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WEAK SEPARATION IN SELF-SIMILAR FRACTALS

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Abstract. We consider the "weak separation property" for graph-directed iterated function systems of similitudes in Euclidean space. The formulations that are known to be equivalent in the case of strongly connected graph are, in general, no longer equivalent when the graph is not strongly connected.

1. Introduction

We consider graph-directed self-similar fractals. The figures illustrate an example. (This is Example 4.6, discussed in a more technical way below.) Figure 1A is a directed multigraph.

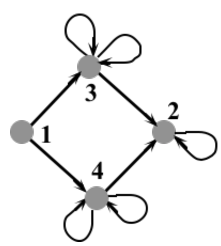

Figure 1A. A directed multigraph

Figure 1B illustrates an ***iterated function system*** (IFS) directed by that graph. There are four sets, and maps sending some

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of the sets inside others. For example, node 1 in the graph has two edges leaving it, one to node 3 and one to node 4. Correspondingly, set 1 contains one image of set 3 and one image of set 4. Similarly for all the other nodes and edges of the graph. We are dealing specifically with "self-similar" sets, so the maps are all similitudes.

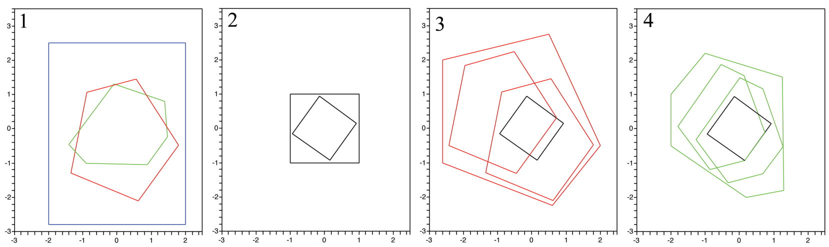

Figure 1B. Four sets mapped according to the graph

Here is a more detailed description of how the similitudes are illustrated in the figures. There are four nodes in the graph in Figure 1A. This means we will be constructing four sets in the plane. But for clarity they are shown in four separate pictures of the plane. In Figures 1B and 1C, the four pictures are labeled 1, 2, 3, and 4. There is an arrow from node 1 to node 3 in Figure 1A. This is supposed to correspond to a similitude of the plane. It maps picture 3 to picture 1 in Figure 1B. In picture 3 there is a large irregular pentagon surrounding everything else. The image of that pentagon is the pentagon shown in picture 1. (Because the pentagon has no symmetries, this uniquely determines the similitude of the plane.) There is an arrow from node 1 to node 4 in Figure 1A. This is supposed to correspond to a similitude of the plane. It maps picture 4 to picture 1 in Figure 1B. In picture 4 there is a large irregular hexagon surrounding everything else. The image of that hexagon is the hexagon shown in picture 1. Departing from node 2 of the graph is only a single arrow, going back to node 2. This indicates a similitude, mapping picture 2 into itself. The image of the large square is the small square. (In fact, because of the summetries of the square, this does not uniquely determine the similitude. For the pictures I chose the one that rotates clockwise by less than 45 degrees.) Departing from node 3 there are three arrows, two going back to 3 itself, and one going to 2. The corresponding similitudes are represented in Figure 1B. The large pentagon in 3 is mapped to each of the smaller pentagons by the first two similitudes, and the large square in picture 2 maps to the square in picture 3. Departing from node 4 there are three arrows, two going back to 4 itself, and one going to 2. These are represented in the same way as before by the two small hexagons and the square in picture 4.

Figure 1C illustrates the "attractors" or "invariant sets" defined by this IFS. (Set 2 is a single point at the origin.) Each of these sets is made up of images of the others as directed by the graph. For example, set 1 is the union of the image of set 3 and the image of set 4 under the similitudes described.

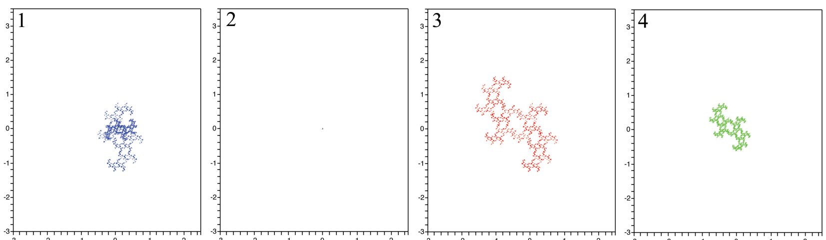

Figure 1C. The four attractors

When we study such self-similar sets, and try to determine a fractal dimension of the attractors (such as Hausdorff dimension or box dimension), the most familiar situation involves the ***open set condition*** (OSC). This example was chosen so that the OSC fails. There is no way to choose four nonempty open sets, so that the images as specified by the IFS are inside the appropriate sets, and disjoint. (For example, the open sets 3 and 4 should have images inside open set 1, but disjoint from each other.) This is described more technically in Section 2, below.

When (as in this case) an IFS fails the OSC, that means there is overlap of the images. Computation of the fractal dimension may be difficult. Two papers that study this problem in one dimension are: Keane, Simon & Solomyak [7] and Sidorov [13]. The ***weak separation property*** (WSP) has been proposed by Lau & Ngai [8] to single out cases where the dimension computation is simplified by the combinatorial nature of the overlaps. We studied this in our previous paper [4] under the assumption that the graph directing the IFS is strongly connected. This work was in turn motivated by Zerner [14].

A special subclass of the IFS with WSP, are those that are of ***finite type***. These were studied in Das & Ngai [5] and Ngai & Wang [10]. An IFS of finite type gives rise to a new graph-directed IFS. Oftentimes, even though the original IFS may be determined by a strongly connected graph, the new IFS may be determined by a graph that is not strongly connected. This raises the question as to which separation condition to use, and what it might even mean. It is worth noting that Bandt & Rao [3] gives an analysis of the relationship between the open set condition and finite type for connected self-similar sets in the plane. The relationship between finite-type and WSP was first proved in Nguyen [11].

So what should the definition be when the graph is not strongly connected? One natural possibility is simply to require WSP separately for each connected component of the graph. In fact, that is probably the best choice. But our concern in this paper is what can be said when the definition of WSP is taken for the entire IFS.

Note that we are adapting only the first part of [4]. Adaptation to non-strongly-connected graphs of the later parts of that paper (such as: similarity and growth dimensions, finite type) is left for the future.

The paper is organized as follows: Section 2 describes the setting. The weak separation property is discussed in Section 3. Finally, in Section 4, we provide several examples, both to illustrate the results and to provide counterexamples.

2. The setting

It may be noted that our setting can be thought of as a generalization of *finitely generated semigroups of contracting homotheties,* which have been studied earlier in the literature. But in that study, there is no mention of the self-similar sets that are generated, whereas for us these sets are the main item of interest.

2.1. **Directed multigraph.** The ***weak separation property*** for iterated function systems with overlap was defined and studied for the one-node setting by Lau & Ngai [8] and Zerner [14], following work by Band & Graf [2] and Schief [12]. In [4], the authors generalized this to the graph-directed setting, but only for strongly connected graphs. In the present paper we consider the definition for general graphs, not necessarily strongly connected.

The setting is the same as in [4], but the definitions are repeated here. Begin with a directed multigraph $G = (V, E)$. So V is a finite set (of "vertices"), E is a finite set (of "edges"), for each $u, v \in V$, $E_{uv} \subseteq E$ (the set of edges "from u to v"). For convenience we assume that E is the disjoint union of the sets E_{uv}. If $e \in E_{uv}$ then e has ***initial vertex*** u and ***final vertex*** v. Again for convenience we assume that every node u is the initial vertex for at least one edge. Write $E_{uv}^{(k)}$ for paths of length k, say $\sigma = e_1 e_2 \cdots e_k$ where e_1 has initial vertex u, the final vertex of each e_i matches the initial vertex of the next one e_{i+1}, and the final vertex of e_k is v. Then $E_{uv}^{(*)} = \bigcup_{k=0}^{\infty} E_{uv}^{(k)}$ is the forest of all paths in G.

2.2. **The IFS.** For each $u \in V$ we associate a complete metric space X_u. For now we will assume each $X_u \subseteq \mathbb{R}^d$ for a certain d. For each $e \in E_{uv}$ we associate a similitude $S_e \colon X_v \to X_u$, with contraction ratio ρ_e:

$$|S_e(x) - S_e(y)| = \rho_e |x - y|.$$

(A similitude defined on a subset of Euclidean space can be extended to a similitude on the whole space, so when convenient we will assume S_e is defined on all of $\mathbb{R}^d$. Also recall that a similitude R on $\mathbb{R}^d$ is ***affine*** so that it has the form $R(x) = Ax + b$ for some matrix A and vector b. When considered defined on all of $\mathbb{R}^d$, the similitude S_e has an inverse from $\mathbb{R}^d$ to itself. If $\sigma \in E_{uv}^{(*)}$ and $\dim X_u = \dim X_v$, then since S_σ maps X_v into X_u it will be bijective and the inverse maps X_u to X_v. But if $\dim X_u > \dim X_v$, of course the inverse will not map X_u into X_v.) Assume $0 < \rho_e < 1$. Write $\rho_{\min} = \min\{\, \rho_e : e \in E \,\}$, $\rho_{\max} = \max\{\, \rho_e : e \in E \,\}$. For $\sigma = e_1 e_2 \cdots e_k$ write $S_\sigma = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$ and $\rho(\sigma) = \rho_{e_1} \cdots \rho_{e_k}$. This formulation is found in [9, 6].

The original version of an IFS, where no graph is specified, can be fit into this scheme by using a graph $G = (V, E)$ where V has exactly one element. Then all edges are loops from that node to itself. To emphasize this case, we will sometimes call it the ***one-node*** case.

There is then a unique family $\{ K_u : u \in V \}$ of nonempty compact sets such that

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(K_v)$$

for all $u \in V$. These are the ***attractors*** or ***invariant sets*** defined by the IFS.

Sometimes it will be convenient to allow $\rho_e \geq 1$ for some edges e. But as long as $\rho(\sigma) < 1$ for all cycles σ, the situation reduces to the previous case by multiplying the metric in each X_u by an appropriate constant [6, Exercise (4.3.9)].

If R is a similitude, write $\rho(R)$ for its contraction ratio. So in our setting, $\rho(S_\sigma) = \rho(\sigma)$.

We will assume that X_u is the affine span of the attractor K_u. This means that if $R_1, R_2 \colon X_u \to \mathbb{R}^d$ are affine, and $R_1(x) = R_2(x)$ for all $x \in K_u$, then $R_1(x) = R_2(x)$ for all $x \in X_u$. If $E_{uv}^{(*)} \neq \varnothing$, so that there is a path from u to v in the graph G, we say u ***precedes*** v. So if u precedes v, then the attractor K_u contains a similar image of K_v, and therefore the affine dimensions satisfy $\dim X_u \geq \dim X_v$. Within a component of G all dimensions $\dim X_u$ are the same, but if G is not strongly connected, then $\dim X_u$ may differ from component to component. If $\dim X_u = \dim X_v$ for all u, v, then we will say that (S_e) has ***uniform affine dimension***. And when this is true, there is no harm in assuming $X_u = \mathbb{R}^d$ for all u.

2.3. **Definitions.** We give next a few definitions formulated in terms of a graph-directed iterated function system. Note that many of these are computed in the examples below, see especially Example 4.3. Reading the examples may help in understanding the definitions.

