Number and Algebra Activities for Middle Grades Teachers

**Bart Snapp, Brad Findell,
Vic Ferdinand, Jenny Sheldon**

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Contents

Part I

Arithmetic and Algebra

1 Home Base

Lothar is recording the number of goats he owns. His recording looks like this.

Lothar's wife, Gertrude, uses the following symbols to record things.

0 1 2 3 4

Gertrude writes the number of Lothar's goats as

234

In case it's helpful information, Gertrude would also write 5 goats as

10

Now answer the following questions about Gertrude's system.

Problem 1 *If Lothar lost a goat, how would Gertrude write this new number of goats?*

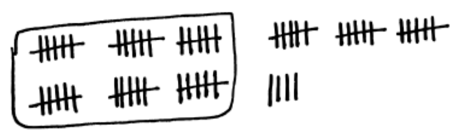
Problem 2 *If Lothar bought another goat (from the original number), how would Gertrude write this new number of goats?*

Problem 3 *If we would write the number of goats as 37, how would Gertrude write the number of goats?*

Learning outcomes:

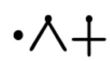
Problem 4 *Explain the rules for Gertrude's counting system. Write out a few (large) numbers in order as an example to show that you understand.*

The aliens living on Omnicron Persei 8 keep track of how many humans they have eaten in the last week. One such alien, Lrrr, tallies the number of humans he has eaten with this picture.



His wife, Ndnd, uses the following symbols to write quantities.

She writes the number of humans that Lrrr ate as follows.



Problem 5 *If Lrrr ate another human, how would Ndnd write this new number of humans?*

Problem 6 *If one of the humans actually escaped from the original number (and therefore was not eaten by Lrrr), how would Ndnd write this new number of humans?*

Problem 7 *If we would write the number of humans as 101, how would Ndnd write this number?*

Problem 8 *Explain the rules for Ndnd's counting system. Write out a few (large) numbers in order as an example to show that you understand.*

2 Shelby and Scotty

We find algorithms for changing bases.

Note: In this activity, we use words (rather than numerals) to indicate bases. And we use use a subscript after a numeral to specify its base.

Shelby and Scotty want to express the (base ten) number 27 in base four. However, they used very different methods to do this. Let's check them out.

Problem 9 *Consider Shelby's work:*

$$\begin{array}{r} 6\ R\,\mathbf{3} \\ 4\,\overline{)27} \end{array} \qquad \begin{array}{r} 1\ R\,\mathbf{2} \\ 4\,\overline{)6} \end{array} \qquad \begin{array}{r} 0\ R\,\mathbf{1} \\ 4\,\overline{)1} \end{array} \qquad \Rightarrow \qquad \boxed{123_{four}}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

Problem 10 *Using the 27 marks below, create an illustration (or series of illustrations) that models Shelby's method for changing bases.*

$$\mid\ \mid$$

Further, explain why Shelby's method works.

Problem 11 *Consider Scotty's work:*

$$\begin{array}{r} 0\ R\,27 \\ 4^3\,\overline{)27} \end{array} \qquad \begin{array}{r} \mathbf{1}\ R\,11 \\ 4^2\,\overline{)27} \end{array} \qquad \begin{array}{r} \mathbf{2}\ R\,\mathbf{3} \\ 4\,\overline{)11} \end{array} \qquad \Rightarrow \qquad \boxed{123_{four}}$$

(a) *Describe how to perform this algorithm.*

(b) *Provide an additional relevant and revealing example demonstrating that you understand the algorithm.*

Learning outcomes: Understand base conversion.

Problem 12 *Using the 27 marks below, create an illustration (or series of illustrations) that models Scotty's method for changing bases.*

$$| \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; | \; |$$

Further, explain why Scotty's method works.

Problem 13 *Use both methods to write* 1644_{ten} *in base seven.*

Problem 14 *Now let's try to be more efficient.*

(a) *Convert* 8630_{ten} *to base thirteen. Use* A *for ten,* B *for eleven, and* C *for twelve.*

(b) *Quickly convert* 2102_{three} *to base nine.*

(c) *Without using base ten, convert* 341_{six} *to base four.*

(d) *Without using base ten, convert* 341_{six} *to base eleven.*

3 Hieroglyphical Arithmetic

We use strange operation tables to solve arithmetic problems.

Note: This activity is based on an activity originally designed by Lee Wayand.

Problem 15 *Suppose the symbol* $\bigstar$ *is used for some operation on mathematical objects* a *and* b.

(a) *What does it mean to say that the operation* $\bigstar$ *is* ***commutative****?*

(b) *Name some familiar operations that are commutative.*

(c) *Name some familiar operations that are* ***not*** *commutative.*

Problem 16 *Suppose the symbol* $\bigstar$ *is used for some operation on mathematical objects* a *and* b.

(a) *What does it mean to say that the operation* $\bigstar$ *is* ***associative****?*

(b) *Name some familiar operations that are associative.*

(c) *Name some familiar operations that are* ***not*** *associative.*

Problem 17 *An operation* $\bigstar$ *is called* ***closed*** *on a set of numbers if for all numbers* a *and* b *in the set:*

$$a \bigstar b \qquad \text{is another number in the set.}$$

(a) *Name an operation and a set that is closed under that operation.*

(b) *Name a set that is* ***not*** *closed under that same operation.*

(c) *Name another operation and a set that is closed under that operation.*

(d) *Name a set that is* ***not*** *closed under that same operation.*

Problem 18 *Suppose some number system has two operations:* $\bigstar$ *and* ✢.

What does it mean for the operation $\bigstar$ *to be* ***distributive*** *over the operation* ✢*?*

Learning outcomes: Learning outcome goes here.

A New Number System

Consider the following addition and multiplication tables:

+	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
fish	fork	balloon	fish	eyeball	cinder-block	skull	lolly-pop	man	DNA
lolly-pop	balloon	eyeball	lolly-pop	fork	fish	man	cinder-block	DNA	skull
skull	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
cinder-block	eyeball	fork	cinder-block	balloon	lolly-pop	DNA	fish	skull	man
DNA	cinder-block	fish	DNA	lolly-pop	man	eyeball	skull	fork	balloon
fork	skull	man	fork	DNA	eyeball	fish	balloon	lolly-pop	cinder-block
man	lolly-pop	cinder-block	man	fish	skull	balloon	DNA	eyeball	fork
balloon	man	DNA	balloon	skull	fork	lolly-pop	eyeball	cinder-block	fish
eyeball	DNA	skull	eyeball	man	balloon	cinder-block	fork	fish	lolly-pop

fish = fish

lolly-pop = lolly-pop

skull = skull

cinder-block = cinder-block

DNA = DNA

fork = fork

man = man

balloon = balloon

eyeball = eyeball

·	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish
fork	fish	lolly-pop	cinder-block	DNA	skull	eyeball	balloon	man	fork
eyeball	lolly-pop	balloon	man	fork	skull	cinder-block	DNA	fish	eyeball
balloon	cinder-block	man	fork	lolly-pop	skull	DNA	fish	eyeball	balloon
man	DNA	fork	lolly-pop	cinder-block	skull	fish	eyeball	balloon	man
skull	skull	skull	skull	skull	skull	skull	skull	skull	skull
lolly-pop	eyeball	cinder-block	DNA	fish	skull	balloon	man	fork	lolly-pop
cinder-block	balloon	DNA	fish	eyeball	skull	man	fork	lolly-pop	cinder-block
DNA	man	fish	eyeball	balloon	skull	fork	lolly-pop	cinder-block	DNA
fish	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish

fish = fish

lolly-pop = lolly-pop

skull = skull

cinder-block = cinder-block

DNA = DNA

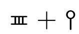
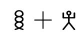
fork = fork

man = man

balloon = balloon

eyeball = eyeball

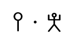
Problem 19 *Use the addition table to compute the following:*

 *and* 

Problem 20 *Do you notice any patterns in the addition table? Tell us about them.*

Problem 21 *Can you tell me which glyph represents* 0*? How did you arrive at this conclusion?*

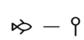
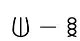
Problem 22 *Use the multiplication table to compute the following:*

 and 

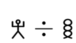
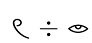
Problem 23 *Do you notice any patterns in the multiplication table? Tell us about them.*

Problem 24 *Can you tell me which glyph represents* 1*? How did you arrive at this conclusion?*

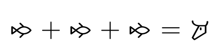
Problem 25 *Compute:*

 *and* 

Problem 26 *Compute:*

 *and* 

Problem 27 *Keen Kelley was working with our tables above. All of a sudden, she writes*



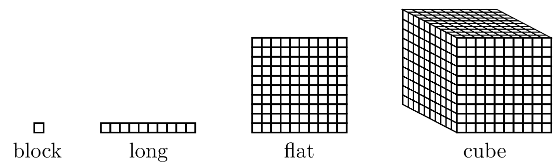
and shouts "Weird!" Why is she so surprised? Try repeated addition with other glyphs. What do you find? Can you explain this?

Problem 28 *Can you find any other oddities of the arithmetic above? Hint: Try repeated multiplication!*

4 Playing with Blocks

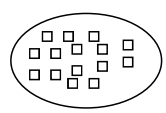
We study addition algorithms.

I always enjoyed blocks quite a bit. Go find yourself some *base-ten blocks*. Just so that we are all on the same page, here are the basic blocks:



Problem 29 *Sketch a model of the number* 247 *with base-ten blocks.*

Problem 30 *Oscar modeled the number* 15 *in the following way:*



What do you think of his model? Can you improve upon it?

Problem 31 *Many problems involving subtraction can be considered one of the following types: take-away, comparison, and missing addend. Write a "word problem" illustrating each of these types.*

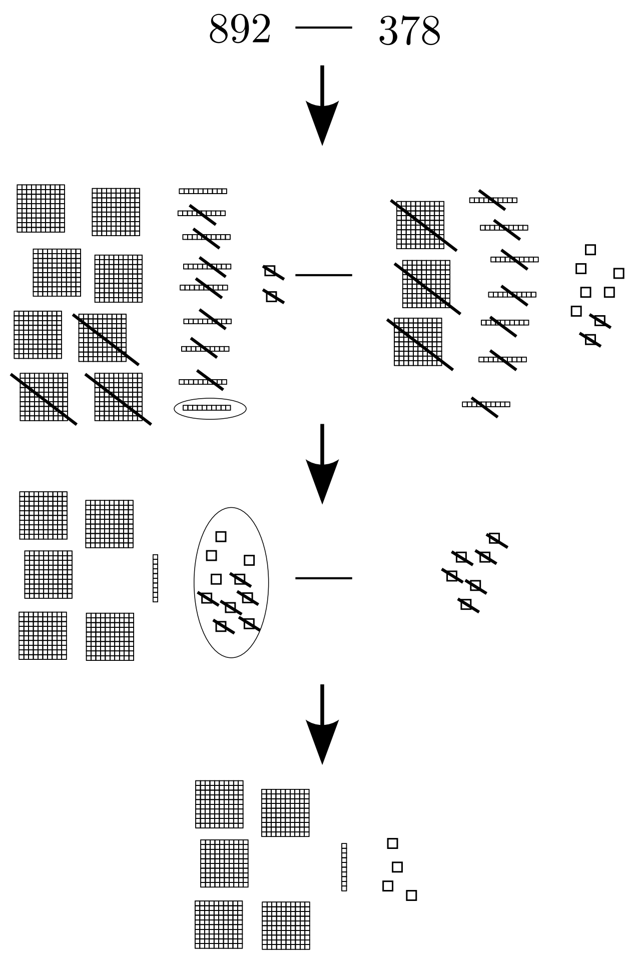
Learning outcomes: Learning outcome goes here.

Problem 32 *Here is a standard subtraction algorithm:*

$$\begin{array}{rrr} & 8 & \\ 8 & \cancel{9} & {}^{1}2 \\ -3 & 7 & 8 \\ \hline 5 & 1 & 4 \end{array}$$

Use base-ten blocks to model this algorithm. Which type of subtraction are you using?

Problem 33 *Oscar uses base-ten blocks to model subtraction.*



Can you explain what is going on? Which type of subtraction is Oscar using?

Problem 34 *Create a “new” subtraction algorithm based on Oscar’s model.*

Problem 35 *Here is an example of a standard addition algorithm:*

$$\begin{array}{r} 11 \\ 892 \\ +398 \\ \hline 1290 \end{array}$$

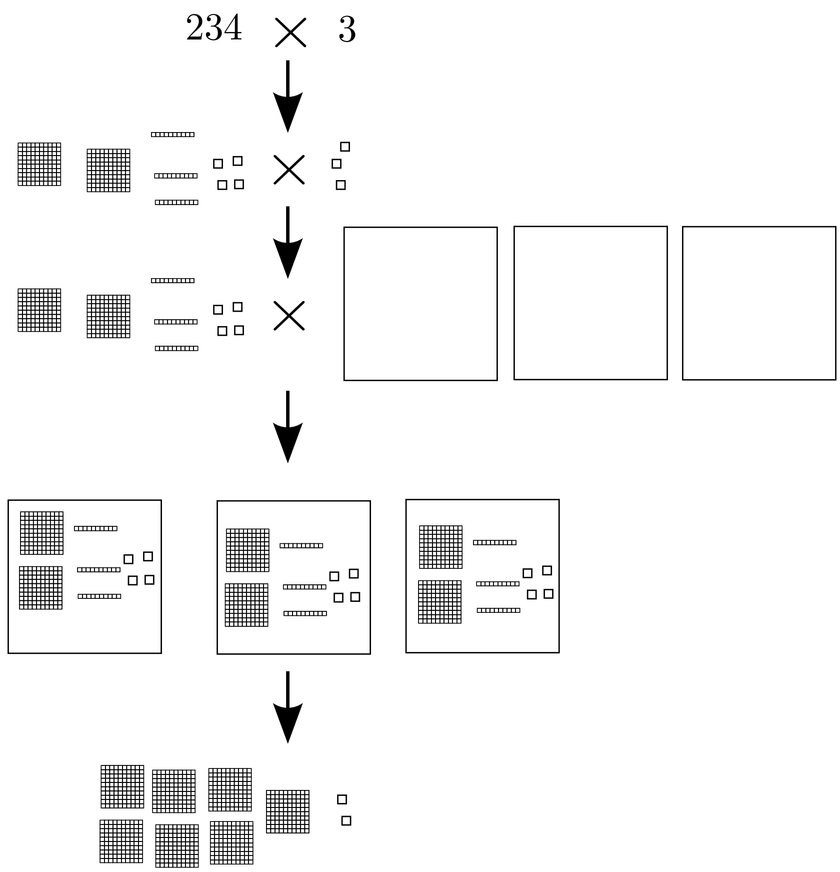
Model this algorithm with base-ten blocks.

5 More Playing with Blocks

We explore multiplication algorithms.

Problem 36 *Now Oscar is modeling the basic multiplication algorithm:*

$$\begin{array}{r} {}^{1}{}^{1} \\ 234 \\ \times\ \ 3 \\ \hline 702 \end{array}$$



Can you explain what is going on? Does his model illustrate the algorithm? If so, explain how. If not, describe how to use base ten blocks to explain the algorithm.

Learning outcomes: Learning outcome goes here.

Problem 37 *Here is an example of the basic division algorithm:*

$$\begin{array}{r@{\,}l} & 67 \text{ R1} \\ 3\,\big) & \overline{202} \\ & \underline{18} \\ & 22 \\ & \underline{21} \\ & 1 \end{array}$$

Explain how to model this algorithm with base-ten blocks, assuming that you start with 202 as two flats and two blocks and that you intend to organize them into three equal piles.

6 Comparative Arithmetic

We compare/contrast arithmetic of polynomials to arithmetic to integers.

Problem 38 *Compute:*

$$\begin{array}{r} 131 \\ \underline{+122} \end{array} \quad \textit{and} \quad \begin{array}{r} x^2+3x+1 \\ \underline{+x^2+2x+2} \end{array}$$

Compare, contrast, and describe your experiences.

Problem 39 *Compute:*

$$\begin{array}{r} 139 \\ \underline{+122} \end{array} \quad \textit{and} \quad \begin{array}{r} x^2+3x+9 \\ \underline{+x^2+2x+2} \end{array}$$

Compare, contrast, and describe your experiences. In particular, discuss how this is different from the first problem.

