Operads

k field $\rightsquigarrow$ $\forall n \geq 0$ $P(n) \in Vect$

Assoc

Lie

Commutative

$P(n) \subseteq$ Free algebra gen by $x_1, \ldots, x_n$

each gen x_i appears once

$\Updownarrow$

universal k-linear ops in algebras A

$A^{\otimes n} \longrightarrow A$

$\dim = n!$ $\quad \mathrm{Assoc}(n) = \left\{ \sum c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \,\middle|\, c_\sigma \in k \right\}$

$= k^{n!}$

$\dim = 1$ $\quad \mathrm{Com}(n) = k \cdot x_1 \ldots x_n$

$\dim = (n-1)!$ $\quad \mathrm{Lie}(n)$ $\quad n=2$ $\quad [x_1, x_2] = -[x_2, x_1]$

$n=3$ $\quad [x_1, [x_2, x_3]], \; [x_2, [x_3, x_1]]$

<u>Def</u> Operad P in symm monoidal cat
$Vect_k$, $\otimes$
is <u>DATA</u> $P(n)$, $n \geq 0$ vector spaces
action of S_n on $P(n)$
$1_P \in P(1)$ $\left(k \to P(1) \right)$
Composition $k \geq 0$, $n_1, \ldots, n_k \geq 0$
$$P(k) \otimes P(n_1) \otimes \ldots \otimes P(n_k) \to P(n_1 + \ldots + n_k)$$

Basic Example
V vector space
$$End(V)(n) := Hom(V^{\otimes n}, V)$$

<u>Def</u> P-algebra is
$V \in Vect$, + maps $P(n) \otimes V^{\otimes n} \to V$
sat axioms S_n-equiv.

$\Longleftrightarrow$ homo of operads
$$P \to End(V)$$

If $P = \text{Assoc}$

P-algebras = assoc algebras

Remarks One can define operads in Symm monoidal cat $\mathcal{C}$

$$\mathcal{C}, \quad \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

Also P-algebras

Can replace Vect, $\otimes$

by $\mathbb{Z}$-graded Vect, $\otimes$ - graded operads

or by cochain complexes, $\otimes$ - dg-operads

or by Sets, $\underset{\text{cartesian product}}{\times}$

X set $\quad \text{End}(X)(n) = \text{Maps}(X^n, X)$

algebras over appropriate set-operads

groups, rings, monoids

but not fields (can't divide by 0), Hopf-algebras

(coproducts can't be formalized)

Back to Vect, $\otimes$

P: operad

V vector space.

$$\bigoplus_{n=0}^{\infty} \left(P(n) \otimes V^{\otimes n}\right)$$ is free P-alg. gen by V

P = assoc: $P(n) \simeq k[S_n]$

$$\bigoplus_{n\geq 0} V^{\otimes n}$$ free assoc.

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, bend left, "\mathrm{Free}"] & P\text{-alg} \arrow[l, bend left, "\mathrm{Forget}"] \end{tikzcd}$$

two adj functors

$$\mathrm{Hom}_{\mathrm{Vect}}(V, \mathrm{Forget}(A)) = \mathrm{Hom}_{P\text{-alg}}(\mathrm{Free}(V), A)$$

$$\begin{tikzcd} \mathrm{Vect} \arrow[loop right, "T"] \end{tikzcd} \quad T = \mathrm{Forget} \circ \mathrm{Free}$$

$\mathrm{id}_{\mathrm{Vect}} \to T$ monad or triple

$T \circ T \to T$

P - operad

A - P-alg. in Vect ($\subseteq$ complexes)

<u>Def</u> Free resolution of A

$\widetilde{A}$: algebra over P in the $\otimes$ category of complexes

$\widetilde{A} \longrightarrow A$ homo of P-alg.

