantor set $C_1$ of dimension $D_1$ plus countably many images of $K_2$. So, unless there is serious overlap of the images, we get

$$\mathcal{H}^{D_1}(K_1) = \mathcal{H}^{D_1}(C_1) + \sum_{n=0}^{\infty} 2^n (\alpha^{D_1})^{n+1} \mathcal{H}^{D_1}(K_2)$$

$$= \mathcal{H}^{D_1}(C_1) + \mathcal{H}^{D_1}(K_2) \left[ \sum_{n=0}^{\infty} \frac{1}{2} \right] = \infty$$

since $\alpha^{D_1} = 1/2$. Presumably for most $\gamma$ there is negligible overlap in this final computation, so $\mathcal{H}^{D_1}(K_1) = \infty$. So in this case the attractor has infinite but sigma-finite measure in its fractal dimension $D_1$. The relevant feature of the graph $G = (V, E)$ is that the maximum dimension occurs in two comparable components. This type of dimension computation was described by Mauldin & Williams [9].

4.4. Example. An example where (4b) holds but (3a) fails.

The graph $G = (V, E)$ is shown: $V = \{1, 2\}$, $E = \{a, b, c, d, e\}$. The two components in the graph are $\{1\}$ and $\{2\}$.

A number $\beta = 0.012\cdots$ will be specified below. The IFS $(S_e)$ is defined with: $X_1 = X_2 = \mathbb{R}$,

$$S_a(x) = 10^{-1}x,$$

$$S_b(x) = 10^{-1}x + 10^{-1},$$

$$S_c(x) = x + \beta,$$

$$S_d(x) = 10^{-1}x,$$

$$S_e(x) = 10^{-1}x + 10^{-1}.$$  

(The figure has ratio $5^{-1}$ instead of $10^{-1}$.)
So $E^{(s)}_{11} = \{ S_{\sigma} : \sigma \in \{a, b\}^{(s)} \}$ corresponds to a Cantor set $C_1$ of dimension $D_1 = \log(2)/\log(10)$. Component $\{1\}$ by itself it satisfies OSC and the identity is isolated in $F_{11}$. Similarly, $E^{(s)}_{22} = \{ S_{\sigma} : \sigma \in \{d, e\}^{(s)} \}$ corresponds to a Cantor set $C_2$ of dimension $D_2 = \log(2)/\log(10)$. Component $\{2\}$ by itself it satisfies OSC and the identity is isolated in $F_{22}$.

The number $\beta$ is defined by a decimal expansion:

$$\beta = 0.012321010001000010000010000001 \cdots$$

all remaining digits are 0 and 1; there are longer and longer blocks of consecutive 0’s separated by single 1’s.

**Theorem 4.4.1.** The identity is not isolated in $F_{12}$.

$$\gamma_{12}(]10^{-1}, 1], \{0\}) < \infty.$$  

Let’s do a few computations. If $\sigma \in E^{(n)}_{11}$, then $\sigma$ is a string of length $n$ made up of the letters $a, b$. By induction on $n$,

$$S_{\sigma}(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i},$$

with all digits $\varepsilon_i \in \{0, 1\}$. In fact, every sequence of 0’s and 1’s occurs for a corresponding string of a’s and b’s.

The description of $E^{(n)}_{22}$ is the same, except that the strings are made up of d’s and e’s.

For $\sigma \in E^{(n+1)}_{12}$, say $\sigma$ consists of a string of a’s and b’s of length $k$, followed by the letter c, followed by a string of d’s and e’s of length $n - k$. Then

$$S_{\sigma}(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k} \beta,$$

where all digits $\varepsilon_i \in \{0, 1\}$.
Here $0 \leq k \leq n$, and every sequence $\varepsilon_i$ of 0’s and 1’s is possible. Taking into account the decimal expansion of $\beta$, we conclude that the constant term

$$S_\sigma(0) = \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k} \beta$$

has a decimal expansion using only digits $0, \ldots, 4$.

**Lemma 4.4.2.** Let

$$x = \sum_{i=N}^{\infty} a_i 10^{-i}, \quad \text{with all digits } a_i \in \{-8, -7, \ldots, 7, 8\}.$$ 

Then $x = 0$ only if all $a_i = 0$. More precisely, if $n$ is the first index with $a_n \neq 0$, then $|x| > 10^{-n-1}$.

**Proof.** Either $a_n > 0$ or $a_n < 0$. Without loss of generality, say $a_n > 0$. Then $a_i = 0$ for all $i < n$, $a_n \geq 1$, and $a_i \geq -8$ for all $i > n$. Therefore

$$x \geq 10^n + \sum_{i=n+1}^{\infty} (-8)10^{-i} = 10^{-n} \left(1 - \frac{8}{9}\right) > 10^{-n-1}.$$ 

□

**Lemma 4.4.3.** Let

$$x = \sum_{i=N}^{\infty} a_i 10^{-i}, \quad x' = \sum_{i=N'}^{\infty} a'_i 10^{-i},$$

with all digits $a_i, a'_i \in \{-4, \ldots, 4\}$. If $x = x'$, then $a_i = a'_i$ for all $i$. More precisely, if $n$ is the first index with $a_n \neq a'_n$, then $|x - x'| > 10^{-n-1}$.