Problem 40 *Compute:*

$$\begin{array}{r} 121 \\ \underline{\times 32} \end{array} \quad \textit{and} \quad \begin{array}{r} x^2+2x+1 \\ \underline{\times \quad 3x+2} \end{array}$$

Compare, contrast, and describe your experiences.

Learning outcomes: Learning outcome goes here.

Problem 41 *Expand:*

$$(x^2 + 2x + 1)(3x + 2)$$

Compare, contrast, and describe your experiences. In particular, discuss how this problem relates to the one above.

Problem 42 *Compute:*

$$\begin{array}{r} 214 \\ \times 53 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{r} 2x^2 + x + 4 \\ \times \quad 5x + 3 \\ \hline \end{array}$$

Compare, contrast, and describe your experiences.

Problem 43 *Use long division to compute* $2785 \div 23$ *and* $(2x^3 + 7x^2 + 8x + 5) \div (2x + 3)$. *Compare, contrast, and describe your experiences.*

Problem 44 *Use long division to compute* $6529 \div 34$ *and* $(6x^3 + 5x^2 + 2x + 9) \div (3x + 4)$. *Compare, contrast, and describe your experiences.*

Part II

Numbers

7 Integer Addition and Subtraction

We explore various models and strategies for making sense of addition and subtraction of integers.

Useful language

Addition and subtraction problems arise in situations where we add to, take from, put together, take apart, or compare quantities.

Recall that addition and subtraction facts are related. For example, if we know that $8+5=13$, then we also know three related facts: $5+8=13$, $13-8=5$, and $13-5=8$. In school mathematics, these are often called *fact families.*

Problem 45 *What are integers? Describe some situations in which both positive and negative integers arise. Use the word "opposite" in your descriptions.*

Red and black chips

Problem 46 *In a red-and-black-chip model of the integers, red and black chips each count for 1, but they are opposites, so that they cancel each other out. Using language from accounting, suppose black chips are assets and red chips are debts. We add by putting chips together. Use red and black chips (or draw the letters R and B) to model the following computations.*

(a) $(-5)+(-3)$

(b) $6+(-4)$.

(c) $(-7)+9$

(d) $2+(-5)$

Problem 47 *In the previous problem, you saw different combinations of red and black chips that had the same numerical value.*

(a) *How many ways are there to represent -3? Draw two different representations.*

Learning outcomes: Learning outcome goes here.

(b) *Use the phrase "zero pairs" to describe how your two representations are related.*

Problem 48 *To subtract in the red-and-black-chip model, we can "take away" chips, as you might expect. When we don't have enough chips of a particular color, we can always add "zero pairs." Use this idea to model the following subtraction problems:*

(a) $6 - 8$

(b) $4 - (-3)$

(c) $(-6) - 5$

(d) $(-3) - (-7)$

Subtraction as missing addend

Problem 49 *To evaluate a subtraction expression, we can solve a related addition equation. For example,* $11 - 7$ *is the solution to* $7 + \_\_ = 11$*. Use this idea to evaluate the subtraction expressions in the previous problem.*

Subtraction as difference on the number line

Problem 50 *Use a number line to reason about* $b - a$ *by asking how to get from* a *to* b*: How far? And in which direction? For example, to evaluate* $11 - 7$*, we can ask how to get from* 7 *to* 11*. We travel* 4 *units to the right. Use this idea to evaluate the subtraction expressions in the previous problems.*

Problem 51 *How is subtraction different from negation?*

Problem 52 *Use what you have learned to explain why* $a - (-b) = a + b$*.*

Other Models

Use the following models for addition and subtraction of integers. Each model requires two decisions: (1) how positive and negative integers are 'opposite' in the situation, and (2) how addition and subtraction are 'opposite' in a different way.

- A postal carrier who brings checks and bills—and who also takes them away.
- Walking on an North-South number line, facing either North or South, and walking either forward or backward.

8 Integer Multiplication

We explore various models and strategies for making sense of multiplication of integers.

Continuing patterns

Problem 53 (a) *Continue the following patterns, and explain why it makes sense to continue them in that way.*

$$\begin{array}{rrr}
4\times 3 = 12 & 3\times 6 = 18 & (-7)\times 3 = -21\\
4\times 2 = & 2\times 6 = & (-7)\times 2 =\\
4\times 1 = & 1\times 6 = & (-7)\times 1 =\\
4\times 0 = & 0\times 6 = & (-7)\times 0 =\\
4\times(-1) = & (-1)\times 6 = & (-7)\times(-1) =\\
4\times(-2) = & (-2)\times 6 = & (-7)\times(-2) =\\
4\times(-3) = & (-3)\times 6 = & (-7)\times(-3) =
\end{array}$$

(b) *What rule of multiplication might a student infer from the first pattern?*

(c) *What rule of multiplication might a student infer from the second pattern?*

(d) *What rule of multiplication might a student infer from the third pattern?*

Using properties of operations

Problem 54 *Suppose* we do not know *how to multiply negative numbers but we do know that $4\times 6 = 24$. We will use this fact and the properties of operations to reason about products involving negative numbers.*

(a) *What do we know about A and B if $A + B = 0$?*

(b) *Use the distributive property to show that the expression $4\times 6+4\times(-6)$ is equal to 0. Then use that fact to reason about what $4\times(-6)$ should be.*

Learning outcomes: Learning outcome goes here.

(c) *Use the distributive property to show that the expression* $4\times(-6)+(-4)\times(-6)$ *is equal to* 0*. Then use that fact to reason about what* $(-4)\times(-6)$ *should be.*

Walking on a number line

Problem 55 *Matt is a member of the Ohio State University Marching Band. Being rather capable, Matt can take* x *steps of size* y *inches for all integer values of* x *and* y*. If* x *is positive it means face North and take* x *steps. If* x *is negative it means face South and take* $|x|$ *steps. If* y *is positive it means your step is a forward step of* y *inches. If* y *is negative it means your step is a backward step of* $|y|$ *inches.*

(a) *Discuss what the expressions* $x \cdot y$ *means in this context. In particular, what happens if* $x = 1$*? What if* $y = 1$*?*

(b) *If* x *and* y *are both positive, how does this fit with the "repeated addition" model of multiplication?*

(c) *Using the context above and specific numbers, demonstrate the general rule:*

$$\text{negative} \cdot \text{positive} = \text{negative}$$

Clearly explain how your problem shows this.

(d) *Using the context above and specific numbers, demonstrate the general rule:*

$$\text{positive} \cdot \text{negative} = \text{negative}$$

Clearly explain how your problem shows this.

(e) *Using the context above and specific numbers, demonstrate the general rule:*

$$\text{negative} \cdot \text{negative} = \text{positive}$$

Clearly explain how your problem shows this.

9 Differentiating Division

Let's consider the division problem $26 \div 4$.

Question 56 *What's the usual answer to this problem? Did your group members have a different idea for what the "usual answer" should be? Why or why not?*

Problem 57 *Give an example of a story or situation in which the answer to* $26 \div 4$ *is "6 remainder 2". Explain how you know this is the appropriate answer in this situation.*

Problem 58 *Give an example of a story or situation in which the answer to* $26 \div 4$ *is 7. Explain how you know this is the appropriate answer in this situation.*

Problem 59 *Give an example of a story or situation in which the answer to* $26 \div 4$ *is 6. Explain how you know this is the appropriate answer in this situation.*

The Division Theorem states: Given any whole positive number n and a nonzero positive whole number d, there exist unique integers q and r such that

$$n = dq + r$$

with $0 \leq r < d$.

Question 60 *What is this theorem saying? What does it have to do with the questions on this page?*

Problem 61 *With your group, try to bring this all together. What have you learned about division? What is the main point of this activity?*

Learning outcomes:

10 What Can Division Mean?

We explore meaning of division.

Solve each of the problems below, explain your reasoning, and indicate whether the problem is asking "**How many in one group?**" or "**How many groups?**" or something else entirely.

Problem 62 *There are a total of* 35 *hard candies. If there are* 5 *boxes with an equal number of candies in each box—and all the candy is accounted for, then how many candies are in each box? What if you had 39 candies?*

Problem 63 *There are a total of* 28 *hard candies. If there are* 4 *candies in each box, how many boxes are there? What if you had 34 candies?*

Problem 64 *There is a total of 29 gallons of milk to be put in 6 containers. If each container holds the same amount of milk and all the milk is accounted for, how much milk will each container hold?*

Problem 65 *There is a total of 29 gallons of milk to be sold in containers holding 6 gallons each. If all the milk is used, how many containers can be sold?*

Problem 66 *There is a total of 29 gallons of milk to be sold in containers holding 6 gallons each. If all the milk is used, how much milk cannot be sold?*

Problem 67 *If there are 29 kids and each van holds 6 kids, how many vans do we need for the field trip?*

Learning outcomes: Learning outcome goes here.

11 Commonality

Problem 68 *You have received a large piece of paper which measures 40 inches by 60 inches. You'd like to draw a large grid (of rectangles, not necessarily squares) on this paper. What size rectangles could you use for your grid? Did you get the same answer as each of your neighbors?*

Problem 69 *At a local carnival, you sneak in behind one of the slot machines and rig it to come up with three cherries (one in each window to hit the jackpot) at a time only known to you. You know the first window will come up with a cherry every 15 times, the second window every 24 times, and the third window every 40 times. When should you step in to win the jackpot? Did you get the same answer as each of your neighbors?*

Learning outcomes:

Problem 70 *Let* $A = 2^{23} \times 3^{87} \times 5^{11} \times 7^{14}$ *and* $B = 2^{18} \times 3^{54} \times 5^{26} \times 7^{16}$.

(a) *Find some common factors of* A *and* B.

(b) *Find some common multiples of* A *and* B.

Problem 71 *The* **Division Theorem** *states that given any (positive) integer* n *and a nonzero (positive) integer* d*, there exist unique integers* q *and* r *such that* $n = dq + r$ *and* .

(a) *What is the missing condition in the theorem? In other words, what could we specify to get a unique answer? How could we get multiple answers?*

(b) *How might you draw a picture to illustrate this theorem?*

(c) *Would any of your answers above change if* n *and* d *were negative integers? Would the theorem still be true?*

12 Divisibility Statements

We explore division.

Let $a|b$ mean $b = aq$ for some integer q. (Read $a|b$ as "a divides b".)

Problem 72 *Using the numbers 56 and 7, make some true statements using the notation above and one or more of the words factor, multiple, divisor, and divides.*

Problem 73 *Use the definition of* divides *to decide which of the following are true and which are false. If a statement is true, find q satisfying the definition of divides. If it is false, give an explanation. (Hint: Try to reason about multiplication without actually multiplying.)*

(a) $21|2121$

(b) $3|(9 \times 41)$

(c) $6|(2^4 \times 3^2 \times 7^3 \times 13^5)$

(d) $100000|(2^3 \times 3^9 \times 5^{11} \times 17^8)$

(e) $6000|(2^{21} \times 3^7 \times 5^{17} \times 29^5)$

(f) $p^3q^5r|(p^5q^{13}r^7s^2t^{27})$

(g) $7|(5 \times 21 + 14)$

Problem 74 *If $a|b$ does $a|(bc)$? Explain.*

Problem 75 *If $a|(bc)$ does $a|b$? Explain.*

Problem 76 *If $a|b$ and $a|c$ does $a|(b+c)$? Explain.*

Learning outcomes: Learning outcome goes here.

Problem 77 *If $a|(b+c)$ does $a|b$ and $a|c$? Explain.*

Problem 78 *If $a|(b+c)$ and $a|c$ does $a|b$? Explain.*

Problem 79 *Suppose that*

$$(3^5 \cdot 7^9 \cdot 11^x \cdot 13^y)|(3^a \cdot 7^b \cdot 11^{19} \cdot 13^7)$$

What values of a, b, x, and y make true statements?

13 Hall of Shoes

We give a physical model for the sieve of Eratosthenes.

Problem 80 *Incognito's Hall of Shoes is a shoe store that just opened in Myrtle Beach, South Carolina. At the moment, they have 100 pairs of shoes in stock. At their grand opening 100 customers showed up. The first customer tried on every pair of shoes, the second customer tried on every 2nd pair, the third customer tried on every 3rd pair, and so on until the 100th customer, who only tried on the last pair of shoes.*

(a) *Which shoes were tried on by only 1 customer?*

(b) *Which shoes were tried on by exactly 2 customers?*

(c) *Which shoes were tried on by exactly 3 customers?*

(d) *Which shoes were tried on by exactly 4 customers?*

(e) *How many customers tried on the 45th pair?*

(f) *How many customers tried on the 81st pair?*

(g) *Challenge: Which shoes were tried on by the most customers?*

In each case, explain your reasoning.

Problem 81 *Which pairs of shoes were tried on by both*

(a) *customers 3 and 5?*

(b) *customers 6 and 8?*

(c) *customers 12 and 30?*

(d) *customers 7 and 13?*

(e) *customers a and b?*

Problem 82 *Which customers tried on both*

Learning outcomes: Learning outcome goes here.

(a) *pairs 24 and 36?*

(b) *pairs 30 and 60?*

(c) *pairs 42 and 12?*

(d) *pairs 28 and 15?*

(e) *pairs a and b?*

14 Sieving It All Out

We do the sieve of Eratosthenes.

Problem 83 *Try to find all the primes from* 1 *to* 120 *without doing any division. Try to circle numbers that are prime and cross out numbers that are not prime. As a gesture of friendship, here are the numbers from* 1 *to* 120.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Describe your method.

Learning outcomes: Learning outcome goes here.

Problem 84 *Now let's be systematic. Ignore 1 (we'll talk about why later). As you identify a prime, first circle it, then cross out its multiples that are not already crossed out. Keep track of your work so that you can answer the following questions:*

(a) *After circling a new prime, note the first number crossed out with that prime. Record your results in a table.*

prime	*first # crossed out*
2	

(b) *What was the biggest prime for which you crossed out at least one multiple?*

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

15 There's Always Another Prime

We think about the number of prime numbers.

We'll start off with easy questions, then move to harder ones.

Problem 85 *Use the Division Theorem to explain why neither* 2 *nor* 3 *divides* $2 \cdot 3 + 1$. *(Hint: Do not multiply and add. Use the expression as written to reason what the quotient and remainder must be.)*

Problem 86 *Use the Division Theorem to explain why neither* 2 *nor* 3 *nor* 5 *divides* $2 \cdot 3 \cdot 5 + 1$.

Problem 87 *Let* $p_1, \dots, p_n$ *be the first* n *primes. Do any of these primes divide*

$$p_1 p_2 \cdots p_n + 1?$$

Explain your reasoning.

Problem 88 *Suppose there were only a finite number of primes, say there were only* n *of them. Call them* $p_1, \dots, p_n$. *Could any of them divide*

$$p_1 p_2 \cdots p_n + 1?$$

what does that mean? Can there really only be a finite number of primes?

Problem 89 *Consider the following:*

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509$$

Does this contradict our work above? If so, explain why. If not, explain why not.

Learning outcomes: Learning outcome goes here.

16 There Are Many Factors to Consider

We count the number of factors of an integer.

Suppose we want to know how many factors 43,560 has, but we don't need to list them all. In this activity we develop a method for computing the number of factors of a number.

Problem 90 *Vic listed the following factors of 80: 1, 2, 4, 5, 8, 10, 20, 40, 80. What is missing? Describe your method of checking the list.*

Problem 91 *Factors of 60.*

(a) *List the factors of 60.*

(b) *Describe your method for ensuring that you have found all of the factors.*

(c) *Consider the prime factorization of 60 and the prime factorization of each of the factors. Describe what you notice.*

Problem 92 *Factors of* 3^7.

(a) *Is 2 a factor of* 3^7*? Say how you know.*

(b) *Is 7 a factor of* 3^7*? Say how you know.*

(c) *List all the factors of* 3^7.

Learning outcomes: Learning outcome goes here.