Let $u, v \in V$, $0 < a < b$, $I \subseteq \mathbb{R}$ an interval, $0 < r < 1$, $U \subseteq X_u$ bounded, $M \subseteq X_v$ nonempty. Define

$$\begin{aligned}
\mathcal{R}_{uv} &= \{ R : R \text{ is a similitude from } X_v \text{ to } X_u \} \\
\mathcal{R}_{uv}(I) &= \{ R \in \mathcal{R}_{uv} : \rho(R) \in I \} \\
E_{uv}^{(*)}\big(]a,b]\big) &= \Big\{ \sigma \in E_{uv}^{(*)} : S_\sigma \in \mathcal{R}_{uv}\big(]a,b]\big) \Big\} \\
F_{uv}\big(]a,b]\big) &= \Big\{ S_\sigma : \sigma \in E_{uv}^{(*)}\big(]a,b]\big) \Big\}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_{uv}\big(]a,b]\big) &= \big\{\, T^{-1}\circ S : \ T,S \in F_{uv}\big(]a,b]\big) \,\big\} \\
&\qquad\qquad \text{(These map } X_v \text{ into } \mathbb{R}^d.) \\
\mathcal{F}_{uv}\big(r\big) &= \bigcup_{b>0} \mathcal{F}_{uv}\big(]rb,b]\big) \\
\mathcal{F}_{uv} &= \bigcup_{0<a<b} \mathcal{F}_{uv}\big(]a,b]\big) = \bigcup_{0<r<1} \mathcal{F}_{uv}(r) \\
&= \Big\{\, S_\tau^{-1}\circ S_\sigma : \ \tau,\sigma \in E_{uv}^{(*)} \,\Big\} \\
F_{uv}(]a,b],U,M) &= \big\{ T \in F_{uv}\big(]a \operatorname{diam} U, b \operatorname{diam} U]\big) : \\
&\qquad T(M)\cap U \neq \varnothing \big\} \\
\gamma_{uv}(]a,b],M) &= \sup\{\, \#F_{uv}(]a,b],U,M) : \ U \subseteq X_u \text{ bounded} \,\}
\end{aligned}
$$

Proposition 2.1. $\mathcal{F}_{uv}(r) = \mathcal{F}_{uv} \cap \mathcal{R}_{uv}\big(]r, r^{-1}[\,\big)$

Proof. Let $R \in \mathcal{F}_{uv}(r)$. Then there is b so that $R = T^{-1}\circ S$ with $T,S \in F_{uv}\big(]rb,b]\big)$. So $\rho(R) = \rho(T)^{-1}\rho(S) < (rb)^{-1}b = r^{-1}$ and $\rho(R) = \rho(T)^{-1}\rho(S) > b^{-1}(rb) = r$. So $R \in \mathcal{F}_{uv} \cap \mathcal{R}_{uv}\big(]r, r^{-1}[\big)$.

Conversely, let $R \in \mathcal{F}_{uv} \cap \mathcal{R}_{uv}\big(]r, r^{-1}[\big)$. Say $R = T^{-1}\circ S$. First take the case $\rho(T) \le \rho(S)$. Let $b = \rho(S)$ so that $T,S \in F_{uv}\big(]rb,b]\big)$ and $R \in \mathcal{F}_{uv}(r)$. For the other case $\rho(T) > \rho(S)$, let $b = \rho(T)$ and again $R \in \mathcal{F}_{uv}(r)$. $\square$

3. The weak separation property

For each $u \in V$ let $X_u \subseteq \mathbb{R}^d$ be the affine span of K_u. For $e \in E_{uv}$ let $S_e \colon \mathbb{R}^d \to \mathbb{R}^d$ be a similitude with ratio ρ_e that maps X_v into X_u. Let $r \in]0, \rho_{\min}]$. We will consider the conditions given below. The length of the list is justified by the fact that it exists in the literature: In [14] all ten of these conditions are proved equivalent in the one-node setting. For the graph-directed setting, in the case of strongly connected graph G, it was proved in [4] that all ten conditions are equivalent.

- **(1a)** For all $v \in V$, there exist $x \in K_v$ and $\varepsilon > 0$ such that for all u preceding v and all $R \in \mathcal{F}_{uv}(r)$, either R is the identity on X_v or $|R(x) - x| \ge \varepsilon$.
- **(1b)** For all $v \in V$ there exist $x \in X_v$ and $\varepsilon > 0$ such that for all u preceding v and all $R \in \mathcal{F}_{uv}(r)$, either $R(x) = x$ or $|R(x) - x| \ge \varepsilon$.

- **(2a)** For all $v \in V$ there are $\varepsilon > 0$ and $\{x_0, \cdots, x_m\} \subseteq X_v$ with affine span X_v such that for all u preceding v, all $R \in \mathcal{F}_{uv}(r)$, and all j, either $R(x_j) = x_j$ or $|R(x_j) - x_j| \geq \varepsilon$.
- **(2b)** For all $v \in V$ there are $\varepsilon > 0$ and $\{x_0, \cdots, x_m\} \subseteq X_v$ with affine span X_v such that for all u preceding v and all $R \in \mathcal{F}_{uv}(r)$, either R is the identity on X_v or $|R(x_j) - x_j| \geq \varepsilon$ for some j.
- **(3a)** For all $u, v \in V$ with u preceding v, the identity is an isolated point of $\mathcal{F}_{uv}$.
- **(3b)** For all $u, v \in V$ with u preceding v, the identity is an isolated point of $\mathcal{F}_{uv}(r)$.
- **(4a)** For all $u, v \in V$ with u preceding v, all bounded $M \subseteq X_v$, and all $b > 0$, we have $\gamma_{uv}(]rb, b], M) < \infty$.
- **(4b)** For all $u, v \in V$ with u preceding v, there exist nonempty $M \subseteq X_v$ and $b > 0$ such that $\gamma_{uv}(]rb, b], M) < \infty$.
- **(5a)** For all $u, v, w \in V$ such that u precedes v precedes w, and all $z \in X_w$, there exists $l \in \mathbb{N}$ such that for any $\tau \in E_{vw}^{(*)}$ and any $b > 0$, every ball in X_u with radius b contains at most l elements of $\left\{ S_{\sigma\tau}(z) : \sigma \in E_{uv}^{(*)}\big(]rb, b]\big) \right\}$.
- **(5b)** For all $u, v, w \in V$ such that u precedes v precedes w, there exist $z \in X_w$ and $l \in \mathbb{N}$ such that for any $\tau \in E_{vw}^{(*)}$ and any $b > 0$, every ball in X_u with radius b contains at most l elements of $\left\{ S_{\sigma\tau}(z) : \sigma \in E_{uv}^{(*)}\big(]rb, b]\big) \right\}$.

We will prove:

$$
\begin{array}{l}
\text{(1a)} \\
\Downarrow \\
\text{(1b)} \iff \text{(2a)} \\
\Downarrow \\
\text{(2b)} \iff \text{(3a)} \iff \text{(3b)} \\
\Downarrow \\
\text{(4a)} \iff \text{(4b)} \iff \text{(5a)} \iff \text{(5b)}
\end{array}
$$

and provide examples showing the vertical arrows cannot be reversed.

Theorem 3.1 proves the implications that do not use the assumption of uniform affine dimension, and Theorem 3.3 proves the implications that do use that assumption. Most of the proofs are taken from [4], where they are carried out in the case of strongly connected graph.

It may also be noted that Theorem 3.1 does not use the system of attractors, so it may be thought of as a result for sets of affine maps of Euclidean space equipped with the compact-open topology which (depending on the situation) are either contracting homotheties or which are homotheties whose dilations are uniformly bounded from above and below. In particular, this point of view easily explains the equivalence of (2b), (3a), (3b).

Theorem 3.1. *Let $G = (V, E)$ be a directed multigraph. Let the iterated function system $(S_e)_{e\in E}$ with invariant list $(K_u)_{u\in V}$ in $\mathbb{R}^d$ be directed by G. Then* (1a) $\Longrightarrow$ (1b) $\Longleftarrow$ (2a) $\Longrightarrow$ (2b) $\Longleftrightarrow$ (3a) $\Longleftrightarrow$ (3b) $\Longrightarrow$ (4b) $\Longleftrightarrow$ (4a) $\Longrightarrow$ (5a) $\Longrightarrow$ (5b).

Proof. (1a) $\Longrightarrow$ (1b), (2a) $\Longrightarrow$ (2b), (3a) $\Longrightarrow$ (3b), (4a) $\Longrightarrow$ (4b), (5a) $\Longrightarrow$ (5b) are trivial.

(2a) $\Longrightarrow$ (1b) is trivial: let $x = x_0$.

(2b) $\Longrightarrow$ (3a): Assume (2b). Let $u, v \in V$ be given with u preceding v. Note that $\mathcal{R}_{uv}\big(]r, r^{-1}[\big)$ is an open neighborhood of the identity in $\mathcal{R}_{uv}$ and $\mathcal{F}_{uv}(r) = \mathcal{F}_{uv} \cap \mathcal{R}_{uv}\big(]r, r^{-1}[\big)$. Let $\{x_0, \cdots, x_d\}$ and ε be as in (2b). The set

$$\{\, R \in \mathcal{F}_{uv}(r) : \ |R(x_j) - x_j| < \varepsilon \text{ for all } j \,\} = \{\mathrm{id}\}$$

is an open neighborhood of the identity in $\mathcal{F}_{uv}(r)$. So the identity is an isolated point of $\mathcal{F}_{uv}$.

(3b) $\Longrightarrow$ (2b): Assume (3b). Let v be given. Let $\{x_0, \cdots, x_d\}$ be any set with affine span X_v. The map $R \mapsto (R(x_0), \cdots, R(x_d))$ is a homeomorphism since it is bijective and affine from one Euclidean space onto another [from the set of affine maps on $\mathbb{R}^d$ to the set $(X_u)^{d+1}$]. But if u precedes v, then the identity is isolated in $\mathcal{F}_{uv}(r)$, so there is $\varepsilon_u > 0$ so that for all $R \in \mathcal{F}_{uv}(r)$ except the identity, $|R(x_j) - x_j| \geq \varepsilon_u$. Let $\varepsilon = \min\{\, \varepsilon_u : \ u \text{ precedes } v \,\}$.

(2b) $\Longrightarrow$ (4a): Assume (2b). Let u precede v, let $M \subseteq X_v$ be bounded, and let $b > 0$. Then apply (2b) with node v to get $\{x_0, \cdots, x_d\}$ with affine span X_v and $\varepsilon > 0$. Let $k = \operatorname{diam}\big(M \cup \{x_0, \cdots, x_d\}\big)$. We must show $\gamma_{uv}(]rb, b], M) < \infty$. Let $U \subseteq X_u$ be bounded. Now if $T, S \in F_{uv}\big(]rb, b], U, M\big)$, and $T \neq S$, then there exists $j = j(S, T) \in \{0, \cdots, d\}$ with $|T^{-1}(S(x_j)) - x_j| \geq \varepsilon$, and thus

$|S(x_j) - T(x_j)| \geq rb\varepsilon \text{diam } U$. This choice of $j(S,T)$ is a "colouring" of all pairs from $F_{uv}\big(]rb,b],U,M\big)$ in $d+1$ colours. From Ramsey's Theorem it follows that if $\sup_U \#F_{uv}\big(]rb,b],U,M\big) = \infty$, then $\sup_U \#F'_U = \infty$ as well, for some choice of $F'_U \subseteq F_{uv}\big(]rb,b],U,M\big)$ such that all pairs $T,S \in F'_U$ have the same colour. But suppose all pairs in F'_U have colour j. Then the balls $B(T(x_j),(rb\varepsilon/2)\text{diam } U)$, $T \in F'_U$, are disjoint, and all their centers have distance at most $bk\text{diam } U$ from U. So these balls are all contained in a ball of radius $(1 + bk + rb\varepsilon/2)\text{diam } U$. Comparing volumes we see that $\#F'_U \leq (1 + bk + rb\varepsilon/2)^d/(rb\varepsilon/2)^d$, where d is the affine dimension of X_u, a bound independent of U. So in fact $\gamma_{uv}(]rb,b],M) = \sup_U \#F_{uv}\big(]rb,b],U,M\big) < \infty$.

