Diagonalizing algebras of operators on infinite-dimensional vector spaces

Manuel L. Reyes
(Joint with: Miodrag Iovanov and Zachary Mesyan)

Bowdoin College, Department of Mathematics

OSU-Denison Mathematics Conference
May 13, 2016
Thank you, George!
1. Classical diagonalization

2. Topology in $\text{End}(V)$

3. An infinite-dimensional Wedderburn-Artin theorem

4. Infinite-dimensional diagonalization
Diagonalization: the basics

Notation fixed throughout:
- \( K = \text{field} \) (no further assumptions, unless explicitly stated)
- \( V = K\)-vector space
- \( \text{End}(V) = \text{ring of } K\)-linear endomorphisms

**Def:** A linear transformation \( T \in \text{End}(V) \) is **diagonalizable** if there is a basis of \( V \) consisting of eigenvectors for \( T \).
Diagonalization: the basics

Notation fixed throughout:

- $K =$ field (no further assumptions, unless explicitly stated)
- $V =$ $K$-vector space
- $\text{End}(V) =$ ring of $K$-linear endomorphisms

**Def:** A linear transformation $T \in \text{End}(V)$ is **diagonalizable** if there is a basis of $V$ consisting of eigenvectors for $T$.

In case $V = K^d$ for $d = \dim(V) < \infty$:

- $\text{End}(V) = \mathbb{M}_d(K)$
- $T \in \mathbb{M}_d(K)$ diagonalizable $\iff$ some $UTU^{-1}$ diagonal, $U \in \mathbb{M}_d(K)$
Classical diagonalization

A blast from the past: some “basic” linear algebra

**Classical diagonalizability (operators):** For $T \in \mathbb{M}_d$, the following are equivalent:

1. $T$ is diagonalizable;
2. The minimal polynomial of $T$ splits into distinct linear factors;
3. $K[T] \cong K^n$ as algebras for some $n \leq d$.

In (2), the linear factors are $x - \lambda_i$ for the distinct eigenvalues $\lambda_i$ of $T$.

In (3), the $n$ orthogonal idempotents in $K[T] \cong K^n$ are projections onto the various eigenspaces.
Def: A subalgebra $A \subseteq \text{End}(V)$ is diagonalizable if there is a basis of $V$ consisting of simultaneous eigenvectors for $A$. ($\iff A$ commutative.)
**Def:** A subalgebra $A \subseteq \text{End}(V)$ is **diagonalizable** if there is a basis of $V$ consisting of simultaneous eigenvectors for $A$. ($\iff A$ commutative.)

**Classical diagonalizability (algebras):** For an algebra $A \subseteq \mathbb{M}_d$, the following are equivalent:

(a) $A$ is diagonalizable;
(b) $A \cong K^n$ for some $n \leq d$;
(c) (when $K = \overline{K}$) $A$ is Jacobson semisimple;
(d) (when $K = \overline{K}$) $A$ is reduced (no nonzero nilpotents).

Diagonalizability of $A$ is determined by its **internal algebraic structure**.
**Def:** A subalgebra \( A \subseteq \text{End}(V) \) is **diagonalizable** if there is a basis of \( V \) consisting of simultaneous eigenvectors for \( A \). (\( \iff A \) commutative.)

**Classical diagonalizability (algebras):** For an algebra \( A \subseteq \mathbb{M}_d \), the following are equivalent:

(a) \( A \) is diagonalizable;

(b) \( A \cong K^n \) for some \( n \leq d \);

(c) (when \( K = \overline{K} \)) \( A \) is Jacobson semisimple;

(d) (when \( K = \overline{K} \)) \( A \) is reduced (no nonzero nilpotents).

Diagonalizability of \( A \) is determined by its **internal algebraic structure**.