(d) *Generalize: If p is a prime number, how many factors does p^n have? Explain your reasoning.*

Problem 93 *List all the factors of $3^2 \cdot 7^3$. Use your example to reason about the number of factors of $p^n q^m$, where p and q are both prime numbers.*

Problem 94 *List all the factors of $3^2 \cdot 7^3 \cdot 11$. Use your example to reason about the number of factors of $p^n q^m r^s$, where p, q, and r are all prime numbers.*

Problem 95 *How many factors does 43,560 have? Use the example of 43,560 to describe and explain a method for computing the number of factors that a number has.*

17 Why Does It Work?

We explain why the Euclidean Algorithm works.

The Euclidean Algorithm is pretty neat. Let's see if we can figure out **why** it works. As a gesture of friendship, I'll compute $\gcd(351, 153)$:

$$\begin{aligned} 351 &= \mathbf{153} \cdot 2 + \mathbf{45} \\ \mathbf{153} &= \mathbf{45} \cdot 3 + \mathbf{18} \\ \mathbf{45} &= \mathbf{18} \cdot 2 + \boxed{\mathbf{9}} \\ 18 &= 9 \cdot 2 + 0 \qquad \boxed{\therefore \gcd(351, 153) = 9} \end{aligned}$$

Let's look at this line-by-line.

The First Line

Problem 96 *Since* $351 = 153 \cdot 2 + 45$, *explain why* $\gcd(153, 45)$ *divides* 351.

Problem 97 *Since* $351 = 153 \cdot 2 + 45$, *explain why* $\gcd(351, 153)$ *divides* 45.

Problem 98 *Since* $351 = 153 \cdot 2 + 45$, *explain why* $\gcd(351, 153) = \gcd(153, 45)$.

The Second Line

Problem 99 *Since* $153 = 45 \cdot 3 + 18$, *explain why* $\gcd(45, 18)$ *divides* 153.

Problem 100 *Since* $153 = 45 \cdot 3 + 18$, *explain why* $\gcd(153, 45)$ *divides* 18.

Problem 101 *Since* $153 = 45 \cdot 3 + 18$, *explain why* $\gcd(153, 45) = \gcd(45, 18)$.

Learning outcomes: Learning outcome goes here.

The Third Line

Problem 102 *Since* $45 = 18 \cdot 2 + 9$, *explain why* $\gcd(18, 9)$ *divides* 45.

Problem 103 *Since* $45 = 18 \cdot 2 + 9$, *explain why* $\gcd(45, 18)$ *divides* 9.

Problem 104 *Since* $45 = 18 \cdot 2 + 9$, *explain why* $\gcd(45, 18) = \gcd(18, 9)$.

The Final Line

Problem 105 *Why are we done? How do you know that the Euclidean Algorithm will* ***always*** *terminate?*

beginproblem What does the final line look like when the GCD is 1? endproblem

18 Prome Factorization

We imagine a system of numbers without unique factorization.

Let's consider a crazy set of numbers—all multiples of 3. Let's use the symbol $3\mathbb{Z}$ to denote the set consisting of all multiples of 3. As a gesture of friendship, I have written down the first 100 nonnegative integers in $3\mathbb{Z}$:

0	3	6	9	12	15	18	21	24	27
30	33	36	39	42	45	48	51	54	57
60	63	66	69	72	75	78	81	84	87
90	93	96	99	102	105	108	111	114	117
120	123	126	129	132	135	138	141	144	147
150	153	156	159	162	165	168	171	174	177
180	183	186	189	192	195	198	201	204	207
210	213	216	219	222	225	228	231	234	237
240	243	246	249	252	255	258	261	264	267
270	273	276	279	282	285	288	291	294	297

Problem 106 *Given any two integers in $3\mathbb{Z}$, will their sum be in $3\mathbb{Z}$? Explain your reasoning.*

Problem 107 *Given any two integers in $3\mathbb{Z}$, will their difference be in $3\mathbb{Z}$? Explain your reasoning.*

Problem 108 *Given any two integers in $3\mathbb{Z}$, will their product be in $3\mathbb{Z}$? Explain your reasoning.*

Learning outcomes: Learning outcome goes here.

Problem 109 *Given any two integers in $3\mathbb{Z}$, will their quotient be in $3\mathbb{Z}$? Explain your reasoning.*

Definition 1. *Call a positive integer **prome** in $3\mathbb{Z}$ if it cannot be expressed as the product of two integers both in $3\mathbb{Z}$.*

As an example, I tell you that 6 is prome number in $3\mathbb{Z}$. You may object because $6 = 2 \cdot 3$, but remember—2 is not in $3\mathbb{Z}$!

Problem 110 *List some of the prome numbers less than 297. Hint: What numbers in $3\mathbb{Z}$* can *be expressed as a product of two integers* both *in $3\mathbb{Z}$?*

Problem 111 *Can you give some sort of algebraic characterization of prome numbers in $3\mathbb{Z}$?*

Problem 112 *Can you find numbers that factor completely into prome numbers in two different ways? How many can you find?*

19 Fractional Thinking

For each of the problems below, solve using a picture. Discuss how our meaning of fractions helped you know what picture to draw, and helped you know from the picture what the answer should be.

Problem 113 *Sammi gave $\frac{1}{3}$ of her jelly beans to Ahmed. Ahmed then gave $\frac{4}{7}$ of the jelly beans he got from Sammi to Tyrone. What fraction of Sammi's jelly beans did Tyrone get?*

Problem 114 *Millie is collecting colorful sand to fill a container. If she has already collected $\frac{8}{3}$ of a pound of sand, and the jar is $\frac{3}{4}$ full, how much sand will she need to fill the jar?*

Problem 115 *Kevin has a recipe for chocolate chip cookies which calls for $\frac{9}{2}$ cups of flour. If Kevin has $\frac{7}{6}$ cups of flour, how much of the recipe can he make?*

Problem 116 *Cameron painted $\frac{3}{5}$ of his wall yesterday. Today, he painted an additional $\frac{1}{3}$ of his wall. How much of the wall has Cameron painted?*

Learning outcomes:

20 Picture Models for Equivalent Fractions

We develop pictures that model equivalent fractions.

Problem 117 *Get out a piece of paper and show* $\frac{3}{8}$*. Explain how you know.*

Problem 118 *Compare the following fractions. Explain how you know:*

(a) $\frac{3}{5}$ $\quad$ $\frac{4}{5}$

(b) $\frac{3}{7}$ $\quad$ $\frac{3}{8}$

(c) $\frac{3}{7}$ $\quad$ $\frac{5}{9}$

(d) $\frac{6}{7}$ $\quad$ $\frac{7}{8}$

(e) $\frac{12}{11}$ $\quad$ $\frac{13}{14}$

Problem 119 *Draw pictures to explain why:*

$$\frac{2}{3} = \frac{4}{6}$$

Explain how your pictures show this.

Problem 120 *Draw pictures to explain why:*

$$\frac{3}{6} = \frac{2}{4}$$

Explain how your pictures show this.

Learning outcomes: Learning outcome goes here.

Problem 121 *Given equivalent fractions with $0 < a \leqslant b$ and $0 < c \leqslant d$:*

$$\frac{a}{b} = \frac{c}{d}$$

Give a procedure for representing this equation with pictures.

Problem 122 *Explain, without cross-multiplication, why if $0 < a \leqslant b$ and $0 < c \leqslant d$:*

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc$$

Feel free to use pictures as part of your explanation.

21 Picture Models for Fraction Operations

We develop picture models for operations between fractions.

Problem 123 *Draw pictures that model:*

$$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

Explain how your pictures show this. Write a story problem whose solution is given by the expression above.

Problem 124 *Draw pictures that model:*

$$\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$

Explain how your pictures model this equation. Be sure to carefully explain how common denominators are represented in your pictures. Write a story problem whose solution is given by the expression above.

Problem 125 *Given $0 < a \leqslant b$ and $0 < c \leqslant d$, explain how to draw pictures that model the sum:*

$$\frac{a}{b} + \frac{c}{d}$$

Use pictures to find this sum and carefully explain how common denominators are represented in your pictures.

Learning outcomes: Learning outcome goes here.

22 Fraction Multiplication

We think about what multiplication of fractions means.

Problem 126 *Suppose x and y are counting numbers.*

(a) *What is our convention for the meaning of xy as repeated addition?*

(b) *In our convention for the meaning of the product xy, which letter describes* how many groups *and which letter describes* how many in one group*?*

(c) *In the product xy, the x is called the* multiplier *and y is called the* multiplicand. *Use these words to describe the meaning of xy as repeated addition.*

Problem 127 *In the Common Core State Standards, fractions and fraction operations are built from* unit fractions*, which are fractions with a* 1 *in the numerator. The meaning of a fraction $\frac{a}{b}$ involves three steps: (1) determining the whole; (2) describing the meaning of $\frac{1}{b}$; and (3) describe the meaning of the fraction $\frac{a}{b}$. Use pictures to illustrate these three steps for the fraction $\frac{3}{5}$.*

Problem 128 *Now we combine the ideas from the previous two problems to describe meanings for simple multiplication of fractions.*

Learning outcomes: Learning outcome goes here.

(a) *Without computing the result, describe the meaning of the product* $5 \times \frac{1}{3}$.

(b) *Without computing the result, describe the meaning of the product* $\frac{1}{3} \times 5$.

(c) *Without using the commutativity of multiplication (which we have not established for fractions), use these meanings and pictures to explain what the products should be.*

Area Models

Problem 129 *Beginning with a unit square, use an area model to illustrate the following:*

(a) $\frac{1}{3} \times \frac{1}{4}$

(b) $\frac{7}{3} \times \frac{5}{4}$

Problem 130 *When computing* $2\frac{1}{3} \times 3\frac{2}{5}$, *Byron says that the answer is* $6\frac{2}{15}$.

(a) *Explain Byron's method.*

(b) *How do you know that he is incorrect?*

(c) *Use what is right about his method to show what he is missing.*

23 Flour Power

We use models to motivate division.

Problem 131 *Suppose a cookie recipe calls for 2 cups of flour. If you have 6 cups of flour total, how many batches of cookies can you make?*

(a) *Draw a picture representing the situation, and use pictures to solve the problem.*

(b) *Identify whether the problem is asking "How many groups?" or "How many in one group?" or something else entirely.*

(c) *You find another recipe that calls for* $1\frac{1}{2}$ *cups per batch. If you have 6 cups of flour, how many batches of these cookies can you make? Again use pictures to solve the problem.*

(d) *Somebody once told you that "to divide fractions, you invert and multiply." Discuss how this rule is manifested in this problem.*

Problem 132 *You have 2 snazzy stainless steel containers (both the same size), which hold a total of 6 cups of flour. How many cups of flour does 1 container hold?*

(a) *Draw a picture representing the situation, and use pictures to solve the problem.*

(b) *Identify whether the problem is asking "How many groups?" or "How many in one group?" or something else entirely.*

(c) *It turned out that the 6 cups of flour filled exactly* $1\frac{1}{2}$ *of your containers. How many cups of flour does 1 container hold? Again use pictures to solve the problem.*

(d) *Somebody once told you that "to divide fractions, you invert and multiply." Discuss how this rule is manifested in this problem.*

Learning outcomes: Learning outcome goes here.

24 Picture Yourself Dividing

We think about division of fractions.

We want to understand how to visualize

$$\frac{a}{b} \div \frac{c}{d}$$

Let's see if we can ease into this.

Problem 133 *Draw a picture that shows how to compute:*

$$10 \div 5$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in one group."

Problem 134 *Try to use a similar process to the one you used in the first problem to draw a picture that shows how to compute:*

$$\frac{1}{4} \div 3$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in one group."

Problem 135 *Try to use a similar process to the one you used in the first two problems to draw a picture that shows how to compute:*

$$3 \div \frac{1}{4}$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in one group."

Learning outcomes: Learning outcome goes here.

Problem 136 *Try to use a similar process to the one you used in the first three problems to draw a picture that shows how to compute:*

$$\frac{7}{5} \div \frac{3}{4}$$

Explain how your picture could be redrawn for other similar numbers. Write two story problems solved by this expression, one asking for "how many groups" and the other asking for "how many in each group."

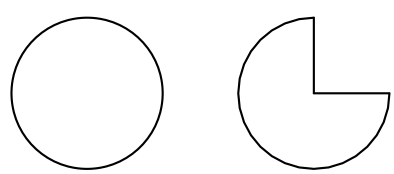
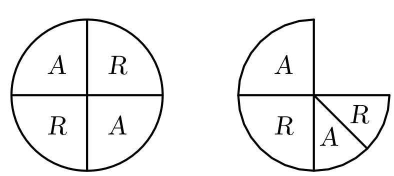
Problem 137 *Explain how to draw pictures to visualize:*

$$\frac{a}{b} \div \frac{c}{d}$$

Problem 138 *Use pictures to explain why:*

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Problem 139 *Alice and Rita wish to model the calculation* $1\frac{3}{4} \div \frac{1}{2}$. *They propose the following story problem:*

> *Alice and Rita have* $1\frac{3}{4}$ *pizzas, as shown below.*
>
> 
>
> *They share the pizza fairly, as follows:*
>
> 
>

So they each get $3\frac{1}{2}$ *pieces.*

(a) *Did they get the right answer for* $1\frac{3}{4} \div \frac{1}{2}$*?*

(b) *Does this context model* $1\frac{3}{4} \div \frac{1}{2}$ *as they claim? Explain?*

25 Cross Something-ing

We analyze different things people might call "cross-multiplication."

Problem 140 *What might someone call the following statements:*

(a) $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$

(b) $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}$

(c) $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

(d) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

(e) $ad < bc \Rightarrow \frac{a}{b} < \frac{c}{d}$

(f) $ad < bc \Rightarrow \frac{c}{d} < \frac{a}{b}$

Problem 141 *Which of the above statements are true? What specific name might you use to describe them?*

Problem 142 *Use pictures to help explain why the true statements above are true and give counterexamples showing that the false statements are false.*

Problem 143 *Can you think of other statements that should be grouped with those above?*

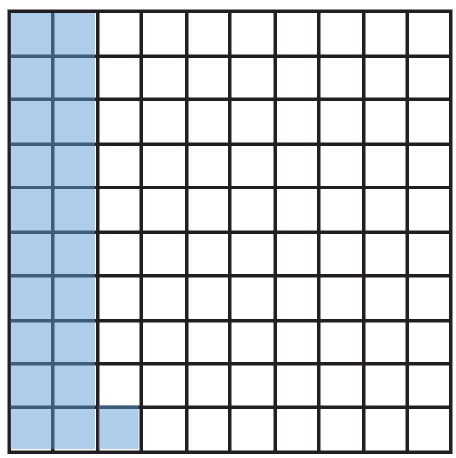
Problem 144 *If mathematics is a subject where you should strive to "say what you mean and mean what you say," what issue might arise with cross-multiplication?*

Learning outcomes: Learning outcome goes here.

26 Hundredths Grids for Rational Numbers

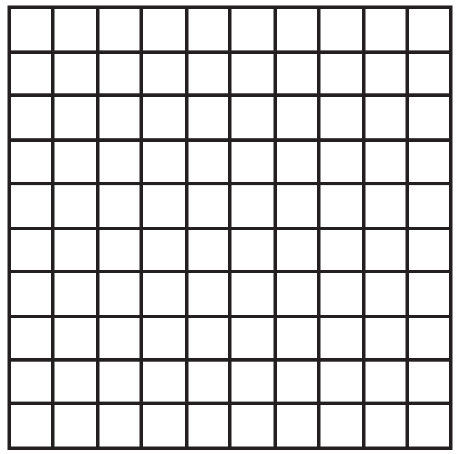
We convert fractions into decimals using hundredths grids.

When a 10×10 square is taken to be 1 whole, it can be used as a "hundredths grid" to represent fractions and decimals between 0 and 1. For example, one of the grids below is shaded to represent $\frac{21}{100}$.