(4a) $\Longrightarrow$ (5a): Assume (4a). Let $u,v,w \in V$ and $z \in X_w$ be given. Then the set

$$M = \left\{ S_\tau(z) : \tau \in E^{(*)}_{vw} \right\}$$

is bounded. Let $l = \gamma_{uv}\big(]r/2,1/2],M\big) < \infty$. Then for any $\tau \in E^{(*)}_{vw}$, any $b > 0$, and any ball U in X_u of radius b (and diameter $2b$):

$$\begin{aligned}
\#\Big(\Big\{ S_{\sigma\tau}(z) : \sigma \in E^{(*)}_{uv}\big(]rb,b]\big) \Big\} \cap U\Big)& \\
\leq \#\left\{ T \in F_{uv}\big(]rb,b]\big) : T(S_\tau(z)) \in U \right\}& \\
\leq \#\left\{ T \in F_{uv}\big(]rb,b]\big) : T(M) \cap U \neq \varnothing \right\}& \\
= \#F_{uv}\big(]r/2,1/2],U,M\big) \leq l.&
\end{aligned}$$

(4b) $\Longrightarrow$ (4a): Assume (4b). Let $u,v \in V$. There exist $M_0 \neq \varnothing$ and $b_0 > 0$ with $\gamma_{uv}\big(]rb_0,b_0],M_0\big) < \infty$. Then since $M_0 \neq \varnothing$, there is $y_0 \in M_0$ with $\gamma_{uv}\big(]rb_0,b_0],\{y_0\}\big) < \infty$.

We claim now that $\gamma_{uv}\big(]rb,b],\{y_0\}\big) < \infty$ for *all* $b > 0$. Indeed, let c be the number of balls of diameter b required to cover a set in X_u of diameter b_0. Given a bounded set $U \subseteq X_u$, write $k = \text{diam } U$, cover U by c balls V_i of diameter kb/b_0. Then

$$\begin{aligned}
F_{uv}\big(]rb,b],U,\{y_0\}\big) &= \left\{ T \in F_{uv}\big(]rbk,bk]\big) : T(y_0) \in U \right\} \\
&= \bigcup_i \left\{ T \in F_{uv}\left(\left]rb_0\frac{kb}{b_0}, b_0\frac{kb}{b_0}\right]\right) : T(y_0) \in V_i \right\},
\end{aligned}$$

so $\#F_{uv}\big(]rb,b],U,\{y_0\}\big) \le c\gamma_{uv}\big(]rb_0,b_0],\{y_0\}\big)$. Taking supremum on U, we conclude

$$\gamma_{uv}\big(]rb,b],\{y_0\}\big) \le c\gamma_{uv}\big(]rb_0,b_0],\{y_0\}\big).$$

Now we are ready to prove (4a). Let $M \subseteq X_v$ be bounded, and let $b > 0$. We claim there exists $b' > 0$ such that

$$\gamma_{uv}\big(]rb,b],M\big) \le \gamma_{uv}\big(]rb',b'],\{y_0\}\big).$$

To see this: let $k = \text{diam }(M \cup \{y_0\})$, and $b' = b/(1+2bk)$. Let $U \subseteq X_u$ be a bounded set. Define $U' = B(U, bk\text{diam }U)$, the open set of all points within distance less than $bk\text{diam }U$ of the set U. So $\text{diam }U' = \text{diam }U + 2bk\text{diam }U = (1+2bk)\text{diam }U$. We claim that

$$F_{uv}\big(]rb,b],U,M\big) \subseteq F_{uv}\big(]rb',b'],U',\{y_0\}\big).$$

Indeed, let $T \in F_{uv}\big(]rb,b],U,M\big)$. So $\rho(T) \in]rb\text{diam }U, b\text{diam }U]$ and $T(M) \cap U \ne \varnothing$. So there exists $y \in M$ with $T(y) \in U$. Now $|y - y_0| \le k$, and $|T(y) - T(y_0)| \le bk\text{diam }U$, so $T(y_0) \in U'$. Also

$$\rho(T) \in \left]\frac{rb}{1+2bk}\text{diam }U', \frac{b}{1+2bk}\text{diam }U'\right].$$

Thus $T \in F_{uv}\big(]rb',b'],U',\{y_0\}\big)$, as required. Now we have

$$\#F_{uv}\big(]rb,b],U,M\big) \le \#F_{uv}\big(]rb',b'],U',\{y_0\}\big) \le \gamma_{uv}\big(]rb',b'],\{y_0\}\big).$$

This is true for all U, so

$$\gamma_{uv}\big(]rb,b],M\big) \le \gamma_{uv}\big(]rb',b'],\{y_0\}\big) < \infty.$$

This completes the proof of (4a). $\square$

Lemma 3.2. *Assume (S_e) is an IFS with uniform affine dimension. Let $u, v \in V$ such that u precedes v. Assume* (5b) *holds. Then there exist w, z, l as in* (5b)*, a constant C, and $\tau \in E^{(*)}_{vw}$ such that v precedes w and for all $y \in X_u$, and all $b > 0$,*

$$\#\left\{\, T \in F_{uv}\big(]br,b]\big) : T(S_\tau(z)) = y \,\right\} \le C.$$

Proof. For each w preceded by v use (5b) with u, v, w to choose $z_w \in X_w$ and $l_w \in \mathbb{N}$. Let $l = \max_w l_w$. Write

$$A = \left\{\, S_\tau(z_w) : v \text{ precedes } w, \tau \in E^{(*)}_{vw} \,\right\}.$$

We claim that the affine span of A is X_v. Indeed, suppose not. Then $A \subseteq L$, where L is a proper affine subspace. But K_v spans

X_v, so there is $x \in K_v \setminus L$. There is w_1 preceded by v and $\tau_1 \in E^{(*)}_{vw_1}$ so that $x \in S_{\tau_1}(K_{w_1})$ and $S_{\tau_1}(K_{w_1}) \cap L = \varnothing$. Then if $d_0 = \max\{\, \operatorname{dist}(z_w, K_w) : u \text{ precedes } w \,\}$, then there is w_2 preceded by w_1 and $\tau_2 \in E^{(*)}_{w_1 w_2}$ as long as $d_0 \rho(\tau_1\tau_2) < \operatorname{dist}(S_{\tau_1}(K_{w_1}), L)$. So $S_{\tau_1\tau_2}(z_{w_2}) \notin L$.

Let d be the dimension of X_v, and let $x_0, \cdots, x_d \in A$ have affine span X_v, with $x_j = S_{\tau_j}(z_{w_j})$. Let $t = \max\{\, |x_j - x_0| : 0 \le j \le d \,\}$, let c_t be the number of balls of radius 1 required to cover a ball in X_u of radius t, write $m = (d+1)c_t l$ and $C = m(m-1)(m-2)\cdots(m-d+1)$.

Now let $y \in X_u$ and $b > 0$ be given. The ball $\overline{B}(y, bt)$ is covered by c_t balls of radius b, so for each $j \in \{0, \cdots, d\}$

$$\begin{aligned} &\#\left\{\, T(x_j) : T \in F_{uv}\big(]br, b]\big), T(x_j) \in \overline{B}(y, bt) \,\right\} \\ &\qquad = \#\left(\left\{\, S_{\sigma\tau_j}(z) : \sigma \in E^{(*)}_{uv}\big(]rb, b]\big) \,\right\} \cap \overline{B}(y, bt)\right) \le c_t l. \end{aligned}$$

If $T \in F_{uv}\big(]br, b]\big)$ and $T(x_0) = y$, then $|T(x_j) - y| = |T(x_j) - T(x_0)| \le bt$ for all j. So

$$\begin{aligned} &\#\left\{\, T(x_j) : T \in F_{uv}\big(]br, b]\big), j \in \{0, \cdots, d\}, T(x_0) = y \,\right\} \\ &\qquad \le (d+1)c_t l = m. \end{aligned}$$

And a similitude is determined by its values on $\{x_0, \cdots, x_d\}$, so

$$\begin{aligned} &\#\left\{\, T \in F_{uv}\big(]br, b]\big) : T(x_0) = y \,\right\} \\ &\qquad \le m(m-1)(m-2)\cdots(m-d+1) = C. \end{aligned}$$

□

Theorem 3.3. *Let the IFS (S_e) be as in Lemma 3.2. Assume also that it has uniform affine dimension. Then* (1b) $\Longrightarrow$ (2a) *and* (5b) $\Longrightarrow$ (4b).

Proof. (1b) $\Longrightarrow$ (2a): Assume (1b). For each $w \in V$, apply (1b): there exist $y_w \in X_w$ and $\varepsilon_w > 0$ so that for all u preceding w and all $R \in \mathcal{F}_{uw}(r)$ either $R(y_w) = y_w$ or $|R(y_w) - y_w| \ge \varepsilon_w$.

Let $v \in V$ be given. Then X_v is the affine span of K_v, so the set

$$\left\{\, S_\tau(y_w) : v \text{ precedes } w, \tau \in E^{(*)}_{vw} \,\right\}$$

also has X_v as affine span. So there exist $w_0, w_1, \cdots, w_d$ preceded by v and $\tau_j \in E^{(*)}_{vw_j}$ such that $\left\{\, S_{\tau_j}(y_{w_j}) : 0 \le j \le d \,\right\}$ spans X_v.

Write $x_j = S_{\tau_j}(y_{w_j})$. Define $\varepsilon = \min_j \varepsilon_{w_j}\rho(\tau_j)$. Let u precede v, let $R \in \mathcal{F}_{uv}(r)$, and $j \in \{0, \cdots, d\}$. Then $R = T^{-1} \circ S$ with $T, S \in F_{uv}\big(]rb, b]\big)$ for some b. So $T \circ S_{\tau_j}, S \circ S_{\tau_j} \in F_{uw_j}\big(]rb\rho(\tau_j), b\rho(\tau_j)]\big)$ and $(T \circ S_{\tau_j})^{-1} \circ (S \circ S_{\tau_j}) \in \mathcal{F}_{uw_j}(r)$, so

$$\left|T^{-1}(S(x_j)) - x_j\right| = \rho(\tau_j)\left|(T \circ S_{\tau_j})^{-1} \circ (S \circ S_{\tau_j})(y_{w_j}) - y_{w_j}\right| \geq \varepsilon$$

if it is not zero.

(5b) $\Longrightarrow$ (4b): Assume (5b). Let $u, v \in V$ be given. Apply Lemma 3.2 to get $x_0 = S_\tau(z)$ and $C > 0$, where w, z, l are as in (5b). Let c be the number of balls of radius 1 required to cover a set of diameter 2 in X_u. We claim that $\gamma_{uv}(]r/2, 1/2], \{x_0\}) \leq cCl$. Indeed, let $U \subseteq X_u$ be a bounded set. Write $b = \operatorname{diam} U$. Now let B be a ball in X_u of radius $b/2$. Write

$$Q = \big\{\, T(x_0) : \ T \in F_{uv}\big(]rb/2, b/2]\big) \,\big\} \cap B.$$

Then $\#Q \leq l$, and

$$\begin{aligned}\#\big\{\, T \in F_{uv}\big(]rb/2, b/2]\big) : \ T(x_0) \in B \,\big\} \\ = \sum_{y \in Q} \#\big\{\, T \in F_{uv}\big(]rb/2, b/2]\big) : \ T(x_0) = y \,\big\} \leq Cl.\end{aligned}$$

Then since U can be covered by at most c balls of radius $b/2$,

$$\#F_{uv}\big(]r/2, 1/2], U, \{x_0\}\big) \leq cCl.$$

This is true for all U, so $\gamma_{uv}(]r/2, 1/2], \{x_0\}) \leq cCl$. ☐

4. Examples

We collect here some examples of iterated function systems (S_e) to illustrate the alternate definitions for WSP. There is a lot of repetition in the descriptions, but we have kept it that way so that the examples can be read independently.

