PARALLELS IN GEOMETRY

MATH 1166: SPRING 2021

THIS DOCUMENT WAS TYPESET ON JANUARY 11, 2021.

This document was typeset on January 11, 2021.

Preface

These notes are designed with future middle grades mathematics teachers in mind. While most of the material in these notes would be accessible to an accelerated middle grades student, it is our hope that the reader will find these notes both interesting and challenging. In some sense we are simply taking the topics from a middle grades class and pushing "slightly beyond" what one might typically see in schools. In particular, there is an emphasis on the ability to communicate mathematical ideas. Three goals of these notes are:

- To enrich the reader's understanding of both numbers and algebra. From the basic algorithms of arithmetic—all of which have algebraic underpinnings—to the existence of irrational numbers, we hope to show the reader that numbers and algebra are deeply connected.
- To place an emphasis on problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." Some worthwhile problems take time to solve and cannot be done in a single sitting.
- To challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and personal discovery followed by justification and explanation. We wish to convey this to the reader, and sincerely hope that the reader will pass this on to others as well.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered; other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

?

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then we encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

`http://www.math.osu.edu/~snapp/1165/`

`http://www.math.osu.edu/~findell.2`

Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at `snapp@math.osu.edu` or Brad Findell at `findell.2@osu.edu`

Thanks and Acknowledgments

This document has a somewhat lengthy history. In the Fall of 2005 and Spring of 2006, Bart Snapp gave a set of lectures at the University of Illinois at Urbana-Champaign. His lecture notes were typed and made available as an open-source textbook. During subsequent semesters, those notes were revised and modified under the supervision of Alison Ahlgren and Bart Snapp. Many people have made contributions, including Tom Cooney, Melissa Dennison, and Jesse Miller. A number of students also contributed to that document by either typing original hand-written notes or suggesting problems. They are: Camille Brooks, Michelle Bruno, Marissa Colatosti, Katie Colby, Anthony 'Tino' Forneris, Amanda Genovise, Melissa Peterson, Nicole Petschenko, Jason Reczek, Christina Reincke, David Seo, Adam Shalzi, Allice Son, Katie Strle, and Beth Vaughn.

In 2009, Greg Williams, a Master of Arts in Teaching student at Coastal Carolina University, worked with Bart Snapp to produce an early draft of the chapter on isometries.

In the Winter of 2010 and 2011, Bart Snapp gave a new set of lectures at the Ohio State University. In this course the previous lecture notes were heavily modified, resulting in a new text *Parallels in Geometry*. Since 2012, Bart Snapp and Brad Findell have continued revising these notes annually. In particular, during 2014 and 2015, exposition and activities were added to address ideas from the Common Core State Standards (CCSS). Many of the individual standards are included as margin notes that begin "CCSS."

Contents

1 Proof by Picture

A picture is worth a thousand words.

—Unknown

1.1 Basic Set Theory

The word *set* has more definitions in the dictionary than any other word. In our case we'll use the following definition:

Definition A **set** is any collection of elements for which we can always tell whether an element is in the set or not.

Question What are some examples of sets? What are some examples of things that are not sets?

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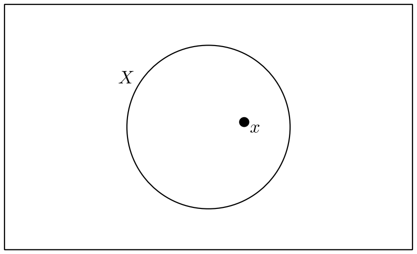
If we have a set X and the element x is inside of X, we write:

$$x \in X$$

This notation is said "x in X." Pictorially we can imagine this as:



Sometimes the elements of a set can be listed or described by words or formulas. In such cases, we often use curly braces { and } to enclose the elements of the set or a description of the set. For example, if $X = \{2, 3, 7\}$, and $Y = \{\text{even numbers}\}$, then each of the following statements are true:

$$2 \in X \qquad 4 \notin X \qquad 6 \in Y \qquad 9 \notin Y.$$

Definition A **subset** Y of a set X is a set Y such that every element of Y is also an element of X. We denote this by:

$$Y \subseteq X$$

If Y is contained in X, we will sometimes loosely say that X is *bigger* than Y.

Question Can you think of a set X and a subset Y where saying X is bigger than Y is a bit misleading?

?

Sometimes it is useful to list a set of sets. For example, if $X = \{2, 3, 7\}$, then

$$Y = \{\{2\}, \{2, 7\}, \{3, 7\}\}$$

is a set containing a few subsets of X.

Question How many elements are in the set Y?

?

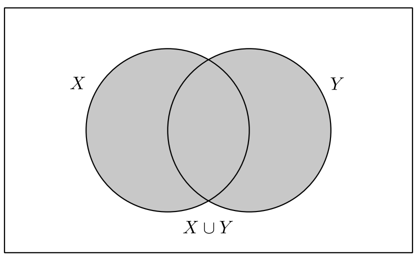
Question How is the meaning of the symbol $\in$ different from the meaning of the symbol $\subseteq$?

?

1.1.1 Union

Definition Given two sets X and Y, X **union** Y is the set of all the elements in X or Y. We denote this by $X \cup Y$.

Pictorially, we can imagine this as:



Warning Note that this definition uses the *inclusive or*. In everyday language, it is common to use the word “or” in an exclusive sense, meaning, “but not both.” But in mathematics, the word “or” is almost always used inclusively. Thus, the phrase “A or B” allows for the possibility of both.

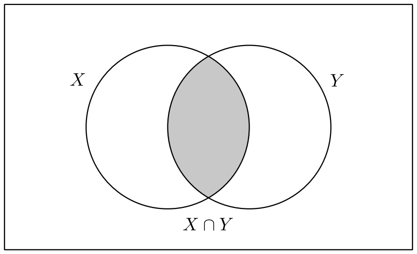
Question What about the above picture shows that “or” is used inclusively in the definition of union?

?

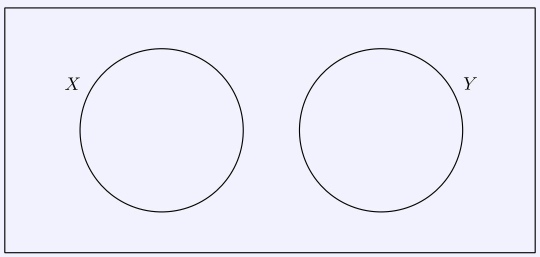
1.1.2 Intersection

Definition Given two sets X and Y, X **intersect** Y is the set of all the elements that are simultaneously in X and in Y. We denote this by $X \cap Y$.

Pictorially, we can imagine this as:



Question Consider the sets X and Y below:



What is $X \cap Y$?

I'll take this one: Nothing! The set with no elements is called the **empty set**. We sometimes denote the empty set as $\{\}$, but it is more common to denote the empty set with the symbol $\varnothing$.

Question How is $\{\varnothing\}$ different from $\varnothing$?

?

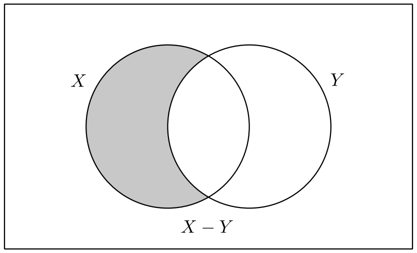
Question The empty set is a subset of every set. Why does this makes sense? Why does it make sense to say *the* empty set rather than *an* empty set?

?

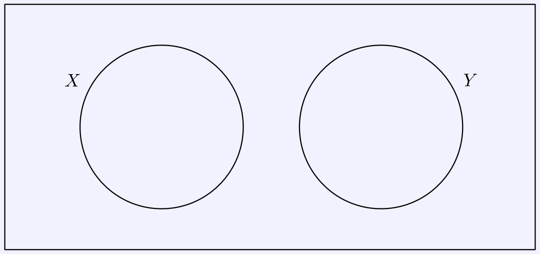
1.1.3 Complement

Definition Given two sets X and Y, X **complement** Y is the set of all the elements that are in X and are not in Y. We denote this by $X - Y$.

Pictorially, we can imagine this as:



Question Check out the two sets below:



What is $X - Y$? What is $Y - X$?

?