**Question:** What happens in case \( \dim(V) \) is infinite?
The trouble with infinite-dimensional operators

**Example:** Assume $\mathbb{Q} \subseteq K$ and $V$ has basis $\{v_1, v_2, v_3, \ldots \}$. Define:

$$S(v_n) = v_{n+1}$$

Right shift operator, has no eigenvalues, not diagonalizable.

$$T(v_n) = n \cdot v_n$$

Infinitely many distinct eigenvalues, is diagonalizable.
The trouble with infinite-dimensional operators

Example: Assume $\mathbb{Q} \subseteq K$ and $V$ has basis $\{v_1, v_2, v_3, \ldots\}$. Define:

$$S(v_n) = v_{n+1}$$

Right shift operator, has no eigenvalues, not diagonalizable.

$$T(v_n) = n \cdot v_n$$

Infinitely many distinct eigenvalues, is diagonalizable.

Each operator generates a subalgebra of $\text{End}(V)$:

$$A := K[S] \cong K[x] \cong K[T] =: B$$

Here $B$ is diagonalizable, $A$ is not, and they are isomorphic!
There are isomorphic subalgebras $A, B \subseteq \text{End}(V)$ with one diagonalizable and the other not.

**Conclusion:** Diagonalizability of $A \subseteq \text{End}(V)$ cannot be determined purely by its internal algebraic structure!

So in order to characterize diagonalizable subalgebras $A \subseteq \text{End}(V)$, we need to take into account some more structure on $A$. 
The trouble with infinite-dimensional operators

There are isomorphic subalgebras $A, B \subseteq \text{End}(V)$ with one diagonalizable and the other not.

**Conclusion:** Diagonalizability of $A \subseteq \text{End}(V)$ cannot be determined purely by its internal algebraic structure!

So in order to characterize diagonalizable subalgebras $A \subseteq \text{End}(V)$, we need to take into account some more structure on $A$.

Taking a cue from operator algebras, we will consider topology on $A$. 
1. Classical diagonalization

2. Topology in $\text{End}(V)$

3. An infinite-dimensional Wedderburn-Artin theorem

4. Infinite-dimensional diagonalization
For sets $X$ and $Y$, we have a topological space of functions:

$$Y^X = \{\text{functions } X \to Y\} \cong \prod_X Y$$

dowed with the product topology ($Y$ has discrete topology).
Topologies on spaces of functions

For sets $X$ and $Y$, we have a topological space of functions:

$$Y^X = \{\text{functions } X \to Y\} \cong \prod_X Y$$

endowed with the product topology ($Y$ has discrete topology).

**Def:** Induced topology on $\text{End}(V) \subseteq V^V$ is the **finite topology**.
Topologies on spaces of functions

For sets $X$ and $Y$, we have a topological space of functions:

$$Y^X = \{ \text{functions } X \rightarrow Y \} \cong \prod_X Y$$

endowed with the product topology ($Y$ has discrete topology).

**Def:** Induced topology on $\text{End}(V) \subseteq V^V$ is the finite topology.

Open neighborhood of $S \in \text{End}(V)$: for given $x_1, \ldots, x_n \in V$,

$$U = \{ T \in \text{End}(V) \mid S(x_1) = T(x_1), \ldots, S(x_n) = T(x_n) \}$$

(\textbf{Note:} This is the topology used to interpret the “density” in the Jacobson Density Theorem.)
The finite topology makes $\text{End}(V)$ a “nice” topological ring. It is:

- Hausdorff
- Left linear (neighborhood basis of 0 consisting of open left ideals)
- Complete (Cauchy nets converge)
The finite topology makes $\text{End}(V)$ a “nice” topological ring. It is:

- Hausdorff
- Left linear (neighborhood basis of 0 consisting of open left ideals)
- Complete (Cauchy nets converge)

**Fact 1:** $C \subseteq \text{End}(V)$ commutative $\implies$ closure $\overline{C}$ is commutative

**Conclusion:** $C \subseteq \text{End}(V)$ is diagonalizable $\iff$ $C$ is diagonalizable.

\[\therefore\] We restrict our classification to closed commutative subalgebras.
Topological algebra of $\text{End}(V)$

The finite topology makes $\text{End}(V)$ a “nice” topological ring. It is:

- Hausdorff
- Left linear (neighborhood basis of 0 consisting of open left ideals)
- Complete (Cauchy nets converge)

**Fact 1:** $C \subseteq \text{End}(V)$ commutative $\implies$ closure $\overline{C}$ is commutative

**Fact 2:** For any basis $B$ of $V$, the algebra of all diagonalizable operators with eigenbasis $B$ forms a maximal (closed) commutative subalgebra, isomorphic as a topological algebra to $K^B = \prod_B K$. 
Topological algebra of \( \text{End}(V) \)

The finite topology makes \( \text{End}(V) \) a “nice” topological ring. It is:
- Hausdorff
- Left linear (neighborhood basis of 0 consisting of open left ideals)
- Complete (Cauchy nets converge)

**Fact 1:** \( C \subseteq \text{End}(V) \) commutative \( \implies \) closure \( \overline{C} \) is commutative

**Fact 2:** For any basis \( B \) of \( V \), the algebra of all diagonalizable operators with eigenbasis \( B \) forms a maximal (closed) commutative subalgebra, isomorphic as a topological algebra to \( K^B = \prod_B K \).

**Conclusion:** \( C \subseteq \text{End}(V) \) is diagonalizable \( \iff \) \( \overline{C} \) is diagonalizable.

\( \therefore \) We restrict our classification to closed commutative subalgebras.
1 Classical diagonalization

2 Topology in End(V)

3 An infinite-dimensional Wedderburn-Artin theorem

4 Infinite-dimensional diagonalization
Recall that Wedderburn-Artin plays a role in classical diagonalization when $K = \overline{K}$. We want a substitute for the artinian (or finite length) property in our topological setting.

A topological module $R$-module $M$ is pseudocompact if it satisfies these equivalent conditions:

1. $M$ is an inverse limit in $R\text{-}\text{Mod}$ of discrete, finite length modules;
2. $M$ is Hausdorff, linearly topologized, complete, and $M/U$ has finite length for all open submodules $U$.

Example: $\mathbb{Z}_p$-adics as a topological $\mathbb{Z}$-module

Example: $\prod (\text{discrete simples})$ over any topological ring

Fact [Gabriel]: The full subcategory $R\text{-}\text{PC}$ of p.c. modules in $R\text{-}\text{Mod}$ is "nice" (abelian, $R\text{-}\text{PC}^{\text{op}}$ is Grothendieck and "locally finite").
Recall that Wedderburn-Artin plays a role in classical diagonalization when $K = \overline{K}$. We want a substitute for the artinian (or finite length) property in our topological setting.

**Def:** A topological module $R M$ is pseudocompact if it satisfies these equivalent conditions:

- $M$ is an inverse limit in $R TMod$ of discrete, finite length modules;
- $M$ is Hausdorff, linearly topologized, complete, and $M/U$ has finite length for all open submodules $U$. 

**Ex:** $p$-adics $\mathbb{Z}_p$ as a topological $\mathbb{Z}$-module

**Ex:** $\prod$ (discrete simples) over any topological ring

**Fact** [Gabriel]: The full subcategory $R PC$ of p.c. modules in $R TMod$ is "nice" (abelian, $R PC$ is Grothendieck and "locally finite").
Pseudocompact modules

Recall that Wedderburn-Artin plays a role in classical diagonalization when $K = \overline{K}$. We want a substitute for the artinian (or finite length) property in our topological setting.

**Def:** A topological module $R M$ is pseudocompact if it satisfies these equivalent conditions:

- $M$ is an inverse limit in $R \text{TMod}$ of discrete, finite length modules;
- $M$ is Hausdorff, linearly topologized, complete, and $M/U$ has finite length for all open submodules $U$.

**Ex:** $p$-adics $\mathbb{Z}_p$ as a topological $\mathbb{Z}$-module

**Ex:** $\prod$ (discrete simples) over any topological ring
Recall that Wedderburn-Artin plays a role in classical diagonalization when $K = \overline{K}$. We want a substitute for the artinian (or finite length) property in our topological setting.