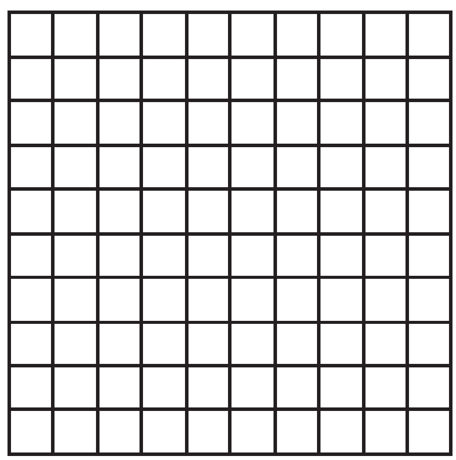


Problem 145 *Shade the hundredths grid below to show* $\frac{3}{20}$. *Use your shading to determine a decimal equivalent for each fraction.*

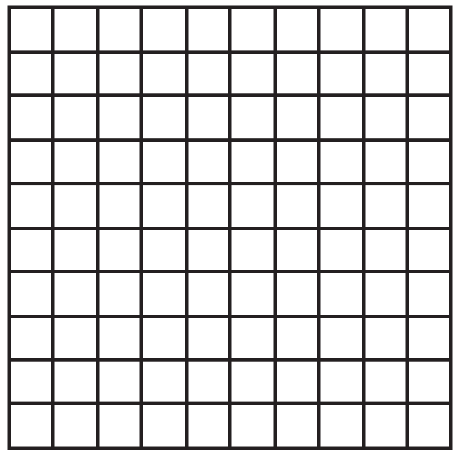
Learning outcomes: Learning outcome goes here.



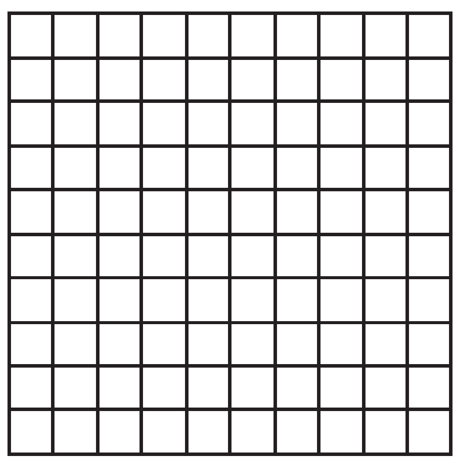
Problem 146 *Shade the hundredths grid below to show* $\frac{1}{8}$. *Use your shading to determine a decimal equivalent for each fraction.*



Problem 147 *Shade the hundredths grid below to show* $\frac{1}{6}$. *Use your shading to determine a decimal equivalent for each fraction.*



Problem 148 *Shade the hundredths grid below to show* $\frac{7}{12}$. *Use your shading to determine a decimal equivalent for each fraction.*



27 Shampoo, Rinse, ...

We think about decimal notation.

We're going to investigate the following question: If a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b?

As a middle school teacher, you should know from memory the decimal equivalents of many fractions, and you should be able to compute others quickly in your head. Use this activity to hone this skill, and use your calculator as backup support.

Problem 149 *Complete the following table. For type, write "T" for "Terminating," and use other letters for other types you observe.*

Learning outcomes: Learning outcome goes here.

Fraction	*Decimal*	*Type*
$\frac{1}{2}$		
$\frac{1}{3}$		
$\frac{1}{4}$		
$\frac{1}{5}$		
$\frac{1}{6}$		
$\frac{1}{7}$		
$\frac{1}{8}$		
$\frac{1}{9}$		
$\frac{1}{10}$		
$\frac{1}{11}$		
$\frac{1}{12}$		
$\frac{1}{13}$		
$\frac{1}{14}$		

Fraction	*Decimal*	*Type*
$\frac{1}{15}$		
$\frac{1}{16}$		
$\frac{1}{20}$		
$\frac{1}{24}$		
$\frac{1}{25}$		
$\frac{1}{28}$		
$\frac{1}{32}$		
$\frac{1}{35}$		
$\frac{1}{40}$		
$\frac{1}{42}$		
$\frac{1}{48}$		
$\frac{1}{64}$		
$\frac{1}{80}$		

Problem 150 *Can you find a pattern from your results from Problem II? Use your pattern to guess whether the following fractions "terminate"?*

$$\frac{1}{61} \qquad \frac{1}{625} \qquad \frac{1}{6251}$$

Problem 151 *Can you explain why your conjecture from Problem II is true?*

Problem 152 *Now let's consider fractions with decimal representations that do not terminate.*

(a) *Use long division to compute* $1/7$.

(b) *State the Division Theorem for integers.*

(c) *How does the Division Theorem for integers appears in your computation for $1/7$?*

(d) *In each instance of the Division Theorem, what is the divisor? And what does this imply about the remainder?*

(e) *Generalize: When a and b are integers with $b \neq 0$, what can you say about the decimal representation of a/b, assuming it does not terminate? Explain your reasoning.*

Problem 153 *Compute* $\frac{1}{9}$, $\frac{1}{99}$, *and* $\frac{1}{999}$. *Can you find a pattern? Can you explain why your pattern holds?*

Problem 154 *Use your work from Problem II to give the fraction form of the following decimals:*

(a) $0.\overline{357}$

(b) $23.\overline{459}$

(c) $0.23\overline{4598}$

(d) $76.3\overline{421}$

Problem 155 *Assuming that the pattern holds, is the number*

$$.123456789101112131415161718192021\ldots$$

a rational number? Explain your reasoning.

28 Decimals Aren't So Nice

We note that every number has an infinite decimal representation.

We will investigate the following question: How is $0.999\dots$ related to 1?

Problem 156 *What symbol do you think you should use to fill in the box below?*

$$.999\dots \boxed{} 1$$

Should you use $<$, $>$, $=$ or something else entirely?

Problem 157 *What is $1 - .999\dots$?*

Problem 158 *How do you write $1/3$ in decimal notation? Express*

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

in both fraction and decimal notation.

Problem 159 *See what happens when you follow the directions below:*

(a) *Set $x = .999\dots$.*

(b) *Compute $10x$.*

(c) *Compute $10x - x$.*

(d) *From the step immediately above, what does $9x$ equal?*

(e) *From the step immediately above, what does x equal?*

Problem 160 *Are there other numbers with this weird property?*

Learning outcomes: Learning outcome goes here.

Part III

Ratios, Functions, and Beyond

29 Strategy For Success

Question 161 *Before we get started, suppose you have two fractions that are equal, like maybe*

$$\frac{3}{5} = \frac{x}{12}.$$

Find the value of x using a picture. Explain your thinking.

Question 162 *In the previous problem, explain how your picture shows you how you could have solved the problem using the equation $3 \cdot 12 = 5x$. (You might have to adjust your picture to see this!) Where do you see the multiplication in your picture? Where do you see the equals sign?*

Question 163 *Summarize your work: why does cross-multiplication make sense to solve such an equation?*

Learning outcomes:

Problem 164 *Now, let's consider a paint problem.*

Jessie is planning to make a lovely shade of purple paint by mixing 5 cups of red paint with 3 cups of blue paint. Suddenly, she realizes she actually has 12 cups of red paint, not 5. How many cups of blue paint should Jessie mix with her 12 cups of red paint to make the same shade of purple paint?

(a) *Explain why the fraction* $\frac{3}{5}$ *is related to this problem. (Explain in terms of the meaning of fractions, not ratios!)*

(b) *If we use* x *to represent the answer to this question, explain why the fraction* $\frac{x}{12}$ *is related to this problem. (Explain in terms of the meaning of fractions, not ratios!)*

(c) *Explain why the fractions in part (a) and (b) are equal to one another in this story.*

Question 165 *Summarize your work: why does it make sense to set up a proportion and cross-multiply to solve the paint problem?*

Question 166 *What other proportions could you set up for the paint problem, and why?*

30 Ratios and Proportional Relationships

We think about ratios and proportional reasoning.

Here begins our work with ratios and proportional reasoning, which are the cornerstone of middle school mathematics. Try to avoid procedural approaches, such as, "set up a proportion and cross multiply." Instead, try to reason from the context and **use pictures and tables to support your reasoning**.

As you solve these problems, note how the problems simultaneously build on understandings of fractions and pave the way for functions.

Stacking Paper

Problem 167 *Suppose you want to know how many sheets are in a particular stack of paper, but don't want to count the pages directly. You have the following information:*

- *The given stack has height 4.50 cm.*
- *A ream of 500 sheets has height 6.25 cm.*

How many sheets of paper do you think are in the given stack?

Problem 168 *In your solution to the previous problem, what did you assume was proportional to what other quantity? Be precise.*

Mixing Punch

Problem 169 *Jenny is mixing punch and is considering two recipes:*

- *Recipe A: 3 parts orange juice for every 5 parts ginger ale*
- *Recipe B: 2 parts orange juice for every 3 parts ginger ale*

(a) *Which recipe will give juice that is the most "orangey"? Explain your reasoning.*

Learning outcomes: Learning outcome goes here.

(b) *Use a table to show various ways to make recipe B.*

(c) *To make 12 gallons of recipe B, how much of each will you need?*

Racing Snails

Problem 170 *Mike is racing snails that move at a constant speed:*

- *Snail A travels 3 inches in 5 minutes.*
- *Snail B travels 2 inches in 3 minutes.*

(a) *Which snail moves faster? Explain your reasoning.*

(b) *Use a table to show other distances and times for snail B.*

31 Poor Old Horatio

We try to understand ratios through mixing problems.

Problem 171 *A shade of orange is made by mixing 3 parts red paint with 5 parts yellow paint. Sam says we can add 4 cups of each color of paint and maintain the same color. Fred says we can quadruple both 3 and 5 and get the same color.*

(a) *Who (if either or both) is correct? Explain your reasoning.*

(b) *Use a table like the one below to show other paint mixtures that are the desired shade of orange.*

Red	*3*						
Yellow	*5*						

Problem 172 *If we wanted to make the same orange paint but were required to use 73 cups of yellow paint, how many cups of red paint would we need? Explain your reasoning.*

Red	*3*				
Yellow	*5*				

Problem 173 *If we wanted to make the same orange paint but were required to use 56 cups of red paint, how many cups of yellow paint would we need? Explain your reasoning.*

Learning outcomes: Learning outcome goes here.

Red	*3*				
Yellow	*5*				

Problem 174 *Generalize your approaches to the previous problems.*

(a) *Give a general formula for computing how much red paint is needed when y cups of yellow paint is used.*

(b) *Give a general formula for computing how much yellow paint is needed when r cups of red paint is used.*

Red	*3*				
Yellow	*5*				

Problem 175 *Now suppose we want to make a* ***different shade*** *of orange, this time made with $\frac{3}{4}$ cup of red paint and $\frac{2}{3}$ cup of yellow paint. How many cups of each color do you need in order to make 15 cups of the mixture? Use the table below.*

Red	$\frac{3}{4}$				
Yellow	$\frac{2}{3}$				
		17	1	15	

Problem 176 *In proportional reasoning problems,* a unit rate *describes the amount of one quantity for 1 unit of another quantity.*

(a) *What are the units for the various numbers in these problems?*

(b) *Identify some unit rates in this activity.*

(c) *In solving the above problems, it is likely that you or your classmates use strategies that made use of unit rates on the way to your solution. Explain why this strategy is sometimes called* going through one.

Problem 177 *If $2\frac{1}{2}$ dollars buys $3\frac{1}{2}$ pounds of bananas, then how many pounds of bananas can you buy for 12 dollars?*

Dollars								
Bananas (lbs.)								

32 Ratio Oddities

We investigate thinking about and adding ratios.

Problem 178 *There are 3 boys for every 4 girls in Mrs. Sanders' class.*

(a) *What fraction of the class are girls?*

(b) *List ratios that can describe this situation.*

(c) *If each of the number of boys and number of girls quadruples, what is the new ratio of girls to boys?*

(d) *Write an equation relating the number of boys in the class to the number of girls in the class.*

(e) *If the number of boys and number of girls each increase by 6, what can you say about the new ratio of boys to girls?*

Problem 179 *Suppose the ratio of girls to boys in Smith's class is 7:3 while the ratio of girls to boys in Jones' class is 6:5.*

(a) *If there are 50 students in Smith's class and 55 students in Jones' class, and both classes get together for an assembly, what is the ratio of girls to boys? Explain your reasoning.*

(b) *What if there are 500 students in Smith's class and still 55 students in Jones' class?*

(c) *What if there are 5000 students in Smith's class and still 55 students in Jones' class?*

(d) *How do the ratios of girls to boys in the combined assembly compare to the ratios of girls to boys in the original classes?*

(e) *Now suppose you don't know how many students are in Smith's class and there are 55 students in Jones' class. What can you say about the ratio of girls to boys at the assembly?*

Learning outcomes: Learning outcome goes here.

Problem 180 *Suppose you are teaching a class, and a student writes*

$$\frac{1}{4}+\frac{3}{5}=\frac{4}{9}$$

(a) *How would you respond to this?*

(b) *This student is most contrary, and presents you with the following problem:*

> *Suppose you have two cars, a 4 seater and a 5 seater. If the first car is 1/4 full and the second car is 3/5 full, how full are they together?*

The student then proceeds to answer their question with "The answer is 4/9." How do you address this?

(c) *This student's reasoning suggest a new kind of "addition" of ratios. Let's use $\oplus$ for this new form of "addition." So*

$$\frac{a}{b}\oplus\frac{c}{d}=\frac{a+c}{b+d}.$$

For which of the previous problems is does this "addition" give the correct answer? What is going on?

(d) *Use the student's context of seats and cars to reason about how $\frac{a}{b}\oplus\frac{c}{d}$ compares with $\frac{a}{b}$ and $\frac{c}{d}$.*

Problem 181 *Let's think a bit more about $\oplus$. If you were going to plot $\frac{a}{b}$ and $\frac{c}{d}$ on a number line, what can you say about the location of $\frac{a}{b}\oplus\frac{c}{d}$? Is this always the case, or does it depend on the values of a, b, c, and d?* Hint*: Assume that all of the letters are positive. Use specific numbers and a context; then try to reason generally.*

33 The Triathlete

We think about average speed.

Problem 182 *On Friday afternoon, just as Laine got off the bus, she realized that she had left her bicycle at school. In order to have her bicycle at home for the weekend, she decided to run to school and then ride her bike back home. If she averaged 6 mph running and 12 mph on her bike, what was her average speed for the round trip? Explain your reasoning.*

Problem 183 *On Saturday, Laine completed a workout in which she split the time evenly between running and cycling. If she again averaged 6 mph running and 12 mph on her bike, what was her average speed for the workout? Explain your reasoning.*

Problem 184 *Why was her average speed on Saturday different from her average speed on Friday? Can you reason, without computation, which average speed should be faster?*

Problem 185 *On Sunday, Laine's workout included swimming. Assuming that she can swim at an average speed of 2 mph, describe two running-cycling-swimming workouts, one similar to Friday's scenario (same distance) and a second similar to Saturday's (same time). Compute the average speed for each and explain your reasoning.*

Problem 186 *Which of the workout scenarios (same distance or same time) most closely resembles an actual triathlon? Why do you think that is the case?*

Problem 187 *After two months of intense training, Laine is able to average s mph swimming, r mph running, and c mph cycling. Again describe two running-cycling-swimming workouts, one similar to each of the two original scenarios, and compute her average speeds.*

Learning outcomes: Learning outcome goes here.

34 The Dreaded Story Problem

We explore ratios through a story problem.

Let's try our hand at a problem involving ratios.

Problem 188 *On orders from his doctor, every day, Marathon Marty must run from his house to a statue of Millard Fillmore and run back home along the same path. So Marty doesn't lollygag, the doctor orders him to average 8 miles per hour for the round trip.*

(a) *On Monday, Marty ran into Gabby Gilly on his way to the statue and averaged only 6 miles per hour for the trip out to the statue.What must Marty do to ensure he's obeyed his doctor's orders?*

(b) *On Tuesday, Marty did not see Gilly on his way to the statue and averaged 9.23 miles per hour for the trip out to the statue. What must Marty do to ensure he's obeyed his doctor's orders?*

(c) *On Wednesday, Gilly talks so much that Marty only averages 4 miles per hour on the way out. What must Marty do to ensure he's obeyed his doctor's orders?*

(d) *Assuming that Marty, for whatever reason, averages r miles per hour on the trip out to the statue. What must Marty do to ensure he's obeyed his doctor's orders?*

Learning outcomes: Learning outcome goes here.

35 I Walk the Line

We solve linear equations, with the context being our guide.

Solve the problems below initially without using letters and without algebraic procedures. Rely on numerical reasoning only, and then generalize your numerical approaches.