4.1. **Example.** *Why we need span K_u to be X_u.*

For this example, let the graph have one node $V = \{2\}$, and two edges $E = \{\mathsf{d}, \mathsf{e}\}$. Let $X_2 = \mathbb{R}$, let $\alpha, \beta \in]0, 1[$ (to be specified later), and define the maps as:

$$S_{\mathsf{d}}(x) = \alpha x, \qquad S_{\mathsf{e}}(x) = \beta x.$$

So in this case we have $K_2 = \{0\}$ with affine span not equal to $X_2 = \mathbb{R}$.

Now for $\sigma \in E_{22}^{(n)}$ we get

$$S_\sigma(x) = \alpha^k \beta^{n-k} x$$

for some k with $0 \le k \le n$. And for $R \in \mathcal{F}_{22}$ we get

$$R(x) = \alpha^n \beta^m x$$

for any $n, m \in \mathbb{N}$. Now of course (1b) is satisfied in this case, for if we let $x_0 = 0$, then $R(x_0) = x_0$ for all R.

Now assume $\log \alpha$ and $\log \beta$ are not commensurable. This means that n, m can be chosen so that $\alpha^n \beta^m$ is not 1 but as close to 1 as we like. So (2a) fails: for any $x_1 \ne 0$, any $r \in]0,1[$, and any $\varepsilon > 0$, we can choose n, m so that $\alpha^n \beta^m \in]r, r^{-1}[$ and $|R(x_1) - x_1| < \varepsilon$. Thus, in the proof that (1b) $\Longrightarrow$ (2a), we really do need the assumption that the affine span of K_u is X_u. Of course we can repair this case by declaring $X_1 = \{0\}$.

4.2. **Example.** *Why we need uniform affine dimension.*

Now consider a larger graph, in which we have the same node 2 as the previous example where K_2 has affine dimension 0, but also another node 1. Let the graph be as shown, $S_{\mathsf{d}}, S_{\mathsf{e}}$ as before, and

$$S_{\mathsf{a}}(x) = \alpha x, \qquad S_{\mathsf{b}}(x) = \alpha(x+1), \qquad S_{\mathsf{c}}(x) = \alpha x.$$

If $\alpha < 1/2$, then the attractor K_1 is a Cantor set (together with countably many images of the point K_2, which are contained in that Cantor set). So K_1 has affine dimension 1. We assume $0 < \beta < \alpha < 1/2$ and $\log \alpha / \log \beta$ is irrational. We claim that (1b) holds but not (2a). The reason that (2a) fails is the same as the previous example. To see (1b), we must do some computations.

As before, maps $S \in F_{22}$ have the form

$$S(x) = \alpha^k \beta^l x.$$

Maps $S \in F_{11}$ have the form

$$S(x) = \alpha^n x + \sum_{i=1}^{n} \theta_i \alpha^i,$$

where $\theta_i \in \{0, 1\}$. Any choice θ_i of digits $0, 1$ can be realized by choosing the appropriate string of a, b. And the attractor K_1 is

$$\left\{ \sum_{i=1}^{\infty} \theta_i \alpha^i \,:\, \theta_i \in \{0, 1\} \right\}.$$

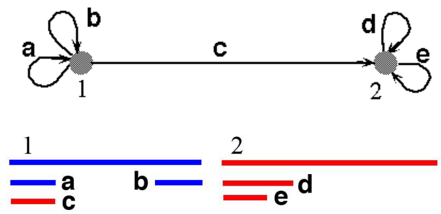

Figure 4.2A. The graph and the IFS

Maps $S \in F_{12}$ have the form

$$S(x) = \alpha^{n+k+1}\beta^l x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \qquad \theta_i \in \{0,1\}.$$

We can always assume $k = 0$ by increasing n and adding extra digits $\theta_i = 0$. Now $R \in \mathcal{F}_{12}$ has the form $R = S'^{-1} \circ S$, where

$$S(x) = \alpha^{n+1}\beta^l x + \sum_{i=1}^{n} \theta_i \alpha^{i+1},\ S'(x) = \alpha^{n'+1}\beta^{l'} x + \sum_{i=1}^{n'} \theta'_i \alpha^{i+1},$$

$\theta_i, \theta'_i \in \{0,1\}$, so that

$$R(x) = \alpha^{n-n'}\beta^{l-l'} x + \sum_{i=1}^{n} \theta_i \alpha^{i+1-n'}\beta^{-l'} - \sum_{i=1}^{n'} \theta'_i \alpha^{i+1-n'}\beta^{-l'}.$$

Now fix $r \in]0,\beta]$. If $R \in \mathcal{F}_{12}(r)$, then $\alpha^{n-n'}\beta^{l-l'} > r$, so $\alpha^{-n'}\beta^{-l'} > r\alpha^{-n}\beta^{-l}$.

Proposition 4.2.1. *The IFS defined above satisfies* (1b).

Proof. We will verify (1b) for node 2. (The argument for node 1 is similar but easier, since no β is involved.) Let $x_0 = 0 \in K_2$ and $\varepsilon = r\alpha(1-2\alpha)/(1-\alpha)$. Let $R \in \mathcal{F}_{12}(r)$, as computed above. Assume $R(0) \neq 0$. We claim that $|R(0)| \geq \varepsilon$. Now

$$R(0) = \sum_{i=1}^{n} \theta_i \alpha^{i+1-n'}\beta^{-l'} - \sum_{i=1}^{n'} \theta'_i \alpha^{i+1-n'}\beta^{-l'}.$$

Since $R(0) \neq 0$, there exist i so that $\theta_i \neq \theta'_i$ [where by convention nonexistent θs are 0]. Let i_0 be the least such i. Then

$$|R(0)| > \alpha^{i_0+1-n'}\beta^{-l'}\left(1 - \sum_{j=1}^{\infty} \alpha^j\right) = \alpha^{i_0+1-n'}\beta^{-l'}\frac{1-2\alpha}{1-\alpha}.$$

If $i_0 < n'$, then $i_0 + 1 - n' \le 0$ so $\alpha^{i_0+1-n'} \ge 1$; also $\beta^{-l'} \ge 1$, so

$$|R(0)| > \frac{1-2\alpha}{1-\alpha} > \varepsilon.$$

On the other hand, if $i_0 \ge n'$, then $i_0 \le n$ and $\alpha^{i_0+1-n'}\beta^{-l'} \ge r\alpha^{i_0+1-n}\beta^{-l}$, $\alpha^{i_0+1-n} \ge \alpha$, $\beta^{-l} \ge 1$, and

$$|R(0)| > \frac{r\alpha(1-2\alpha)}{1-\alpha} = \varepsilon.$$

This completes the proof of (1b). □

Again this time we can try to get out of the difficulty by declaring $X_2 = \{0\}$. But then we do not know how to define $\mathcal{F}_{12}$ and related constructions. We must have $X_1 = \mathbb{R}$ since the Cantor set K_1 is not a single point. If we allow $\dim X_2 = 0 \ne 1 = \dim X_1$, what should we do? In order to define maps like $R = S_\tau^{-1} \circ S_\sigma$, we need S_{c} to have an inverse S_{c}^{-1} of some kind. And whatever that inverse is, if $S_{\mathsf{b}}(0) \ne 0$, then we need $S_{\mathsf{c}}^{-1}(S_{\mathsf{b}}(0)) \ne S_{\mathsf{c}}^{-1}(0)$. Our solution in this paper is to require all similitudes be defined on $\mathbb{R}^d$, and then maps $R = S_\tau^{-1} \circ S_\sigma$ exist and have range in $\mathbb{R}^d$, but need not map X_u into itself. Even in classical cases like the middle-thirds Cantor set in $\mathbb{R}$, maps $S_\tau^{-1} \circ S_\sigma$ do not map the attractor K_u into itself.

4.3. **Example.** An example with a graph that is not strongly connected. We will compute some of the conditions related to the WSP.

Let $\alpha, \beta \in]0, 1/2[$ and $\gamma > 0$. The case $\alpha = \beta = 1/3$ is interesting, but for this example we will want $\alpha \ne \beta$, and in fact $\log\alpha, \log\beta$ incommensurable. The graph $G = (V, E)$ is shown; $V = \{1, 2\}$, $E = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$. The two components in the graph are $\{1\}$ and $\{2\}$.

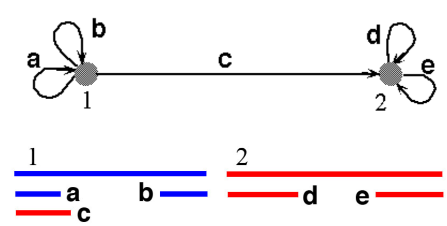

Figure 4.3A. The graph and the IFS

The IFS (S_e) is defined with: $X_1 = X_2 = \mathbb{R}$,

$$\begin{aligned} S_{\mathsf{a}}(x) &= \alpha x, \\ S_{\mathsf{b}}(x) &= \alpha x + \gamma, \\ S_{\mathsf{c}}(x) &= \alpha x, \\ S_{\mathsf{d}}(x) &= \beta x, \\ S_{\mathsf{e}}(x) &= \beta x + (1-\beta). \end{aligned}$$

So $E_{11}^{(*)} = \left\{ S_\sigma : \sigma \in \{\mathsf{a},\mathsf{b}\}^{(*)} \right\}$ corresponds to a Cantor set C_1 of dimension $D_1 = \log 2/\log(1/\alpha)$. Component $\{1\}$ by itself it satisfies OSC and the identity is isolated in $\mathcal{F}_{11}$. Similarly, $E_{22}^{(*)} = \left\{ S_\sigma : \sigma \in \{\mathsf{d},\mathsf{e}\}^{(*)} \right\}$ corresponds to a Cantor set C_2 of dimension $D_2 = \log 2/\log(1/\beta)$. Component $\{2\}$ by itself it satisfies OSC and the identity is isolated in $\mathcal{F}_{22}$.

Proposition 4.3.1. *Let $\log\alpha, \log\beta$ be incommensurable. Then the identity is not isolated in $\mathcal{F}_{12}$.*

Proof. Consider $T = S_\tau, S = S_\sigma \in F_{12}$ defined by $\tau = \mathsf{c}\mathsf{d}^n$, $\sigma = \mathsf{a}^m\mathsf{c}$. Then $T^{-1} \circ S(x) = \beta^{-n}\alpha^m x$. Choose sequences $n_k, m_k \nearrow \infty$ with $m_k \log\alpha - n_k \log\beta \to 0$. Then $\beta^{-n_k}\alpha^{m_k}x \to x$. So the identity is not isolated. ☐

Next we investigate $\gamma_{uv}(]a,b], M)$. Of course, by the result for the one-node case, we have $\gamma_{11}(]a,b], M) < \infty$ and $\gamma_{22}(]a,b], M) < \infty$ for all nonempty M. And of course $\gamma_{21}(]a,b], M) = 0$. For γ_{12} we need a lemma.

