d-algebras $\quad d \geq 0$

Topological operad $\mathcal{C}_d$

$\mathcal{C}_d(0) = \emptyset$

$\mathcal{C}_d(1) = \{id\}$

$n \geq 2 \quad \mathcal{C}_d(n) = \Big\{$  $\Big|$ n disjoint disks in unit disk $\Big\}$

Functor: Chains : Spaces $\longrightarrow$ complexes of ab groups

need $(X_i)_{i \in I}$ $\quad \bigotimes_i$: Chains $(X_i) \to$ Chains $(\prod X_i)$

I finite

usual singular chain complex doesn't have this

use cubical chain complexes modded out by action of symmetric groups

<u>Def.</u> d-algebra is alg. in $\otimes$ of complexes over Chains $(\mathcal{C}_d)$

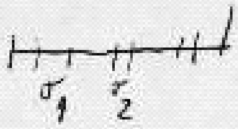
$d = 0 \quad \mathcal{C}_d$ trivial

0-algebra is just complex

<u>Remark</u>

Chains $(\mathcal{C}_d) \otimes \mathbb{Q}$ is free as graded operad

$d=1$



$\sigma \in S_n$

$C_1(n) = \coprod_{n!}$ contractible spaces

$$\begin{tikzcd} \text{Chains}(C_1) \arrow[r] \arrow[dr] & \text{Assoc} \\ & \text{Chains}_0(C_1) \arrow[u] \end{tikzcd}$$

(operad of assoc. non-unital algebras)

<u>Picture</u> <u>Def</u> A d-algebra

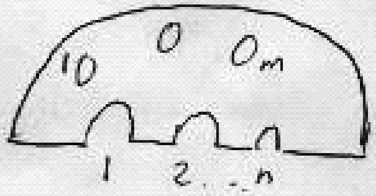
B $(d+1)$-algebra

<u>action</u> of B on A is structure of

Chains$(S.C)$

$\uparrow$

swiss cheese oper.

$SC_{m,n} = \{$  $\}$

colored operad with two colors

$$\text{Chains}(S.C_{m,n}) \otimes B^{\otimes m} \otimes A^{\otimes n} \longrightarrow A$$

Claim ① A: d-algebra

then $A[d-1]$ is L_∞ algebra

($\leftrightarrow$ on formal completion of $A[d]$ at 0 will be vector Q $\deg Q = +1$

$A[d-1]^k := A^{d-1+k}$ $Q^2 = 0$

Claim ② A d-algebra

$\downarrow$

Hochschild complex of A

Hoch(A) which is (d+1)-algebra

as L_∞-algebra Hoch(A) $\longrightarrow$ $A[d-1]$

Deformation complex of A

Claim ③ From the point of view of deformation theory

$$\left(\underset{d\text{-alg}}{A},\ \underset{d+1\text{-alg}}{B},\ \text{action of } B \text{ on } A \right)$$

$$\simeq \left(A, B, \underset{\text{of } (d+1)\text{alg}}{\text{hom}}\ B \to \mathrm{Hoch}(A) \right)$$

<u>Example</u> $d=0$ A - vector space
(complex in $\deg = 0$)

Def. complex of $A = \mathrm{End}(A)$
Lie alg (in deg 0)

$\mathrm{Hoch}(A) =$ Lie alg. of affine transf. of A
$\mathrm{End}(A) \oplus A$

it is not only Lie algebra, it is assoc algebra
(in particularly, 1-algebra)

$$\mathrm{Lie}(\mathrm{Aff}(A)) = \{ (M, a) \}$$

$$M \in \mathrm{End}\, A$$
$$a \in A$$

$$(M, a)(M', a') = (MM', Ma')$$

assoc. alg. without 1

What are d-algebra

Theorem (Tamarkin) $\mathrm{Chains}(C_d)$ is quasi-isom to $H(C_d)$ with zero differential

For $d \geq 2$ operad $H_*(C_d)$ is generated by

$$H_*(C_d(2)) = H_*(S^{d-1})$$

$H_*(C_d)$ algebra is

dg. comm assoc. algebra A

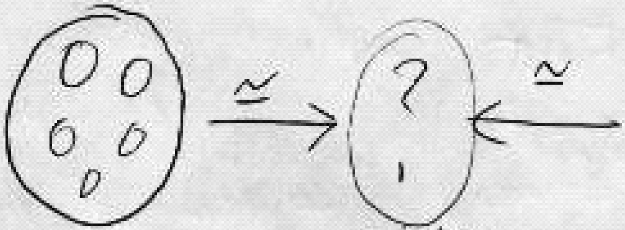
$[\,,\,] : A \otimes A \to A$ of deg $1-d$

graded skew-symmetric Jacobi identity

Leibnitz rule $[a, bc] = \pm [a,b]c \pm b[a,c]$

Proof of formality of Chains(C_d)

