Meaningful Mathematics from Fractions to Linear Functions

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Outline of Activities

- Equivalent fractions (grade 4)
- Fraction multiplication (grades 4 and 5)
- Note: division with remainder, division in context, division as fraction
- Fraction division (Grades 5 and 6)
- Note: Division involving zero
- Decimals as fractions: Hundredths grids (grades 5-8)
- Note: Decimal operations
- Ratios and proportional relationships (grades 6-7)
- Note: Percent increase/decrease (grade 7)
- Linear functions (grade 8+)

Equivalent Fractions

Grade 4

Important Standard

- **4.NF.1** Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

- Note: *Ready Math* misses the point of this standard.

Explaining Fraction Equivalence

- Draw pictures to explain why $\frac{3}{4} = \frac{5\times 3}{5\times 4}$.
- Generalize so that your explanation shows that $\frac{3}{4} = \frac{n\times 3}{n\times 4}$, where n is any positive integer.

- Note: Because students in grade 4 do not yet know how to multiply fractions, you should use the following foundational standard:
  - **3.NF.1** Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

Fraction Multiplication

Grades 4-5

Meanings of Multiplication

- What is our convention for the meaning of $a \times b$ as repeated addition?
- Which one is called the **multiplier**? And which is the **multiplicand**?
- Note the following standard:
  - **3.OA.1.** Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. (Note: These standards are written with the convention that $a \times b$ means a groups of b objects each; however, because of the commutative property, students may also interpret 5×7 as the total number of objects in 7 groups of 5 objects each).

Whole Number Times Fraction, vice versa

- Without computing the result, describe (in words) the meaning of the product $5 \times \frac{1}{3}$. Then draw a picture.
- Without computing the result, describe (in words) the meaning of the product $\frac{1}{3} \times 5$. Then draw a picture.

- Note: Multiplication turns out to be commutative, so the answers should be the same. But for understanding fractions and their arithmetic, it is important to work through **both** lines of reasoning.

Explaining Multiplication of Fractions

- Beginning with a unit square, use an area model to illustrate $\frac{1}{3} \times \frac{1}{4}$.
  - Use the meaning of fractions to explain the result.
- Beginning with a unit square, use an area model to illustrate $\frac{2}{3} \times \frac{5}{4}$.
  - Use the meaning of fractions to explain the result.

- Can we reason generally from the specific numbers?
- Use an area model to illustrate $\frac{a}{b} \times \frac{c}{d}$.
  - Use the meaning of fractions to explain the result.

Multiplying Mixed Numbers

- When computing $2\frac{1}{3} \times 3\frac{2}{5}$, Byron says that the answer is $6\frac{2}{15}$.
- Describe Byron's method.
- How can you know quickly that the answer is incorrect?
- Use what is right about his method to show what he is missing.

- Note: Some teachers insist that mixed numbers must first be converted to improper fractions before multiplying.
  - But this is false.
  - And perhaps debilitating.

Fraction Division

Grades 5-6

Different Kinds of Division Answers

- Write four word problems for $17 \div 5$ with four different answers:
  - Quotient and remainder
  - Round up
  - Round down
  - Completed division (fraction or decimal)

- Note the following standard:
  - **5.NF.3** Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

Different Meanings for Division

- Many division problems are asking one of two questions:
  - How many groups?
  - How many in one group?

- Note the following standard:
  - **3.OA.2.** Interpret whole number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.

Word Problems for Fraction Division

- 4 yards of ribbon, 1/3 yard per bow. How many bows?
- 1/3 pan of brownies, 4 people. How much brownie for each?

- 4 students are 1/3 of the class. How many students in the whole class?
- 1/3 cups of flour, 4 cups per batch. How many batches?

Word Problems for Fraction Division

- 5 lbs hamburger, 2/3 lb per burger. How many burgers?
- 5 lbs hamburger is enough for 2/3 of the meatloaf. How much for the whole meatloaf?

- 7/4 lbs hamburger, 2/3 lb per burger. How many burger?
- 7/4 lbs hamburger is enough for 2/3 of the meatloaf. How much for the whole meatloaf?

Division Involving Zero

- We often say, “Division by zero is undefined.” But why?
- The previous contexts can be used to explain why division by zero is undefined.
- Note:
  - The explanation for $2 \div 0$ is different from the the explanation for $0 \div 0$.