**Def:** A topological module $R M$ is **pseudocompact** if it satisfies these equivalent conditions:

- $M$ is an inverse limit in $R \text{TMod}$ of discrete, finite length modules;
- $M$ is Hausdorff, linearly topologized, complete, and $M/U$ has finite length for all open submodules $U$.

**Ex:** $p$-adics $\mathbb{Z}_p$ as a topological $\mathbb{Z}$-module

**Ex:** $\prod$ (discrete simples) over any topological ring

**Fact [Gabriel]:** The full subcategory $R \text{PC}$ of p.c. modules in $R \text{TMod}$ is “nice” (abelian, $R \text{PC}^{op}$ is Grothendieck and “locally finite”).
“Topologically left artinian” rings

A topological replacement for the left artinian property:

**Def:** A topological ring \( R \) is **left pseudocompact** if the topological module \( R \mathcal{R} \) is pseudocompact.

**Example 1:** Discrete, left artinian rings are left pseudocompact.
Def: A topological ring $R$ is **left pseudocompact** if the topological module $R^R$ is pseudocompact.

Example 1: Discrete, left artinian rings are left pseudocompact.

Example 2: $E = \text{End}(V_D)$ with its finite topology is left pseudocompact, since $E E \cong E V^{\dim(V)}$ is a product of discrete simples (Think of "matrix columns" of operators.)
“Topologically left artinian” rings

A topological replacement for the left artinian property:

**Def:** A topological ring $R$ is **left pseudocompact** if the topological module $RR$ is pseudocompact.

**Example 1:** Discrete, left artinian rings are left pseudocompact.

**Example 2:** $E = \text{End}(V_D)$ with its finite topology is left pseudocompact, since $E E \cong E V^{\dim(V)}$ is a product of discrete simples (Think of “matrix columns” of operators.)

**Example 3:** A noetherian complete local ring $(R, m)$ with its $m$-adic topology is pseudocompact:

$$R \cong \lim_{\leftarrow} R/m^n$$

is an inverse limit of discrete, finite length modules
Infinite-Dimensional Wedderburn-Artin Theorem

For a topological ring $R$, the following are equivalent:

1. $R$ is a product of discrete simple left modules in $\mathcal{R} \text{TMod}$
2. $R$ is left pseudocompact and every closed left ideal has a closed complement
3. $R \cong \prod \text{End}_{D_i}(V_i)$ for right vector spaces $V_i$ over division rings $D_i$
4. $R$ is left pseudocompact and Jacobson semisimple
5. Every short exact sequence in $\mathcal{R} \text{PC}$ splits
6. Every object in $\mathcal{R} \text{PC}$ is a product of (discrete) simples

Proof utilizes P. Gabriel’s duality theory (especially for the conditions on $\mathcal{R} \text{PC}$), as it’s often easier to reason about objects in the locally finite category $\mathcal{R} \text{PC}^\text{op}$. 
Infinite-Dimensional Wedderburn-Artin Theorem

For a topological ring $R$, the following are equivalent:

1. $R$ is a product of discrete simple left modules in $\mathcal{R}T\text{Mod}$
2. $R$ is left pseudocompact and every closed left ideal has a closed complement
3. $R \cong \prod \text{End}_{D_i}(V_i)$ for right vector spaces $V_i$ over division rings $D_i$
4. $R$ is left pseudocompact and Jacobson semisimple
5. Every short exact sequence in $\mathcal{R}\text{PC}$ splits
6. Every object in $\mathcal{R}\text{PC}$ is a product of (discrete) simples

Proof utilizes P. Gabriel’s duality theory (especially for the conditions on $\mathcal{R}\text{PC}$), as it’s often easier to reason about objects in the locally finite category $\mathcal{R}\text{PC}^{op}$.
When $R$ is a commutative topological ring, what form does the WA theorem take?