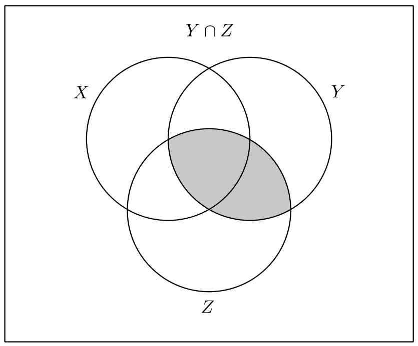
1.1.4 Putting Things Together

OK, let's try something more complex:

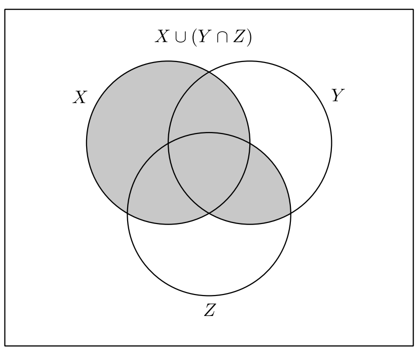
Question Prove that:

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

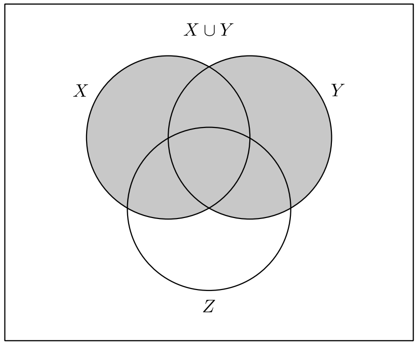
Proof *Look at the left-hand side of the equation first. We can represent the elements in $Y \cap Z$ with shaded region in the following diagram:*



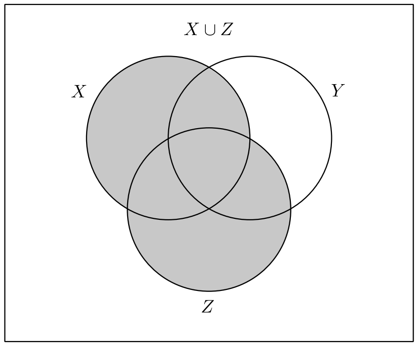
So the union of this region with X is represented the shaded region in this diagram.



Now, looking at the right-hand side of the equation, $X \cup Y$ is represented by this *shaded region:*

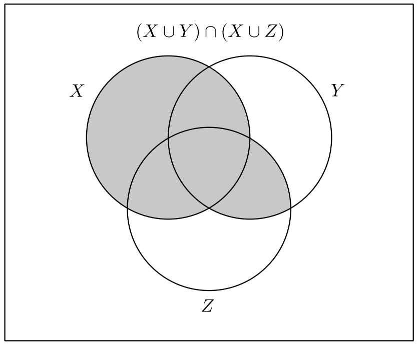


And $X \cup Z$ is represented by this shaded region:



The region shaded in both of the diagrams, which is the intersection of $X \cup Y$

and $X \cup Z$, is represented by the shaded region below.



Comparing the diagrams representing the left-hand and right-hand sides of the equation, we see that the same regions are shaded, and so we are done.

Problems for Section 1.1

(1) Given two sets X and Y, explain what is meant by $X \cup Y$.

(2) Given two sets X and Y, explain what is meant by $X \cap Y$.

(3) Given two sets X and Y, explain what is meant by $X - Y$.

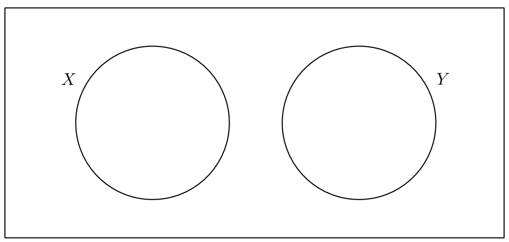
(4) Explain the difference between the symbols $\in$ and $\subseteq$.

(5) How is $\{\varnothing\}$ different from $\varnothing$?

(6) List all the subsets of the set $X = \{2, 3, 5, 7\}$. In general, how many subsets are there of an n-element set? Explain why this makes sense. Does your formula work for a 0-element set? Explain.

(7) Draw a Venn diagram for the set of elements that are in X or Y *but not both*. How does it differ from the Venn diagram for $X \cup Y$?

(8) If we let X be the set of "right triangles" and we let Y be the set of "equilateral triangles" does the picture below show the relationship between these two sets?



Explain your reasoning.

(9) If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{3, 4, 5, 6\}$ find:

(a) $X \cup Y$
(b) $X \cap Y$
(c) $X - Y$
(d) $Y - X$

In each case explain your reasoning.

(10) Let $n\mathbb{Z}$ represent the integer multiples of n. So for example:

$$3\mathbb{Z} = \{\dots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$$

Compute the following:

(a) $3\mathbb{Z} \cap 4\mathbb{Z}$
(b) $2\mathbb{Z} \cap 5\mathbb{Z}$
(c) $3\mathbb{Z} \cap 6\mathbb{Z}$
(d) $4\mathbb{Z} \cap 6\mathbb{Z}$
(e) $4\mathbb{Z} \cap 10\mathbb{Z}$

In each case explain your reasoning.

(11) Make a general rule for intersecting sets of the form $n\mathbb{Z}$ and $m\mathbb{Z}$. Explain why your rule works.

(12) Prove that:
$$X = (X \cap Y) \cup (X - Y)$$

(13) Prove that:
$$X - (X - Y) = (X \cap Y)$$

(14) Prove that:
$$X \cup (Y - X) = (X \cup Y)$$

(15) Prove that:
$$X \cap (Y - X) = \varnothing$$

(16) Prove that:
$$(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$$

(17) Prove that:
$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

(18) Prove that:
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

(19) Prove that:
$$X - (Y \cap Z) = (X - Y) \cup (X - Z)$$

(20) Prove that:
$$X - (Y \cup Z) = (X - Y) \cap (X - Z)$$

(21) If $X \cup Y = X$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

(22) If $X \cap Y = X$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

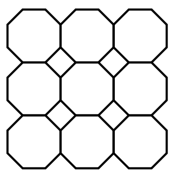
(23) If $X - Y = \varnothing$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

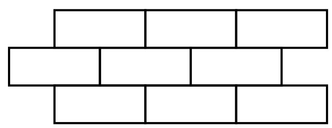
1.2 Tessellations

Go to the Internet and look up M.C. Escher. He was an artist. Look at some of his work. When you do your search be sure to include the word "tessellation" Back already? Very good. Sometimes Escher worked with tessellations. What's a tessellation? I'm glad you asked:

Definition A **tessellation** is a pattern of polygons fitted together to cover the entire plane without overlapping.

While it is impossible to actually cover the entire plane with shapes, if we give you enough of a tessellation, you should be able to continue it's pattern indefinitely. Here are pieces of tessellations:

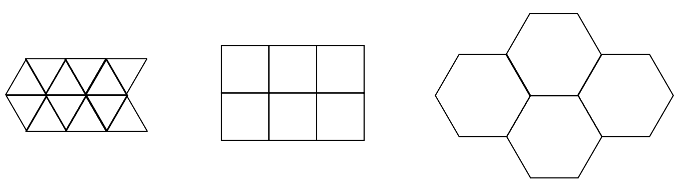




On the left we have a tessellation of a square and an octagon. On the right we have a "brick-like" tessellation.

Definition A tessellation is called a **regular tessellation** if it is composed of copies of a single regular polygon and these polygons meet vertex to vertex.

Example 1.2.1) Here are some examples of regular tessellations:



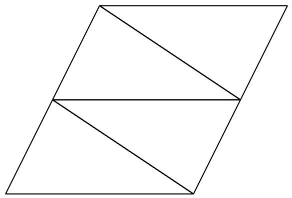
Johannes Kepler, who lived from 1571–1630, was one of the first people to study tessellations. He certainly knew the next theorem:

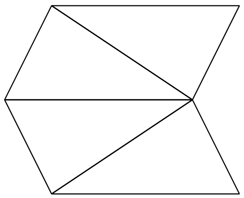
Theorem 1.2.2 *There are only* 3 *regular tessellations.*

Question Why is the theorem above true?

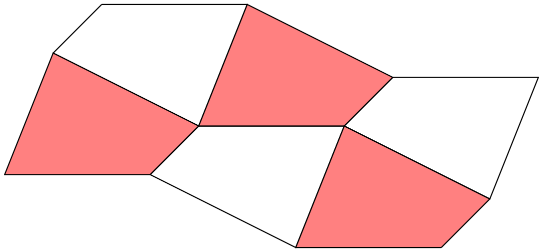
?