Lemma 4.3.2. *Let $A, B > 0$, A/B irrational. Then*

$$\limsup_{k\to\infty} \#\left\{ (n,m) \in \mathbb{N}\times\mathbb{N} : \ k-1 < nA + mB < k \right\} = \infty.$$

Proof. For $k \in \mathbb{N}$ write

$$S_k = \left\{ (n,m) \in \mathbb{N}\times\mathbb{N} : \ k-1 < nA + mB < k \right\}.$$

Since A/B is irrational, each line

$$L_k = \left\{ (x,y) \in \mathbb{R}\times\mathbb{R} : \ xA + yB = k \right\}$$

can have at most one lattice point on it. The rectangle

$$\left\{ (x,y) : \ 0 < x < \frac{k}{A}, 0 < y < \frac{k}{B} \right\}$$

contains at least

$$\left(\frac{k}{A}-2\right)\left(\frac{k}{B}-2\right)$$

lattice points and is covered by the closures of $2k$ strips $S_1, \cdots, S_{2k}$. At most $2k+1$ of the lattice lattice points lie on the lines L_k, so there is at least one (open) strip with at least

$$\frac{\left(\frac{k}{A}-2\right)\left(\frac{k}{B}-2\right)-(2k+1)}{2k}$$

lattice points in it. This goes to ∞. $\square$

Proposition 4.3.3. *Assume* $\log\alpha, \log\beta$ *are incommensurable and* $0<a<b$. *Then* $\gamma_{12}(]a,b],\{0\})=\infty$.

Proof. Write

$$q=b\left(\frac{1}{\beta}-1\right), \qquad p=q-1, \qquad a'=\max\left\{a, \frac{p}{\frac{1}{\beta}-1}\right\}.$$

So $0<a'<b$, $a\le a'$, $p\le a'(\frac{1}{\beta}-1)$, $0<a'/b<1$. The two numbers

$$A=\frac{\log\frac{1}{\alpha}}{\log\frac{b}{a'}}>0 \qquad \text{and} \qquad B=\frac{\log\frac{1}{\beta}}{\log\frac{b}{a'}}>0$$

are incommensurable. Given $C>0$, there exists $k\in\mathbb{N}$ so that

$$\#\left\{(m,n):\ k-1<(m+1)\frac{\log\frac{1}{\alpha}}{\log\frac{b}{a'}}+(n+1)\frac{\log\frac{1}{\beta}}{\log\frac{b}{a'}}<k\right\}>C.$$

So for all (m,n) in the set,

$$\begin{aligned}
(k-1)\log\frac{b}{a'}\le &\quad (m+1)\log\tfrac{1}{\alpha}+(n+1)\log\tfrac{1}{\beta} &&\le k\log\frac{b}{a'},\\
\frac{b^{k-1}}{{a'}^{k-1}}< &\quad \tfrac{1}{\alpha^{m+1}\beta^{n+1}} &&<\frac{b^k}{{a'}^k},\\
\frac{{a'}^{k-1}}{b^{k-1}}> &\quad \alpha^{m+1}\beta^{n+1} &&>\frac{{a'}^k}{b^k},\\
b\varepsilon> &\quad \alpha^{m+1}\beta^{n+1} &&>a'\varepsilon,
\end{aligned}$$

where $\varepsilon={a'}^{k-1}/b^k$ goes to 0 as $k\to\infty$.

So let $\tau = \mathsf{a}^m\mathsf{c}\mathsf{d}^n\mathsf{e}$, $T = S_\tau$, $M = \{0\}$, $U =]p\varepsilon, q\varepsilon[$; so diam $U = \varepsilon$, $\rho(T) = \alpha^{m+1}\beta^{n+1}$, $T \in F_{12}(]a\mathrm{diam}\, U, b\mathrm{diam}\, U])$,

$$T(0) = \alpha^{m+1}\beta^n(1-\beta) = \alpha^{m+1}\beta^{n+1}\left(\frac{1}{\beta} - 1\right),$$

so $T(0) < b\varepsilon(\frac{1}{\beta} - 1) = q\varepsilon$, $T(0) > a'\varepsilon(\frac{1}{\beta} - 1) \geq p\varepsilon$. So $T(0) \in U$. Different (m,n) yield different T. So $\#F_{12}(]a,b], U, \{0\}) > C$, and therefore $\gamma_{12}(]a,b], \{0\}) = \infty$. □

Proposition 4.3.4. *Assume* $\log\alpha, \log\beta$ *are incommensurable and* $0 < a < b$ *and* $M \neq \varnothing$. *Then* $\gamma_{12}(]a,b], M) = \infty$.

Proof. Let $y \in M$. If $y = 0$, use the preceding. We do the case $y > 0$ here. The case $y < 0$ is similar.

Let $p = by$, $q = p - 1$, $a' = \max\{a, q/y\}$. Choose (m,n) as before, so $a'\varepsilon < \alpha^{m+1}\beta^{n+1} < b\varepsilon$ where $\varepsilon = {a'}^{k-1}/b^k$ goes to 0 as $k \to \infty$. Let $U =]q\varepsilon, p\varepsilon[$, diam $U = \varepsilon$, let $\tau = \mathsf{a}^m\mathsf{c}\mathsf{d}^{n+1}$, $T = S_\tau$. Then $\rho(T) = \alpha^{m+1}\beta^{n+1}$,

$$\begin{aligned} T(y) &= \alpha^{m+1}\beta^{n+1}y < b\varepsilon y = p\varepsilon, \\ T(y) &= \alpha^{m+1}\beta^{n+1}y > a'\varepsilon y \geq \frac{q}{y}\varepsilon y = q\varepsilon, \end{aligned}$$

so $T(y) \in U$ for all these T. Therefore $\#F_{12}(]a,b], U, M) > C$ so $\gamma_{12}([a,b], M) = \infty$. □

What of the attractors (invariant sets) for this IFS? The dimension is the maximum of D_1 and D_2. If $\alpha < \beta$, then the dimension is $D_2 = \log 2/\log(1/\beta)$. And K_2 is a Cantor set of that dimension,

$$0 < \mathcal{H}^{D_2}(K_2) < \infty.$$

But K_1 is a Cantor set C_1 of dimension $D_1 < D_2$ plus countably many images of K_2. In fact, there are 2^n images of size α^{n+1} for $n = 0, 1, \cdots$. Then

$$\mathcal{H}^{D_2}(K_1) \leq \mathcal{H}^{D_2}(C_1) + \sum_{n=0}^{\infty} 2^n \left(\alpha^{D_2}\right)^{n+1} \mathcal{H}^{D_2}(K_2).$$

The first term is 0, and since $\alpha^{D_1} = 2, \beta^{D_2} = 2$, we have $1/2 = \beta^{D_2} > \alpha^{D_2}$, so $2\alpha^{D_2} < 1$ and the series is a convergent geometric series. Therefore

$$0 < \mathcal{H}^{D_2}(K_1) < \infty.$$

What about the case $\alpha > \beta$, so the dimension is D_1? Again K_2 is a Cantor set of dimension $D_2 < D_1$, so $\mathcal{H}^{D_1}(K_2) = 0$. And K_1 is a Cantor set C_1 of dimension D_1 plus countably many images of K_2.

$$\mathcal{H}^{D_1}(K_1) \leq \mathcal{H}^{D_1}(C_1) + \sum_{n=0}^{\infty} 2^n \left(\alpha^{D_2}\right)^{n+1} \mathcal{H}^{D_1}(K_2).$$

Now all the terms in the series are zero, so we get just $\mathcal{H}^{D_1}(C_1)$, which is positive and finite.

Finally consider the case $\alpha = \beta$. So $D_1 = D_2$. Now K_2 is a Cantor set of that dimension, $0 < \mathcal{H}^{D_1}(K_2) < \infty$. And K_1 consists of a Cantor set C_1 of dimension D_1 plus countably many images of K_2. So, unless there is serious overlap of the images, we get

$$\begin{aligned} \mathcal{H}^{D_1}(K_1) &= \mathcal{H}^{D_1}(C_1) + \sum_{n=0}^{\infty} 2^n \left(\alpha^{D_1}\right)^{n+1} \mathcal{H}^{D_1}(K_2) \\ &= \mathcal{H}^{D_1}(C_1) + \mathcal{H}^{D_1}(K_2) \left[\sum_{n=0}^{\infty} \frac{1}{2}\right] = \infty \end{aligned}$$

since $\alpha^{D_1} = 1/2$. Presumably for most γ there is negligible overlap in this final computation, so $\mathcal{H}^{D_1}(K_1) = \infty$. So in this case the attractor has infinite but sigma-finite measure in its fractal dimension D_1. The relevant feature of the graph $G = (V, E)$ is that the maximum dimension occurs in two comparable components. This type of dimension computation was described by Mauldin & Williams [9].

4.4. **Example.** An example where (4b) holds but (3a) fails.

The graph $G = (V, E)$ is shown; $V = \{1, 2\}$, $E = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$. The two components in the graph are $\{1\}$ and $\{2\}$.

A number $\beta = 0.012\cdots$ will be specified below. The IFS (S_e) is defined with: $X_1 = X_2 = \mathbb{R}$,

$$\begin{aligned} S_{\mathsf{a}}(x) &= 10^{-1}x, \\ S_{\mathsf{b}}(x) &= 10^{-1}x + 10^{-1}, \\ S_{\mathsf{c}}(x) &= x + \beta, \\ S_{\mathsf{d}}(x) &= 10^{-1}x, \\ S_{\mathsf{e}}(x) &= 10^{-1}x + 10^{-1}. \end{aligned}$$

(The figure has ratio 5^{-1} instead of 10^{-1}.)

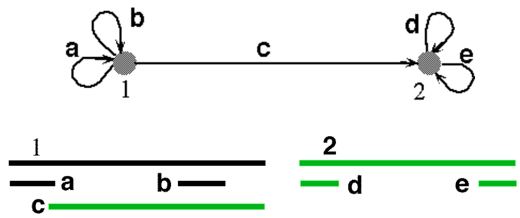

Figure 4.4A The graph and the IFS

So $E_{11}^{(*)} = \{ S_\sigma : \sigma \in \{\mathsf{a}, \mathsf{b}\}^{(*)} \}$ corresponds to a Cantor set C_1 of dimension $D_1 = \log(2)/\log(10)$. Component $\{1\}$ by itself it satisfies OSC and the identity is isolated in $\mathcal{F}_{11}$. Similarly, $E_{22}^{(*)} = \{ S_\sigma : \sigma \in \{\mathsf{d}, \mathsf{e}\}^{(*)} \}$ corresponds to a Cantor set C_2 of dimension $D_2 = \log(2)/\log(10)$. Component $\{2\}$ by itself it satisfies OSC and the identity is isolated in $\mathcal{F}_{22}$.

The number β is defined by a decimal expansion:

$$\beta = 0.01232101001000100001000001000000 1\cdots$$

all remaining digits are 0 and 1; there are longer and longer blocks of consecutive 0's separated by single 1's.

Theorem 4.4.1. *The identity is not isolated in $\mathcal{F}_{12}$.*

$$\gamma_{12}\big(]10^{-1}, 1], \{0\}\big) < \infty.$$

Let's do a few computations. If $\sigma \in E_{11}^{(n)}$, then σ is a string of length n made up of the letters a, b. By induction on n,

$$S_\sigma(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i},$$

with all digits $\varepsilon_i \in \{0, 1\}$. In fact, every sequence of 0's and 1's occurs for a corresponding string of a's and b's.

The description of $E_{22}^{(n)}$ is the same, except that the strings are made up of d's and e's.

For $\sigma \in E_{12}^{(n+1)}$, say σ consists of a string of a's and b's of length k, followed by the letter c, followed by a string of d's and e's of length $n-k$. Then

$$S_\sigma(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k}\beta, \quad \text{where all digits } \varepsilon_i \in \{0, 1\}.$$

Here $0 \le k \le n$, and every sequence ε_i of 0's and 1's is possible. Taking into account the decimal expansion of β, we conclude that the constant term

$$S_\sigma(0) = \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k}\beta$$

has a decimal expansion using only digits $0, \cdots, 4$.