Problem 189 *Slimy Sam is on the lam from the law. Being not-too-smart, he drives the clunker of a car he stole east on I-70 across Ohio. Because the car can only go a maximum of 52 miles per hour, he floors it all the way from where he stole the car (just now at the Rest Area 5 miles west of the Indiana line) and goes as far as he can before running out of gas 3.78 hours from now.*

(a) *At what mile marker will he be 3 hours after stealing the car?*

(b) *At what mile marker will he be when he runs out of gas and is arrested?*

(c) *At what mile marker will he be x hours after stealing the car?*

(d) *At what time will he be at mile marker 99 (east of Indiana)?*

(e) *At what time will he be at mile marker 71.84?*

(f) *At what time will he be at mile marker y?*

(g) *Do parts (c) and (f) supposing that the car goes m miles per hour and Sam started b miles east of the Ohio-Indiana border.*

(h) *What "form" of an equation for a line does this problem motivate?*

Problem 190 *Free-Lance Freddy works for varying hourly rates, depending on the job. He also carries some spare cash for lunch. To make his customers sweat, Freddy keeps a meter on his belt telling how much money they currently owe (with his lunch money added in).*

(a) *On Monday, 3 hours into his work as a gourmet burger flipper, Freddy's meter reads \$42. 7 hours into his work, his meter reads \$86. If he works for 12 hours, how much money will he have? When will he have \$196? Solve this problem* ***without*** *finding his lunch money.*

Learning outcomes: Learning outcome goes here.

(b) *On Tuesday, Freddy is CEO of the of We Say So Company. After 2.53 hours of work, his meter reads \$863.15 and after 5.71 hours of work, his meter reads \$1349.78. If he works for 10.34 hours, how much money will he have? How much time will he be in office to have \$1759.21?*

(c) *On Wednesday, Freddy is starting goalie for the Columbus Blue Jackets. After x_1 hours of work, his meter reads y_1 dollars and after x_2 hours of work, his meter reads y_2 dollars. Without finding his amount of lunch money, if he works for x hours, how much money will he have? How much time will he be in front of the net to have y dollars?*

(d) *What "form" of an equation for a line does this problem motivate?*

Problem 191 *Counterfeit Cathy buys two kinds of fake cereal: Square Cheerios for \$4 per pound and Sugarless Sugar Pops for \$5 per pound.*

(a) *If Cathy's goal for today is to buy \$1000 of cereal, how much of each kind could she purchase? Give five possible answers.*

(b) *Plot your answers. What does the slope represent in this situation? What do the points where your curve intercepts the axes represent?*

(c) *If she buys Square Cheerios for a dollars per pound and Sugarless Sugar Pops for b dollars per pound and she wants to buy c dollars of cereal, write an equation that relates the amount of Sugar Pops Cathy buys to the amount of Cheerios she buys. What "form" of the equation of a line does this problem motivate?*

(d) *Write a function in the form*

$$\text{pounds of Sugar Pops} = f(\text{pounds of Cheerios}).$$

Problem 192 *Given points $p = (3, 7)$ and $q = (4, 9)$, find the formula for the line that connects these points.*

Problem 193 *In each of the situations above, write an equation relating the two variables (hours and position, hours and current financial status, pounds of Square Cheerios and pounds of Sugarless Sugar Pops) and answer the following questions:*

(a) *How did (or could) the equations help you solve the problems above? What about a table or a graph?*

(b) *Organize the information in each problem into a table and then into a graph. What patterns do you see, if any?*

(c) *What do the different features of your graph represent for each situation?*

36 Constant Amount Changes

We explore sequences and functions and the fact that sequences are functions.

Sometimes you compute the output value of a function from a previous output value. This is called a *recursive* representation of the function. Other times, you compute the output value directly from the input value. This is called a *closed form* representation of the function. Both approaches are important, as they provide different insights.

Problem 194 *We can use function notation for sequences, with $f(n)$ representing the n^{th} term of a sequence. Here is an example of a sequence specified recursively:*

$$f(0) = 1,\ f(1) = 1,\ \text{and } f(n) = f(n-1) + f(n-2) \text{ for } n \geqslant 2.$$

(a) *Find $f(6)$ and explain your reasoning.*

(b) *Why was it important to give the values $f(0) = 1$ and $f(1) = 1$?*

Problem 195 *Gertrude the Gumchewer has an addiction to Xtra Sugarloaded Gum, and it's getting worse. At the beginning of her habit, on day 0, she chews 3 pieces and then, each day afterward, she chews 8 more pieces than she chewed the day before.*

(a) *Gertrude's friend Wanda notices that Gertrude chewed 35 pieces on day 4. Wanda claims that, because Gertrude is increasing the number of pieces she chews at a constant rate, we can just use proportions with the given piece of information to find out how many pieces Gertrude chewed on any other day. Is Wanda correct or not? Explain.*

(b) *Make a table of how many pieces of gum Gertrude chewed on each of the first 10 days of her addiction.*

(c) *Think of what a 4^{th} grader would do to predict the next day's number of pieces given the previous day's number of pieces. Use the variables $Next$ and Now to write an equation that describes the thinking.*

(d) *Write a recursive specification for a function $g(n)$ that gives the number of pieces of gum Gertrude chewed on the n^{th} day.*

Learning outcomes: Learning outcome goes here.

(e) *How many pieces of gum Gertrude did chew on the* 793^{rd} *day of her habit? Explain your reasoning.*

(f) *How would the* 4^{th} *grader answer the previous question? How does this differ from how you solved it?*

(g) *Write a closed formula for computing* $g(n)$ *directly from* n.

(h) *Make a graph of your data about Gertrude's gum chewing. Which variable do you plot on the horizontal axis? Explain.*

(i) *Does it make sense to connect the dots on your graph? Explain.*

(j) *Locate the values* $g(6)$ *and* $g(5)$ *in your table from above, compute* $g(6) - g(5)$, *and interpret your result. How might you have known the answer without doing any calculation?*

Problem 196 *Slimy Sam steals a car from a rest area 3 miles east of the Indiana-Ohio state line and starts heading east along the side of I-70. Because the car is a real clunker, it can only go 8 miles per hour.*

(a) *Assuming the police are laughing too hard to arrest Sam, describe Sam's position on I-70 (via mile markers)* t *hours after stealing the car.*

(b) *Make a graph of your data about Sam's travel. Which variable do you plot on the horizontal axis? Explain.*

(c) *Does it make sense to connect the dots on your graph? Explain your reasoning.*

(d) *Write a recursive specification for a function* $s(t)$ *that gives Sam's position on I-70 at hour* t.

(e) *Write closed formula for* $s(t)$.

(f) *How is this problem the same and how is it different from the Gertrude problem?*

(g) *Dumb Question: At any specific time, how many positions could Sam be in?*

37 Constant Percentage (Ratio) Changes

We explore geometric series.

Problem 197 *Billy is a bouncing ball. He is dropped from a height of 13 feet and each bounce goes up 92% of the bounce before it. Assume that the first time Billy hits the ground is bounce 1.*

(a) *Make a table of how high Billy bounced after each of the first 10 times he hit the ground. Be sure to indicate the arithmetic process you go through for each bounce (i.e., not just the final height). Find a pattern that will predict an answer.*

Bounce	*Height*
0	*13*

(b) *Think of what a 7^{th} grader would do to predict the next bounce's height given the previous bounce's height. How would the 7^{th} grader answer the previous question? How does this differ from how you solved it?*

(c) *Make a graph of your data about Billy. Which variable do you plot on the horizontal axis? Explain.*

(d) *Does it make sense to connect the dots on your graph? Explain your reasoning.*

(e) *How high will Billy bounce after the 38th bounce? How high will Billy bounce after the n^{th} bounce? Explain your reasoning.*

Learning outcomes: Learning outcome goes here.

(f) *Use function notation, $f(n)$, and a recursive formula to specify the height of Billy's bounces, including the initial condition and general term.*

(g) *Use function notation, $f(n)$, and an explicit formula to specify the height of Billy's bounces. Indicate the domain of the function.*

(h) *Using your table from above, compute the differences between the heights on successive bounces (e.g., $f(1) - f(0)$, $f(2) - f(1)$, etc.). What do you notice? Why does this happen?*

(i) *Compare and contrast the explicit and recursive representations from Billy and from Gertrude. How do the role(s) of the operations and initial values differ, remain the same, or relate?*

Problem 198 *Supppose 13 mg of a drug is administered to a patient once, and the amount of the drug in the patient's body decreases by 8% each hour.*

(a) *Describe the amount of the drug in the patient's body t hours after it was administered.*

(b) *Make a graph of your data about the amount of drug in the body over time. Which variable do you plot on the horizontal axis? Explain.*

(c) *Does it make sense to connect the dots on your graph? Explain your reasoning.*

(d) *Use function notation, $g(t)$, and an explicit formula to specify the the amount of drug remaining in the body after t hours. Indicate the domain of the function.*

(e) *How is this problem fundamentally different from the Billy problem? What is the same and different about the functions f and g?*

(f) *Dumb Question: At any one time, how many different amounts of the drug are possible in the patient's body?*

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38 Meanings of Exponents

We look for patterns in exponents.

Students in grades 3–7 can use their understanding of counting number arithmetic to build understandings of the arithmetic of negative integers and rational numbers. Here are the key ideas:

- The properties of operations (commutative, associative, and distributive properties) are established for counting numbers based on meanings of operations.
- As we extend arithmetic to negative integers and rational numbers, we want the properties of operations to continue to hold.

This activity follows an analogous process for exponents: Students use their understanding of counting number exponents to build an understanding of negative integer and rational exponents. Here are the key ideas:

- The rules of exponents are established for counting number exponents based on the meaning of an exponent.
- As we extend to negative and rational exponents, we want the rules of exponents to continue to hold.

Problem 199 *Students sometimes say that a^n means "a multiplied by itself n times." But for counting number exponents, this is not correct. For example, how many multiplications are there in 3^5? Write a better definition for a^n, where n is a counting number.*

Problem 200 *Why is x^3 not the same function as 3^x? We often think of multiplication as "repeated addition," and we find that adding a copies of b gives the same result as adding b copies of a. Does this idea work for thinking of exponentiation as "repeated multiplication"? Explain.*

Problem 201 *If you do not know (or do not remember) the rules for exponents, you can still use your definition of a^n to figure out other ways of writing expressions with exponents. Use* ***specific values*** *for letters in expressions of the form $a^n a^m$, a^n/a^m, $(a^n)^m$, and $(ab)^n$ for counting-number exponents, to*

Learning outcomes: Learning outcome goes here.

explain what the rules must be. Choose specific values that help you explain generally.

Problem 202 ***Patterns.*** *One way to reason about the meanings of zero and negative exponents is to use patterns. As you complete the following table,* ***imagine that you know nothing about zero and negative exponents.*** *Instead, use the patterns in the values for positive exponents to reason about what the values should be for zero and negative exponents. Then reason generally about the meaning of a^0 and a^{-n}, where n is a counting number and a is a real number. Are there any values of a for which your reasoning is not valid? Explain.*

$2^3 =$	$3^3 =$	$(-2)^3 =$	$\left(\frac{1}{2}\right)^3 =$
$2^2 =$	$3^2 =$	$(-2)^2 =$	$\left(\frac{1}{2}\right)^2 =$
$2^1 =$	$3^1 =$	$(-2)^1 =$	$\left(\frac{1}{2}\right)^1 =$
$2^0 =$	$3^0 =$	$(-2)^0 =$	$\left(\frac{1}{2}\right)^0 =$
$2^{-1} =$	$3^{-1} =$	$(-2)^{-1} =$	$\left(\frac{1}{2}\right)^{-1} =$
$2^{-2} =$	$3^{-2} =$	$(-2)^{-2} =$	$\left(\frac{1}{2}\right)^{-2} =$
$2^{-3} =$	$3^{-3} =$	$(-2)^{-3} =$	$\left(\frac{1}{2}\right)^{-3} =$

Problem 203 ***Extending the rules.*** *A careful way to approach zero and negative integer exponents is to use the rules of exponents (which you established above for counting-number exponents) to determine what 0 and negative integer exponents must mean if the exponent rules continue to hold in this extended domain.*

(a) *Use the exponent rules to provide two explanations for a sensible definition of a^0, being clear about why your definition makes sense. Note any restrictions on a.*

(b) *Use the exponent rules to provide two explanations for a sensible definition of a^{-n}, where n is a counting number. Again, note any restrictions on a.*

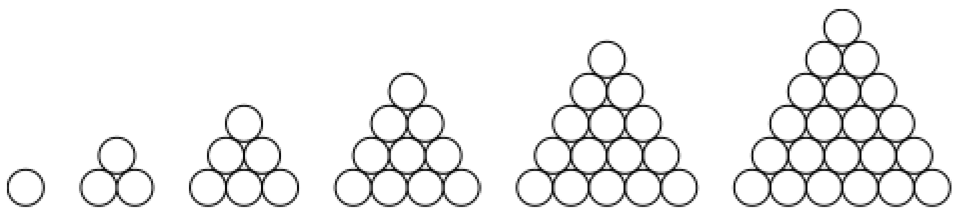
Problem 204 *While trying to decide what $3^{\frac{2}{5}}$ should mean, Katie wondered about the expression $\left(3^{\frac{2}{5}}\right)^5$. What should Katie's expression be equal to? Explain, using rules of exponents. Then use Katie's idea to determine a value for $3^{\frac{2}{5}}$.*

39 Arithmetic Series

We study arithmetic series.

In this activity, we explore *arithmetic series*, which are sums of consecutive terms from an arithmetic sequence.

Ms. Nguyen's math class has been looking at "triangular numbers." The first 6 triangular numbers are shown below.



Problem 205 *Blair wanted to find the* 551^{st} *triangular number. She used a table and looked for a pattern in the* sequence of partial sums*:* $1, 1+2, 1+2+3, \dots$. *Help her finish her idea.*

Problem 206 *Kaley realized the the* 551^{st} *triangular number would be the sum*

$$1+2+3+4+\cdots+548+549+550+551$$

She started pairing the first with the last number; the second with the second-to-last; the third with the third-to-last; and so on. She saw that the averages are always the same. Help her finish her idea.

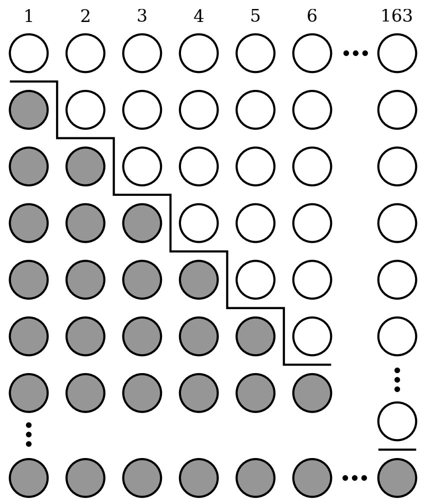
Problem 207 *Ali begin by writing out the sum forward and backward and follows:*

$$\begin{array}{c}1\ +\ 2\ +\ 3\ +\ 4\ +\ 5\ +\ 6\ +\cdots+546+547+548+549+550+551\\ 551+550+549+548+547+546+\cdots+\ 6\ +\ 5\ +\ 4\ +\ 3\ +\ 2\ +\ 1\end{array}$$

Help her finish her idea. Be sure to explain clearly what happens "in the dots." Does it matter whether there are an even or an odd number of terms?

Learning outcomes: Sum arithmetic series.

Problem 208 *Cooper was interested in a different triangular number and drew the following picture:*



Which triangular number was he finding? Help him finish his idea. Be sure to explain clearly what happens "in the dots."