Since one can prove that there are only three regular tessellations, and we have shown three above, then that is all of them. On the other hand there are lots of nonregular tessellations. Here are two different ways to tessellate the plane with a triangle:





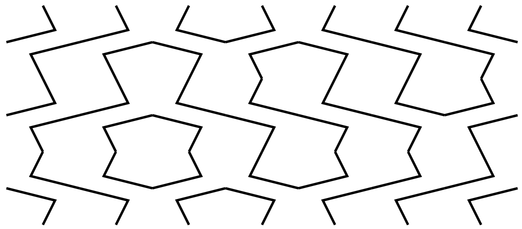
Here is a way that you can tessellate the plane with any old quadrilateral:



1.2.1 Tessellations and Art

How does one make art with tessellations? To start, a little decoration goes a long way. Check this out: Decorate two squares as such:

Tessellate them randomly in the plane to get this lightning-like picture:



Question What sort of picture do you get if you tessellate these decorated squares randomly in a plane?

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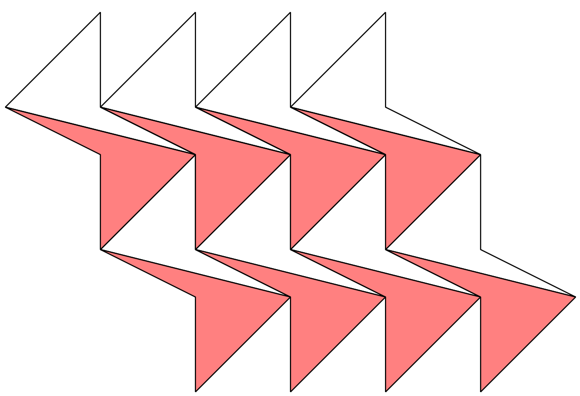
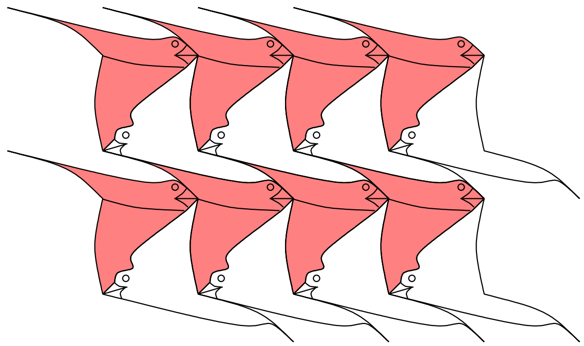
Another way to go is to start with your favorite tessellation:



Then you modify it a bunch to get something different:



Question What kind of art can you make with tessellations?

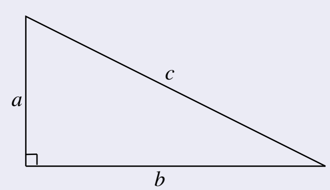
?

I'm not a very good artist, but I am a mathematician. So let's use a tessellation to give a proof! Let me ask you something:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

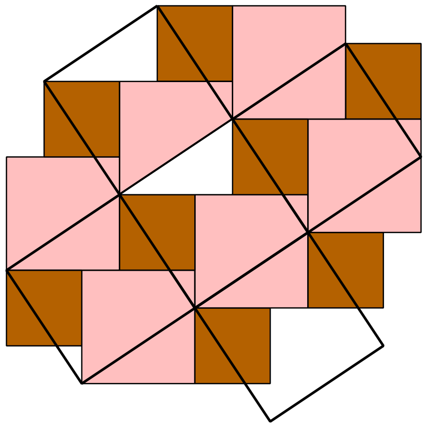
Theorem 1.2.3 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is the length of the hypotenuse,*



then

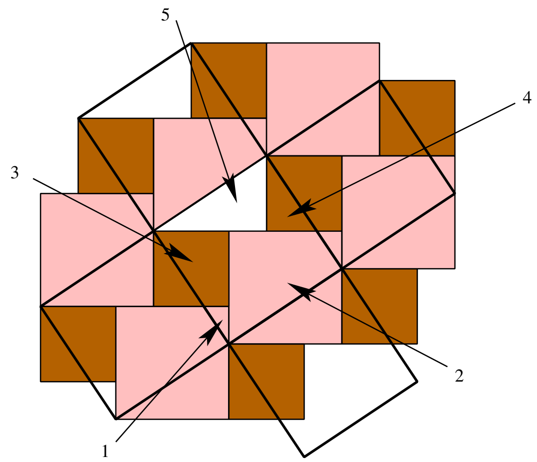
$$a^2 + b^2 = c^2.$$

Let's give a proof! Check out this tessellation involving 2 squares:



Question How does the picture above "prove" the Pythagorean Theorem?

Proof (Solution) *The white triangle is our right triangle. The area of the middle overlaid square is c^2, the area of the small dark squares is a^2, and the area of the medium lighter square is b^2. Now label all the "parts" of the large overlaid* *square:*



From the picture we see that

$$a^2 = \{3 \text{ and } 4\}$$
$$b^2 = \{1, 2, \text{ and } 5\}$$
$$c^2 = \{1, 2, 3, 4, \text{ and } 5\}$$

Hence

$$c^2 = a^2 + b^2$$

Since we can always put two squares together in this pattern, this proof will work for any right triangle.

Problems for Section 1.2

(1) Show two different ways of tessellating the plane with a given scalene triangle. Label your picture as necessary.

(2) Show how to tessellate the plane with a given quadrilateral. Label your picture.

(3) Show how to tessellate the plane with a nonregular hexagon. Label your picture.

(4) Give an example of a polygon with 9 sides that tessellates the plane.

(5) Give examples of polygons that tessellate and polygons that do not tessellate.

(6) Give an example of a triangle that tessellates the plane where both 4 and 8 angles fit around each vertex.

(7) True or False: Explain your conclusions.

- (a) There are exactly 5 regular tessellations.
- (b) Any quadrilateral tessellates the plane.
- (c) Any triangle will tessellate the plane.
- (d) If a triangle is used to tessellate the plane, then it is always the case that exactly 6 angles will fit around each vertex.
- (e) If a polygon has more than 6 sides, then it cannot tessellate the plane.

(8) Given a regular tessellation, what is the sum of the angles around a given vertex?

(9) Given that the regular octagon has 135 degree angles, explain why you cannot give a regular tessellation of the plane with a regular octagon.

(10) Fill in the following table:

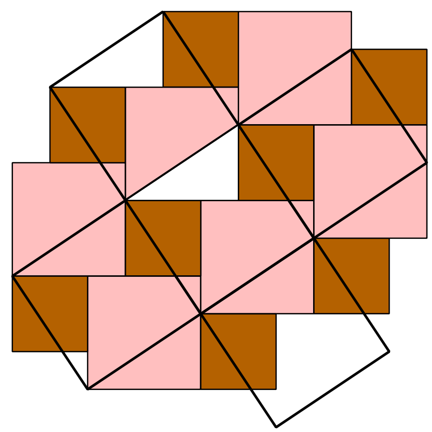
Regular n-gon	Does it tessellate?	Measure of an angle	If it tessellates, how many surround each vertex?
3-gon			
4-gon			
5-gon			
6-gon			
7-gon			
8-gon			
9-gon			
10-gon			

Hint: A regular n-gon has interior angles of $180(n-2)/n$ degrees.

- (a) What do the shapes that tessellate have in common?
- (b) Make a graph with the number of sides of an n-gon on the horizontal axis and the measure of a single angle on the vertical axis. Briefly describe the relationship between the number of sides of a regular n-gon and the measure of one of its angles.
- (c) What regular polygons *could* a bee use for building hives? Give some reasons that bees seem to use hexagons.

(11) Considering that the regular n-gon has interior angles of $180(n-2)/n$ degrees, and Problem (10) above, prove that there are only 3 regular tessellations of the plane.

(12) Explain how the following picture "proves" the Pythagorean Theorem.



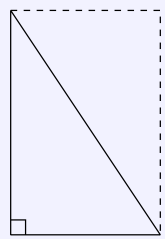
1.3 Proof by Picture

Pictures generally do not constitute a proof on their own. However, a good picture can show insight and communicate concepts better than words alone. In this section we will show you pictures giving the idea of a proof and then ask you to supply the words to finish off the argument.