Lemma 4.4.2. *Let*

$$x = \sum_{i=N}^{\infty} a_i 10^{-i}, \qquad \textit{with all digits } a_i \in \{-8, -7, \cdots, 7, 8\}.$$

Then $x = 0$ only if all $a_i = 0$. More precisely, if n is the first index with $a_n \neq 0$, then $|x| > 10^{-n-1}$.

Proof. Either $a_n > 0$ or $a_n < 0$. Without loss of generality, say $a_n > 0$. Then $a_i = 0$ for all $i < n$, $a_n \ge 1$, and $a_i \ge -8$ for all $i > n$. Therefore

$$x \ge 10^n + \sum_{i=n+1}^{\infty} (-8)10^{-i} = 10^{-n}\left(1 - \frac{8}{9}\right) > 10^{-n-1}.$$

□

Lemma 4.4.3. *Let*

$$x = \sum_{i=N}^{\infty} a_i 10^{-i}, \quad x' = \sum_{i=N'}^{\infty} a'_i 10^{-i},$$

with all digits $a_i, a'_i \in \{-4, \cdots, 4\}$. If $x = x'$, then $a_i = a'_i$ for all i. More precisely, if n is the first index with $a_n \neq a'_n$, then $|x - x'| > 10^{-n-1}$.

Proof. Subtract $x - x'$ to get a decimal of the form in Lemma 4.4.2. □

Proposition 4.4.4. *The identity is not isolated in $\mathcal{F}_{12}$.*

Proof. For $n \in \mathbb{N}, n \ge 5$, let $S, T \in \mathcal{F}_{12}$ be

$$\begin{aligned} S(x) &= 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-1}\beta, \\ T(x) &= 10^{-n}x + \sum_{i=1}^{n} \varepsilon'_i 10^{-i} + 10^{0}\beta, \end{aligned}$$

$$T^{-1}(S(x)) \quad = \quad x + \sum_{i=1}^{n} (\varepsilon_i - \varepsilon_i')10^{n-i} + 10^{n-1}\beta - 10^n\beta.$$

When we do the subtraction $10^{n-1}\beta - 10^n\beta$ (without borrowing) we get a decimal with digits in $\{-1, 0, 1\}$, so we can choose $\varepsilon_i, \varepsilon_i'$ to cancel all of the digits to the left of the decimal point. If n is chosen so that a string of k zeros begins in β in place 10^{-n-1}, we will have the constant term

$$\sum_{i=1}^{n} (\varepsilon_i - \varepsilon_i')10^{-i} + 10^{n-1}\beta - 10^n\beta$$

with all 0's to the left of the decimal, a string of 0's of length $k-1$ to the right of the decimal, followed by other digits in $\{-1, 0, 1\}$. So the constant term has absolute value $< 10^{-k+2}$. We can do this for k as large as we like, so we can arrange $|T^{-1}(S(x)) - x|$ as small as we like. $\square$

Proposition 4.4.5. $\gamma_{12}\big(]10^{-1}, 1], \{0\}\big) \leq 2 \cdot 10^4$.

Proof. Let U be an interval in $\mathbb{R}$. There is a unique n with $10^{-n} \leq \operatorname{diam} U < 10^{-n+1}$, so that $10^{-n} \in]10^{-1}\operatorname{diam} U, \operatorname{diam} U]$. But the only ratios in the IFS are powers of 10^{-1}, so all

$$S \in F_{12}\big(]10^{-1}\operatorname{diam} U, \operatorname{diam} U]\big)$$

must have the same ratio 10^{-n}. I claim that if V is any interval of length 10^{-n-3}, then

$$\#\left\{\, S \in F_{12}\big(]10^{-1}\operatorname{diam} U, \operatorname{diam} U]\big) : \; S(0) \in V \,\right\} \leq 2.$$

To see this, recall that such S has the form

$$S(x) = 10^{-n}x + \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k}\beta.$$

By Lemma 2, different maps S have different constant terms $S(0)$. Indeed, given $S(0)$, we may determine k by looking at the digits to the right of the decimal, and then subtract $10^{-k}\beta$ to determine the ε_i. How could two different $S(0)$'s be in V? Suppose

$$S(0) = \sum_{i=1}^{n} \varepsilon_i 10^{-i} + 10^{-k}\beta, \qquad S'(0) = \sum_{i=1}^{n} \varepsilon_i' 10^{-i} + 10^{-k'}\beta,$$

where $S \neq S'$ and $|S(0) - S'(0)| < 10^{-n-3}$. First, if $k = k'$, then we have $\varepsilon_i \neq \varepsilon_i'$ for some i, so that by Lemma 2, $|S(0) - S'(0)| > 10^{-n-1}$, which is too large. Next, if $k \leq k' - 2$, then $10^{-k}\beta$ has a digit 3 in a place where $10^{-k'}\beta$ has digit ≤ 1 and also $|\varepsilon_i - \varepsilon_i'| \leq 1$, so $S(0) - S'(0)$ has nonzero digit in that place, no further to the right than the 10^{-n+2-4} place, so $|S(0) - S'(0)| > 10^{-n-3}$, again too large. Finally, if $k = k'-1$, then $10^{-k}\beta - 10^{-k'}\beta$ has only digits $-1, 0, 1$; if ε_i are all given, then ε_i' are uniquely determined by the condition that all digits of $S(0) - S'(0)$ up to place 10^{-n} must be 0. Therefore: if three different maps $S \in F_{12}\big(]10^{-1}\text{diam } U, \text{diam } U]\big)$ all have $S(0) \in V$, then the three corresponding values of k must be pairwise consecutive, which is impossible.

The set U has length $< 10^{-n+1}$, so it can be covered by at most 10^4 invervals V of length 10^{-n-3} and each of these intervals V can contain at most two $S(0)$. So in all there are at most $2 \cdot 10^4$ different $S \in F_{12}\big(]10^{-1}\text{diam } U, \text{diam } U]\big)$ with $S(0) \in U$. This means: $\gamma_{12}\big(]10^{-1}, 1], \{0\}\big) \leq 2 \cdot 10^4$. $\square$

4.5. **Example.** This is an example where (2b) holds but (2a) fails.

We begin with a general description of the IFS. The details will be filled in below. The directed multigraph G is (V, E). There is a "contraction factor" $\alpha \in]0, 1[$. There are two digit-sets (finite sets of real numbers) A and D. There are two numbers $\beta, \gamma \in [0, 1]$.

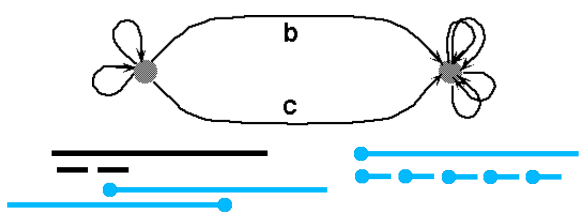

Figure 4.5A. The graph and the IFS

The vertex set is $V = \{1, 2\}$. For edges, we will have $\#E_{11} = \#A$, $\#E_{22} = \#D$, $\#E_{12} = 2$, and $\#E_{21} = 0$. All ratios in E_{11} and E_{22} are α, and all ratios in E_{12} are 1. The IFS (S_e) will have $X_1 = X_2 = \mathbb{R}$ and: maps S_e for $e \in E_{11}$ are the maps

$$\alpha(x + \varepsilon), \qquad \varepsilon \in A;$$

maps S_e for $e \in E_{22}$ are the maps

$$\alpha(x+\theta), \qquad\qquad \theta \in D;$$

maps S_e for $e \in E_{12}$ are the maps

$$S_{\mathsf{b}}(x) = x + \beta, \qquad S_{\mathsf{c}}(x) = -x + \gamma.$$

In this IFS, component $\{1\}$ alone has an attractor consisting of all points with expansion in base α using digit set A. It could be either an interval or a Cantor set, depending on the choice of digits. In our example, it will be a Cantor set. For $\sigma \in E_{11}^{(n)}$ compute

$$S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \varepsilon_i \, \alpha^i,$$

where the digits $\varepsilon_i \in A$. Any choice of ε_i may be obtained by using the proper choice of σ.

Similarly, for $\sigma \in E_{22}^{(n)}$,

$$S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \theta_i \, \alpha^i,$$

where $\theta_i \in D$.

Of course $E_{21}^{(*)} = \varnothing$. Elements of $E_{12}^{(*)}$ are of two types, depending on whether the string contains letter b or letter c. In the first case, an element $\sigma \in E_{12}^{(n+1)}$ consists of a string in $E_{11}^{(k)}$, then b, then a string in $E_{22}^{(n-k)}$. Here, $0 \le k \le n$. The result is a map

$$S(x) = \alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \qquad \varepsilon_i \in A, \theta_i \in D.$$

Similarly, when the letter c appears, the result is a map

$$T(x) = -\alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i - \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \gamma, \qquad \varepsilon_i \in A, \theta_i \in D.$$

Now let $r = \rho_{\min} = \alpha$. Since all ratios $\rho(R)$ for maps $R \in \mathcal{F}_{uv}$ are integer powers of α, the interval $]r^{-1}, r[$ contains no such ratio except 1. Therefore all $R \in \mathcal{F}_{uv}(r)$ have $\rho(R) = 1$.

Next we compute the sets $\mathcal{F}_{uv}(r)$ as follows. Let $R \in \mathcal{F}_{11}(r)$; then $R = S_{\sigma'}^{-1} \circ S_\sigma$, where for some n,

$$S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \varepsilon_i \, \alpha^i, S_{\sigma'}(x) = \alpha^n x + \sum_{i=1}^{n} \varepsilon'_i \, \alpha^i.$$

Thus

$$R(x) = x + \sum_{i=1}^{n} (\varepsilon_i - \varepsilon'_i)\alpha^{i-n}, \qquad \varepsilon_i, \varepsilon'_i \in A.$$

Similarly, $R \in \mathcal{F}_{22}(r)$ is of the form

$$R(x) = x + \sum_{i=1}^{n} (\theta_i - \theta'_i)\alpha^{i-n}, \qquad \theta_i, \theta'_i \in D.$$

For $R \in \mathcal{F}_{12}(r)$ there are four cases. Case 1 is $R = S'^{-1} \circ S$ where for some n and some $k, k' \in \{0, \cdots, n\}$

$$\begin{aligned} S(x) &= \alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \qquad \varepsilon_i \in A, \theta_i \in D; \\ S'(x) &= \alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i + \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \beta, \qquad \varepsilon'_i \in A, \theta'_i \in D; \end{aligned}$$

so

$$\begin{aligned} R(x) = x + \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} - \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} + \sum_{j=k+1}^{n} \theta_i \alpha^{i-n} \\ - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} + \alpha^{k-n}\beta - \alpha^{k'-n}\beta. \end{aligned}$$

Case 2 is $R = T'^{-1} \circ T$ where

$$T(x) = -\alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i - \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \gamma, \qquad \varepsilon_i \in A, \theta_i \in D.$$

$$T'(x) = -\alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i - \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \gamma, \qquad \varepsilon'_i \in A, \theta'_i \in D.$$

so

$$R(x) = x - \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} + \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} + \sum_{j=k+1}^{n} \theta_i \alpha^{i-n} - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} + \alpha^{k-n}\gamma - \alpha^{k'-n}\gamma.$$

Case 3 is $R = T'^{-1} \circ S$ where

$$S(x) = \alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i + \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \beta, \qquad \varepsilon_i \in A, \theta_i \in D;$$

$$T'(x) = -\alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i - \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \gamma, \qquad \varepsilon'_i \in A, \theta'_i \in D.$$

so

$$R(x) = -x - \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} + \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} - \sum_{j=k+1}^{n} \theta_i \alpha^{i-n} - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} - \alpha^{k-n}\beta + \alpha^{k'-n}\gamma.$$