**Proof.** Subtract $x - x'$ to get a decimal of the form in Lemma 4.4.2. □

**Proposition 4.4.4.** The identity is not isolated in $\mathcal{F}_{12}$.

**Proof.** For $n \in \mathbb{N}, n \geq 5$, let $S, T \in \mathcal{F}_{12}$ be

$$S(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-1} \beta,$$

$$T(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon'_i 10^{-i} + 10^0 \beta,$$
\[ T^{-1}(S(x)) = x + \sum_{i=1}^{n} (\varepsilon_i - \varepsilon'_i)10^{n-i} + 10^{n-1}\beta - 10^n\beta. \]

When we do the subtraction \(10^{n-1}\beta - 10^n\beta\) (without borrowing) we get a decimal with digits in \([-1,0,1]\), so we can choose \(\varepsilon_i, \varepsilon'_i\) to cancel all of the digits to the left of the decimal point. If \(n\) is chosen so that a string of \(k\) zeros begins in \(\beta\) in place \(10^{-n-1}\), we will have the constant term
\[
\sum_{i=1}^{n} (\varepsilon_i - \varepsilon'_i)10^{-i} + 10^{n-1}\beta - 10^n\beta
\]
with all 0’s to the left of the decimal, a string of 0’s of length \(k - 1\) to the right of the decimal, followed by other digits in \([-1,0,1]\). So the constant term has absolute value \(<10^{-k+2}\). We can do this for \(k\) as large as we like, so we can arrange \(|T^{-1}(S(x)) - x|\) as small as we like. \(\square\)

**Proposition 4.4.5.** \(\gamma_{12}(\[10^{-1},1]\), \{0\}) \leq 2 \cdot 10^4.\)

*Proof.* Let \(U\) be an interval in \(\mathbb{R}\). There is a unique \(n\) with \(10^{-n} \leq \text{diam } U < 10^{-n+1}\), so that \(10^{-n} \in \text{diam } U, \text{diam } U]\). But the only ratios in the IFS are powers of \(10^{-1}\), so all
\[ S \in F_{12}(\[10^{-1}\text{diam } U, \text{diam } U]) \]
must have the same ratio \(10^{-n}\). I claim that if \(V\) is any interval of length \(10^{-n-3}\), then
\[ \# \{ S \in F_{12}(\[10^{-1}\text{diam } U, \text{diam } U]) : S(0) \in V \} \leq 2. \]

To see this, recall that such \(S\) has the form
\[ S(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k}\beta. \]

By Lemma 2, different maps \(S\) have different constant terms \(S(0)\). Indeed, given \(S(0)\), we may determine \(k\) by looking at the digits to the right of the decimal, and then subtract \(10^{-k}\beta\) to determine the \(\varepsilon_i\). How could two different \(S(0)\)'s be in \(V\)? Suppose
\[ S(0) = \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k}\beta, \quad S'(0) = \sum_{i=1}^{n} \varepsilon'_i 10^{-i} + 10^{-k'}\beta, \]
where $S \neq S'$ and $|S(0) - S'(0)| < 10^{-n-3}$. First, if $k = k'$, then we have $\varepsilon_i \neq \varepsilon'_i$ for some $i$, so that by Lemma 2, $|S(0) - S'(0)| > 10^{-n-1}$, which is too large. Next, if $k \leq k' - 2$, then $10^{-k}\beta$ has a digit 3 in a place where $10^{-k'}\beta$ has digit $\leq 1$ and also $|\varepsilon_i - \varepsilon'_i| \leq 1$, so $S(0) - S'(0)$ has nonzero digit in that place, no further to the right than the $10^{-n+2-4}$ place, so $|S(0) - S'(0)| > 10^{-n-3}$, again too large. Finally, if $k = k' - 1$, then $10^{-k}\beta - 10^{-k'}\beta$ has only digits $-1, 0, 1$; if $\varepsilon_i$ are all given, then $\varepsilon'_i$ are uniquely determined by the condition that all digits of $S(0) - S'(0)$ up to place $10^{-n}$ must be 0. Therefore: if three different maps $S \in F_{12}([10^{-1}\text{diam } U, \text{diam } U])$ all have $S(0) \in V$, then the three corresponding values of $k$ must be pairwise consecutive, which is impossible.