1.3.1 Proofs Involving Right Triangles

Let's start with something easy:

Question Explain how the following picture "proves" that the area of a right triangle is half the base times the height.



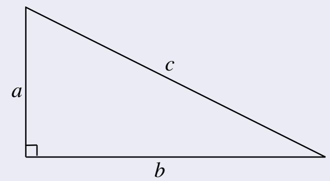
?

That wasn't so bad was it? Now for a game of *whose-who*:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

Theorem 1.3.1 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is* *the length of the hypotenuse,*



then

$$a^2 + b^2 = c^2.$$

Question What is the converse to the Pythagorean Theorem? Is it true? How do you prove it?

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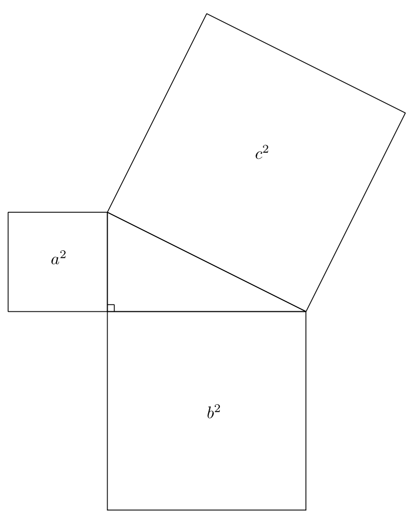
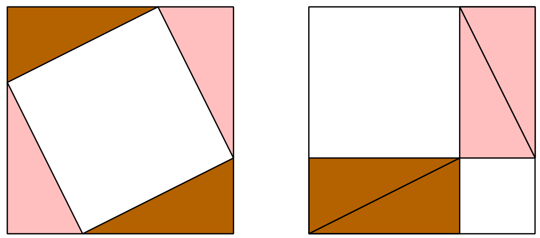
While everyone may know the Pythagorean Theorem, not as many know how to prove it. Euclid's proof goes kind of like this:

Consider the following picture:



Now, cut up the squares a^2 and b^2 in such a way that they fit into c^2 perfectly. When you give a proof that involves cutting up the shapes and putting them back together, it is called a **dissection proof**. The trick to ensure that this is actually a proof is in making sure that your dissection will work no matter what right triangle you are given. Does it sound complicated? Well it can be.

Is there an easier proof? Sure, look at:



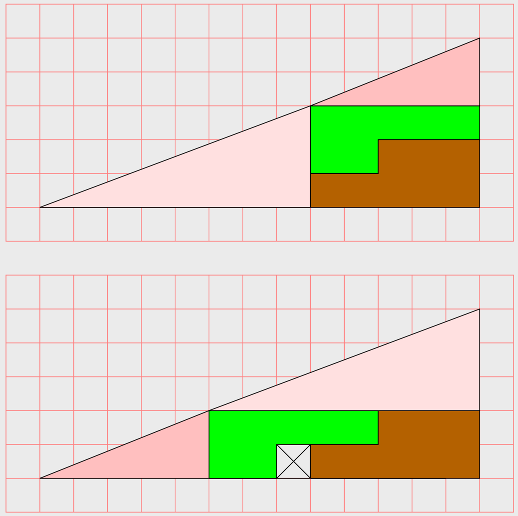
Question How does the picture above "prove" the Pythagorean Theorem?

Proof (Solution) *Both of the large squares above are the same size. Inside the large squares, the shaded triangles have been rearranged. Thus, the unshaded regions of the two figures above must have the same area. The large white square on the left has an area of* c^2 *and the two white squares on the right have a combined area of* $a^2 + b^2$. *Thus we see that:*

$$c^2 = a^2 + b^2$$

Now a paradox:

Paradox What is wrong with this picture?



Question How does this happen?

?

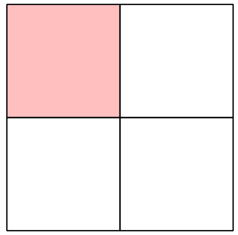
1.3.2 Proofs Involving Boxy Things

Consider the problem of *Doubling the Cube*. If a mathematician asks us to double a cube, he or she is asking us to double the **volume** of a given cube. One may be tempted to merely double each side, but this doesn't double the volume!

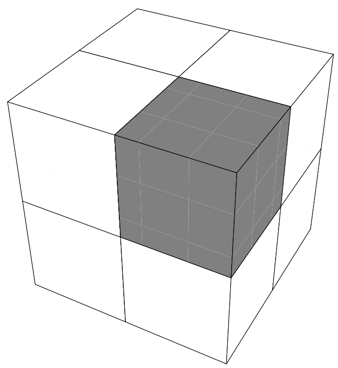
Question Why doesn't doubling each side of the cube double the volume of the cube?

?

Well, let's answer an easier question first. How do you double the area of a square? Does taking each side and doubling it work?



No! You now have four times the area. So you **cannot** double the area of a square merely by doubling each side. What about for the cube? Can you double the volume of a cube merely by doubling the length of every side? Check this out:



Ah, so the answer is again no. If you double each side of a cube you have 8 times the volume.

> **Question** What happens to the area of a square if you multiply the sides by an arbitrary integer? What about the volume of a cube? Can you explain what is happening here?

?

1.3.3 Proofs Involving Infinite Sums

As is our style, we will start off with a question:

Question Can you add up an infinite number of terms and still get a finite number?

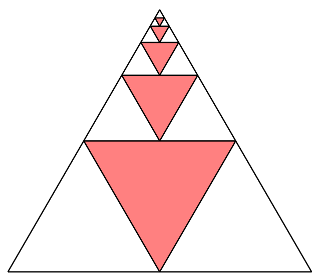
Consider 1/3. Actually, consider the decimal notation for 1/3:

$$\frac{1}{3} = .33333333333333333333333333333\ldots$$

But this is merely the sum:

$$.3 + .03 + .003 + .0003 + .00003 + .000003 + \cdots$$

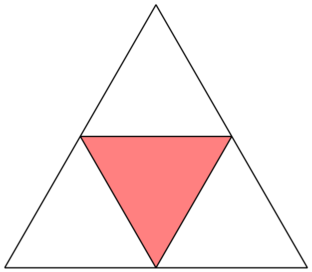
It stays less than 1 because the terms get so small so quickly. Are there other infinite sums of this sort? You bet! Check out this picture:



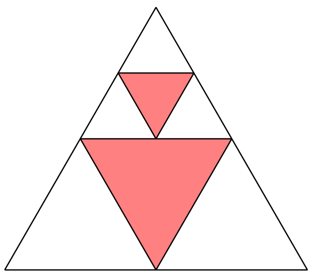
Question Explain how the picture above "proves" that:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots = \frac{1}{3}$$

Proof (Solution) *Let's take it in steps. If the big triangle has area 1, the area* *of the shaded region below is* $1/4$.



We also see that the area of the shaded region below



is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2$$

Continuing on in this fashion we see that the area of all the shaded regions is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots$$

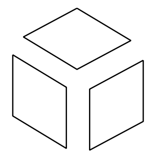
But look, the unshaded triangles have twice as much area as the shaded triangle. Thus the shaded triangles must have an area of $1/3$.

1.3.4 Thinking Outside the Box

A *calisson* is a French candy that sort of looks like two equilateral triangles stuck together. They usually come in a hexagon-shaped box.

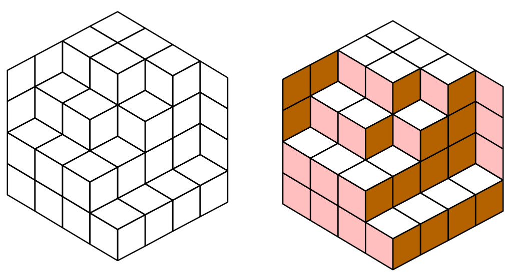
Question How do the calissons fit into their hexagon-shaped box?

If you start to put the calissons into a box, you quickly see that they can be placed in there with exactly three different orientations:



Theorem 1.3.2 *In any packing, the number of calissons with a given orientation is exactly one-third the total number of calissons in the box.*

Look at this picture:



Question How does the picture above "prove" Theorem 1.3.2? Hint: Think outside the box!

?