Case 4 is $R = S'^{-1} \circ T$ where

$$T(x) = -\alpha^n x + \sum_{i=1}^{k} \varepsilon_i \alpha^i - \sum_{i=k+1}^{n} \theta_i \alpha^i + \alpha^k \gamma, \qquad \varepsilon_i \in A, \theta_i \in D.$$

$$S'(x) = \alpha^n x + \sum_{i=1}^{k'} \varepsilon'_i \alpha^i + \sum_{i=k'+1}^{n} \theta'_i \alpha^i + \alpha^{k'} \beta, \qquad \varepsilon'_i \in A, \theta'_i \in D;$$

so

$$R(x) = -x + \sum_{i=1}^{k} \varepsilon_i \alpha^{i-n} - \sum_{i=1}^{k'} \varepsilon'_i \alpha^{i-n} - \sum_{j=k+1}^{n} \theta_i \alpha^{i-n} - \sum_{i=k'+1}^{n} \theta'_i \alpha^{i-n} + \alpha^{k-n}\gamma - \alpha^{k'-n}\beta.$$

Now we specialize the choices. Let $\alpha = 1/9$, $A = \{-1, 0\}$, $D = \{0, 1, 2, 3, 4\}$. Let $q \in]4/9, 1/2[$ be "normal" base 9 in the sense that the expansion in base 9 contains all finite strings from the alphabet $\{0, \cdots, 8\}$ (with the appropriate frequency). Let $\beta \in [2/9, 3/9[$ and $\gamma \in [6/9, 7/9[$ be such that $\gamma - \beta = q$. (In fact, $\beta = 2/9$, $\gamma = q + 2/9$ will do.)

Proposition 4.5.1. *For the IFS defined above, components* $\{1\}$ *and* $\{2\}$ *both satisfy WSP.*

Proof. Component $\{2\}$ satisfies the open set condition using open set $]0, 1[$. Component $\{1\}$ satisfies the open set conditions using open set $]-1, 0[$. And OSC implies WSP.

Alternatively, we may consider the maps in $\mathcal{F}_{22}(r)$ as computed before:

$$x + \sum_{i=1}^{n} (\theta_i - \theta_i') 9^{n-i}, \qquad \theta_i, \theta_i' \in D.$$

These have the form $x + b$ where b is an integer. So the identity is isolated in $\mathcal{F}_{22}(r)$. Component $\{1\}$ is done in the same way. $\square$

Proposition 4.5.2. *The IFS defined above satisfies* (2b)*.*

Proof. We know (2b), (3a), (3b) are equivalent. By Proposition 4.5.1, the identity is isolated in $\mathcal{F}_{11}$ and in $\mathcal{F}_{22}$. So we must show that the identity is isolated in $\mathcal{F}_{12}$. Let $R \in \mathcal{F}_{12}$. Then R has the form $ax + b$. Because all ratios a are powers of $\alpha = 1/9$, it follows that if R is close enough to the identity, then $a = 1$. Write $\varepsilon_0 = 1/24$. We will show that if $R(x) = x + b$ is in $\mathcal{F}_{12}$ and $|b| < \varepsilon_0$, then $b = 0$. That will show that the identity is isolated.

So assume $R \in \mathcal{F}_{12}$ is of the form $R(x) = x + b$ with $|b| < \varepsilon_0$. There are two possibilities for b. Case 1:

$$b = \sum_{i=1}^{k} \varepsilon_i 9^{n-i} - \sum_{i=1}^{k'} \varepsilon_i' 9^{n-i} + \sum_{i=k+1}^{n} \theta_i 9^{n-i} \\ - \sum_{i=k'+1}^{n} \theta_i' 9^{n-i} + 9^{n-k}\beta - 9^{n-k'}\beta.$$

First we claim that $k = k'$. Indeed, suppose $k < k'$. (The case $k > k'$ is similar.) Now the sum $\sum_{i=1}^{k} (\varepsilon_i - \varepsilon_i') 9^{n-i}$ is 9^{n-k} times an integer K. If $K \geq 0$, then

$$\begin{aligned}
-\sum_{i=k+1}^{k'} \varepsilon'_i 9^{n-i} &\geq 0,\\
+\sum_{i=k+1}^{n} \theta_i 9^{n-i} &\geq 0,\\
-\sum_{i=k'+1}^{n} \theta'_i 9^{n-i} &\geq 9^{n-k'}\left(-4\sum_{j=1}^{\infty} 9^{-j}\right) = -9^{n-k'}\,\frac{1}{2}\\
&\geq -9^{n-k}\,\frac{1}{18},\\
+(9^{n-k}-9^{n-k'})\beta &\geq 9^{n-k}\left(1-\frac{1}{9}\right)\frac{2}{9} \geq 9^{n-k}\,\frac{16}{81}.
\end{aligned}$$

Therefore

$$b \geq 9^{n-k}\left(-\frac{1}{18}+\frac{16}{81}\right) = 9^{n-k}\,\frac{23}{162} \geq \varepsilon_0.$$

On the other hand, if $K < 0$, then $K \leq -1$ and

$$\begin{aligned}
-\sum_{i=k+1}^{k'} \varepsilon'_i 9^{n-i} &\leq 9^{n-k}\left(\sum_{j=1}^{\infty} 9^{-j}\right) = 9^{n-k}\,\frac{1}{8},\\
+\sum_{i=k+1}^{n} \theta_i 9^{n-i} &\leq 9^{n-k}\left(4\sum_{j=1}^{\infty} 9^{-j}\right) = 9^{n-k}\,\frac{1}{2},\\
-\sum_{i=k'+1}^{n} \theta'_i 9^{n-i} &\leq 0,\\
+(9^{n-k}-9^{n-k'})\beta &\leq 9^{n-k}(1-0)\,\frac{3}{9} = 9^{n-k}\,\frac{1}{3}.
\end{aligned}$$

Therefore

$$b \leq 9^{n-k}\left(-1+\frac{1}{8}+\frac{1}{2}+\frac{1}{3}\right) = -9^{n-k}\,\frac{1}{24} \leq -\frac{1}{24} \leq -\varepsilon_0.$$

So we must have $k = k'$. Then

$$b = \sum_{i=1}^{k}(\varepsilon_i - \varepsilon'_i)9^{n-i} + \sum_{i=k+1}^{n}(\theta_i - \theta'_i)9^{n-i}.$$

This is an integer with $|b| < 1$ so $b = 0$ as claimed.

Case 2:

$$\begin{aligned} b = -\sum_{i=1}^{k} \varepsilon_i 9^{n-i} + \sum_{i=1}^{k'} \varepsilon_i' 9^{n-i} + \sum_{i=k+1}^{n} \theta_i 9^{n-i} \\ - \sum_{i=k'+1}^{n} \theta_i' 9^{n-i} - 9^{n-k}\gamma + 9^{n-k'}\gamma. \end{aligned}$$

First we claim that $k = k'$. Indeed, suppose $k < k'$. (The case $k > k'$ is similar.) Now the sum $\sum_{i=1}^{k} (\varepsilon_i - \varepsilon_i')9^{n-i}$ is 9^{n-k} times an integer K. If $K > 0$, then $K \geq 1$ and

$$+\sum_{i=k+1}^{k'} \varepsilon_i' 9^{n-i} > 9^{n-k}\left(-\sum_{j=1}^{\infty}\right) = -9^{n-k}\,\frac{1}{8},$$

$$+\sum_{i=k+1}^{n} \theta_i 9^{n-i} \geq 0,$$

$$-\sum_{i=k'+1}^{n} \theta_i' 9^{n-i} \geq 9^{n-k'}\left(-4\sum_{j=1}^{\infty} 9^{-j}\right) = -9^{n-k'}\,\frac{1}{2} \geq -9^{n-k}\,\frac{1}{18},$$

$$(-9^{n-k} + 9^{n-k'})\gamma \geq 9^{n-k}(-1+0)\,\frac{7}{9} = 9^{n-k}\,\frac{7}{9}.$$

Therefore

$$b \geq 9^{n-k}\left(1 - \frac{1}{8} - \frac{1}{18} - \frac{7}{9}\right) = 9^{n-k}\,\frac{1}{24} \geq \frac{1}{24} \geq \varepsilon_0.$$

On the other hand, if $K \leq 0$, then

$$\begin{aligned} +\sum_{i=k+1}^{k'} \varepsilon_i' 9^{n-i} &\leq 0, \\ +\sum_{i=k+1}^{n} \theta_i 9^{n-i} &\leq 9^{n-k}\left(4\sum_{j=1}^{\infty} 9^{-j}\right) = 9^{n-k}\,\frac{1}{2}, \\ -\sum_{i=k'+1}^{n} \theta_i' 9^{n-i} &\leq 0, \end{aligned}$$

$$(-9^{n-k}+9^{n-k'})\gamma \quad\leq\quad 9^{n-k}\left(-1+\frac{1}{9}\right)\frac{2}{3}=-9^{n-k}\,\frac{16}{27}.$$

Therefore

$$b\leq 9^{n-k}\left(\frac{1}{2}-\frac{16}{27}\right)=9^{n-k}\frac{5}{54}\leq-\frac{5}{54}\leq-\varepsilon_0$$

So we must have $k=k'$. Then

$$b=\sum_{i=1}^{k}(\varepsilon'_i-\varepsilon_i)9^{n-i}+\sum_{i=k+1}^{n}(\theta_i-\theta'_i)9^{n-i}.$$

This is an integer with $|b|<1$ so $b=0$ as claimed. □

Proposition 4.5.3. *For the IFS defined above,* (2a) *fails.*

Proof. Let $x_0\in X_2=\mathbb{R}$ and let $\varepsilon>0$. We claim that there is $R\in\mathcal{F}_{12}(r)$ such that $R(x_0)\neq x_0$ but $|R(x_0)-x_0|<\varepsilon$. Consider $R=S'^{-1}\circ T$ where for some n (and $k=k'=0$)

$$R(x)=-x-\sum_{i=1}^{n}(\theta_i+\theta'_i)9^{n-i}+9^n(\gamma-\beta).$$

We claim that n,θ_i,θ'_i may be chosen so that $R(x)=-x+b$ with $0<2x_0-b<\varepsilon$. Note that $\theta_i+\theta'_i$ can have any value in $\{0,\cdots,8\}$, so that $\sum_{i=1}^{n}(\theta_i+\theta'_i)9^{n-i}$ can have any integer value from 0 to 9^n-1. Also, $9^n(\gamma-\beta)=9^nq$ is between $(4/9)9^n$ and $(1/2)9^n$. Choose n so that $(4/9)9^n>|2x_0|$ and the fractional part of $2x_0-9^nq$ lies in $]0,\varepsilon[$. This is possible since the base 9 expansion of q is normal. Then choose θ_i,θ'_i so that $-\sum_{i=1}^{n}(\theta_i+\theta'_i)9^{n-i}$ is the greatest integer $\leq 2x_0-9^nq$. Then $b=-\sum_{i=1}^{n}(\theta_i+\theta'_i)9^{n-i}+9^nq$ satisfies $0<2x_0-b<\varepsilon$. So $R(x_0)-x_0=-x_0+b-x_0=b-2x_0$. Thus we have $R(x_0)\neq x_0$ and $|R(x_0)-x_0|<\varepsilon$ as required. □

4.6. **Example.** This is an example where (1b) holds but (1a) fails. Such a counterexample is impossible in one dimension, so we do it in the plane, which we identify with the set $\mathbb{C}$ of complex numbers. However, we have not kept the condition of uniform affine dimension. [In this case the problem is with $K_2=\{0\}$.] We haven't found a way to adapt this example, yet still keep uniform affine dimension.