(complex in deg = 0) <u>quasi</u>-isomorphis

$\widetilde{A}$ free as g-algebra (forgetting differenti

<u>thm</u> : Free resolutions exist

A algebra

$A \xleftarrow{epi} T(A)$ take kernel $= A_1$

$$\text{Free}\left(\begin{array}{cc} -1 & \text{in deg } (0) \\ A_1 & \text{Forget}(A) \end{array}\right)$$

P assoc^{n} $\quad A = k[x,y]$

$\widetilde{A} = k\langle x, y, \xi\rangle$ free assoc

$\deg x = \deg y = 0$ $\quad d(x) = 0$

$\deg \xi = -1$ $\quad d(y) = 0$

$d(\xi) = xy - yx$

Deformation Theory

char $k=0$

$A \rightsquigarrow$ formal Q-manifold / quasi-isomorphis

2-constructions (1) $\tilde{A} \to A$
free resolution

$\mathbb{Z}$-graded (super) Lie-algebra
$\underline{\mathrm{Der}}(\tilde{A})$
As $\mathbb{Z}$-graded space
$= \underline{\mathrm{Hom}}(\text{gens of } \tilde{A}, A)$

$\mathrm{Der}(\tilde{A})^1 \ni d_{\tilde{A}}$, $[d_{\tilde{A}}, d_{\tilde{A}}] = 0$

$\Downarrow$

dg Lie algebra

$(\mathrm{Der}(\tilde{A}), [d_{\tilde{A}}, \cdot])$ dg Lie alg
differential

This is dg Lie algebra for deformation of A as P-alg.

② What is an operad

$$P(n) = \bigoplus_{\substack{\lambda \text{ partition} \\ \text{of } n}} \rho_\lambda \otimes P_\lambda$$

ρ_λ irred repr of S_n

algebra over typed / colored operad to $\mathcal{O}$perad

Example of typed operad
A Assoc alg
M A-module
$A \otimes A \to A$
$A \otimes M \to M$

$\exists$ dg operad, $\tilde{P} \xrightarrow[\text{iso}]{\text{quasi-}} P$

$\tilde{P}$ free resolution

$\tilde{P} \to P \to End(A)$ A is also $\tilde{P}$ algebra

Forget about $d_{\tilde{P}}$
Consider $\mathbb{Z}$-graded (super) manifold $\mathcal{M}$ parametrizing actions of $\tilde{P}$ as ($\mathbb{Z}$-graded operad on A)

$\mathcal{M} \ni$ pt : our action initial

$d_{\tilde{P}}$ infinitesimal auto of $\tilde{P}$

$\Rightarrow Q$ on {actions}

Formal completion of $\mathcal{M}$ at this point

$P = $ Assoc for non-unital alg

$\tilde{P}$ is freely generated by
$\deg 0$ — $m_2 : A \otimes A \to A$
$\deg -1$ — $m_3 : A \otimes A \otimes A \to A$
-2 — $m_3 : A \otimes A \otimes A \otimes A \to A$

We get Q-manifold $\hat{M}$, Q

$\underset{\text{vect}}{End(A)}$ as Lie-alg acts on $\hat{M}$

Can make new Q-manifold

$$\tilde{M}' = \hat{M} \times End(A)[-1]$$

$\hat{M}$ = completion at zero of $\mathbb{Z}$-graded vector sp

$$\begin{array}{ccc} 0 & 1 & \\ Hom(A^{\otimes 2}, A) & Hom(A^{\otimes 3}, A) & \ldots \end{array}$$

$\tilde{M}$

$$\begin{array}{ccc} Hom(A, A) & Hom(A^{\otimes 2}, A) & \\ -1 & 0 & 1 \end{array}$$

Thm ① st const (Free resol of alg

& ② nd const (Free resol of operad)

give quasi-iso Q-mflds

Deformation theory of

Assoc

Comm is trivial

Lie

Operad

Poisson alg $*, [\ ,\ ]$ Def theory $\neq 0$

$$\overset{-2}{A} \quad \overset{-1}{Hom(A,A)} \rightsquigarrow \overset{0}{Hom(A^{\otimes 2}, A)} \quad \overset{1}{Hom(A^{\otimes 3}, A)} \ \ldots$$