Problems for Section 1.3

(1) Explain the rule

even + even = even

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(2) Explain the rule

odd + even = odd

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(3) Explain the rule

odd + odd = even

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(4) Explain the rule

even · even = even

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(5) Explain the rule

odd · odd = odd

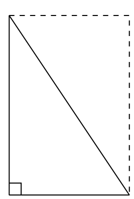
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(6) Explain the rule

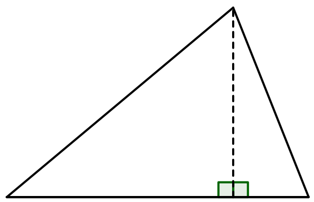
odd · even = even

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

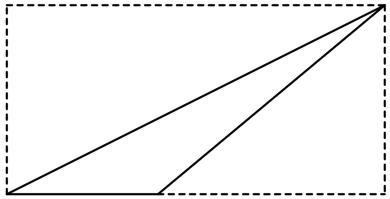
(7) Explain how the following picture "proves" that the area of a right triangle is half the base times the height.



(8) Suppose you know that the area of a **right** triangle is half the base times the height. Explain how the following picture "proves" that the area of **every** triangle is half the base times the height.

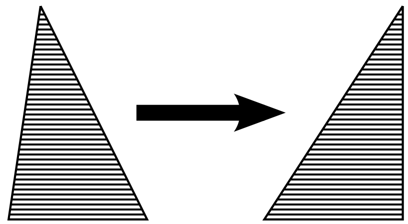


Now suppose that *Geometry Giorgio* attempts to solve a similar problem. Again knowing that the area of a right triangle is half the base times the height, he draws the following picture:

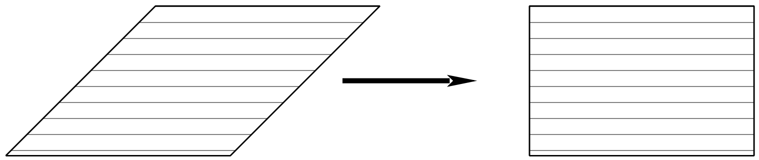


Geometry Giorgio states that the diagonal line cuts the rectangle in half, and thus the area of the triangle is half the base times the height. Is this correct reasoning? If so, give a complete explanation. If not, give correct reasoning based on *Geometry Giorgio*'s picture.

(9) Suppose you know that the area of a **right** triangle is half the base times the height. Explain how the following picture "proves" that the area of any triangle is half the base times the height. Note, this way of thinking is the basis for Cavalieri's Principle.

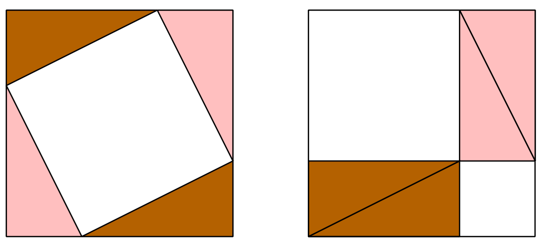


(10) Explain how the following picture "proves" that the area of any parallelogram is base times height. Note, this way of thinking is the basis for Cavalieri's Principle.

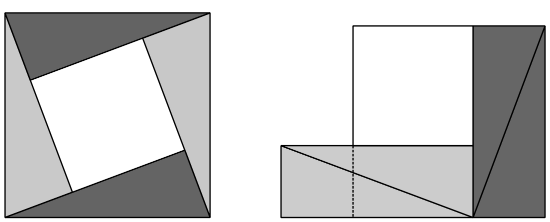


(11) Explain how to use a picture to "prove" that a triangle of a given area could have an arbitrarily large perimeter.

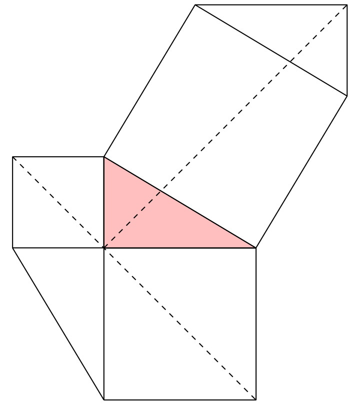
(12) Give two explanations of how the following picture "proves" the Pythagorean Theorem, one using algebra and one without algebra.



(13) Give two explanations of how the following picture "proves" the Pythagorean Theorem, one using algebra and one without algebra.

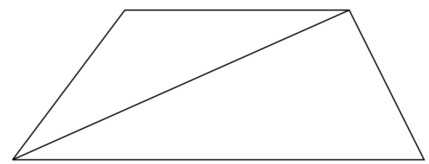


(14) Explain how the following picture "proves" the Pythagorean Theorem.



Note: This proof is due to Leonardo da Vinci.

(15) Recall that a trapezoid is a quadrilateral with two parallel sides. Consider the following picture:

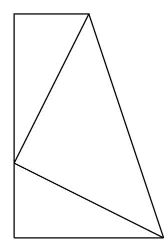


How does the above picture prove that the area of a trapezoid is

$$\text{area} = \frac{h(b_1 + b_2)}{2},$$

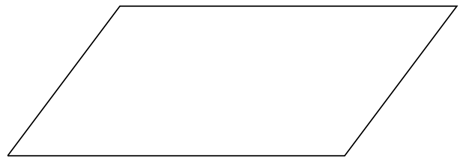
where h is the height of the trapezoid and b_1, b_2, are the lengths of the parallel sides?

(16) Explain how the following picture "proves" the Pythagorean Theorem.



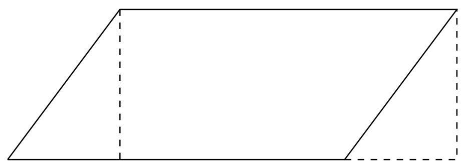
Note: This proof is due to James A. Garfield, the 20th President of the United States.

(17) Look at Problem (15). Can you use a similar idea to prove that the area of a parallelogram

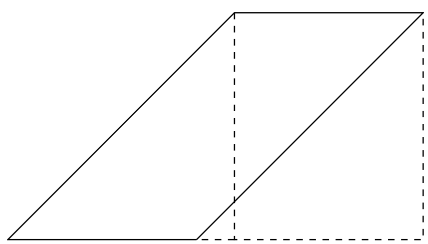


is the length of the base times the height?

(18) Explain how the following picture "proves" that the area of a parallelogram is base times height.



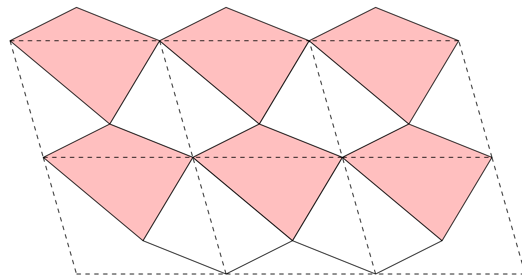
Now suppose that *Geometry Giorgio* attempts to solve a similar problem. In an attempt to prove the formula for the area of a parallelogram, he draws the following picture:



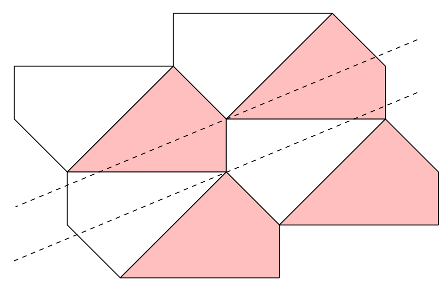
At this point *Geometry Giorgio* says that he has proved the formula for area of a parallelogram. What do you think of his picture? Give a complete argument based on his picture, adding labels to support your reasoning.

(19) Which of the above "proofs" for the formula for the area of a parallelogram is your favorite? Explain why.

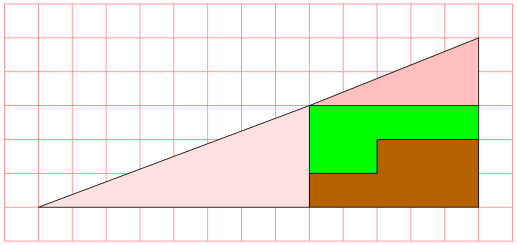
(20) Explain how the following picture "proves" that the area of a quadrilateral is equal to half of the area of the parallelogram whose sides are parallel to and equal in length to the diagonals of the original quadrilateral.