Problem 209 *Sum the numbers:*

$$106 + 112 + 118 + \cdots + 514$$

Problem 210 *Sum the numbers:*

$$2.2 + 2.9 + 3.6 + 4.3 + \cdots + 81.3$$

Problem 211 *Suppose you have an arithmetic sequence beginning with a, with a constant difference of d and with n terms.*

(a) *What is the n^{th} term of the sequence?*

(b) *Use dots to write the series consisting of the first n terms of this sequence.*

(c) *Find the sum of this series.*

40 Geometric Series

We explore geometric series*, which are sums of consecutive terms from an geometric sequence.*

Ms. Radigan's math class has been trying to compute the following sums:

$$1 + 2 + 4 + 8 + \cdots + 2^{19}$$

$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{13}}$$

Problem 212 *Kelsey used tables and looked for pattern in the* sequence of partial sums*:* $1, 1+2, 1+2+4, \ldots$. *Help her finish her idea for both sequences.*

Problem 213 *For the sum beginning with* $\frac{2}{3}$*, Erin started by drawing a large square (which she imagined as having area 1), and she shaded in* $\frac{2}{3}$ *of it. Then she shaded in* $\frac{2}{9}$ *more, and so on. Help her finish her idea.*

Problem 214 *Ryan wrote out all of the terms in the first sum, represented as powers of 2, beginning with* $1 + 2 + 2^2 + 2^3$*. Then he realized that because the terms formed a geometric sequence, he could multiply the series by the common ratio of 2, and the resulting series would be almost identical to the first, differing only at the beginning and the end. By subtracting the first series from the second, all of the middle terms would cancel. Help him finish his idea.*

Problem 215 *Ali said, "Here is a thought experiment. I take a sheet of paper, rip it perfectly into thirds, place one piece to start a pile that I will call A, another piece to start a pile I will call B, and I keep the third piece in my hands. I then rip that piece into thirds, place one piece on pile A, one piece on pile B, and keep the third. Notice that each of pile A and pile B have* $\frac{1}{3} + \frac{1}{9}$ *of a sheet of paper, and I still have* $\frac{1}{9}$ *of a sheet in my hands. I continue this process until I place* $\frac{1}{3^{13}}$ *of a sheet on each pile and still have* $\frac{1}{3^{13}}$ *of a sheet in my hands.*

Learning outcomes: Learning outcome goes here.

Help Ali finish her idea.

Problem 216 *Sum the expression:*

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{2^n}{3^n}$$

What happens to this sum as n gets really large?

Problem 217 *Consider the expression:*

$$\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots + \frac{7}{10^n}$$

(a) *Find the sum of the expression.*

(b) *What happens to this sum as n gets really large?*

(c) *How does this help you explain why a particular repeating decimal is a particular rational number? Be sure to indicate what repeating decimal and what rational number you are talking about.*

Problem 218 *Suppose you have an geometric sequence beginning with a, with a constant ratio of r and with n terms.*

(a) *What is the n^{th} term of the sequence?*

(b) *Use dots to write the series consisting of the first n terms of this sequence.*

(c) *Find the sum of this series.*

41 Second Differences

We use arithmetic series to develop a formula for a sequence that has constant second differences.

In a previous activity, we developed strategies for finding the sum of arithmetic series. In this activity, we use arithmetic series to develop a formula for a sequence that has constant second differences. Then we demonstrate that all quadratic sequences have constant second differences.

Problem 219 *Consider the sequence $f(n)$ given in the table below. In the rightmost column, Δ ("delta") means difference, computed by subtracting the current value of $f(n)$ from the next.*

n	$f(n)$	Δ
0	4	3
1	7	3
2	10	3
3	13	3
4	16	3
5	19	

(a) *Explain how $f(5)$ can be computed from the shaded cells in the table.*

(b) *Generalize your method to develop and explain a formula for $f(n)$.*

(c) *What was it about the differences that made this problem easy?*

Learning outcomes: Learning outcome goes here.

Problem 220 *Consider the sequence $g(n)$ given in the table below.*

n	$g(n)$	Δ	$\Delta\Delta$
0	1		
1	-2		
2	1		
3	10		
4	25		
5	46		
6	73		

(a) *Compute Δ by subtracting the current value of $g(n)$ from the next.*

(b) *Explain the formula $\Delta(n) = g(n+1) - g(n)$.*

(c) *Check that the shaded cells sum to $g(5)$, and explain how that makes sense based upon how the Δ values were calculated.*

(d) *Because the Δ values ("first differences") are not constant, use the $\Delta\Delta$ column to compute the "differences of the differences" (also called "second differences").*

(e) *From the fact that the second differences are constant, develop an explicit formula for Δ in terms of n.*

Problem 221 *The same sequence $g(n)$ is given below, this time with a formula for Δ in terms of n.*

n	$g(n)$	$\Delta(n) = 6n - 3$
0	1	-3
1	-2	3
2	1	9
3	10	15
4	25	21
5	46	27
6	73	

(a) *Explain each of the following steps:*

$$\begin{aligned} g(5) &= 1 + \Delta(0) + \Delta(1) + \Delta(2) + \Delta(3) + \Delta(4) \\ &= 1 + (6 \cdot 0 - 3) + (6 \cdot 1 - 3) + (6 \cdot 2 - 3) + (6 \cdot 3 - 3) + (6 \cdot 4 - 3) \\ &= 1 + 6 \cdot (0 + 1 + 2 + 3 + 4) + (-3 + -3 + -3 + -3 + -3) \\ &= 1 + 6 \cdot \frac{5 \cdot 4}{2} + 5 \cdot (-3) \end{aligned}$$

(b) *Where do you see arithmetic series in the calculations you just explained?*

(c) *Generalize the above approach to yield an expression for $g(n)$.*

(d) *What kind of sequence is $g(n)$?*

Problem 222 *A general quadratic sequence $h(n)$ is given below.*

n	$h(n) = an^2 + bn + c$	Δ	$\Delta\Delta$
0			
1			
2			
3			

(a) *Compute the values of $h(n)$.*

(b) *Compute Δ by subtracting the next value of $h(n)$ from the current.*

(c) *Use the $\Delta\Delta$ column to compute the second differences.*

(d) *Generalize the result for first differences by computing $\Delta(n) = h(n+1) - h(n)$.*

(e) *Generalize the result for second differences by computing $\Delta\Delta(n) = \Delta(n+1) - \Delta(n)$.*

(f) *Explain how your work demonstrates that, for any quadratic sequence, the second differences must be constant.*

Part IV

Solving Equations

42 Hieroglyphical Algebra

We use a strange multiplication table to solve algebra problems.

Note: This activity is based on an activity originally designed by Lee Wayand.

Consider the following addition and multiplication tables:

+	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
fish	fork	balloon	fish	eyeball	cinder-block	skull	lolly-pop	man	DNA
lolly-pop	balloon	eyeball	lolly-pop	fork	fish	man	cinder-block	DNA	skull
skull	fish	lolly-pop	skull	cinder-block	DNA	fork	man	balloon	eyeball
cinder-block	eyeball	fork	cinder-block	balloon	lolly-pop	DNA	fish	skull	man
DNA	cinder-block	fish	DNA	lolly-pop	man	eyeball	skull	fork	balloon
fork	skull	man	fork	DNA	eyeball	fish	balloon	lolly-pop	cinder-block
man	lolly-pop	cinder-block	man	fish	skull	balloon	DNA	eyeball	fork
balloon	man	DNA	balloon	skull	fork	lolly-pop	eyeball	cinder-block	fish
eyeball	DNA	skull	eyeball	man	balloon	cinder-block	fork	fish	lolly-pop

fish = fish

lolly-pop = lolly-pop

skull = skull

cinder-block = cinder-block

DNA = DNA

fork = fork

man = man

balloon = balloon

eyeball = eyeball

Learning outcomes: Learning outcome goes here.

·	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish
fork	fish	lolly-pop	cinder-block	DNA	skull	eyeball	balloon	man	fork
eyeball	lolly-pop	balloon	man	fork	skull	cinder-block	DNA	fish	eyeball
balloon	cinder-block	man	fork	lolly-pop	skull	DNA	fish	eyeball	balloon
man	DNA	fork	lolly-pop	cinder-block	skull	fish	eyeball	balloon	man
skull	skull	skull	skull	skull	skull	skull	skull	skull	skull
lolly-pop	eyeball	cinder-block	DNA	fish	skull	balloon	man	fork	lolly-pop
cinder-block	balloon	DNA	fish	eyeball	skull	man	fork	lolly-pop	cinder-block
DNA	man	fish	eyeball	balloon	skull	fork	lolly-pop	cinder-block	DNA
fish	fork	eyeball	balloon	man	skull	lolly-pop	cinder-block	DNA	fish

= fish

= lolly-pop

= skull

= cinder-block

= DNA

= fork

= man

= balloon

= eyeball

Problem 223 *Can you tell me which glyph represents* 0*? How did you arrive at this conclusion?*

Problem 224 *Can you tell me which glyph represents* 1*? How did you arrive at this conclusion?*

Problem 225 *A number* x *has an additive inverse if you can find another number* y *with*

$$x + y = 0.$$

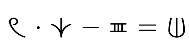
and we say that "y *is the additive inverse for* x*." If possible, find the additive inverse of every number in the table above.*

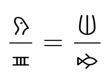
Problem 226 *A number* x *has a multiplicative inverse if you can find another number* y *with*

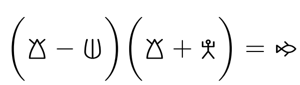
$$x \cdot y = 1.$$

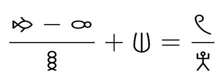
and we say that "y *is the multiplicative inverse for* x*." If possible, find the multiplicative inverse of every number in the table above.*

Problem 227 *If possible, solve the following equations:*

(a) 

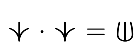
(b) 

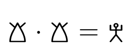
(c) 

(d) 

In each case explain your reasoning.

Problem 228 *If possible, solve the following equations:*

(a) 

(b) 

(c) 

(d) 

In each case explain your reasoning.

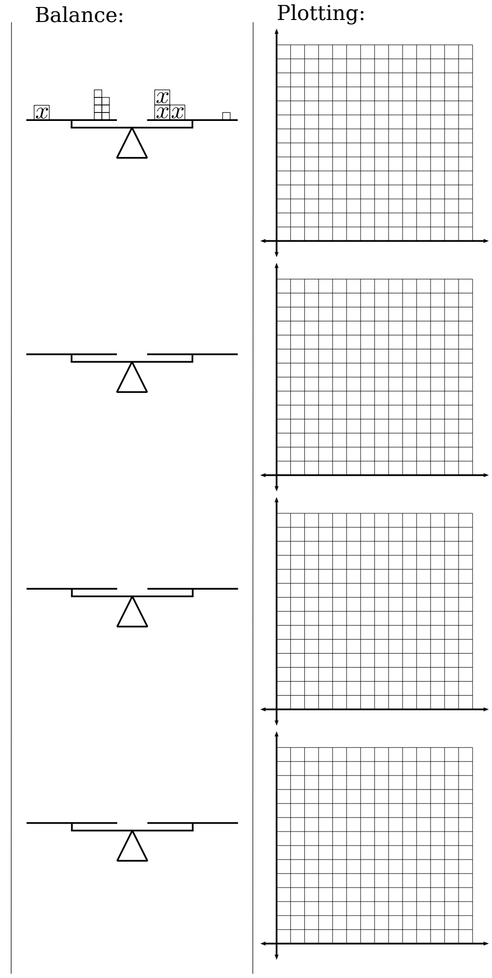
43 The Other Side—Solving Equations

We explore ideas related to solving equations.

Problem 229 *Solve the following equation three ways: Using algebra, using the balance, and with the graph. At each step, the three models should be in complete alignment.*

Equations: Balance: Plotting:

$$x + 7 = 3x + 1$$



Learning outcomes: Learning outcome goes here.

Problem 230 *Critically analyze the three "different" methods of solving equations, noting the advantages and disadvantages of each.*

Problem 231 *Can you solve quadratic equations using the methods above? If so give an example. If not, explain why not.*

Problem 232 *Can you think of an example when the undoing via algebraic manipulation would fail?*

While sometimes we solve equations via a process of algebraic manipulation, other times we have a formula.

Problem 233 *Give a formula for solving linear equations of the form* $ax+b = 0$.

44 Solving Quadratics

We solve quadratic equations.

Here we explore various methods for solving quadratic equations in one variable. **Please read all instructions carefully.**

Problem 234 *Is* $\sqrt{4} = \pm 2$*? Explain.*

Problem 235 *Suppose that* $\sqrt{4} = \pm 2$*? Then evaluate* $\sqrt{4} + \sqrt{9}$.

Problem 236 *What does your calculator say about* $\sqrt{4} + \sqrt{9}$*?*

Problem 237 *In the following problems, you* ***may not use the quadratic formula****. But just for the record, write down the quadratic formula.*

Problem 238 *In the following list of equations, solve those that are* ***easy*** *to solve.*

(a) $(x-3)(x+2) = 0$

(b) $(x-3)(x+2) = 1$

(c) $(2x-5)(3x+1) = 0$

(d) $(x-a)(x-b) = 0$

(e) $(x-1)(x-3)(x+2)(2x-3) = 0$

Learning outcomes: Learning outcome goes here.

Problem 239 *Regarding the previous problem, state the property of numbers that made all but one of the equations easy to solve.*

Problem 240 *For each part below, write a quadratic equation with the stated solution(s) and no other solutions.*

(a) $x = 7$ *or* $x = -4$

(b) $x = p$ *or* $x = q$

(c) $x = 3$

(d) $x = \dfrac{1 \pm \sqrt{5}}{2}$

Problem 241 *Find all solutions to* $x^3 - 3x^2 + x + 1 = 0$. *Hint: One solution is* $x = 1$.

45 Complete Squares

We think about completing the square.

Problem 242 *In the following list of equations, solve those that are* ***easy*** *to solve.*

(a) $x^2 = 5$

(b) $x^2 - 4 = 2$

(c) $x^2 - 4x = 2$

(d) $2x^2 = 1$

(e) $(x - 2)^2 = 5$

Problem 243 *Regarding the previous problem, state the property of numbers that made all but one of the equations easy to solve.*

Problem 244 *Although* 160 *is not a square in base ten, what could you add to* 160 *so that the result would be a square number?*

Problem 245 *Consider the polynomial expression* $x^2 + 6x$ *to be a number in base* x*. We want to add to this polynomial so that the result is a square in base* x*.*

(a) *Use "flats" and "longs" to draw a picture of this polynomial as a number in base* x*, adding enough "ones" so that you can arrange the polynomial into a square.*

Learning outcomes: Learning outcome goes here.

(b) *What "feature" of the square does the new polynomial expression represent?*

(c) *Why does it make sense to call this technique "completing the square"?*

(d) *Use your picture to help you solve the equation* $x^2 + 6x = 5$.

Problem 246 *Complete the square to solve the following equations:*

(a) $x^2 + 3x = 4$

(b) $x^2 + bx = q$

(c) $2x^2 + 8x = 12$

(d) $ax^2 + bx + c = 0$

Problem 247 *Solve the following equation*

$$x^5 - 4x^4 - 18x^3 + 64x^2 + 17x - 60 = 0$$

assuming you know that 1, -1, *and* 3 *are roots.*

46 Maximums and Minimums

We think about different forms of quadratic equations.

In high school mathematics, you saw three different forms for quadratic functions. In this activity, we explore the advantages and disadvantages of each.

Note: We use only real numbers for x. And we begin by agreeing that the shape of the graph of a quadratic function is a parabola.

Problem 248 *Consider the function $f(x) = x^2 - 3$. What are the maximum/minimum value(s) of $f(x)$, and for what x values do they occur? Explain your reasoning. Use this information to sketch a graph.*

Problem 249 *Consider the function $f(x) = 3(x-5)^2 + 7$. What are the maximum/minimum value(s) of $f(x)$, and for what x values do they occur? Explain your reasoning. Use this information to sketch a graph.*

Problem 250 *Consider the function $f(x) = -2(x+3)^2 + 7$. What are the maximum/minimum value(s) of $f(x)$, and for what x values do they occur? Explain your reasoning. Use this information to sketch a graph.*

Problem 251 *What are the advantages of the form $f(x) = a(x-h)^2 + k$ for a quadratic function? Why is it called vertex form? What do the values of a, h, and k tell you about the graph?*

Problem 252 *Consider the function $f(x) = x^2 + 4x + 2$. Complete the square to put this function into vertex form, and sketch a graph.*

Problem 253 *Consider the function $f(x) = 2x^2 - 8x + 6$. Complete the square to put this function into vertex form, and sketch a graph.*

Learning outcomes: Learning outcome goes here.