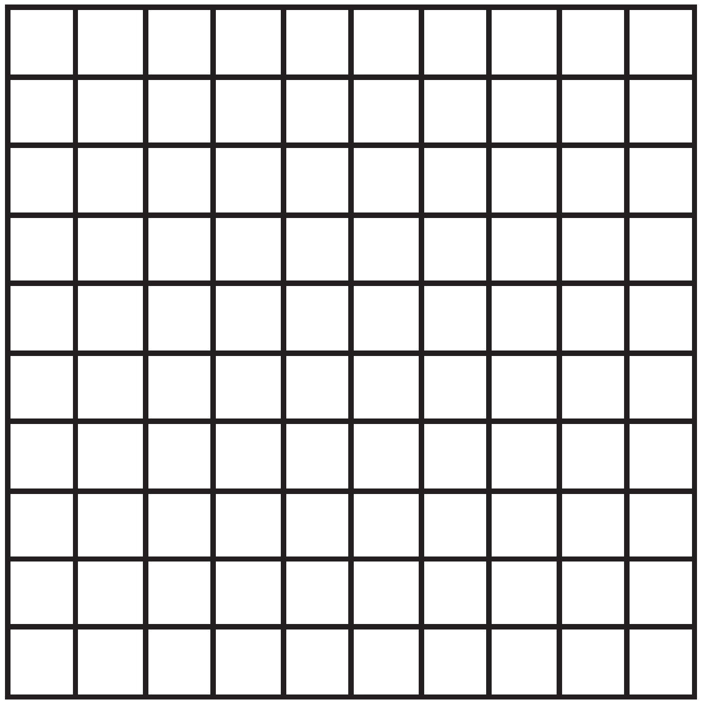
Decimals as Fractions

Grades 5-8

What Is This?

- A 10×10 square, sometimes called a "flat" when using base-ten blocks.
- When the 10×10 square is taken to be a whole, each small square represents $\frac{1}{100}$.
- Shade a "hundredths grid" to show each of the given fractions.

  (a) 3/20 (b) 1/8 (c) 1/6
- Then *use your shading to determine a decimal equivalent* for each fraction.



Targeted Standards

- **5.NBT.7.** Solve real-world problems by adding, subtracting, multiplying, and dividing decimals using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction, or multiplication and division; relate the strategy to a written method and explain the reasoning used.
- **5.NBT.1.** Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
- **MP.1.** Make sense of problems and persevere in solving them.
- **MP.3.** Construct viable arguments and critique the reasoning of others.

Decimals and Their Arithmetic

- For decimals, the most important standard is this one:
  - **5.NBT.1.** Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
- Thus, whole-number algorithms can work for finite decimals by reframing the problem with the smallest decimal unit as 1.
  - For example, 3.4 + 5.78 can be accomplished as 340 (hundredths) + 578 (hundredths) = 918 (hundredths).
- Then use estimation to get the decimal point in the right place in the answer.
  - The problem is approximately 3 + 6, which is 9. So the answer must be 9.18.

Ratios and Proportional Relationships

Grades 6-7

A Rich Problem

- A hen-and-a-half lays an egg-and-a-half in a day-and-a-half. How many eggs would 6 hens lay in 4 days?

  - Think
  - Draw a picture
  - Make a table

A Pictorial Solution

6 hens

Hh Hh Hh Hh

Ee Ee Ee Ee 1.5 days

Ee Ee Ee Ee 1.5 days

So 6 hens lay 12 eggs in 3 days

We need one more day

A Pictorial Solution

6 hens

<u>Hh Hh Hh Hh</u>

Ee	Ee	Ee	Ee	1.5 days
Ee	Ee	Ee	Ee	1.5 days
E	E	E	E	1 day

So 6 hens lay 16 eggs in 4 days

A Tabular Solution

Hens	Days	Eggs
1.5	1.5	1.5
3	1.5	3
3	3	6
3	1	2
6	1	4
6	4	16

Observations

- Proportional relationships typically involve **two** quantities
- … but here we have **three** quantities, so standard methods fail

- When hens are constant, eggs and days are proportional
- When days are constant, eggs and hens are proportional
- When eggs are constant, hens and days are inversely proportional

- At what grade can students do this kind of reasoning?

Let’s Start Again

- We are beginning a new unit in Math 1 (i.e., grade 9)
- Today’s standards:

QUANTITIES N.Q

Reason quantitatively and use units to solve problems.

N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★

N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.★

Chicken and Egg Problems

- A hen-and-a-half lays an egg-and-a-half in a day-and-a-half. How many eggs would 6 hens lay in 4 days?