The set $U$ has length $< 10^{-n+1}$, so it can be covered by at most $10^4$ intervals $V$ of length $10^{-n-3}$ and each of these intervals $V$ can contain at most two $S(0)$. So in all there are at most $2 \cdot 10^4$ different $S \in F_{12}([10^{-1}\text{diam } U, \text{diam } U])$ with $S(0) \in U$. This means: $\gamma_{12}([10^{-1}, 1], \{0\}) \leq 2 \cdot 10^4$. □

4.5. Example. This is an example where (2b) holds but (2a) fails.

We begin with a general description of the IFS. The details will be filled in below. The directed multigraph $G$ is $(V, E)$. There is a “contraction factor” $\alpha \in [0, 1]$. There are two digit-sets ($\text{finite sets of real numbers}$) $A$ and $D$. There are two numbers $\beta, \gamma \in [0, 1]$.

The vertex set is $V = \{1, 2\}$. For edges, we will have $\#E_{11} = \#A$, $\#E_{22} = \#D$, $\#E_{12} = 2$, and $\#E_{21} = 0$. All ratios in $E_{11}$ and $E_{22}$ are $\alpha$, and all ratios in $E_{12}$ are 1. The IFS $(S_e)$ will have $X_1 = X_2 = \mathbb{R}$ and: maps $S_e$ for $e \in E_{11}$ are the maps

$$\alpha(x + \varepsilon), \quad \varepsilon \in A;$$

Figure 4.5A. The graph and the IFS
maps $S_e$ for $e \in E_{22}$ are the maps

$$\alpha(x + \theta), \quad \theta \in D;$$

maps $S_e$ for $e \in E_{12}$ are the maps

$$S_b(x) = x + \beta, \quad S_c(x) = -x + \gamma.$$ 

In this IFS, component $\{1\}$ alone has an attractor consisting of all points with expansion in base $\alpha$ using digit set $A$. It could be either an interval or a Cantor set, depending on the choice of digits. In our example, it will be a Cantor set. For $\sigma \in E^{(n)}_{11}$ compute

$$S_{\sigma}(x) = \alpha^n x + \sum_{i=1}^{n} \varepsilon_i \alpha^i,$$

where the digits $\varepsilon_i \in A$. Any choice of $\varepsilon_i$ may be obtained by using the proper choice of $\sigma$.

Similarly, for $\sigma \in E^{(n)}_{22}$,

$$S_{\sigma}(x) = \alpha^n x + \sum_{i=1}^{n} \theta_i \alpha^i,$$

where $\theta_i \in D$.

Of course $E^{(*)}_{21} = \emptyset$. Elements of $E^{(*)}_{12}$ are of two types, depending on whether the string contains letter $b$ or letter $c$. In the first case, an element $\sigma \in E^{(n+1)}_{12}$ consists of a string in $E^{(k)}_{11}$, then $b$, then a string in $E^{(n-k)}_{22}$. Here, $0 \leq k \leq n$. The result is a map

$$S(x) = \alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \quad \varepsilon_i \in A, \theta_i \in D.$$

Similarly, when the letter $c$ appears, the result is a map

$$T(x) = -\alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i - \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \gamma, \quad \varepsilon_i \in A, \theta_i \in D.$$

Now let $r = r_{\min} = \alpha$. Since all ratios $\rho(R)$ for maps $R \in \mathcal{F}_{uv}$ are integer powers of $\alpha$, the interval $]r^{-1}, r[\} \}$ contains no such ratio except 1. Therefore all $R \in \mathcal{F}_{uv}(r)$ have $\rho(R) = 1$. 


Next we compute the sets \( F_{uv}(r) \) as follows. Let \( R \in F_{11}(r) \); then \( R = S_{\sigma'}^{-1} \circ S_{\sigma} \), where for some \( n \),

\[
S_{\sigma}(x) = \alpha^nx + \sum_{i=1}^{n} \varepsilon_i \alpha^i, S_{\sigma'}(x) = \alpha^nx + \sum_{i=1}^{n} \varepsilon'_i \alpha^i.
\]

Thus

\[
R(x) = x + \sum_{i=1}^{n} (\varepsilon_i - \varepsilon'_i) \alpha^{i-n}, \quad \varepsilon_i, \varepsilon'_i \in A.
\]

Similarly, \( R \in F_{22}(r) \) is of the form

\[
R(x) = x + \sum_{i=1}^{n} (\theta_i - \theta'_i) \alpha^{i-n}, \quad \theta_i, \theta'_i \in D.
\]

For \( R \in F_{12}(r) \) there are four cases. Case 1 is \( R = S'^{-1} \circ S \) where for some \( n \) and some \( k, k' \in \{0, \cdots, n\} \)

\[
S(x) = \alpha^nx + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \quad \varepsilon_i \in A, \theta_i \in D;
\]

\[
S'(x) = \alpha^nx + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i + \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \beta, \quad \varepsilon'_i \in A, \theta'_i \in D;
\]

so

\[
R(x) = x + \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} - \sum_{i=k+1}^{n} \varepsilon'_i \alpha^{i-n} + \sum_{j=k+1}^{n} \theta_j \alpha^{j-n}
- \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} + \alpha^{k-n} \beta - \alpha^{k'-n} \beta.
\]