Problem 254 *Consider the function $f(x) = (x+1)(x+5)$.*

(a) *What points on the graph are easy to locate?*

(b) *How can you use those points to find the vertex?*

(c) *Sketch the graph.*

Problem 255 *Consider the function $f(x) = -2(x-2)(x+3)$.*

(a) *What points on the graph are easy to locate?*

(b) *How can you use those points to find the vertex?*

(c) *Sketch the graph.*

Problem 256 *Consider the function $f(x) = a(x-r_1)(x-r_2)$.*

(a) *What do the values of a, r_1, and r_2 tell you about the graph?*

(b) *How can you use that information to find the vertex?*

(c) *What is this form called and why?*

Problem 257 *Consider the function $f(x) = ax^2 + bx + c$.*

(a) *What do the values of a, b, and c tell you about the graph?*

(b) *What are the advantages and disadvantages of this form?*

47 Solving Cubic Equations

We find a method for solving cubic equations.

To solve the cubic equation $x^3 + px + q = 0$, we use methods that were discovered and advanced by various mathematicians, including Ferro, Tartaglia, and Cardano. The approach is organized in three steps. **Make notes in the margin as you follow along.**

Step 1: Replace x with $u + v$

In $x^3 + px + q = 0$, let $x = u + v$. Show that the result can be written as follows:

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$

Step 2: Set uv to eliminate terms

If $3uv + p = 0$, then all of the terms are eliminated except for u^3, v^3, and constant terms. Explain why the equation simplifies nicely to:

$$u^3 + v^3 + q = 0.$$

Solve $3uv + p = 0$ for v, substitute, and show that we have:

$$u^3 + \left(\frac{-p}{3u}\right)^3 + q = 0.$$

Step 3: Recognize the equation as a quadratic in u^3 and solve

By multiplying by u^3, show that we get a quadratic in u^3:

$$u^6 + qu^3 + \left(\frac{-p}{3}\right)^3 = 0.$$

Show that this has solutions:

Learning outcomes: Learning outcome goes here.

$$u^3 = \frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}.$$

Now, use the facts $v = -p/(3u)$ and $x = u + v$ to write a formula for x:

$$x = \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}} + \frac{-p}{3\sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4\left(\frac{-p}{3}\right)^3}}{2}}}.$$

Problem 258 *How many values does this formula give for x? From the original equation $x^3 + px + q = 0$, how many solutions should we expect?*

Problem 259 *Use the above formula to solve the specific equation $x^3 - 15x - 4 = 0$. Show that*

$$x = \sqrt[3]{2 \pm \sqrt{-121}} + \frac{5}{\sqrt[3]{2 \pm \sqrt{-121}}}.$$

Are these values of x real numbers?

Problem 260 *Use technology to graph $y = x^3 - 15x - 4$. According to the graph, how many real roots does the polynomial have? What is going on?*

Problem 261 *Choose "plus" in the $\pm$, and check that $2 + \sqrt{-1}$ is a cube root of $2 + \sqrt{-121}$. Use that fact to simplify the above expression for x. What do you notice?*

Problem 262 *Now choose "minus" in the $\pm$ above, and find the value of x. What do you notice?*

In both cases, the formula requires computations with square roots of negative numbers, but the result is a real solution. These kinds of occurrences were the historical impetus behind the gradual acceptance of complex numbers.

48 Sketching Roots

We seek to understand the connection between roots and the plots of polynomials.

In this activity we seek to better understand the connection between roots and the plots of polynomials. We will restrict our attention to polynomials with real coefficients.

First, we need to be precise about the correct usage of some important language:

- Expressions have *values*.
- Equations have *solutions*: values of the variables that make the equation true.
- Functions have *zeros*: input values that give output values of 0.
- Polynomials (i.e., polynomial expressions) have *roots*.

These ideas are related, of course, as follows: A zero of a polynomial function, $p(x)$, is a root of the polynomial $p(x)$ and a solution to the equation $p(x) = 0$.

Please try to use this language correctly: Equations do not have zeros, and functions do not have solutions.

Problem 263 *Give an example of a polynomial, and write a true sentence about related equations, functions, zeros, equations, and roots.*

Problem 264 *Sketch the plot of a quadratic polynomial with real coefficients that has:*

(a) *Two real roots.*

(b) *One repeated real root.*

(c) *No real roots.*

In each case, give an example of such a polynomial.

Problem 265 *Can you have a quadratic polynomial with exactly one real root and 1 complex root? Explain why or why not.*

Learning outcomes: Learning outcome goes here.

Problem 266 *Sketch the plot of a cubic polynomial with real coefficients that has:*

(a) *Three distinct real roots.*

(b) *One real root and two complex roots.*

In each case, give an example of such a polynomial.

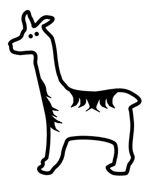
Problem 267 *Can you have a cubic polynomial with no real roots? Explain why or why not. What about two distinct real roots and one complex root?*

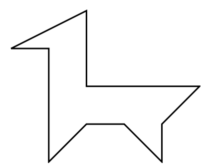
Problem 268 *For polynomials with real coefficients of degree* 1 *to* 5, *classify exactly which types of roots can be found. For example, in our work above, we classified polynomials of degree* 2 *and* 3.

49 Geometry and Adding Complex Numbers

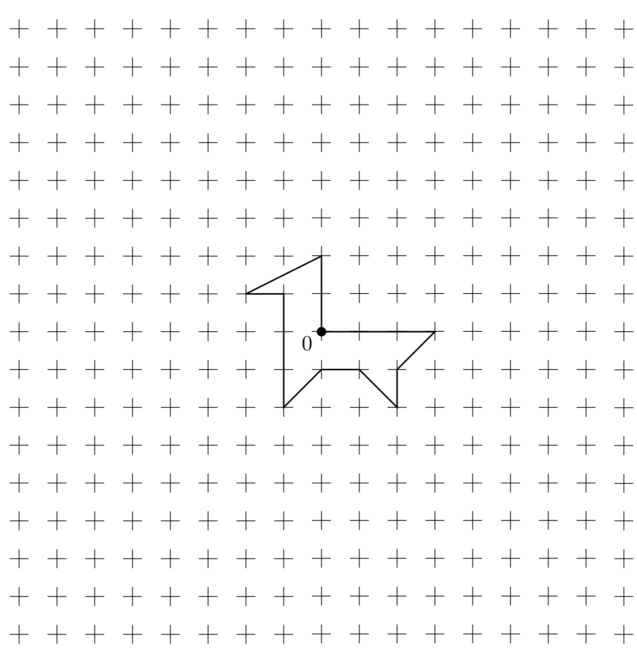
We investigate how to add complex numbers.

Let's think about the geometry of adding complex numbers. We won't be alone on our journey—Louie Llama is here to help us out:


Louie Llama


how we'll draw him

Problem 269 *Here's Louie Llama hanging out near the point 0 in the complex plane. Add $4 + 4i$ to him. Make a table and show in the plane below what happens.*



Learning outcomes: Learning outcome goes here.

Problem 270 *Explain what it means to "add" a complex number to Louie Llama. Describe the process(es) used when doing this.*

Problem 271 *Put Louie Llama back where he started, now add* $1-5i$ *to him. Make a table and show what happens in the plane.*

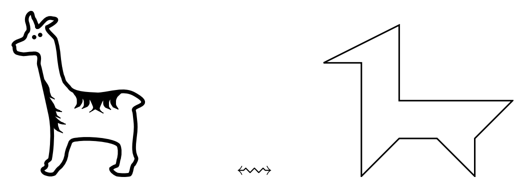
Problem 272 *Geometrically speaking, what does it mean to "add" complex numbers?*

50 Geometry and Multiplying Complex Numbers

We investigate how to multiply complex numbers.

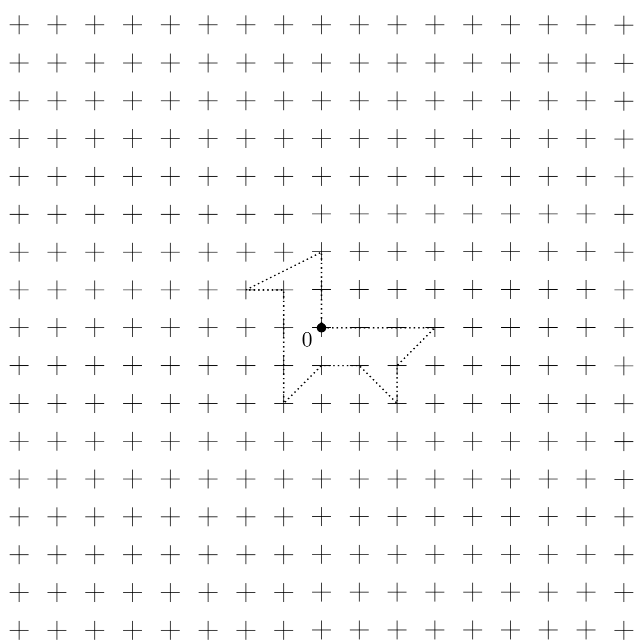
Now we'll investigate the geometry of multiplying complex numbers. In each case, specify the transformation. For example, if you see a rotation, specify the angle and the center of rotation.

Louie Llama is here to help us out:



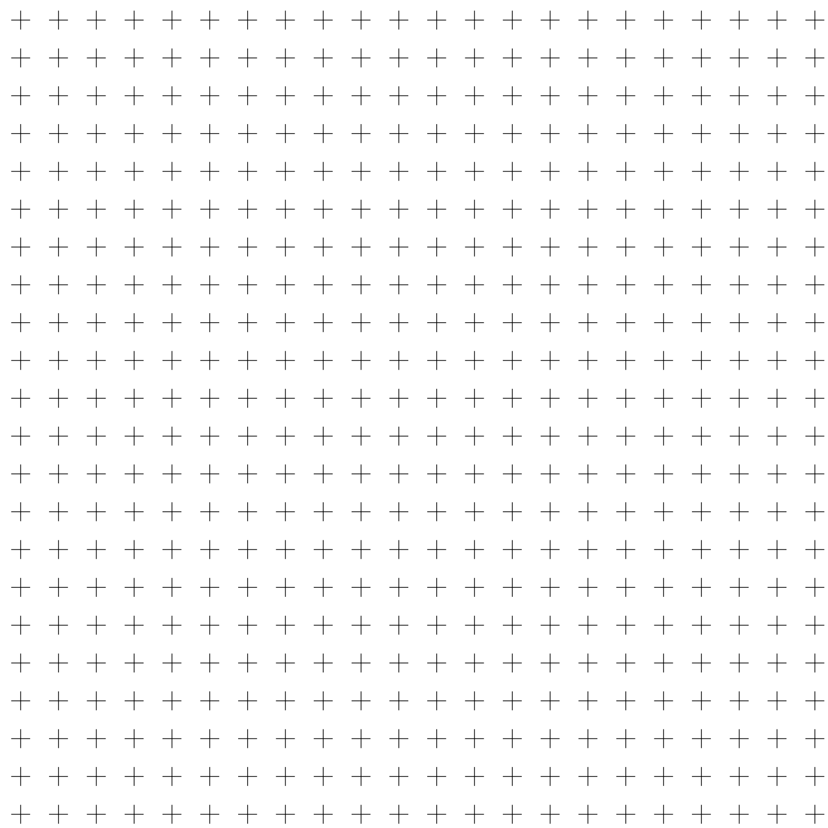
Louie Llama how we'll draw him

Problem 273 *Here's Louie Llama hanging out near the point* 0 *in the complex plane. Multiply him by* 2*. Make a table and show in the plane below what happens. What transformation do you see?*

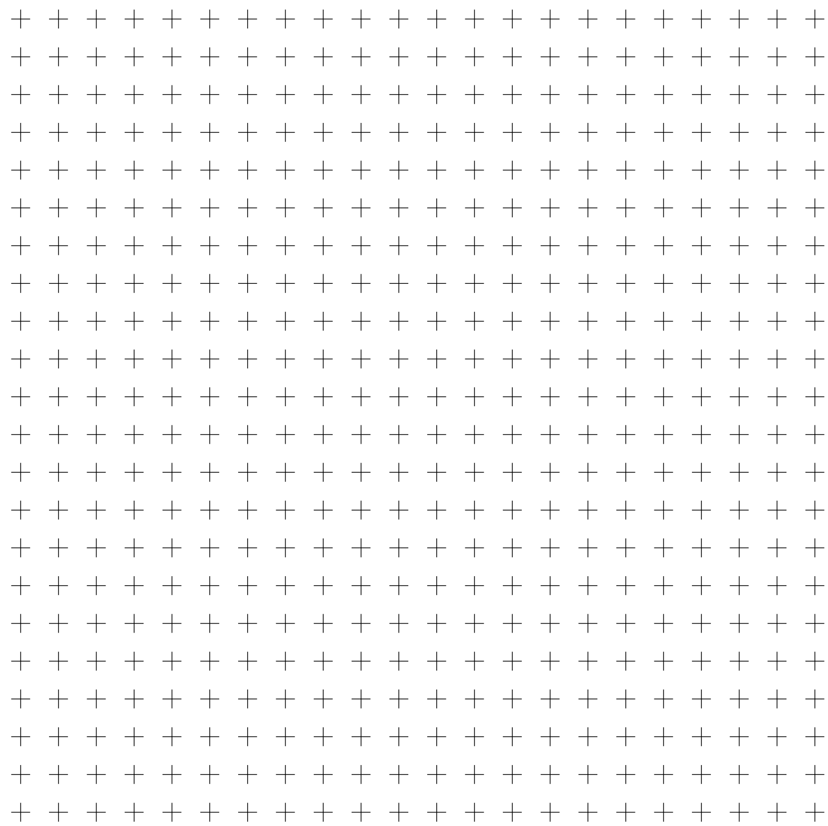


Learning outcomes: Learning outcome goes here.

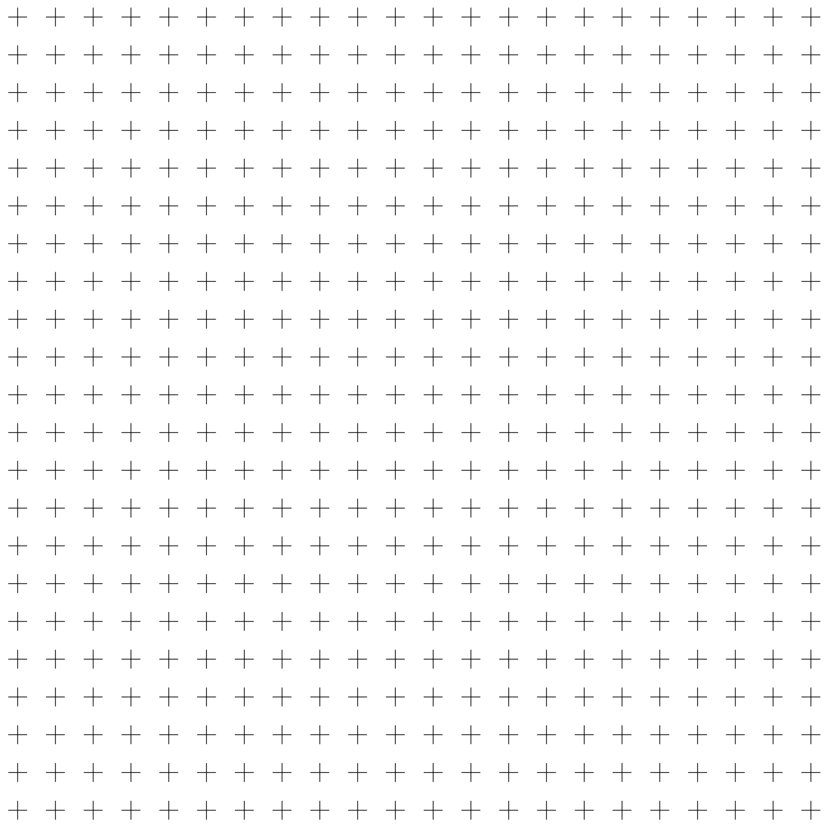
Problem 274 *Now multiply him by i. Make a table and show in the plane below what happens. What transformation do you see?*



Problem 275 *Now multiply Louie Llama by $1+i$. Make a table and show in the plane below what happens. What transformation do you see?*



Problem 276 *Now multiply Louie Llama by $1 - 2i$. Make a table and show in the plane below what happens. What transformation do you see?*



Problem 277 *Make a table to summarize your results from the previous problems. Then describe what happens geometrically when we "multiply" by a complex number.*

Part V

Harmony of Numbers

51 On the Road

We find binomial coefficients in unexpected places.

