Kontsevich – T – Lecture #1

X, α Poisson manifold

$\alpha \in \Gamma(X, \Lambda^2 T)$

$[\alpha, \alpha] = 0 \in \Gamma(X, \Lambda^3 T)$

$*$ product

$C^\infty(X)[[T]]^{\otimes 2} \longrightarrow C^\infty(X)[[T]]$

$f * g = fg + tr\{f,g\} + \cdots$

D. Tamarkin (student of Tsygan) near proof

Today 1) Deformation theory

2) Operads

Object in math / field k, Char $k = 0$

$\begin{cases} A \text{ assoc alg}/k \\ S \text{ alg variety}/k \\ X \text{ complex manifold } k = \mathbb{C} \end{cases}$

$\rightsquigarrow$ Deformation functor

finite dim com ass. non-unital algebras m nilpotent m^N $\longrightarrow$ Sets

A assoc alg/k

$m \longmapsto$ equiv. classes

frees ass

$\tilde{A}/_{1\cdot k \oplus m}$ $1k \oplus m$ -module

ass. alg.

+ identification

$\tilde{A}/m\tilde{A} \cong A$

fixed

X compl mfld

$m \longmapsto$ eq. classes

complex spaces

$\tilde{X} \xrightarrow[\text{flat}]{} \mathrm{Spec}(\mathbb{C}+m)$

special fiber $\cong X$

Old theory (Deligne, Drinfeld, Millson, ...)

Object $\longrightarrow$ differential graded Lie algebra

$\mathfrak{g}$

$\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^n$

$[\ ,\ ] : \mathfrak{g}^n \otimes \mathfrak{g}^m \to \mathfrak{g}^{n+m}$

$d : \mathfrak{g}^n \longrightarrow \mathfrak{g}^{n+1}$

$[\alpha, \beta] = (-1)^{\bar\alpha \bar\beta} [\beta, \alpha]$

Jacobi identity ..

$$d^2=0$$

$$d[\alpha,\beta]=[d\alpha,\beta]+(-1)^{\bar\alpha}[\alpha,d\beta]$$

$m \longmapsto \mathcal{G}\otimes m$ d.g. Lie algebra nilpoten

$$[\gamma_1\otimes a_1, \gamma_2\otimes a_2] = [\gamma_1,\gamma_2]\otimes(a_1 a_2)$$

---

d.g. nilp Lie-alg $\longmapsto$ Set

$$\left\{ \gamma\in(\mathcal{G}\otimes m)^1 \;\middle|\; d\gamma+\frac{1}{2}[\gamma,\gamma]=0 \right\} \Big/ \begin{matrix}\text{group}\\ \operatorname{Exp}(\mathcal{G}^0\otimes m\end{matrix}$$

On $\mathcal{G}\otimes m^1$ acts $\mathcal{G}^0\otimes m$ by affine vector fields

$\varphi\in\mathcal{G}^0\otimes m$

$\longrightarrow$ v. field $= d\varphi$

$\gamma+[\varphi,\gamma]$

$\in$

$\mathcal{G}^1\otimes m$

A assoc alg

$\mathfrak{g}^n := Hom_{vect}(A^{\otimes(n+1)}, A)$

$$\begin{array}{cc} 0 & 1 \\ Hom(A,A) & Hom(A^{\otimes 2}, A), \end{array}$$

Gerstenhaber bracket

$\varphi \in \mathfrak{g}^n$

$\psi \in \mathfrak{g}^m$

not associative

$[\varphi, \psi] := \varphi \circ \psi - (-1)^{nm} \psi \circ \varphi$

$(\varphi \circ \psi)(a_0 \otimes \ldots \otimes a_{n+m})$

$= \sum (-1)^{mi\,?} \varphi(a_0 \otimes \ldots \otimes a_{i-1} \otimes \psi(a_i \otimes \ldots \otimes a_{i+m})$
$\otimes a_{i+m+1} \otimes \ldots \otimes a_{m+n}$

$[\,,\,]$ satisfies Jacobi even though $\circ$ not assoc.