Case 2 is \( R = T'^{-1} \circ T \) where

\[
T(x) = -\alpha^nx + \sum_{i=1}^{k} \varepsilon_i \alpha^i - \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \gamma, \quad \varepsilon_i \in A, \theta_i \in D.
\]

\[
T'(x) = -\alpha^nx + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i - \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \gamma, \quad \varepsilon'_i \in A, \theta'_i \in D.
\]
so

\[ R(x) = x - \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} + \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} - \sum_{j=k+1}^{n} \theta_j \alpha^{i-n} \]

\[ - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} + \alpha^{k-n} \gamma - \alpha^{k'-n} \gamma. \]

Case 3 is \( R = T' \circ S \) where

\[ S(x) = \alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \quad \varepsilon_i \in A, \theta_i \in D; \]

\[ T'(x) = -\alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i - \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \gamma, \quad \varepsilon'_i \in A, \theta'_i \in D. \]

so

\[ R(x) = \alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \quad \varepsilon_i \in A, \theta_i \in D; \]

\[ S'(x) = \alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i + \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \beta, \quad \varepsilon'_i \in A, \theta'_i \in D; \]

so

\[ R(x) = -x - \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} + \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} - \sum_{j=k+1}^{n} \theta_j \alpha^{i-n} \]

\[ - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} - \alpha^{k-n} \beta + \alpha^{k'-n} \gamma. \]

Case 4 is \( R = S' \circ T \) where

\[ T(x) = -\alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i - \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \gamma, \quad \varepsilon_i \in A, \theta_i \in D. \]

\[ S'(x) = \alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i + \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \beta, \quad \varepsilon'_i \in A, \theta'_i \in D; \]

so

\[ R(x) = -x + \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} - \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} - \sum_{j=k+1}^{n} \theta_j \alpha^{i-n} \]

\[ - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} + \alpha^{k-n} \gamma - \alpha^{k'-n} \beta. \]

Now we specialize the choices. Let \( \alpha = 1/9, A = \{-1, 0\}, D = \{0, 1, 2, 3, 4\}. \) Let \( q \in ]4/9, 1/2[ \) be “normal” base 9 in the sense that
the expansion in base 9 contains all finite strings from the alphabet \{0, \cdots, 8\} (with the appropriate frequency). Let \( \beta \in [2/9, 3/9] \) and \( \gamma \in [6/9, 7/9] \) be such that \( \gamma - \beta = q \). (In fact, \( \beta = 2/9, \gamma = q + 2/9 \) will do.)

**Proposition 4.5.1.** For the IFS defined above, components \{1\} and \{2\} both satisfy WSP.

**Proof.** Component \{2\} satisfies the open set condition using open set \([0, 1]\). Component \{1\} satisfies the open set conditions using open set \([-1, 0]\). And OSC implies WSP.

Alternatively, we may consider the maps in \( F_{22}(r) \) as computed before:

\[
x + \sum_{i=1}^{n} (\theta_i - \theta'_i) 9^{n-i}, \quad \theta_i, \theta'_i \in D.
\]

These have the form \( x + b \) where \( b \) is an integer. So the identity is isolated in \( F_{22}(r) \). Component \{1\} is done in the same way. \( \Box \)

**Proposition 4.5.2.** The IFS defined above satisfies (2b).

**Proof.** We know (2b), (3a), (3b) are equivalent. By Proposition 4.5.1, the identity is isolated in \( F_{11} \) and in \( F_{22} \). So we must show that the identity is isolated in \( F_{12} \). Let \( R \in F_{12} \). Then \( R \) has the form \( ax + b \). Because all ratios \( a \) are powers of \( \alpha = 1/9 \), it follows that if \( R \) is close enough to the identity, then \( a = 1 \). Write \( \varepsilon_0 = 1/24 \). We will show that if \( R(x) = x + b \) is in \( F_{12} \) and \( |b| < \varepsilon_0 \), then \( b = 0 \). That will show that the identity is isolated.