Use the Units

$$\left(\tfrac{3}{2}\,\text{egg}\right)/\left(\tfrac{3}{2}\,\text{hen}\right)/\left(\tfrac{3}{2}\,\text{day}\right) = \left(1\ \text{egg/hen}\right)/\left(\tfrac{3}{2}\,\text{day}\right)$$

$$= \frac{2}{3}\,\text{egg}/\text{hen}/\text{day}$$

$$= \frac{2}{3}\,\frac{\text{egg}}{\text{hen}\cdot\text{day}}$$

6 hens, 4 days. How many eggs?

$$\frac{2}{3}\,\frac{\text{egg}}{\text{hen}\cdot\text{day}}\cdot 6\,\text{hens}\cdot 4\ \text{days} = 16\ \text{eggs}$$

- The homework is 1-49 odd. You may get started now.

*** *Stop* ***

Questions

- What is a hen-day?
- What is 12 hen-days?
- Did the lesson structure seem familiar?

Extension Questions

a) 6 hens, 4 days. How many eggs?

b) 8 hens, 16 eggs. How many days?

c) 12 eggs, 3 days. How many hens?

d) 12 eggs, 1 week. How many hens?

e) 8 hens. How many eggs and days?

e) 48 eggs. How many hens and days?

Some Answers

b) 8 hens, 16 eggs.
How many days?

$$\frac{16\text{ eggs}}{8\text{ hens}} \cdot \frac{3}{2}\frac{\text{hen}\cdot\text{day}}{\text{egg}} = 3\text{ days}$$

c) 12 eggs, 3 days.
How many hens?

$$\frac{12\text{ eggs}}{3\text{ days}} \cdot \frac{3}{2}\frac{\text{hen}\cdot\text{day}}{\text{egg}} = 6\text{ hens}$$

d) 12 eggs, 7 days.
How many hens?

$$\frac{12\text{ eggs}}{7\text{ days}} \cdot \frac{3}{2}\frac{\text{hen}\cdot\text{day}}{\text{egg}} = \frac{18}{7}\text{ hens}$$

Stacking Paper

- Suppose you want to know how many sheets are in a particular stack of paper, but don't want to count the pages directly. You have the following information:
  - The given stack has height 4.50 cm.
  - A ream of 500 sheets has height 6.25 cm.
- How many sheets of paper do you think are in the given stack?

- From Stanley, 2014. See more at:
  - http://blogs.ams.org/matheducation/2014/11/20/proportionality-confusion/

Mixing Punch

- Jenny is mixing punch and is considering two recipes:
  - Recipe A: 3 parts orange juice for every 5 parts ginger ale
  - Recipe B: 2 parts orange juice for every 3 parts ginger ale
- Which recipe will give juice that is the most "orangey"?
  - Explain your reasoning.
- To make 12 gallons of recipe B, how much of each will you need?

Racing Snails

- Mike is racing snails who move at a constant speed:
  - Snail A travels 3 inches in 5 minutes
  - Snail B travels 2 inches in 3 minutes
- Which snail moves faster? Explain your reasoning.
- How far will snail B go in 10 minutes?