The parameters in this case are as follows. A complex number α in polar coordinates $\alpha=se^{2\pi i\phi}$; we will use $s=1/10$ and $\phi\in]0,1/2[$

irrational. So if $\omega = e^{2\pi i\phi}$, then $\{\, \omega^k : k \in \mathbb{N} \,\}$ is dense in the unit circle $\{\, z \in \mathbb{C} : |z| = 1 \,\}$. [In the preceding, we have used "i" for $\sqrt{-1}$, but from now on "i" is merely an index.] Two digit-sets $A = \{0, 1\}$, $D = \{0, 2\}$.

The vertex set is $V = \{1, 2, 3, 4\}$. For edges, we will have $\#E_{13} = 1$, $\#E_{32} = 1$, $\#E_{14} = 1$, $\#E_{42} = 1$, $\#E_{33} = 2$, $\#E_{44} = 2$, $\#E_{22} = 1$, and all others empty. All ratios are $s = |\alpha|$. The IFS (S_e) will have $X_1 = X_2 = X_3 = X_4 = \mathbb{C}$ and: the maps (S_e) for $e \in E_{33}$ are the maps $\alpha(x + \theta)$, where $\theta \in \{0, 2\}$; the maps (S_e) for $e \in E_{44}$ are the maps $\alpha(x + \varepsilon)$, where $\varepsilon \in \{0, 1\}$; the maps (S_e) for $e \in E_{13} \cup E_{32} \cup E_{42} \cup E_{22}$ are all equal to the map αx; the map (S_e) for $e \in E_{14}$ is the map $\alpha \overline{x}$. The overline $\overline{x}$ denotes the complex conjugate of x.

Figures illustrating this example were given in the Introduction, above. The figures use ratio $2/3$ rather than $1/9$ and $\phi = 0.4$.

In this IFS, component $\{3\}$ alone has an attractor consisting of all points with expansion in base α using digit-set $\{0, 2\}$. It is a Cantor set. For $\sigma \in E_{33}^{(n)}$ compute

$$S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \theta_i \, \alpha^i,$$

where the digits $\theta_i \in \{0, 2\}$. Any choices of θ_i from the digit-set $\{0, 2\}$ can be obtained by using the proper choice of σ.

Similarly, for $\sigma \in E_{44}^{(n)}$,

$$S_\sigma(x) = \alpha^n x + \sum_{i=1}^{n} \varepsilon_i \, \alpha^i,$$

where $\varepsilon_i \in \{0, 1\}$.

Of course, for $\sigma \in E_{22}^{(n)}$ there is only $S_\sigma(x) = \alpha^n x$. The attractor for component $\{2\}$ is the single point 0.

Elements of $E_{12}^{(*)}$ are of two types, depending on whether the string passes through node 3 or node 4. In the first case, an element $\sigma \in E_{12}^{(n+2)}$ consists of the edge from 1 to 3 followed by a string in $E_{33}^{(n-k)}$, then the edge from 3 to 2, then a string in $E_{22}^{(k)}$. Here

$0 \le k \le n$. The result is a map

$$S(x) = \alpha^{n+2}x + \sum_{i=1}^{n-k} \theta_i \alpha^{i+1}, \qquad \theta_i \in \{0, 2\}.$$

This may be written as

$$S(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \qquad \theta_i \in \{0, 2\},$$

where the sum goes all the way to n, with the proviso that, when $k > 0$, then some of the θ_i are required to be 0. But by taking $k = 0$ we may realize any sequence of $\theta_i \in \{0, 2\}$. Similarly, when the path passes through node 4, the result is a map

$$T(x) = \alpha\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\varepsilon_i}\alpha\overline{\alpha}^{i}, \qquad \varepsilon_i \in \{0, 1\}.$$

Again, we may realize any sequence of $\varepsilon_i \in \{0, 1\}$.

Now let $r = \rho_{\min} = s$. Since all ratios $\rho(R)$ for maps $R \in \mathcal{F}_{uv}$ are integer powers of s, the interval $]r^{-1}, r[$ contains no such ratio except 1. Therefore all $R \in \mathcal{F}_{uv}(r)$ have $\rho(R) = 1$.

Next we compute the sets $\mathcal{F}_{uv}(r)$ as follows. Let $R \in \mathcal{F}_{12}(r)$; then there are four cases. Case 1 is $R = S'^{-1} \circ S$ where

$$S(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \qquad S'(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta'_i \alpha^{i+1},$$

so

$$R(x) = x + \sum_{i=1}^{n} \theta_i \alpha^{i-1-n} - \sum_{i=1}^{n} \theta'_i \alpha^{i-1-n}.$$

Case 2 is $R = T'^{-1} \circ T$ where

$$T(x) = \alpha\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\varepsilon_i}\alpha\overline{\alpha}^{i}, \qquad T'(x) = \alpha\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\varepsilon'_i}\alpha\overline{\alpha}^{i},$$

so

$$R(x) = x + \sum_{i=1}^{n} \varepsilon_i \alpha^{i-1-n} - \sum_{i=1}^{n} \varepsilon'_i \alpha^{i-1-n}.$$

Case 3 is $R = T'^{-1} \circ S$ where

$$S(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i \alpha^{i+1}, \qquad T'(x) = \alpha\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\varepsilon'_i}\alpha\overline{\alpha}^{i},$$

so

$$R(x) = \alpha^{-n-1}\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\theta_i}\alpha^{-n-1}\overline{\alpha}^{i} - \sum_{i=1}^{n} \varepsilon_i'\alpha^{i-n-1}.$$

Case 4 is $R = S'^{-1} \circ T$ where

$$T(x) = \alpha\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\varepsilon_i}\alpha\overline{\alpha}^{i}, \qquad S'(x) = \alpha^{n+2}x + \sum_{i=1}^{n} \theta_i'\alpha^{i+1},$$

so

$$R(x) = \alpha^{-n-1}\overline{\alpha}^{n+1}\overline{x} + \sum_{i=1}^{n} \overline{\varepsilon_i}\alpha^{-n-1}\overline{\alpha}^{i} - \sum_{i=1}^{n} \theta_i'\alpha^{i-n-1}.$$

If we write $\alpha = s\omega$ as above, these four cases become:

$$\begin{aligned}
S'^{-1}\big(S(x)\big) &= x + \sum_{i=1}^{n} s^{i-1-n}\big(\omega^{i-1-n}\theta_i - \omega^{i-1-n}\theta_i'\big), \\
T'^{-1}\big(T(x)\big) &= x + \sum_{i=1}^{n} s^{i-1-n}\big(\omega^{i-1-n}\varepsilon_i - \omega^{i-1-n}\varepsilon_i'\big), \\
T'^{-1}\big(S(x)\big) &= \omega^{-2n-2}\overline{x} + \sum_{i=1}^{n} s^{i-1-n}\big(\omega^{-i-1-n}\overline{\theta_i} - \omega^{i-1-n}\varepsilon_i'\big), \\
S'^{-1}\big(T(x)\big) &= \omega^{-2n-2}\overline{x} + \sum_{i=1}^{n} s^{i-1-n}\big(\omega^{-i-1-n}\overline{\varepsilon_i} - \omega^{i-1-n}\theta_i'\big).
\end{aligned}$$

Proposition 4.6.1. *The IFS defined above satisfies* (1b).

Proof. Let $x_0 = 0$, $\varepsilon_0 = 1$. We will show that if $R \in \mathcal{F}_{12}(r)$, then either $R(0) = 0$ or $|R(0)| \geq \varepsilon_0$. There are different cases depending on the cases for the map R defined above.

Let $R \in \mathcal{F}_{12}(r)$ fall in Case 1:

$$R(0) = \sum_{i=1}^{n} 10^{n+1-i}\big(\omega^{i-1-n}\theta_i - \omega^{i-1-n}\theta_i'\big).$$

Suppose $|R(0)| < \varepsilon_0$. We must show that $R(0) = 0$. In fact we claim that $\theta_i = \theta_i'$ for all i. Assume not: then let i_0 be the least i with $\theta_i \neq \theta_i'$. Now $\theta_i - \theta_i'$ is 2, 0, or -2, and

$$|R(0)| \geq 10^{n+1-i_0}\left(2 - 2\sum_{j=1}^{\infty} 10^{-j}\right) \geq 10\left(2 - \frac{2}{9}\right) = \frac{160}{9} \geq \varepsilon_0.$$

Let $R \in \mathcal{F}_{12}(r)$ fall in Case 2:

$$R(0) = \sum_{i=1}^{n} 10^{n+1-i}\big(\omega^{i-1-n}\varepsilon_i - \omega^{i-1-n}\varepsilon_i'\big).$$

Suppose $|R(0)| < \varepsilon_0$. We must show that $R(0) = 0$. In fact we claim that $\varepsilon_i = \varepsilon_i'$ for all i. Assume not: then let i_0 be the least i with $\varepsilon_i \neq \varepsilon_i'$. Now $\varepsilon_i - \varepsilon_i'$ is 1, 0, or -1, and

$$|R(0)| \geq 10^{n+1-i_0}\left(1 - \sum_{j=1}^{\infty} 10^{-j}\right) \geq 10\left(1 - \frac{1}{9}\right) = \frac{80}{9} \geq \varepsilon_0.$$

Let $R \in \mathcal{F}_{12}(r)$ fall in Case 3:

$$R(0) = \sum_{i=1}^{n} 10^{n+1-i}\big(\omega^{-i-1-n}\overline{\theta_i} - \omega^{i-1-n}\varepsilon_i'\big).$$

Suppose $|R(0)| < \varepsilon_0$. We must show that $R(0) = 0$. In fact we claim that $\theta_i = \varepsilon_i' = 0$ for all i. Assume not: then let i_0 be the least i where it fails. We take two subcases: If $\theta_{i_0} = 2$, then

$$\begin{aligned} |R(0)| &\geq 10^{n+1-i_0}\left(2 - 1 - 3\sum_{j=1}^{\infty} 10^{-j}\right) \\ &\geq 10\left(2 - 1 - \frac{3}{9}\right) = \frac{20}{3} \geq \varepsilon_0. \end{aligned}$$

On the other hand, if $\theta_{i_0} = 0$, then $\varepsilon_{i_0} = 1$ and

$$|R(0)| \geq 10^{n+1-i_0}\left(1 - 3\sum_{j=1}^{\infty} 10^{-j}\right) \geq 10\left(1 - \frac{3}{9}\right) = \frac{20}{3} \geq \varepsilon_0.$$

Case 4 is similar.

So in every case, if $|R(0)| < \varepsilon_0$, then $R(0) = 0$. This completes the verification of (1b). $\square$

Proposition 4.6.2. *For the IFS defined above,* (1a) *fails.*

Proof. Let $x \in \mathbb{C}$ and $\varepsilon > 0$. We claim that there exists $R \in \mathcal{F}_{12}(r)$, $R \neq \mathrm{id}$, $|R(x) - x| < \varepsilon$. To do this, we will take R in Case 3 with all $\theta_i = \varepsilon_i' = 0$. Then $R(x) = \omega^{-2n-2}\overline{x}$. This is not the identity. Also, $R(0) = 0$, so if $x = 0$ we are done. So assume $x \neq 0$. Since $\left\{\omega^k : k = 1, 2, 3, \cdots\right\}$ is dense in the unit circle, there exist

$n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$ such that $\omega^{-2n_k-2} \to x/\overline{x}$ and therefore $\omega^{-2n_k-2}\overline{x} \to x$. So for some k, $|R(x) - x| < \varepsilon$. □

Open question. Is (1a) equivalent to (1b) when the IFS has uniform affine dimension? We have not answered this.

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