Connections Between Regularity, Unit-Regularity, Cleanness and Strong Cleanness

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joint work with

Pace Nielsen,
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$R$ a (non-commutative) ring with 1

$U(R)$ units

idem($R$) idempotents
Basic Definitions and Facts

- $a \in R$ is regular if $\exists x \in R : axa = a$
- $R$ is regular if every element is regular (von Neumann)
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  - $R$ is regular if every element is regular (von Neumann)
- $a \in R$ is unit-regular if $\exists u \in U(R) :avanaugh = a$
  - $R$ is unit-regular if every element is unit-regular (Ehrlich 1968)

All these classes are subclases of exchange rings.
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- \( a \in R \) is clean if \( \exists e \in \text{idem}(R), \exists u \in U(R) : a = e + u \)
  - \( R \) is clean if every element is clean (Nicholson 1977)

Connections Between (Unit)-Regular & (Strongly) Clean
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  - $R$ is clean if every element is clean (Nicholson 1977)
- $a \in R$ is **strongly clean** if
  - $\exists e \in \text{idem}(R), \exists u \in U(R): a = e + u$ and $eu = ue$
  - $R$ is strongly clean if every element is strongly clean (Nicholson 1999)

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All these classes are subclasses of exchange rings.
Known facts:

- Regular, commutative (or abelian) \( \Rightarrow \) unit-regular

- There exist regular rings which are not unit-regular, e.g. endomorphism ring of an infinite dimensional vector space

- There exists a regular ring which is not clean (Bergman's example)

- Unit-regular \( \Rightarrow \) clean (Camillo-Yu 1994, Camillo-Khurana 2001)

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Unit-Regular Ring Which Is not Strongly Clean

Field, \( F \) (\( (t) \)) the field of formal Laurent series over \( F \);
the set of all \( \mathbb{Z} \times \mathbb{Z} \) matrices
\( A = (a_{ij}) \) \( i, j \in \mathbb{Z} \) over \( F \) such that
\( \exists m, n \in \mathbb{Z} \) and \( \exists f(t) = \sum_{k=k_0}^{\infty} a_k t^k \in F((t)) \) with the following properties:

1. if \( i \geq m \) or \( j < n \) then \( a_{ij} = a_{j-i,n} \),

2. the submatrix \( A_0 = (a_{ij}) \) \( i < m, j \geq n \) has finite rank.

\[ A = \begin{pmatrix}
    a_{n-m} & a_{n-m-1} & \ldots & a_{n-m-1} \\
    a_{n-m} & a_{n-m-1} & \ldots & a_{n-m-1} \\
    \vdots & \vdots & \ddots & \vdots \\
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Connections Between (Unit)-Regular & (Strongly) Clean
The set $R$ is a ring under usual matrix operations. The ring $R$ was originally defined by G. Bergman in a different way, as a unit-regular ring with a regular subring which is not unit-regular.

Proposition

The ring $R$ is unit-regular.

Proof.

Let $A \in R$, and let $f(t)$ denote the corresponding Laurent series. We consider two cases, $f(t) \neq 0$ and $f(t) = 0$. In the case $f(t) = 0$ we need that $A_0$ has finite rank.
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The matrix \[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -2 & -1 \end{bmatrix} \] is not strongly clean in \( \mathbb{R} \). Therefore, \( \mathbb{R} \) is unit-regular but not strongly clean. (In fact, very few elements commute with \( A \).)
Theorem

The matrix

\[ A = \begin{pmatrix}
-1 & -2 & -1 & 0 \\
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Element-Wise Equivalence Between Clean and Unit-Regular Elements

Theorem (Camillo-Khurana, 2001)
Every unit-regular ring is clean. In fact,
\[ R \text{ unit-regular} \iff \forall a \in R \exists e \in \text{idem}(R), \exists u \in \text{U}(R) \text{ such that } a = e + u \text{ and } aR \cap eR = 0. \]

The unit-regular \( \Rightarrow \) clean implication does not hold element-wise:
Example (Khurana-Lam, 2004)
There exist matrices in \( M_2(\mathbb{Z}) \) which are unit-regular but not clean.

What do we need to assume for the element \( a \in R \), besides the unit-regularity, to obtain that \( a \) is clean?
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Theorem

Let \( a \) be an element of a ring \( R \). The following are equivalent:

1. \( \exists u \in U(R) \) with \( au = a \), such that, writing \( e := ua \in \text{idem}(R) \), \( eae \) is unit-regular in \( eRe \);

2. \( \exists r \in R \) with \( ara = a \), such that, writing \( e := ra \in \text{idem}(R) \), \( eae \) is unit-regular in \( eRe \);

3. \( \exists e \in \text{idem}(R) \) and \( u \in U(R) \) such that \( a = e + u \), \( aR \cap eR = (0) \), and \( a^2 R \cap aeR = (0) \);