def of assoc on given vector space

get $\tilde{M}$

get $\tilde{M}'$: def of assoc alg on vect space A

deformation of abelian cat A-module

Hochschild complex C

$C[-1]$ is d.g. Lie algebra

Question: What is "natural" dg. operad acting on $C = C^*(A, A)$ for assoc alg A

$$\begin{array}{ccccc} C^k & \otimes & C^l & \longrightarrow & C^{k+l} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ \varphi & & \psi & & \end{array}$$

$$\varphi : A^{\otimes k} \to A \qquad \psi : A^{\otimes l} \to A$$

$$(\varphi \circ \psi)(a_1 \otimes \ldots \otimes a_{k+l}) := (-1)^{k(l-1)} \varphi(a_1 \otimes \ldots \otimes a_k) \cdot_A \psi(a_{k+1} \otimes \ldots \otimes a_{k+l})$$

compatible with d_C

commutative on $H^\cdot(A, A)$

Why commutative up to homotopy?

$$H^\cdot(A, A) = Ext_{A\text{-mod-}A}(A, A)$$

A - unital

$$\wr\wr$$

$$Ext_{\underbrace{Funct(A\text{-mod}, A\text{-mod})}_{\text{monoidal category}}\, 1}(Id, Id)$$

A. Bondal

$M \in A\text{-mod-}A$

$A\text{-mod} \to A\text{-mod}$

$M \otimes_A -$

Picture: $\mathcal{C}$, $\otimes$ monoidal abelian cat
idem

1_C

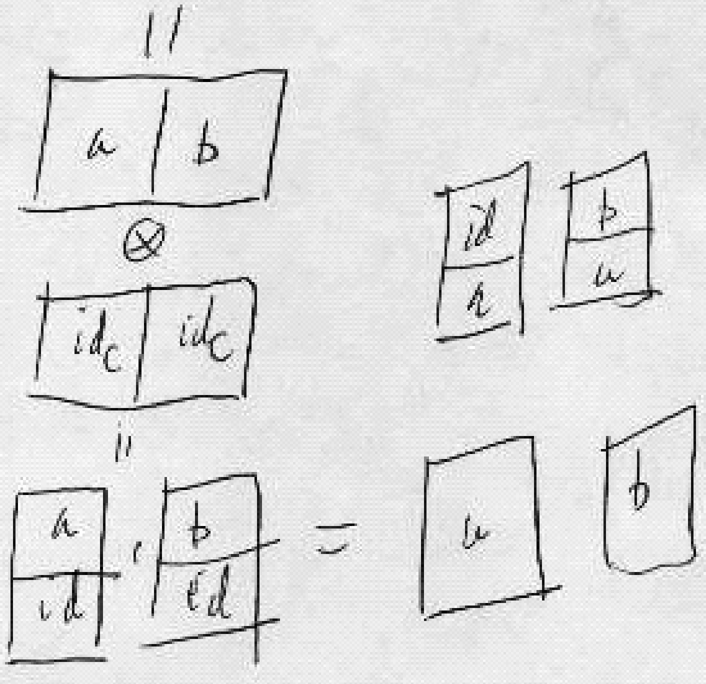
$\mathrm{RHom}(1_C, 1_C)$ is commutative

$E \boxed{a} F \qquad F \boxed{b} G$ compose $E \boxed{a \mid b} G$

$\otimes$

$E' \boxed{a'} F'$

$1_C \boxed{a} 1_C \quad 1_C \boxed{} 1_C = \boxed{b \circ a}$



Observation

operad acting on $H^*(A,A)$

gen by $[\,,\,]$, $*$

n-th space $= H_*(\mathbb{C}^n \setminus \mathrm{diag})$

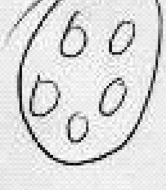
$\mathbb{Z}$

Topological operad

C_2

$C_2(0) = \phi$

$C_2(1) = id$

$n \geq 2 \quad C_2(n) =$  n-little disks in $\{x^2+y^2 \leq 1\}$

$H_*(C_2)$

<u>Conj</u> Chains of C_2 acts as dg operad on $C(A,A)$