So assume \( R \in F_{12} \) is of the form \( R(x) = x + b \) with \( |b| < \varepsilon_0 \). There are two possibilities for \( b \). Case 1:

\[
b = \sum_{i=1}^{k} \varepsilon_i 9^{n-i} - \sum_{i=1}^{k'} \varepsilon'_i 9^{n-i} + \sum_{i=k+1}^{n} \theta_i 9^{n-i} \\
- \sum_{i=k'+1}^{n} \theta'_i 9^{n-i} + 9^{n-k} \beta - 9^{n-k'} \beta.
\]

First we claim that \( k = k' \). Indeed, suppose \( k < k' \). (The case \( k > k' \) is similar.) Now the sum \( \sum_{i=1}^{k} (\varepsilon_i - \varepsilon'_i) 9^{n-i} \) is \( 9^{n-k} \) times an
integer $K$. If $K \geq 0$, then

\[- \sum_{i=k+1}^{k'} \varepsilon_i' g^{n-i} \geq 0,\]

\[+ \sum_{i=k+1}^{n} \theta_i g^{n-i} \geq 0,\]

\[- \sum_{i=k'+1}^{n} \theta_i' g^{n-i} \geq g^{n-k'} \left( -\frac{4}{9} \sum_{j=1}^{\infty} 9^{-j} \right) = -g^{n-k'} \frac{1}{2},\]

\[\geq -g^{n-k'} \frac{1}{18},\]

\[+(g^{n-k} - g^{n-k'}) \beta \geq g^{n-k} \left( 1 - \frac{1}{9} \right) \frac{2}{9} > g^{n-k} \frac{16}{81}.\]

Therefore

\[b \geq g^{n-k} \left( -\frac{1}{18} + \frac{16}{81} \right) = g^{n-k} \left( \frac{23}{162} \right) \geq \varepsilon_0.\]

On the other hand, if $K < 0$, then $K \leq -1$ and

\[- \sum_{i=k+1}^{k'} \varepsilon_i' g^{n-i} \leq g^{n-k} \left( \sum_{j=1}^{\infty} 9^{-j} \right) = g^{n-k} \frac{1}{8},\]

\[+ \sum_{i=k+1}^{n} \theta_i g^{n-i} \leq g^{n-k} \left( \frac{4}{9} \sum_{j=1}^{\infty} 9^{-j} \right) = g^{n-k} \frac{1}{2},\]

\[\leq \sum_{i=k'+1}^{n} \theta_i' g^{n-i} \leq 0,\]

\[+(g^{n-k} - g^{n-k'}) \beta \leq g^{n-k} \left( 1 - 0 \right) \frac{3}{9} = g^{n-k} \frac{1}{3}.\]

Therefore

\[b \leq g^{n-k} \left( -1 + \frac{1}{8} + \frac{1}{2} + \frac{1}{3} \right) = -g^{n-k} \frac{1}{24} \leq -\frac{1}{24} \leq -\varepsilon_0.\]

So we must have $k = k'$. Then

\[b = \sum_{i=1}^{k} (\varepsilon_i - \varepsilon_i') g^{n-i} + \sum_{i=k+1}^{n} (\theta_i - \theta_i') g^{n-i}.\]
This is an integer with $|b| < 1$ so $b = 0$ as claimed.

Case 2:

$$b = -\sum_{i=1}^{k} \varepsilon_i 9^{n-i} + \sum_{i=1}^{k'} \varepsilon'_i 9^{n-i} + \sum_{i=k+1}^{n} \theta_i 9^{n-i}$$

$$- \sum_{i=k'+1}^{n} \theta'_i 9^{n-i} - 9^{n-k} \gamma + 9^{n-k'} \gamma.$$

First we claim that $k = k'$. Indeed, suppose $k < k'$. (The case $k > k'$ is similar.) Now the sum $\sum_{i=1}^{k}(\varepsilon_i - \varepsilon'_i) 9^{n-i}$ is $9^{n-k}$ times an integer $K$. If $K > 0$, then $K \geq 1$ and

$$+ \sum_{i=k+1}^{k'} \varepsilon'_i 9^{n-i} > 9^{n-k} \left( -\sum_{j=1}^{\infty} \right) = -9^{n-k} \frac{1}{8},$$

$$+ \sum_{i=k+1}^{n} \theta_i 9^{n-i} \geq 0,$$

$$- \sum_{i=k'+1}^{n} \theta'_i 9^{n-i} \geq 9^{n-k'} \left( -4 \sum_{j=1}^{\infty} 9^{-j} \right) = -9^{n-k'} \frac{1}{2} \geq -9^{n-k} \frac{1}{18},$$

$$(-9^{n-k} + 9^{n-k'}) \gamma \geq 9^{n-k}(-1 + 0) \frac{7}{9} = 9^{n-k} \frac{7}{9}.$$

Therefore

$$b \geq 9^{n-k} \left( 1 - \frac{1}{8} - \frac{1}{18} - \frac{7}{9} \right) = 9^{n-k} \frac{1}{24} \geq \frac{1}{24} \geq \varepsilon_0.$$

On the other hand, if $K \leq 0$, then

$$+ \sum_{i=k+1}^{k'} \varepsilon'_i 9^{n-i} \leq 0,$$

$$+ \sum_{i=k+1}^{n} \theta_i 9^{n-i} \leq 9^{n-k} \left( 4 \sum_{j=1}^{\infty} 9^{-j} \right) = 9^{n-k} \frac{1}{2},$$

$$- \sum_{i=k'+1}^{n} \theta'_i 9^{n-i} \leq 0,$$
Proposition 4.5.3. For the IFS defined above, (2a) fails.