4. \( \exists e \in \text{idem}(R) \) and \( u \in U(R) \) such that \( a = e + u \) with \( au - 1 = a \) and \( a^2 u - 2a = a^2 \).
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Let $a$ be an element of a ring $R$. The following are equivalent:

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Element-Wise Equivalence Between Clean and Unit-Regular Elements

**Corollary**
In a unit-regular ring $R$, for every $a \in R$ exists $u \in U(R)$ such that $a = au = a^2u$. Can this be further generalized?
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Special Inner Inverses

**Theorem**

Let \( R \) be a ring and \( a \in R \) such that \( a, a^2, \ldots, a^n \) are regular. Then there exists \( w \in R \) such that \( a_j w = a_j \) for all \( j = 1, \ldots, n \).

Moreover, if \( a \) is unit-regular, then we can take \( w \in U(R) \).

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Examples and Counterexamples

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In a regular ring $R$, for every $a \in R$ and every $n \geq 1$ there exists $w \in R$ such that $a^j w a^j = a^j$ for all $0 \leq j \leq n$.

Can we find $w$ such that these equations hold for arbitrary (unbounded) $j$?

Example
There exists a ring $R$ and $a \in R$ such that all powers of $a$ are regular but there is no $w \in R$ satisfying $a^j w a^j = a^j$ for all $j \geq 1$.

Proof.
Take the algebra $R = \langle a, x_i \mid (i \geq 1) : a^i x_i a^i = a^i \rangle$.
Corollary

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**Proof.**

Take the algebra $R = F\langle a, x_i (i \geq 1) : a^i x_i a^i = a^i \rangle$. 

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Connections Between (Unit)-Regular & (Strongly) Clean
Examples and Counterexamples

What about if the whole ring \( R \) is regular, do the unbounded equations hold?

**Proposition**

If \( R \) is a regular (right or left) self-injective ring, then for every \( a \in R \) there exists \( w \in R \) such that

\[
ajwja = aj
\]

for all \( j \).

The unbounded equations also hold in some other special cases (e.g. when the ring has bounded index of nilpotence).

G. Bergman recently found a unit-regular ring \( R \) and \( a \in R \) such that there is no \( w \in R \) satisfying

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2. $\exists r \in R$ with $ara = a$, such that, writing $e := ra \in \text{idem}(R)$, $eae$ is unit-regular in $eRe$;
3. $\exists e \in \text{idem}(R)$ and $u \in U(R)$ such that $a = e + u$, $aR \cap eR = (0)$, and $a^2R \cap aeR = (0)$;
4. $\exists e \in \text{idem}(R)$ and $u \in U(R)$ such that $a = e + u$ with $au - 1a = a$ and $a^2u - 2a^2 = a^2$.

($1) \iff (2)$ of this theorem implies that if $a$ is regular and $a^2 = 0$ then $a$ is unit-regular.

Is every regular nilpotent unit-regular?
In an exchange ring, the answer is yes (Ara 1996).
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Connections Between (Unit)-Regular & (Strongly) Clean

Janez Šter, University of Ljubljana
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Is every regular nilpotent unit-regular?

In an exchange ring, the answer is yes (Ara 1996).
Examples and Counterexamples

Example

There exists a ring $R$ and $a \in R$ with $a^3 = 0$ such that $a$ is regular but not unit-regular.

Proof.

Let $S = F \langle x, y : x^2 = 0 \rangle$, and $I = S(1 - yx)$. Define $R = (S I S F + I)$. Then $A = (x 0 1 0) \in R$ is the desired element.
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Proof. Let $S = \mathbb{F} \langle x, y : x^2 = 0 \rangle$, and $I = S(1 - yx)$. Define $R = (S I S \mathbb{F} + I)$. Then $A = (x^0 1 0)$ is the desired element.
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Let $S = F\langle x, y : x^2 = 0 \rangle$, and $I = S(1 - yx)$. Define

$$R = \begin{pmatrix} S & I \\ S & F + I \end{pmatrix}.$$  

Then $A = \begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix} \in R$ is the desired element.
The power inner inverse condition does not imply the property, even in exchange rings.

Example

There exists a regular ring $R$, $a \in R$ and $w \in U(R)$, such that $a^j w^j a^j = a^j$ for all $j$, but $a$ is not clean in $R$.

Proof.

Let $F$ be a field, let $S$ be Bergman's example of a non-clean exchange ring. Denote the canonical map $\psi \colon S \to F((t))$ and $I = \ker(\psi)$. Then $R = (S I I S)$ and $a = (\alpha 0 0 0)$ $\in R$ (where $\alpha$ denotes the right shift operator) satisfy the desired properties.
Examples and Counterexamples

The “power inner inverse” condition does not imply the clean property, even in exchange rings.
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R = \begin{pmatrix} S & I \\ I & S \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in R
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Thank you for your attention.