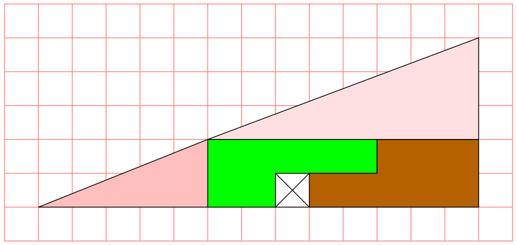


(21) Explain how the following picture "proves" that if a quadrilateral has two opposite angles that are equal, then the bisectors of the other two angles are parallel or on top of each other.

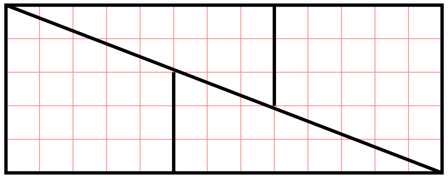


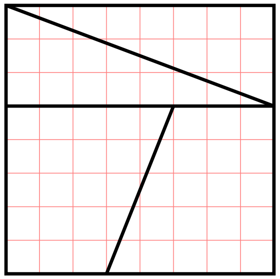
(22) Why might someone find the following picture disturbing? How would you assure them that actually everything is good and well in the geometrical world?





(23) Why might someone find the following picture disturbing? How would you assure them that actually everything is good and well in the geometrical world?

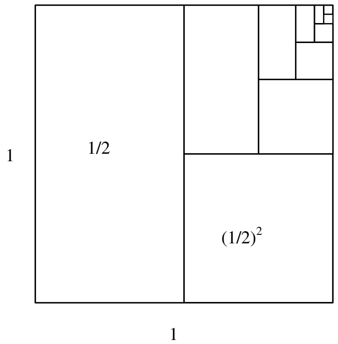




(24) How could you explain to someone that doubling the lengths of each side of a cube does not double the volume of the cube?

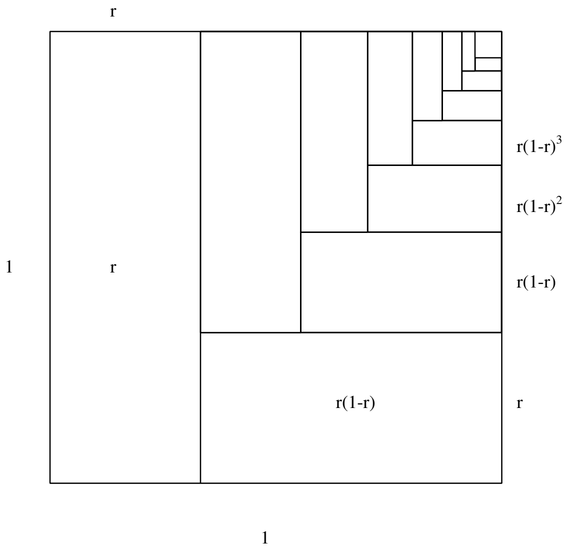
(25) Explain how the following picture "proves" that:

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots = 1$$



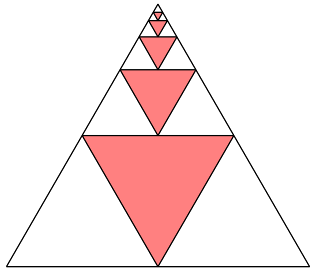
(26) Explain how the following picture "proves" that if $0 < r < 1$:

$$r + r(1-r) + r(1-r)^2 + r(1-r)^3 + \cdots = 1$$



(27) Explain how the following picture "proves" that:

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\cdots=\frac{1}{3}$$

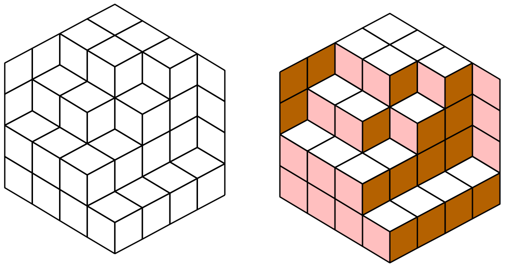


(28) Considering Problem (25), Problem (26), and Problem (27) can you give a new picture "proving" that:

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\cdots=\frac{1}{3}$$

Carefully explain the connection between your picture and the mathematical expression above.

(29) Explain how the following picture "proves" that in any packing, the number of calissons with a given orientation is exactly one-third the total number of calissons in the box.



2 Compass and Straightedge Constructions

Mephistopheles:	I must say there is an obstacle That prevents my leaving: It's the pentagram on your threshold.
Faust:	The pentagram impedes you? Tell me then, you son of hell, If this stops you, how did you come in?
Mephistopheles:	Observe! The lines are poorly drawn; That one, the outer angle, Is open, the lines don't meet.

—Göthe, *Faust* act I, scene III

2.1 Constructions

About a century before the time of Euclid, Plato—a student of Socrates—declared that the compass and straightedge should be the only tools of the geometer. Why would he do such a thing? For one thing, both the the compass and straightedge are fairly simple instruments. One draws circles, the other draws lines—what else could possibly be needed to study geometry? Moreover, rulers and protractors are far more complex in comparison and people back then couldn't just walk to the campus bookstore and buy whatever they wanted. However, there are other reasons:

(1) Compass and straightedge constructions are **independent of units**.

(2) Compass and straightedge constructions are **theoretically correct**.

(3) Combined, the compass and straightedge seem like **powerful tools**.

Compass and straightedge constructions are **independent of units**. Whether you are working in centimeters or miles, compass and straightedge constructions work just as well. By not being locked to set of units, the constructions given by a compass and straightedge have certain generality that is appreciated even today.

Compass and straightedge constructions are **theoretically correct**. In mathematics, a correct method to solve a problem is more valuable than a correct solution. In this sense, the compass and straightedge are ideal tools for the mathematician. Easy enough to use that the rough drawings that they produce can be somewhat relied upon, yet simple enough that the tools themselves can be described theoretically. Hence it is usually not too difficult to connect a given construction to a formal proof showing that the construction is correct.

Combined, the compass and straightedge seem like **powerful tools**. No tool is useful unless it can solve a lot of problems. Without a doubt, the compass and straightedge combined form a powerful tool. Using a compass and straightedge, we are able to solve many problems exactly. Of the problems that we cannot solve exactly, we can always produce an approximate solution.

We'll start by giving the rules of compass and straightedge constructions:

Rules for Compass and Straightedge Constructions

(1) You may only use a compass and straightedge.

(2) You must have two points to draw a line.

(3) You must have a point and a line segment to draw a circle. The point is the center and the line segment gives the radius.

(4) Points can only be placed in two ways:

    (a) As the intersection of lines and/or circles.

    (b) As a **free point**, meaning the location of the point is not important for the final outcome of the construction.

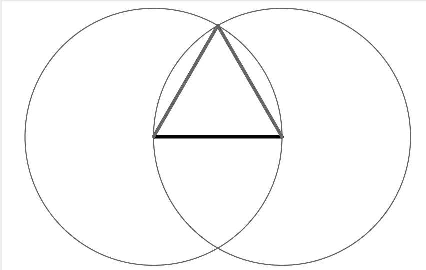
Our first construction is also Euclid's first construction:

Construction (Equilateral Triangle) We wish to construct an equilateral triangle given the length of one side.

(1) Open your compass to the width of the line segment.

(2) Draw two circles, one with the center being each end point of the line segment.

(3) The two circles intersect at two points. Choose one and connect it to both of the line segment's endpoints.



Euclid's second construction will also be our second construction:

Construction (Transferring a Segment) Given a segment, we wish to move it so that it starts on a given point, on a given line.

(1) Draw a line through the point in question.

(2) Open your compass to the length of the line segment and draw a circle with the given point as its center.

(3) The line segment consisting of the given point and the intersection of the circle and the line is the transferred segment.

If you read *The Elements*, you'll see that Euclid's construction is much more complicated than ours. Apparently, Euclid felt the need to justify the ability to move a distance. Many sources say that Euclid used what is called a *collapsing compass*, that is a compass that collapsed when it was picked up. However, I do not believe that such an invention ever existed. Rather this is something that lives in the conservative geometer's head.

Regardless of whether the difficulty of transferring distances was theoretical or physical, we need not worry when we do it. In fact, Euclid's proof of the above theorem proves that our modern way of using the compass to transfer distances is equivalent to using the so-called collapsing compass.

Question Exactly how would one prove that the modern compass is equivalent to the collapsing compass? Hint: See Euclid's proof.

?