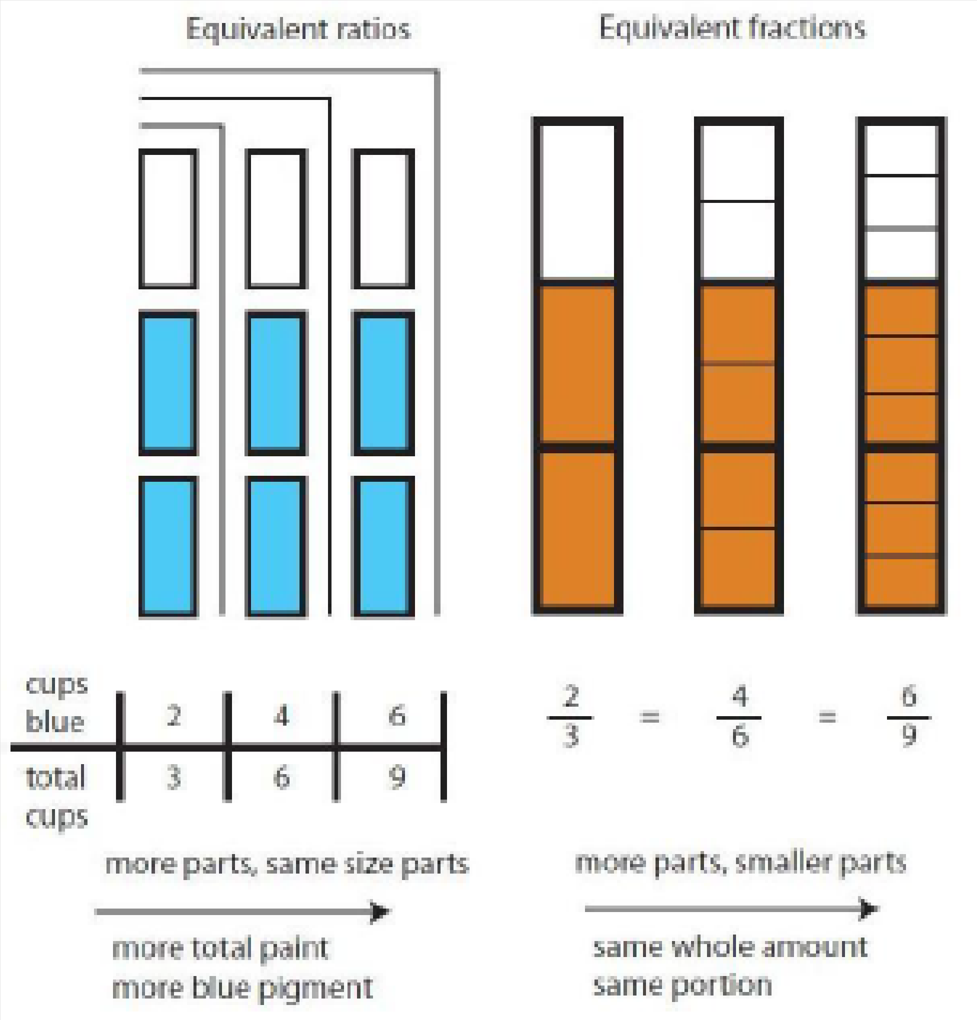
Reflection Questions

- How are these situations the same and different?
- How might they be modified for grade 7?
- How might they be used to reason about division involving zero?

Tools for Proportional Reasoning

- Pictures
- Ratio tables
- Equivalent fractions
- Equivalent ratios
- Tape diagrams
- Double number lines
- Unit rates (as fractions, decimals, or percents)
- Graphs
- Equations

Equivalent ratios versus equivalent fractions



Fractions, Ratios, and Rates

- Three connected ideas with differing histories and differing usage
  - Fractions are numbers, often used to express results of sharing, cutting
  - Ratios have historically been used to compare "like" quantities
    - Often expressed as pairs of counting numbers, without units, e.g., 3:2
  - Rates have historically been used to compare different quantities
    - Often expressed as quotients of quantities (e.g., meters and seconds)
- Ultimately, we want students to see all of these as quotients (i.e., the result of a division)
  - In high school and beyond, the distinctions become blurred
- Sometimes it is useful to attend only to the numbers
  - Two apparently different problems might be abstractly "the same"
- It is important to use the units to interpret numeric "answers" in context

Ratio Addition?

- On Saturday, Jenna got 2 hits in 3 at bats. On Sunday, she got 4 hits in 5 at bats. She computed her weekend batting average as follows:

$$\frac{2}{3} + \frac{4}{5} = \frac{6}{8} = 0.750$$

- Comment on her calculation.
- Her answer is correct, but it should not be written as fraction addition.
- Before this weekend, her batting average was 0.429. What can you say about her average after the weekend?
- We don't know how many at-bats. Could have been 3/7, 6/14, 9/21, etc.
- But the new average must be between 0.429 and 0.750.

Percent increase and decrease

- Proportional reasoning can be useful in percent problems
- For percent increase (or decrease), use pictures and the distributive property to make sense of $1 + r$ (or $1 - r$).
  - For example, if a price increases 15%, the new price will be

$$1 + 0.15 = 1.15 = 115\%$$

  - of the original price.
- This sort of reasoning is more efficient in middle grades and necessary for exponential functions in high school.

Proportional Relationships?

- Which of the following are proportional relationships? Explain.
  - If Josh is 10 and Reina is 7, how old will Reina be when Josh is 20?
  - If 2 people can paint the fence in 5 hours, how long will it take 4 people?
  - Celsius-Fahrenheit conversion
  - Price of a pizza vs. diameter
  - Shoe size vs. length of foot in inches.

Linear Functions

Grade 8+

Stacking Cups

Plastic cups come in bags of 50, in two stacks. One plastic cup is 12.3 cm high. A stack of 5 cups is 15.6 cm high.

a) Find the height of each stack.

b) Find a general formula for the height of n cups.

c) Graph the relationship.

d) How many will fit in a stack on a shelf that is 25.8 cm high?