Proof. Let \( x_0 \in X_2 = \mathbb{R} \) and let \( \varepsilon > 0 \). We claim that there is \( R \in \mathcal{F}_{12}(r) \) such that \( R(x_0) \neq x_0 \) but \( |R(x_0) - x_0| < \varepsilon \). Consider \( R = S_{r^{-1}} \circ T \) where for some \( n \) (and \( k = k' = 0 \))

\[
R(x) = -x - \sum_{i=1}^{n} (\theta_i + \theta'_i)9^{n-i} + 9^n(\gamma - \beta).
\]

We claim that \( n, \theta_i, \theta'_i \) may be chosen so that \( R(x) = -x + b \) with \( 0 < 2x_0 - b < \varepsilon \). Note that \( \theta_i + \theta'_i \) can have any value in \{0, \cdots, 8\}, so that \( \sum_{i=1}^{n} (\theta_i + \theta'_i)9^{n-i} \) can have any integer value from 0 to \( 9^n - 1 \). Also, \( 9^n(\gamma - \beta) = 9^n q \) is between \((4/9)9^n\) and \((1/2)9^n\). Choose \( n \) so that \((4/9)9^n > |2x_0|\) and the fractional part of \( 2x_0 - 9^n q \) lies in \([0, \varepsilon]\). This is possible since the base 9 expansion of \( q \) is normal. Then choose \( \theta_i, \theta'_i \) so that \( -\sum_{i=1}^{n} (\theta_i + \theta'_i)9^{n-i} \) is the greatest integer \( \leq 2x_0 - 9^n q \). Then \( b = -\sum_{i=1}^{n} (\theta_i + \theta'_i)9^{n-i} + 9^n q \) satisfies \( 0 < 2x_0 - b < \varepsilon \). So \( R(x_0) - x_0 = -x_0 + b - x_0 = b - 2x_0 \). Thus we have \( R(x_0) \neq x_0 \) and \( |R(x_0) - x_0| < \varepsilon \) as required. \( \square \)

4.6. Example. This is an example where (1b) holds but (1a) fails. Such a counterexample is impossible in one dimension, so we do it in the plane, which we identify with the set \( \mathbb{C} \) of complex numbers. However, we have not kept the condition of uniform affine dimension. [In this case the problem is with \( K_2 = \{0\} \).] We haven't found a way to adapt this example, yet still keep uniform affine dimension.

The parameters in this case are as follows. A complex number \( \alpha \) in polar coordinates \( \alpha = se^{2\pi i \phi} \); we will use \( s = 1/10 \) and \( \phi \in [0, 1/2[ \)
irrational. So if \( \omega = e^{2\pi i \phi} \), then \( \{ \omega^k : k \in \mathbb{N} \} \) is dense in the unit circle \( \{ z \in \mathbb{C} : |z| = 1 \} \). [In the preceding, we have used “i” for \( \sqrt{-1} \), but from now on “i” is merely an index.] Two digit-sets \( A = \{0, 1\}, D = \{0, 2\} \).

The vertex set is \( V = \{1, 2, 3, 4\} \). For edges, we will have \#\( E_{13} = 1, \#E_{32} = 1, \#E_{14} = 1, \#E_{42} = 1, \#E_{33} = 2, \#E_{44} = 2, \#E_{22} = 1 \), and all others empty. All ratios are \( s = |\alpha| \). The IFS \( (S_e) \) will have \( X_1 = X_2 = X_3 = X_4 = \mathbb{C} \) and: the maps \( (S_e) \) for \( e \in E_{33} \) are the maps \( \alpha(x + \theta) \), where \( \theta \in \{0, 2\} \); the maps \( (S_e) \) for \( e \in E_{44} \) are the maps \( \alpha(x + \varepsilon) \), where \( \varepsilon \in \{0, 1\} \); the maps \( (S_e) \) for \( e \in E_{13} \cup E_{32} \cup E_{42} \cup E_{22} \) are all equal to the map \( \alpha x \); the map \( (S_e) \) for \( e \in E_{14} \) is the map \( \alpha \overline{x} \). The overline \( \overline{x} \) denotes the complex conjugate of \( x \).

Figures illustrating this example were given in the Introduction, above. The figures use ratio 2/3 rather than 1/9 and \( \phi = 0.4 \).

In this IFS, component \( \{3\} \) alone has an attractor consisting of all points with expansion in base \( \alpha \) using digit-set \( \{0, 2\} \). It is a Cantor set. For \( \sigma \in E_{33}^{(n)} \) compute

\[
S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \theta_i \alpha^i,
\]

where the digits \( \theta_i \in \{0, 2\} \). Any choices of \( \theta_i \) from the digit-set \( \{0, 2\} \) can be obtained by using the proper choice of \( \sigma \).