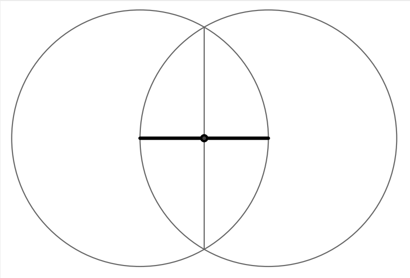
Construction (Bisecting a Segment) Given a segment, we wish to cut it in half.

(1) Open your compass to the width of the segment.

(2) Draw two circles, one with the center being at each end point of the line segment.

(3) The circles intersect at two points. Draw a line through these two points.

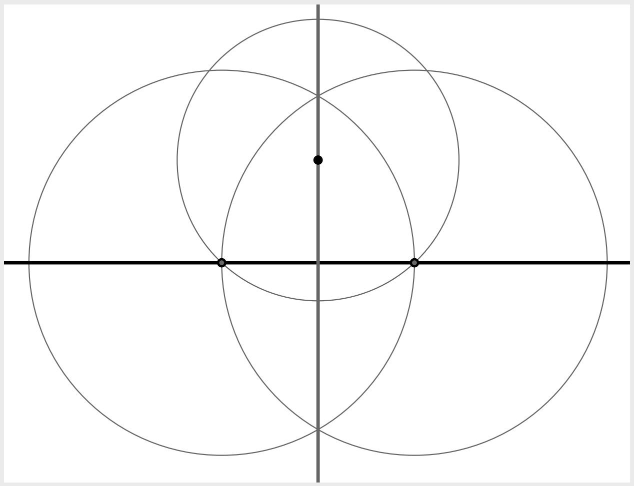
(4) The new line bisects the original line segment.



Construction (Perpendicular to a Line through a Point) Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.

(1) Draw a circle centered at the point large enough to intersect the line in two distinct points.

(2) Bisect the line segment. The line used to do this will be the desired line.

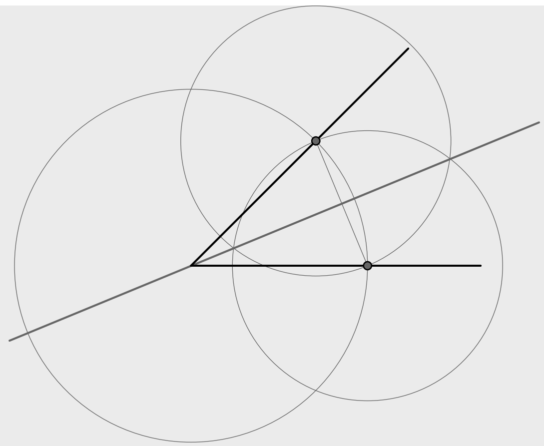


Construction (Bisecting an Angle) We wish to divide an angle in half.

(1) Draw a circle with its center being the vertex of the angle.

(2) Draw a line segment where the circle intersects the lines.

(3) Bisect the new line segment. The bisector will bisect the angle.



We now come to a very important construction:

Construction (Copying an Angle) Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.

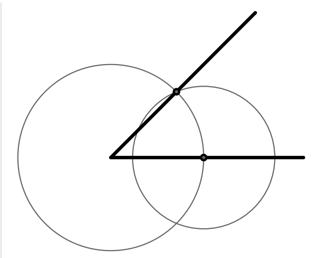
(1) Open the compass to a fixed width and make a circle centered at the vertex of the angle.

(2) Make a circle of the same radius on the line with the point.

(3) Open the compass so that one end touches the 1st circle where it hits an edge of the original angle, with the other end of the compass extended to where the 1st circle hits the other edge of the original angle.

(4) Draw a circle with the radius found above with its center where the second circle hits the line.

(5) Connect the point to where the circles meet. This is the other leg of the angle we are constructing.

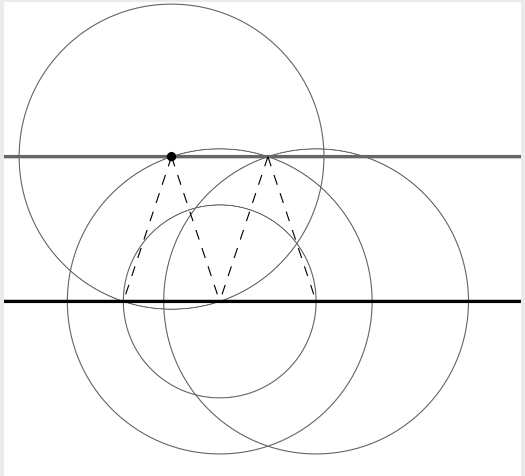


Construction (Parallel to a Line through a Point) Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.

(1) Draw a circle centered at the given point and passing through the given line at two points.

(2) We now have an isosceles triangle, duplicate this triangle.

(3) Connect the top vertexes of the triangles and we get a parallel line.



Question Can you give another different construction?

?

Problems for Section 2.1

(1) What are the rules for compass and straightedge constructions?

(2) What is a collapsing compass? Why don't we use them or worry about them any more?

(3) Prove that the collapsing compass is equivalent to the modern compass.

(4) Given a line segment, construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction and how you know it works.

(5) Use a compass and straightedge to bisect a given line segment. Explain the steps in your construction and how you know it works.

(6) Given a line segment with a point on it, construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction and how you know it works.

(7) Use a compass and straightedge to bisect a given angle. Explain the steps in your construction and how you know it works.

(8) Given an angle and a ray, use a compass and straightedge to copy the angle so that the new angle has the ray as one side. Explain the steps in your construction and how you know it works.

(9) Given a point and line, construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction and how you know it works.

(10) Given a point and line not containing the point, construct a line parallel to the given line that passes through the given point. Explain the steps in your construction and how you know it works.

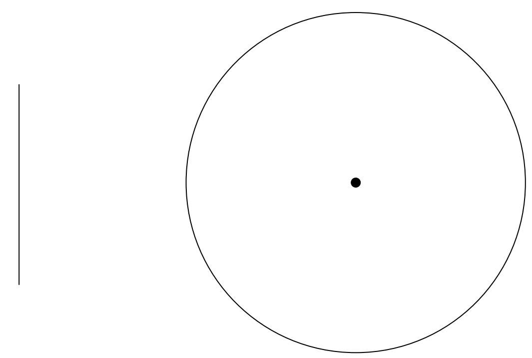
(11) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction and how you know it works.

(12) Construct a 30-60-90 right triangle. Explain the steps in your construction and how you know it works.

(13) Given a length of 1, construct a triangle with a perimeter of $3 + \sqrt{5}$. Explain the steps in your construction and how you know it works.

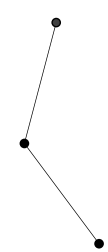
(14) Given a length of 1, construct a triangle with a perimeter that is a multiple of $2 + \sqrt{2}$. Explain the steps in your construction and how you know it works.

(15) Here is a circle and here is the side length of an inscribed regular 5-gon.



Construct the regular 5-gon. Explain the steps in your construction and how you know it works.

(16) Here is a piece of a regular 7-gon.



Construct the entire regular 7-gon. Explain the steps in your construction and how you know it works.

2.2 Anatomy of Figures

In studying geometry we seek to discover the points that can be obtained given a set of rules. In our case the set of rules consists of the rules for compass and straightedge constructions.

Question In regards to compass and straightedge constructions, what is a *point*?

?

Question In regards to compass and straightedge constructions, what is a *line*?

?

Question In regards to compass and straightedge constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them:

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

2.2.1 Lines Related to Triangles

Believe it or not, in mathematics we often try to study the simplest objects as deeply as possible. After the objects listed above, triangles are among the most basic of geometric figures, yet there is much to know about them. There are several lines that are commonly associated to triangles. Here they are:

- Perpendicular bisectors of the sides.
- Bisectors of the angles.
- Altitudes of the triangle.
- Medians of the triangle.

The first two lines above are self-explanatory. The next two need definitions.

Definition An **altitude** of a triangle is a line segment originating at a vertex of the triangle that meets the line containing the opposite side at a right angle.

Definition A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.

Question The intersection of any two lines containing the altitudes of a triangle is called an **orthocenter**. How many orthocenters does a given triangle have?

?

Question The intersection of any two medians of a triangle is called a **centroid**. How many centroids does a given triangle have?

?

Question What is the physical meaning of a centroid?

?

