Meaningful Mathematics from Fractions to Linear Functions

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Multiplication and Division (grade 3+)

1. Repeated addition, sets, arrays, area (rectangles preferred)
2. How many groups? How many in one group?
3. Division with remainder (grade 4) vs. division as fraction (grade 5)
4. Division involving 0: When is it undefined? And why? (grade 5 or 7?)

Fractions (grades 3-6)

1. Fraction meaning (grade 3):
    a. Know the whole, the size of the pieces, the number of piece (in that order)
    b. Unit fractions and same-size pieces
2. Fraction models: length (esp. number line), area, set
3. Equivalent fractions (grade 4, initially **not** because of "multiplication by 1")
4. Fraction addition and subtraction: need same-size pieces
5. Fraction multiplication
    a. Initially with whole numbers and unit fractions (grade 4)
    b. Length (number line), area, and set models
    c. Area model to explain algorithm (grade 5)
    d. Multiplying mixed numbers: okay with distributive property
6. Showing that fractions can mean division and vice versa (grade 5)
7. Fraction division models:
    a. Initially with whole numbers and unit fractions (grade 5)
    b. How many groups? Repeated subtraction strategy
    c. How many in one group? What is one whole group?
    d. Why invert and multiply? (grade 6)
8. Emphasize meaning: What are story problems that model this operation?
    a. The whole can change (esp. with multiplication and division)
    b. What are the knowns and unknowns?
9. Do we always have to write the answers as a single fraction?

Decimals (grades 4-8)

1. Decimals as fractions, hundredths grids
2. Decimals as an extension of base-ten place value
3. Decimal operations: Use base-ten algorithms and estimation
4. Repeating decimals: division algorithm, reasoning about remainders

Ratios and Proportional Relationships (grades 6-7)

1. Ratios and proportional relationships extend multiplication, division, and measurement
2. Ratio contexts: slopes of lines, similar figures, recipes, measurement conversions
3. Some authors needlessly distinguish ratios (same units) from rates (different units)
4. Fractions are numbers; ratios associate two **or more** quantities
5. Equivalent ratios vs. equivalent fractions (Figure 1)

6. Comparing ratios: part:part and part:whole comparisons (for same type units)
7. Proportional relationships are collections of equivalent ratios
8. Ratios of two quantities have associated rates (fractions).
    a. Unit rates: “For each,” “for every,” “for every 1,” “per”
    b. Move from “3/2 miles in every 1 hour” to 1.5 mi/hr (derived quantities)
9. Percent as ratio “for every 100” or “per 100”
10. Ratio as composed unit or “batch” (sizes and units given), to be repeated or subdivided
    a. Additive reasoning (iteration, skip counting, mostly whole numbers)
11. Ratio as fixed numbers of parts (which can be any size)
    a. Multiplicative reasoning and non-integer scaling
12. Problem-solving strategies (Figures 2-3):
    a. Representations: pictures, ratio tables (connect to multiplication table), tape diagrams (same units), double number lines (different units), graphs ($y = kx$)
    b. Going through 1, unit rates
    c. Write an equation and solve
    d. Note: Proportions and cross-multiplication are never needed!
13. Compare non-proportional contexts
    a. Graphs are not lines through the origin
14. Grade 7: Same representations and strategies; more challenging numbers.
15. Unit rate is the slope of the graph of $y = kx$.
16. Unit rates are obscured by setting up proportions and cross-multiplying.
17. Percent increase and decrease
    a. Goal: Multiply by $1 + r$ or $1 - r$
    b. For subsequent percent-change events, the whole is different
18. Connect to geometry: scale drawings, similarity, scaling
    a. Watch what happens to area and volume
19. Connect to probability and statistics
    a. Descriptive statistics, inferences about a population

Linear functions (grade 8+)

1. Direct proportions vs. linear functions that are not
2. The rule of 4: Explore relationships algebraically (with symbols), graphically, numerically (in tables), and through verbal descriptions in context
3. Slope-intercept form: starting point + (unit rate)*change
4. Can we connect the dots? If not, why not? If so, how?

Themes

1. Number sense with fractions and decimals
2. Using a context for meaning
3. Quantities: numbers with units, in context
4. Drawing pictures (often) to develop meaning and to support sense-making
5. Simplest form depends on the context
    a. Unsimplified expressions that yield the answer illustrate (algebraic) reasoning
    b. Different expressions can show different reasoning about the same situation

Figures

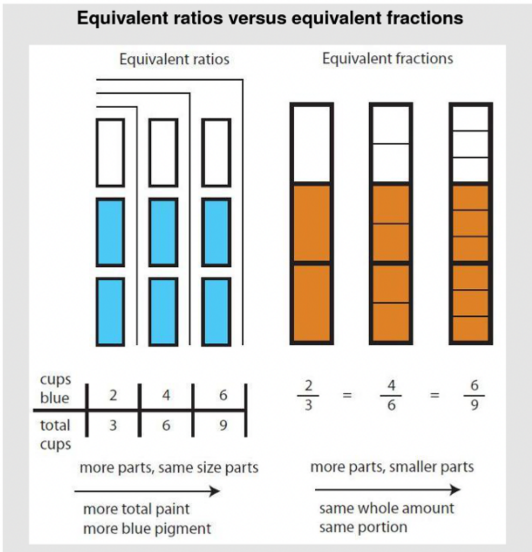
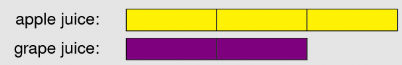