Similarly, for \( \sigma \in E_{44}^{(n)} \),

\[
S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \varepsilon_i \alpha^i,
\]

where \( \varepsilon_i \in \{0, 1\} \).

Of course, for \( \sigma \in E_{22}^{(n)} \) there is only \( S_\sigma(x) = \alpha^n x \). The attractor for component \( \{2\} \) is the single point 0.

Elements of \( E_{12}^{(n)} \) are of two types, depending on whether the string passes through node 3 or node 4. In the first case, an element \( \sigma \in E_{12}^{(n+2)} \) consists of the edge from 1 to 3 followed by a string in \( E_{33}^{(n-k)} \), then the edge from 3 to 2, then a string in \( E_{22}^{(k)} \). Here
0 \leq k \leq n. The result is a map
\[ S(x) = \alpha^{n+2}x + \sum_{i=1}^{n-k} \theta_i \alpha^{i+1}, \quad \theta_i \in \{0, 2\}. \]
This may be written as
\[ S(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \quad \theta_i \in \{0, 2\}, \]
where the sum goes all the way to \( n \), with the proviso that, when \( k > 0 \), then some of the \( \theta_i \) are required to be 0. But by taking \( k = 0 \) we may realize any sequence of \( \theta_i \in \{0, 2\} \).

Similarly, when the path passes through node 4, the result is a map
\[ T(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \varepsilon_i \alpha^{i+1}, \quad \varepsilon_i \in \{0, 1\}. \]
Again, we may realize any sequence of \( \varepsilon_i \in \{0, 1\} \).

Now let \( r = \rho_{\text{min}} = s \). Since all ratios \( \rho(R) \) for maps \( R \in \mathcal{F}_{uv} \) are integer powers of \( s \), the interval \( [r^{-1}, r] \) contains no such ratio except 1. Therefore all \( R \in \mathcal{F}_{uv}(r) \) have \( \rho(R) = 1 \).

Next we compute the sets \( \mathcal{F}_{uv}(r) \) as follows. Let \( R \in \mathcal{F}_{uv}(r) \); then there are four cases. Case 1 is \( R = S' \circ S \) where
\[ S(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \quad S'(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta'_i \alpha^{i+1}, \]
so
\[ R(x) = x + \sum_{i=1}^{n} \theta_i \alpha^{i-1-n} - \sum_{i=1}^{n} \theta'_i \alpha^{i-1-n}. \]
Case 2 is \( R = T' \circ T \) where
\[ T(x) = \alpha^{n+1}\bar{x} + \sum_{i=1}^{n} \varepsilon_i \alpha \bar{\alpha}^i, \quad T'(x) = \alpha^{n+1}\bar{x} + \sum_{i=1}^{n} \varepsilon'_i \alpha \bar{\alpha}^i, \]
so
\[ R(x) = x + \sum_{i=1}^{n} \varepsilon_i \alpha^{i-1-n} - \sum_{i=1}^{n} \varepsilon'_i \alpha^{i-1-n}. \]
Case 3 is \( R = T' \circ S \) where
\[ S(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \quad T'(x) = \alpha^{n+1}\bar{x} + \sum_{i=1}^{n} \varepsilon'_i \alpha \bar{\alpha}^i, \]
so

\[ R(x) = \alpha^{-n-1}\alpha^{n+1}x + \sum_{i=1}^{n} \theta_i \alpha^{-n-1} \alpha^i - \sum_{i=1}^{n} \varepsilon'_i \alpha^{i-n-1}. \]

Case 4 is \( R = S' \circ T \) where

\[
T(x) = \alpha \alpha^{n+1}x + \sum_{i=1}^{n} \varepsilon_i \alpha \alpha^i, \quad S'(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta'_i \alpha^{i+1},
\]

so

\[ R(x) = \alpha^{-n-1}\alpha^{n+1}x + \sum_{i=1}^{n} \varepsilon_i \alpha^{-n-1} \alpha^i - \sum_{i=1}^{n} \theta'_i \alpha^{i-n-1}. \]

If we write \( \alpha = s\omega \) as above, these four cases become:

\[
S'^{-1}(S(x)) = x + \sum_{i=1}^{n} s^{i-1-n} (\omega^{i-1-n} \theta_i - \omega^{i-1-n} \theta'_i),
\]

\[
T'^{-1}(T(x)) = x + \sum_{i=1}^{n} s^{i-1-n} (\omega^{i-1-n} \varepsilon_i - \omega^{i-1-n} \varepsilon'_i),
\]

\[
T'^{-1}(S(x)) = \omega^{-2n-2}x + \sum_{i=1}^{n} s^{i-1-n} (\omega^{-i-1-n} \theta_i - \omega^{-i-1-n} \theta'_i),
\]

\[
S'^{-1}(T(x)) = \omega^{-2n-2}x + \sum_{i=1}^{n} s^{i-1-n} (\omega^{-i-1-n} \varepsilon_i - \omega^{-i-1-n} \varepsilon'_i).
\]

**Proposition 4.6.1.** The IFS defined above satisfies (1b).

**Proof.** Let \( x_0 = 0, \varepsilon_0 = 1 \). We will show that if \( R \in F_{12}(r) \), then either \( R(0) = 0 \) or \( |R(0)| \geq \varepsilon_0 \). There are different cases depending on the cases for the map \( R \) defined above.