**Def:** $\text{Spec}_0(R)$ is the set of open prime ideals of $R$

**Fact:** When $R$ is pseudocompact, each $m \in \text{Spec}_0(R)$ is maximal

**Corollary** For a commutative topological ring $R$, the following are equivalent:

- $R$ is pseudocompact semisimple
- $R$ is a (topological) direct product of discrete fields
- $R \cong \prod_{m \in \text{Spec}_0(R)} R/m$ as topological rings
1. Classical diagonalization

2. Topology in $\text{End}(V)$

3. An infinite-dimensional Wedderburn-Artin theorem

4. Infinite-dimensional diagonalization
Further restrict the WA theorem to characterize algebras like $K^B$.

**Corollary:** For a commutative topological $K$-algebra, TFAE:
- $A$ is p.c. semisimple, with all open maximal ideals of codimension 1
- $A \cong K^X$ for some set $X$
- $A \cong K^{\text{Spec}_0(A)}$

Call such $A$ a **function algebra**. These form a category $\text{Func}(K)$. 
Pseudocompactness for diagonalizable subalgebras

Further restrict the WA theorem to characterize algebras like $K^B$.

**Corollary:** For a commutative topological $K$-algebra, TFAE:

- $A$ is p.c. semisimple, with all open maximal ideals of codimension 1
- $A \cong K^X$ for some set $X$
- $A \cong K^{\text{Spec}_0(A)}$

Call such $A$ a *function algebra*. These form a category $\text{Func}(K)$.

**Fact:** Closed subalgebras of function algebras are again function algebras.
Further restrict the WA theorem to characterize algebras like $K^B$.

**Corollary:** For a commutative topological $K$-algebra, TFAE:

- $A$ is p.c. semisimple, with all open maximal ideals of codimension 1
- $A \cong K^X$ for some set $X$
- $A \cong K^{\text{Spec}_0(A)}$

Call such $A$ a *function algebra*. These form a category $\text{Func}(K)$.

**Fact:** Closed subalgebras of function algebras are again function algebras.

**Thm:** $\text{Spec}_0 : \text{Func}(K)^{op} \to \text{Set}$ is a contravariant equivalence.
Pseudocompactness for diagonalizable subalgebras

Further restrict the WA theorem to characterize algebras like $K^B$.

**Corollary:** For a commutative topological $K$-algebra, TFAE:

- $A$ is p.c. semisimple, with all open maximal ideals of codimension 1
- $A \cong K^X$ for some set $X$
- $A \cong K^{\text{Spec}_0(A)}$

Call such $A$ a **function algebra**. These form a category $\text{Func}(K)$.

**Fact:** Closed subalgebras of function algebras are again function algebras.

**Thm:** $\text{Spec}_0: \text{Func}(K)^{\text{op}} \to \text{Set}$ is a contravariant equivalence.

**Cor:** Any closed subalgebra $A \subseteq K^X$ is isomorphic to $K^Y$ for some set $Y$ with $|Y| \leq |X|$. (**Proof:** $K^Y \hookrightarrow K^X \iff X \to Y$)
For $K = \overline{K}$, the algebras $K^n$ are characterized as the f.d. semisimple algebras. What is the topological analogue?

Def: A topological $K$-algebra is $K$-pseudocompact if it is an inverse limit of discrete algebras that have finite $K$-dimension. (These are “pro-finite-dimensional” algebras)

Ex: $K[[x]]$ (adic topology) is $K$-pseudocompact.

Proposition: For a commutative topological algebra over an algebraically closed field $K$, the following are equivalent:

- $A \cong K$ is a function algebra
- $A$ is $K$-p.c. and Jacobson semisimple
- $A$ is $K$-p.c. and topologically reduced (i.e., $x^n \to 0 \implies x = 0$)
For $K = \overline{K}$, the algebras $K^n$ are characterized as the f.d. semisimple algebras. What is the topological analogue?

**Def:** A topological $K$-algebra is \textit{$K$-pseudocompact} if it is an inverse limit of discrete algebras that have finite $K$-dimension.