Problem 278 *Steve likes to drive the city roads. Suppose he is driving down a road with three traffic lights. For this activity, we will ignore yellow lights, and pretend that lights are either red or green.*

(a) *How many ways could he see one red light and two green lights?*

(b) *How many ways could he see one green light and two red lights?*

(c) *How many ways could he see all red lights?*

Problem 279 *Now suppose Steve is driving down a road with four traffic lights.*

(a) *How many ways could he see two red light and two green lights?*

(b) *How many ways could he see one green light and three red lights?*

(c) *How many ways could he see all green lights?*

Problem 280 *In the following chart let n be the number of traffic lights and k be the number of green lights seen. In each square, write the number of ways this number of green lights could be seen while Steve drives down the street.*

Learning outcomes: Learning outcome goes here.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$							
$n=1$							
$n=2$							
$n=3$							
$n=4$							
$n=5$							
$n=6$							

Describe any patterns you see in your table and try to explain them in terms of traffic lights.

52 Pascal's Triangle: Fact or Fiction?

We investigate the connection between binomial coefficients and Pascal's triangle.

Problem 281 *Consider the numbers* $\binom{n}{k}$*. These numbers can be arranged into a "triangle" form that is popularly called "Pascal's Triangle". Assuming that the "top" entry is* $\binom{0}{0} = 1$*, we write the numbers row by row, with* n *fixed for each row. Write out the first 7 rows of Pascal's Triangle.*

Note that there are many patterns to be found. Your job is to justify the following patterns in the context of relevant models. Here are three patterns. Can you explain them?

(a) $\binom{n}{k} = \binom{n}{n-k}$.

(b) *The sum of the entries in each row is* 2^n.

(c) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

Learning outcomes: Learning outcome goes here.

53 You Can Count on It!

We count the ways selections can be made.

Problem 282 *The Diet-Lite restaurant offers 3 entrées, 4 side dishes, 2 desserts and 5 kinds of drinks.*

(a) *If you were going to select a dinner with one entrée and one side dish, how many different dinners could you order? Explain your reasoning.*

(b) *If you were going to select a dinner with one entrée, one side dish, one dessert, and one drink, how many different dinners could you order?*

Problem 283 *Suppose an Ohio license plate consists of two letters followed by two digits followed by two letters. How many different license plates can be made if:*

(a) *There are no more restrictions on the numbers or letters.*

(b) *There are no repeats of numbers or letters.*

Problem 284 *Naming officers and choosing a committee.*

(a) *How many ways can a chairperson, secretary, and treasurer be named in a club of 10 people?*

(b) *How many ways can a committee of 3 people be chosen from this same club?*

(c) *Explain using how the answer to (b) makes sense by beginning with the answer to (a) and then "adjusting" for overcounting.*

(d) *Generalize part (c) to explain a formula for the number of ways that a committee of k people can be chosen from a club of n members, where k and n are counting numbers with $k < n$.*

Learning outcomes: Learning outcome goes here.

Problem 285 *Six coins are flipped separately (e.g., HTHHHT is different from THHHTH). How many different results are possible?*

Problem 286 *A pizza shop always puts cheese on their pizzas. If the shop offers n additional toppings, how many different pizzas can be ordered? (Note: A plain cheese pizza is an option.)*

54 Which Road Should We Take?

We explore distributions of dice rolls.

Problem 287 *Consider a six-sided die. Without actually rolling a die, guess the number of 1's, 2's, 3's, 4's, 5's, and 6's you would obtain in 50 rolls. Record your predictions in the chart below:*

Predictions

# of 1's	*# of 2's*	*# of 3's*	*# of 4's*	*# of 5's*	*# of 6's*	*Total*

Now roll a die 50 times and record the number of 1's, 2's, 3's, 4's, 5's, and 6's you obtain.

Experimental Results

# of 1's	*# of 2's*	*# of 3's*	*# of 4's*	*# of 5's*	*# of 6's*	*Total*

How did you come up with your predictions? How do your predictions compare with your actual results? Now make a chart to combine your data with that of the rest of the class.

Experiment 1 We investigated the results of throwing one die and recording what we saw (a 1, a 2, ..., or a 6). We said that the probability of an event (for example, getting a "3" in this experiment) predicts the frequency with which we expect to see that event occur in a large number of trials. You argued the $P(\text{seeing } 3) = 1/6$ (meaning we expect to get a 3 in about 1/6 of our trials) because there were six different outcomes, only one of them is a 3, and you expected each outcome to occur about the same number of times.

Experiment 2 We are now investigating the results of throwing two dice and recording the sum of the faces. We are trying to analyze the probabilities associated with these sums. Let's focus first on $P(sum = 2) =?$. We might have some different theories, such as the following:

Learning outcomes: Learning outcome goes here.

Theory 1 $P(\text{sum} = 2) = 1/11$.

It is proposed that a sum of 2 was 1 out of the 11 possible sums $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Theory 2 $P(\text{sum} = 2) = 1/21$.

It is proposed that a sum of 2 was 1 of 21 possible results, counting $1 + 3$ as the same as $3 + 1$:

$$\begin{array}{cccccc}
1+1 & - & - & - & - & - \\
2+1 & 2+2 & - & - & - & - \\
3+1 & 3+2 & 3+3 & - & - & - \\
4+1 & 4+2 & 4+3 & 4+4 & - & - \\
5+1 & 5+2 & 5+3 & 5+4 & 5+5 & - \\
6+1 & 6+2 & 6+3 & 6+4 & 6+5 & 6+6
\end{array}$$

Problem 288 *Propose your own Theory 3.*

Problem 289 *Test all theories by computing $P(2)$, $P(3)$, ..., $P(12)$ for each theory and comparing to the dice rolls recorded by the class. What do you notice?*

Problem 290 *Which theory do you like best? Why?*

Problem 291 *How could we test our theory further?*

55 Lumpy and Eddie

We think about probabilities.

Two ancient philosophers, Lumpy and Eddie, were sitting on rocks flipping coins.

Problem 292 *Lumpy and Eddie wondered about the probability of obtaining both a head and a tail. Here is how it went:*

> *Eddie argued the following: "Look Lumpy, it's clear to me that when we flip two coins, we should get one of each about half the time because there are two possibilities: They're either the same or different." Lumpy, on the other hand, argued this way: "Eddie, stop being a wise guy! If we flipped two coins, we should expect both a head and tail to come up about a third of the time because there are only three possibilities: two heads, two tails, and one of each."*

Which, if any, of these two guys is right? Is there another answer?

Problem 293 *Next Lumpy and Eddie threw a third coin in the mix and wondered about the probability of obtaining 2 heads and a tail or 2 tails and a head.*

(a) *What would Lumpy say in this case?*

(b) *What would Eddie say in this case?*

Be sure to clearly explain why you think they would answer in the way you suggest.

Learning outcomes: Learning outcome goes here.

56 Probably More Probability

Problem 294 *The Diet-Lite restaurant offers 3 entrées, 4 side dishes, 2 desserts, and 5 kinds of drinks.*

(a) *If you picked a dish randomly, what is the probability you would select a dessert?*

(b) *If you randomly selected a dinner with one entrée, one side dish, one dessert, and one drink, what is the probability that your dinner would include orange juice as the beverage? (Assume only one of the drink choices is OJ.)*

Problem 295 *Suppose an Ohio License plate consists of two letters followed by two digits followed by two letters. Also assume that license plates are distributed randomly, that no license plates are forbidden, and that repeats of letters or digits are allowed (some of these things are not true in life, but assuming them makes our calculations easier).*

(a) *What is the probability that you will get the license plate $AA00AA$?*

(b) *What is the probability that the first letter of your license plate will be J?*

(c) *What is the probability that your license plate will have the letter Z somewhere on it?*

Learning outcomes:

Problem 296 *A class has 10 children, three of which are named Jay.*

(a) *What is the probability that if you choose a group of three children, exactly one of them will be named Jay?*

(b) *What is the probability that if you choose a group of three children, at least one of them will be named Jay?*

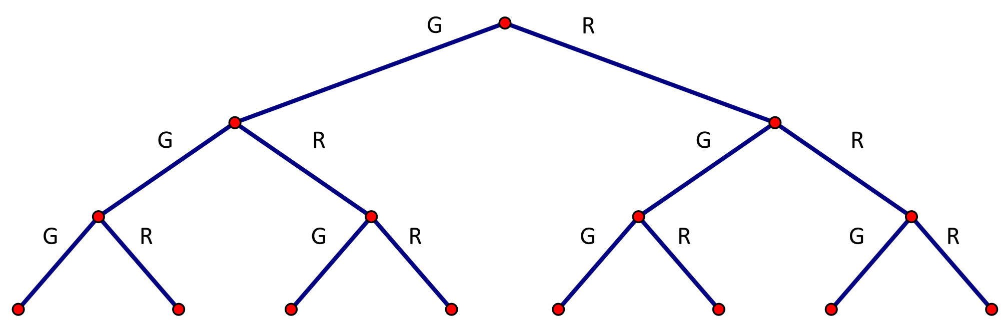
57 Go Climb a Tree!

We study probability using tree diagrams.

In this activity, we'll evaluate the probabilities of complex events using tree diagrams, fraction arithmetic, and counting.

Problem 297 *Place* ***three green*** *cubes and* ***three red*** *cubes in a bag. Without looking, draw three cubes, one at a time,* ***without replacement****.*

(a) *Write probabilities in the tree diagram below to organize the possible outcomes and compute their probabilities.*



(b) *What is the probability of three reds?*

(c) *What is the probability of two reds and a green?*

(d) *What is the probability of one red and two greens?*

(e) *What is the probability of three greens?*

(f) *If the first cube is green, what is the probability that the second cube is red?*

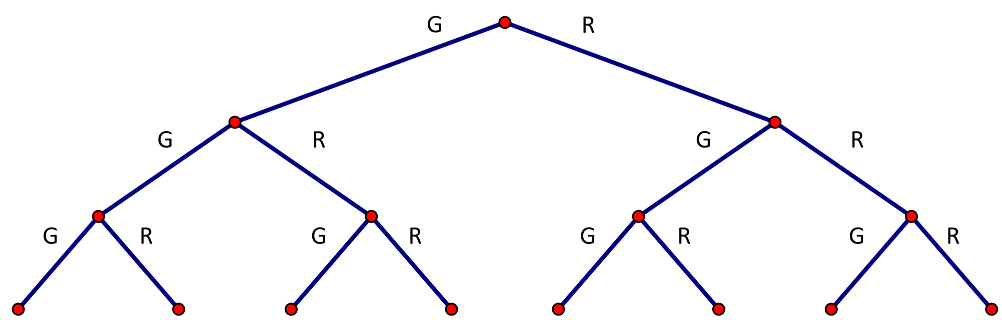
(g) *If the first cube is red, what is the probability that the second cube is red?*

(h) *What is the probability that the second cube is red?*

Learning outcomes: Learning outcome goes here.

Problem 298 *Place **three green** cubes and **three red cubes** in a bag. Without looking, draw three cubes, one at a time, **this time with replacement**.*

(a) *Write probabilities in the tree diagram below to organize the possible outcomes and compute their probabilities.*



(b) *What is the probability of three reds?*

(c) *What is the probability of two reds and a green?*

(d) *What is the probability of one red and two greens?*

(e) *What is the probability of three greens?*

(f) *If the first cube is green, what is the probability that the second cube is red?*

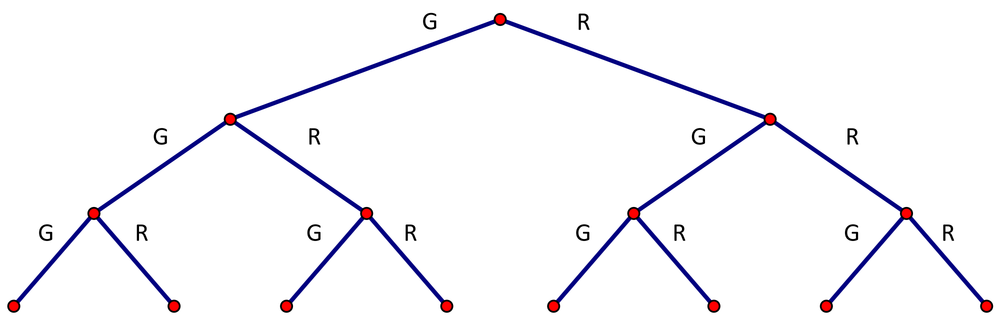
(g) *If the first cube is red, what is the probability that the second cube is red?*

(h) *What is the probability that the second cube is red?*

(i) *Use the words "dependent" and "independent" to contrast the probability that the second cube is red in this and the previous scenario.*

Problem 299 *This time, place* ***two green*** *cubes and* ***three red*** *cubes in a bag. Without looking, draw three cubes, one at a time,* ***again with replacement****.*

(a) *Write probabilities in the tree diagram below to organize the possible outcomes and compute their probabilities.*



(b) *What is the probability of three reds?*

(c) *What is the probability of two reds and a green?*

(d) *What is the probability of one red and two greens?*

(e) *What is the probability of three greens?*

(f) *If the first cube is green, what is the probability that the second cube is red?*

(g) *If the first cube is red, what is the probability that the second cube is red?*

(h) *What is the probability that the second cube is red?*

(i) *Is the probability that the second cube is red independent of the outcome of the first cube? Explain.*

(j) *In the previous scenario, some outcomes are equally likely. Is that true here? Why or why not?*

Problem 300 *Suppose the Indians and the Yankees are to face each other in a best-of-three series. Suppose the probability that the Indians will win any game is 60%, and suppose the outcomes of the games are independent of one another.*

(a) *Draw a tree diagram for the series.*

(b) *What is the probability that the Indians win games 1 and 3 to win the series?*

(c) *What is the probability that the Indians win the series in exactly 3 games?*

(d) *What is the probability that the Indians win the series?*

Problem 301 *Fred the Slob has an unreliable car that starts only 60% of the days. If the car doesn't start, poor Fred must walk the one block to work. This week, he is slated to work 5 days (Monday through Friday).*

(a) *What is the probability that Fred will walk on Monday and Wednesday and drive the other days?*

(b) *What is the probability that Fred will drive on exactly 4 of the days?*

(c) *What is the probability that poor Fred will have to walk on at least two of the days?*

58 They'll Fall for Anything!

We explore some common misconceptions.

What is incorrect about the following reasoning? Be specific!

Problem 302 *Herman says that if you pick a United States citizen at random, the probability of selecting a citizen from Indiana is because Indiana is one of 50 equally likely states to be selected.*

Problem 303 *Jerry has set up a game in which one wins a prize if he/she selects an orange chip from a bag. There are two bags to choose from. One has 2 orange and 4 green chips. The other bag has 7 orange and 7 green chips. Jerry argues that you have a better chance of winning by drawing from the second bag because there are more orange chips in it.*

Problem 304 *Gil the Gambler says that it is just as likely to flip 5 coins and get exactly 3 heads as it is to flip 10 coins and get exactly 6 heads because*

$$\frac{3}{5} = \frac{6}{10}$$

Problem 305 *We draw 4 cards without replacement from a deck of 52. Know-it-all Ned says the probability of obtaining all four 7's is* $\frac{4}{\binom{52}{4}}$ *because there are ways to select the* $\binom{52}{4}$ *4 cards and there are four 7's in the deck.*

Problem 306 *At a festival, Stealin' Stan gives Crazy Chris the choice of one of three prizes—each of which was hidden behind a door. One of the doors has a fabulous prize behind it while the other two doors each have a "zonk" (a free used tube of toothpaste, etc.). Crazy Chris chooses Door #1. Before opening that door, Stealin' Stan shows Chris that hidden behind Door #3 is a zonk and gives Chris the option to keep Door #1 or switch to Door #2. Chris says, "Big deal. It doesn't help my chances of winning to switch or not switch."*

Learning outcomes: Learning outcome goes here.