Let \( R \in F_{12}(r) \) fall in Case 1:

\[ R(0) = \sum_{i=1}^{n} 10^{n+1-i} (\omega^{i-1-n} \theta_i - \omega^{i-1-n} \theta'_i). \]

Suppose \( |R(0)| < \varepsilon_0 \). We must show that \( R(0) = 0 \). In fact we claim that \( \theta_i = \theta'_i \) for all \( i \). Assume not: then let \( i_0 \) be the least \( i \) with \( \theta_i \neq \theta'_i \). Now \( \theta_i - \theta'_i \) is 2, 0, or -2, and

\[
|R(0)| \geq 10^{n+1-i_0} \left( 2 - 2 \sum_{j=1}^{\infty} 10^{-j} \right) \geq 10 \left( 2 - \frac{2}{9} \right) = \frac{160}{9} \geq \varepsilon_0.
\]
Let $R \in \mathcal{F}_{12}(r)$ fall in Case 2:

$$R(0) = \sum_{i=1}^{n} 10^{n+1-i} (\omega^{i-1-n} \varepsilon_i - \omega^{i-1-n} \varepsilon'_i).$$

Suppose $|R(0)| < \varepsilon_0$. We must show that $R(0) = 0$. In fact we claim that $\varepsilon_i = \varepsilon'_i$ for all $i$. Assume not: then let $i_0$ be the least $i$ with $\varepsilon_i \neq \varepsilon'_i$. Now $\varepsilon_i - \varepsilon'_i$ is 1, 0, or $-1$, and

$$|R(0)| \geq 10^{n+1-i_0} \left(1 - \sum_{j=1}^{\infty} 10^{-j}\right) \geq 10 \left(1 - \frac{1}{9}\right) = \frac{80}{9} \geq \varepsilon_0.$$

Let $R \in \mathcal{F}_{12}(r)$ fall in Case 3:

$$R(0) = \sum_{i=1}^{n} 10^{n+1-i} (\omega^{i-1-n} \theta_i - \omega^{i-1-n} \varepsilon'_i).$$

Suppose $|R(0)| < \varepsilon_0$. We must show that $R(0) = 0$. In fact we claim that $\theta_i = \varepsilon'_i = 0$ for all $i$. Assume not: then let $i_0$ be the least $i$ where it fails. We take two subcases: If $\theta_{i_0} = 2$, then

$$|R(0)| \geq 10^{n+1-i_0} \left(2 - 1 - 3 \sum_{j=1}^{\infty} 10^{-j}\right)$$

$$\geq 10 \left(2 - 1 - \frac{3}{9}\right) = \frac{20}{3} \geq \varepsilon_0.$$

On the other hand, if $\theta_{i_0} = 0$, then $\varepsilon_{i_0} = 1$ and

$$|R(0)| \geq 10^{n+1-i_0} \left(1 - 3 \sum_{j=1}^{\infty} 10^{-j}\right) \geq 10 \left(1 - \frac{3}{9}\right) = \frac{20}{3} \geq \varepsilon_0.$$

Case 4 is similar.

So in every case, if $|R(0)| < \varepsilon_0$, then $R(0) = 0$. This completes the verification of (1b). \hfill \Box

**Proposition 4.6.2.** For the IFS defined above, (1a) fails.

**Proof.** Let $x \in \mathbb{C}$ and $\varepsilon > 0$. We claim that there exists $R \in \mathcal{F}_{12}(r)$, $R \neq \text{id}$, $|R(x) - x| < \varepsilon$. To do this, we will take $R$ in Case 3 with all $\theta_i = \varepsilon'_i = 0$. Then $R(x) = \omega^{-2n-2} x$. This is not the identity. Also, $R(0) = 0$, so if $x = 0$ we are done. So assume $x \neq 0$. Since $\{\omega^k : k = 1, 2, 3, \ldots\}$ is dense in the unit circle, there exist
Let $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$ such that $\omega^{-2n_k-2} \to x/\overline{x}$ and therefore $\omega^{-2n_k-2}\overline{x} \to x$. So for some $k$, $|R(x) - x| < \varepsilon$. □

**Open question.** Is (1a) equivalent to (1b) when the IFS has uniform affine dimension? We have not answered this.

**References**


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