(These are “pro-finite-dimensional” algebras)

**Ex:** $K[[x]]$ (adic topology) is $K$-pseudocompact.
For \( K = \overline{K} \), the algebras \( K^n \) are characterized as the f.d. semisimple algebras. What is the topological analogue?

**Def:** A topological \( K \)-algebra is \( K \)-pseudocompact if it is an inverse limit of discrete algebras that have finite \( K \)-dimension.

(These are “pro-finite-dimensional” algebras)

**Ex:** \( K[[x]] \) (adic topology) is \( K \)-pseudocompact.

**Proposition:** For a commutative topological algebra over an algebraically closed field \( K \), the following are equivalent:

- \( A \cong K^X \) is a function algebra
- \( A \) is \( K \)-p.c. and Jacobson semisimple
- \( A \) is \( K \)-p.c. and topologically reduced (i.e., \( x^n \to 0 \iff x = 0 \))
Infinite-Dimensional Diagonalization Theorem

For any $K$-vector space $V$ and any closed subalgebra $A \subseteq \text{End}(V)$, the following are equivalent:

(a) $A$ is diagonalizable
(b) $A \cong K^\Omega$ as topological algebras for some cardinal $\Omega \leq \dim(V)$
(c) (when $K = \bar{K}$) $A$ is $K$-pseudocompact and Jacobson semisimple
(d) (when $K = \bar{K}$) $A$ is $K$-pseudocompact and topologically reduced

Proof:
(a) $\Rightarrow$ (b): $A$ diagonalizable w.r.t. basis $B$ implies $A \subseteq K^B$, closed inside function algebra. So $A \cong K^\Omega$ with $\Omega \leq |B| = \dim(V)$.

(b) $\Rightarrow$ (a): $A \cong K^\Omega$ generated by orthogonal idempotents $e_i$ whose net of finite sums converges to 1. In $\text{End}(V)$, these project onto subspaces $V_i$ with $V = \bigoplus V_i$. The $V_i$ are simultaneous eigenspaces, so $A$ is diagonalizable.

(Already discussed (b) $\iff$ (c) $\iff$ (d).)
Infinite-Dimensional Diagonalization Theorem

For any $K$-vector space $V$ and any closed subalgebra $A \subseteq \text{End}(V)$, the following are equivalent:

(a) $A$ is diagonalizable

(b) $A \cong K^\Omega$ as topological algebras for some cardinal $\Omega \leq \dim(V)$

(c) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and Jacobson semisimple

(d) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and topologically reduced

Proof: (a) $\Rightarrow$ (b): $A$ diagonalizable w.r.t. basis $B$ implies $A \subseteq K^B$, closed inside function algebra. So $A \cong K^\Omega$ with $\Omega \leq |B| = \dim(V)$. 

Manuel Reyes  (Bowdoin)
Infinite-Dimensional Diagonalization Theorem

For any $K$-vector space $V$ and any closed subalgebra $A \subseteq \text{End}(V)$, the following are equivalent:

(a) $A$ is diagonalizable
(b) $A \cong K^\Omega$ as topological algebras for some cardinal $\Omega \leq \dim(V)$
(c) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and Jacobson semisimple
(d) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and topologically reduced

Proof: (a) $\Rightarrow$ (b): $A$ diagonalizable w.r.t. basis $B$ implies $A \subseteq K^B$, closed inside function algebra. So $A \cong K^\Omega$ with $\Omega \leq |B| = \dim(V)$.

(b) $\Rightarrow$ (a): $A \cong K^\Omega$ generated by orthogonal idempotents $e_i$ whose net of finite sums converges to 1.
Infinite-Dimensional Diagonalization Theorem

For any $K$-vector space $V$ and any closed subalgebra $A \subseteq \text{End}(V)$, the following are equivalent:

(a) $A$ is diagonalizable
(b) $A \cong K^\Omega$ as topological algebras for some cardinal $\Omega \leq \dim(V)$
(c) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and Jacobson semisimple
(d) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and topologically reduced

Proof: (a) $\Rightarrow$ (b): $A$ diagonalizable w.r.t. basis $B$ implies $A \subseteq K^B$, closed inside function algebra. So $A \cong K^\Omega$ with $\Omega \leq |B| = \dim(V)$.