$m_A : A \otimes A \to A \quad \in \mathfrak{g}^1$

assoc $\Leftrightarrow [m_A, m_A] = 0$

$d = [m_A, \cdot\,]$

⑤

$$\{\gamma \in (\mathfrak{g} \otimes \mathfrak{m})^1 \mid d\gamma + \tfrac{1}{2}[\gamma, \gamma] = 0\}$$

Maurer-Cartan eqn

$$[m_A + \gamma, m_A + \gamma] = 0 \quad A$$

X complex mfld

$$\mathfrak{g}^n = \Gamma(X, T^{1,0} \otimes \Omega^{0,n})$$

$[\ ,\ ]$ $d = \bar\partial$

$[\ ,\ ] \otimes \wedge$

Better language

Q-manifolds

Formal (∞)-dim manifld/k

finite dim manifold $k[[x_1, \dots, x_n]]$

formal ∞ is top $\ldots/k$ $\cong$

$$k[[x_1,\ldots,x_n]]^*_{top} = k \oplus V \oplus S^2V \oplus \cdots$$

cofree cocomm coassoc coalgebra with n gens

Def Formal manifold

= cofree ... coalgebra (possibly ∞ many gens)

in

Symmetric monoidal cats of vect/k

$\mathbb{Z}$-graded (super-) formal mfld

— " — in symm monoid cat. of

$\mathbb{Z}$-graded v spaces.

Formal Q manifold

formal $\mathbb{Z}$-graded mfld coalg. $C \cong k + V \oplus S^2V$

together with coderiv $Q : C \to C$

$\deg Q = +1$

$Q^2 = 0$

Q preserves $(V \oplus S^2V \oplus \ldots)$

Choose affine structure

i.e. identify $C \cong k + V + S^2V + \dots$

Q Taylor coeffs

$Q_1 : V \longrightarrow V$ deg $+1$

$Q_2 : S^2V \to V$ deg $+1$

$Q^2 = 0 \Longleftrightarrow$ system of eqns

$Q_1^2 = 0$

Q_2 is morphism of complexes

Assume

then $Q_3 = Q_4 = \dots = 0$

$Q^2 = 0 \Longleftrightarrow \mathfrak{g} := V[-1]$

$\mathfrak{g}^k = V^{k-1}$

is d.g. Lie algebra

Q-mfld $\rightsquigarrow$ Functor: nilp algebras $\mathfrak{m}$ $\to$ Sets

Supermfld Q odd vector field

$$Q^2=0$$

$Q|_x \neq 0$ in local coords Q is constant $\frac{\partial}{\partial \xi}$

Z = Subscheme $\{Q=0\}$

~~(M, ξ)~~, {Maurer-Cartan eq}

$[V, Q]$ preserves Z

$V \in T(X)$

"Sing foliation on Z"

equiv. relation

9

Thm $M, Q_{(1)} \ni pt_1$ $\odot$

$$\begin{array}{c} M, Q_{(1)} \ni pt_1 \\ \text{equivar map} \big\downarrow \\ M_{(2)}, Q_{(2)} \ni pt_2 \end{array}$$

induces $T_{p_1} M_{(1)} \to T_{p_2} M_{(2)}$

homomorphism of complexes

Thm If this $\uparrow$, the quasi-isom, ie

induces iso of $H^*(\ )$

$\Rightarrow$ functors : nilp alg $\longrightarrow$ sets

are equiv.

Pf. Sketch $\forall$ Q-mfld $M \cong M_{min} \times M_{contractible}$

$Q_{min,1} = 0$ $\quad V \oplus V[1]$

$\sum x_i \frac{\partial}{\partial \xi_i}$

Formality Theorem

A alg. $\longrightarrow$ Hochschild comple

degree: $\quad -1 \qquad\qquad 0 \qquad\qquad 1$

$$\mathrm{Hom}(A^{\otimes 0}, A) \to \mathrm{Hom}(A, A) \to \mathrm{Hom}(A^{\otimes 2}, A) \to \cdots$$

d.g. Lie algebra

$\longrightarrow$ formal Q manifold

$A := k[x_1, \ldots, x_n]$

Thm This ~~is~~ Q manifold is quasi-isomorphic to Q-mfld ~~of~~ Lie alg of poly vector fields in k^n

$$0 \to T(A^n) \to \Lambda^2 T($$

$d = 0$

$[\gamma, \gamma] = 0$