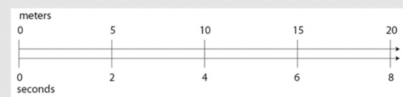
Figure 1

Representing ratios with tape diagrams



This diagram can be interpreted as representing any mixture of apple juice and grape juice with a ratio of 3 to 2. The total amount of juice is represented as partitioned into 5 parts of equal size, represented by 5 rectangles. For example, if the diagram represents 5 cups of juice mixture, then each of these rectangles represents 1 cup. If the total amount of juice mixture is 1 gallon, then each part represents $\frac{1}{5}$ gallon and there are $\frac{3}{5}$ gallon of apple juice and $\frac{2}{5}$ gallon of grape juice.

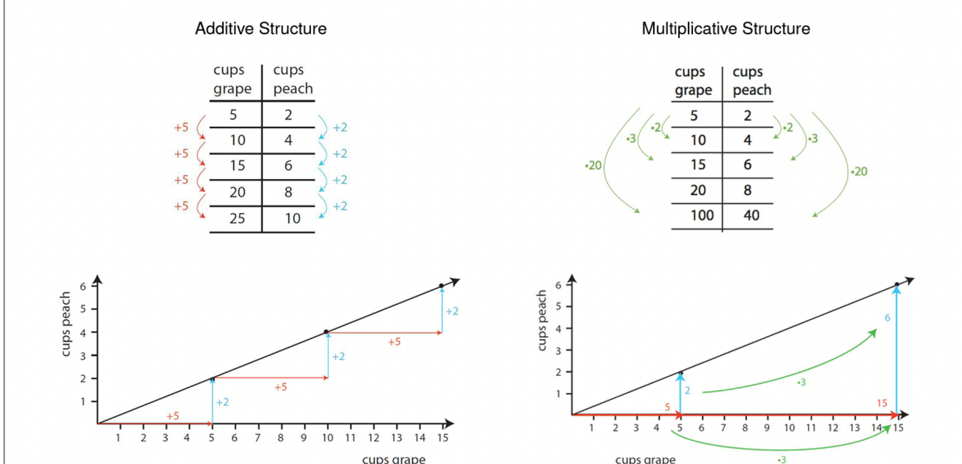
Representing ratios with double number line diagrams



On double number line diagrams, if A and B are in the same ratio, then A and B are located at the same distance from 0 on their respective lines. Multiplying A and B by a positive number p results in a pair of numbers whose distance from 0 is p times as far. So, for example, 3 times the pair 2 and 5 results in the pair 6 and 15 which is located at 3 times the distance from 0.

Figure 2

Showing structure in tables and graphs



In the tables, equivalent ratios are generated by repeated addition (left) and by scalar multiplication (right). Students might be asked to identify and explain correspondences between each table and the graph beneath it (MP1).

Figure 3

Outline of Activities

- Equivalent fractions (grade 4)
    - Use the meaning of fraction (Ready Math misleads)
- Multiply fraction by whole number and vice versa (grade 4)
    - Use multiplication as repeated addition (multiplier versus multiplicand)
- Fraction multiplication (grades 4 and 5)
    - Use area model to explain algorithm
    - Use area model for multiplying mixed numbers
- Fraction division (Grades 5 and 6)
    - Meanings: How many groups? How many in one group?
    - Unit fraction divided by whole number and vice versa
    - Use both meanings to explain invert and multiply
- Note: Division with remainder, division in context, division as fraction,
- Note: Division involving zero
- Decimals as fractions: Hundredths Grids (grades 5-8)
- Note: Decimal operations
- Ratios and proportional relationships (grades 6-7)
    - Hen-and-a-half
    - Mixing punch: tape diagrams, ratio tables;
        - part:part vs. part:whole; Batch reasoning vs. scaling
        - Fractions, decimals, and percents
    - Racing snails: double number lines, graphs
    - Stacking paper: unit rates, graphs, equations ($y = kx$)
    - Unit conversions
    - Non-proportional contexts
    - Ratio addition? Contexts in which equivalent ratios are misleading
- Note: Percent increase/decrease (grade 7+)
    - Ratio representations for $1 + r$ and $1 - r$
- Linear functions (grade 8+)
    - Stacking paper cups. Domain, graph not through origin

Related and Future Content

1. Arithmetic of signed numbers
2. Irrational numbers
3. Quadratic functions
4. Exponential functions
5. Function domain, range; function notation
6. Distinguish function from formula: some functions not given by a formula