C_d



mixture of C_d & C_d'

C_d' operad of top manifold with boundaries

$$C_d'(0) = \phi$$

$$C_d'(1) = \{id\}$$

$$n \geq 2 \quad C_d'(n) := \overline{\left((\mathbb{R}^d)^n - diag\right) / \mathbb{R}^d \rtimes \mathbb{R}_+^*}$$

Fulton Compactification

$$\dim C_d'(n) = \dim - (d+1)$$

$C_d'(2) = S^{d-1}$

$C_d'(3)$

$\downarrow$

$C_d'(2) = S^{d-1}$ fiber before compactifying is $\mathbb{R}^d \setminus$ 2 points

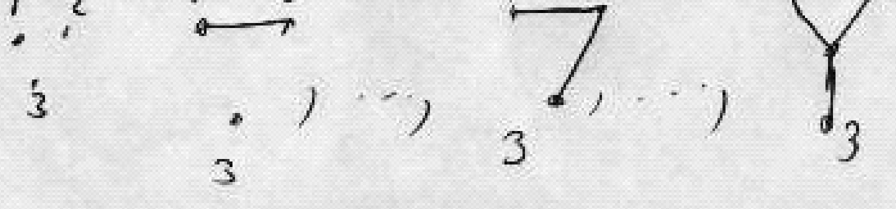
add spheres at infinity

Special Diff forms on C_d'

F : Forest (graph # trees) with n-vertices labelled $\{1, \dots, n\}$ + other valencies have valency ≥ 3

$n = 2$

$n = 3$



$$F \rightsquigarrow \omega_F \in \text{``}\Omega^k\text{''}\left(C_d'(n)\right)$$

on open part $\mathrm{Conf}_n(\mathbb{R}^d)/\mathbb{R}^d \rtimes \mathbb{R}_+^*$

$$\begin{tikzcd} \mathrm{Conf}_{n+k}(\mathbb{R}^d)/\mathbb{R}^d \rtimes \mathbb{R}_+^* \arrow[r, "p_2"] \arrow[d, "p_1"] & \prod_{\text{edges}} S^{d-1} \\ \mathrm{Conf}_n(\mathbb{R}^d)/\mathbb{R}^d \rtimes \mathbb{R}_+^* & \end{tikzcd}$$

$k = \#$ of additional vertices

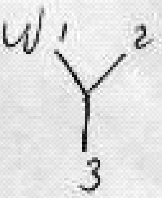
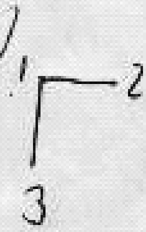
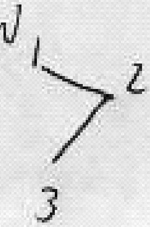
$\omega_F = (p_1)_* \, p_2^* \prod$ standard volume element on S^{d-1} $SO(d)$ invariant

not C^∞ – bad singularities at boundary

Main Formula

$$d\omega_F = \sum_{\substack{e \text{ edges} \\ \text{at least one} \\ \text{end of } e \text{ is} \\ \text{unnumbered}}} \pm \, \omega_{F/e}$$

collapse edge

Eg. $d\omega$  $= \omega$  $+ \, \omega$  $+ \, \omega$ 

$$H^{*}(\text{Forest})^{*} \longrightarrow (\text{Forests})^{*} \xleftarrow[\text{iso}]{\text{quasi}} \text{Chains}(C_d)$$
$$\| $$
$$H^{*}(C^{d})$$

Not possible to prove this combinatorially
values of $\zeta(\ )$ enter into formula

Deform quantization

$$\underset{\text{assoc alg}}{A} \longrightarrow \underset{2\text{-alg}}{\text{Hoch}(A)}$$

L_∞-morphism

$$\text{Hoch}(A) \longrightarrow \text{Hoch}(\text{Hoch}(A))$$

for $A = k[x_1, \ldots, x_n]$
this quasi-iso

deform assoc alg $\approx$ def. 2-alg $\tilde{\approx}$ def Gerstenhaber alg. formality

k^* acts here
$[\ ,\ ] \to \lambda[\ ,\ ]$