(b) $\Rightarrow$ (a): $A \cong K^\Omega$ generated by orthogonal idempotents $e_i$ whose net of finite sums converges to 1. In $\text{End}(V)$, these project onto subspaces $V_i$ with $V = \bigoplus V_i$. 
Infinite-Dimensional Diagonalization Theorem

For any $K$-vector space $V$ and any closed subalgebra $A \subseteq \text{End}(V)$, the following are equivalent:

(a) $A$ is diagonalizable
(b) $A \cong K^\Omega$ as topological algebras for some cardinal $\Omega \leq \dim(V)$
(c) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and Jacobson semisimple
(d) (when $K = \overline{K}$) $A$ is $K$-pseudocompact and topologically reduced

Proof: (a) $\Rightarrow$ (b): $A$ diagonalizable w.r.t. basis $B$ implies $A \subseteq K^B$, closed inside function algebra. So $A \cong K^\Omega$ with $\Omega \leq |B| = \dim(V)$.

(b) $\Rightarrow$ (a): $A \cong K^\Omega$ generated by orthogonal idempotents $e_i$ whose net of finite sums converges to 1. In $\text{End}(V)$, these project onto subspaces $V_i$ with $V = \bigoplus V_i$. The $V_i$ are simultaneous eigenspaces, so $A$ is diagonalizable. (Already discussed (b) $\iff$ (c) $\iff$ (d).)
**Question:** Which operators $T \in \text{End}(V)$ can be “approximated” by diagonalizable operators?

I.e., for $D = \{\text{diagonalizable operators}\}$, when is $T \in \overline{D}$?
**Question:** Which operators $T \in \text{End}(V)$ can be "approximated" by diagonalizable operators?

i.e., for $D = \{\text{diagonalizable operators}\}$, when is $T \in \overline{D}$? Define:

$$H(T) := \{ v \in V \mid p(T)v = 0 \text{ for some } p \in K[x] \}$$

$$= \sum (\text{f.d. } T\text{-invariant subspaces})$$
**Question:** Which operators \( T \in \text{End}(V) \) can be “approximated” by diagonalizable operators?

I.e., for \( \mathcal{D} = \{ \text{diagonalizable operators} \} \), when is \( T \in \overline{\mathcal{D}} \)? Define:

\[
H(T) := \{ v \in V \mid p(T)v = 0 \text{ for some } p \in K[x] \}
\]

\[
= \sum \text{(f.d. } T\text{-invariant subspaces)}
\]

**Theorem:** (1) If \( K \) is an infinite field, then \( T \in \overline{\mathcal{D}} \) if and only if its restriction to \( H(T) \) is diagonalizable. (2) If \( K \) is finite, then \( \overline{\mathcal{D}} = \mathcal{D} \).
**Question:** Which operators $T \in \text{End}(V)$ can be “approximated” by diagonalizable operators?

I.e., for $\mathcal{D} = \{\text{diagonalizable operators}\}$, when is $T \in \overline{\mathcal{D}}$? Define:

$$H(T) := \{v \in V \mid p(T)v = 0 \text{ for some } p \in K[x]\}$$

$$= \sum (\text{f.d. } T\text{-invariant subspaces})$$

**Theorem:** (1) If $K$ is an infinite field, then $T \in \overline{\mathcal{D}}$ if and only if its restriction to $H(T)$ is diagonalizable. (2) If $K$ is finite, then $\overline{\mathcal{D}} = \mathcal{D}$.

**Ex:** The right-shift operator $S(v_n) = v_{n+1}$ on $V = \bigoplus K v_n$ has $H(S) = 0$. It is in $\overline{\mathcal{D}}$ when $K$ is infinite, but not in $\overline{\mathcal{D}}$ when $K$ is finite.
Thank you!