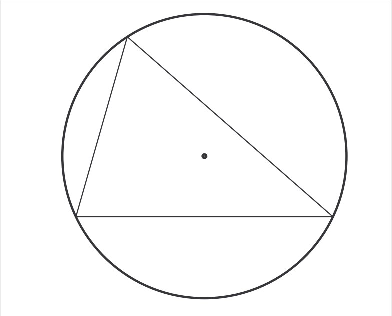
2.2.2 Circles Related to Triangles

There are also two circles that are commonly associated to triangles. Here they are:

- The circumcircle.
- The incircle.

These aren't too bad. Check out the definitions.

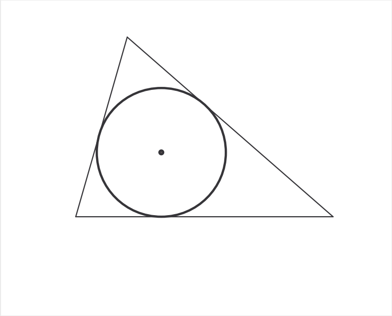
Definition The **circumcircle** of a triangle is the circle that contains all three vertexes of the triangle. Its center is called the **circumcenter** of the triangle.



Question Does every triangle have a circumcircle?

?

Definition The **incircle** of a triangle is the largest circle that will fit inside the triangle. Its center is called the **incenter** of the triangle.



Question Does every triangle have an incircle?

?

Question Are any of the lines described above related to these circles and/or centers? Clearly articulate your thoughts.

?

Problems for Section 2.2

(1) Compare and contrast the idea of “intersecting sets” with the idea of “intersecting lines.”

(2) Place three points in the plane. Give a detailed discussion explaining how they may or may not be on a line.

(3) Place three lines in the plane. Give a detailed discussion explaining how they may or may not intersect.

(4) Explain how a perpendicular bisector is different from an altitude. Draw an example to illustrate the difference.

(5) Explain how a median is different from an angle bisector. Draw an example to illustrate the difference.

(6) What is the name of the point that is the same distance from all three sides of a triangle? Explain your reasoning.

(7) What is the name of the point that is the same distance from all three vertexes of a triangle? Explain your reasoning.

(8) Could the circumcenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(9) Could the orthocenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(10) Could the incenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(11) Could the centroid be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(12) Are there shapes that do not contain their centroid? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(13) Draw an equilateral triangle. Now draw the lines containing the altitudes of this triangle. How many orthocenters do you have as intersections of lines in your drawing? Hints:

- (a) More than one.
- (b) How many triangles are in the picture you drew?

(14) Given a triangle, construct the circumcenter. Explain the steps in your construction.

(15) Given a triangle, construct the orthocenter. Explain the steps in your construction.

(16) Given a triangle, construct the incenter. Explain the steps in your construction.

(17) Given a triangle, construct the centroid. Explain the steps in your construction.

(18) Given a triangle, construct the incircle. Explain the steps in your construction.

(19) Given a triangle, construct the circumcircle. Explain the steps in your construction.

(20) Given a circle, give a construction that finds its center.

(21) Where is the circumcenter of a right triangle? Explain your reasoning.

(22) Where is the orthocenter of a right triangle? Explain your reasoning.

(23) Can you draw a triangle where the circumcenter, orthocenter, incenter, and centroid are all the same point? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(24) True or False: Explain your conclusions.

- (a) An altitude of a triangle is always perpendicular to a line containing some side of the triangle.
- (b) An altitude of a triangle always bisects some side of the triangle.
- (c) The incenter is always inside the triangle.
- (d) The circumcenter, the centroid, and the orthocenter always lie in a line.
- (e) The circumcenter can be outside the triangle.
- (f) The orthocenter is always inside the triangle.
- (g) The centroid is always inside the incircle.

(25) Given 3 distinct points not all in a line, construct a circle that passes through all three points. Explain the steps in your construction.

2.3 Trickier Constructions

Question How do you construct regular polygons? In particular, how do you construct regular: 3-gons, 4-gons, 5-gons, 6-gons, 7-gons, 8-gons, 10-gons, 12-gons, 17-gons, 24-gons, and 144-gons?

?

Well the equilateral triangle is easy. It was the first construction that we did. What about squares? What about regular hexagons? It turns out that they aren't too difficult. What about pentagons? Or say n-gons? We'll have to think about that. Let's leave the difficult land of n-gons and go back to thinking about nice, three-sided triangles.

Construction (SAS Triangle) Given two sides with an angle between them, we wish to construct the triangle with that angle and two adjacent sides.

(1) Transfer the one side so that it starts at the vertex of the angle.

(2) Transfer the other side so that it starts at the vertex.

(3) Connect the end points of all moved line segments.

The "SAS" in this construction's name spawns from the fact that it requires two sides with an angle *between* them. The SAS Theorem states that we can obtain a unique triangle given two sides and the angle between them.

Construction (SSS Triangle) Given three line segments we wish to construct the triangle that has those three sides, if it exists.

(1) Choose a side and select one of its endpoints.

(2) Draw a circle of radius equal to the length of the second side around the chosen endpoint.

(3) Draw a circle of radius equal to the length of the third side around the other

endpoint.

(4) Connect the end points of the first side and the intersection of the circles. This is the desired triangle.

Question Can this construction fail to produce a triangle? If so, show how. If not, why not?

?

Question Remember earlier when we asked about the converse to the Pythagorean Theorem? Can you use the construction above to prove the converse of the Pythagorean Theorem?

?

Question Can you state the SSS Theorem?

?

Construction (SAA Triangle) Given a side and two angles, where the given side does not touch one of the angles, we wish to construct the triangle that has this side and these angles if it exists.

(1) Start with the given side and place the adjacent angle at one of its endpoints.

(2) Move the second angle so that it shares a leg with the leg of the first angle—not the leg with the given side.

(3) Extend the given side past the first angle, forming a new angle with the leg of the second angle.

(4) Move this new angle to the other endpoint of the side, extending the legs of

this angle and the first angle will produce the desired triangle.

Question Where does your construction use parallel lines?

?

Question Can this construction fail to produce a triangle? If so, show how. If not, why not?

?

Question Can you state the SAA Theorem?

?

Question What about other combinations of S's and A's?

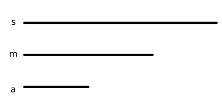
SSS, SSA, SAS, SAA, ASA, AAA

?

2.3.1 Challenge Constructions

Question How can you construct a triangle given the length of one side s, the length of the median to that side m, and the length of the altitude from the opposite angle a?

Proof (Follow-Along) *Use these lengths and follow the directions below.*



(1) *Start with the given side.*

(2) *Since the median hits our side at the center, bisect the given side.*

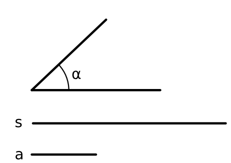
(3) *Make a circle of radius equal to the length of the median centered at the bisector of the given side.*

(4) *Construct a line parallel to our given line of distance equal to the length of the given altitude away.*

(5) *Where the line and the circle intersect is the third point of our triangle. Connect the endpoints of the given side and the new point to get the triangle we want.*

Question How can you construct a triangle given one angle α, the length of an adjacent side s, and the altitude to that side a?

Proof (Follow-Along) *Use these and follow the directions below.*



(1) *Start with a line containing the side.*

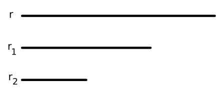
(2) *Put the angle at the end of the side.*

(3) *Draw a parallel line to the side of the length of the altitude away.*

(4) *Connect the angle to the parallel side. This is the third vertex. Connect the endpoints of the given side and the new point to get the triangle we want.*

Question How can you construct a circle with a given radius tangent to two other circles?

Proof (Follow-Along) *Use these and follow the directions below.*



(1) *Let r be the given radius, and let r_1 and r_2 be the radii of the given circles.*

(2) *Draw a circle of radius $r_1 + r$ around the center of the circle of radius r_1.*

(3) *Draw a circle of radius $r_2 + r$ around the center of the circle of radius r_2.*

(4) *Where the two circles drawn above intersect is the center of the desired circle.*

Question Place two tacks in a wall. Insert a sheet of paper so that the edges hit the tacks and the corner passes through the imaginary line between the tacks. Mark where the corner of the piece of paper touches the wall. Repeat this process, sliding the paper around. What curve do you end up drawing?

?

Question How can you construct a triangle given an angle and the length of the opposite side?

Proof (Solution) *We really can't solve this problem completely because the information given doesn't uniquely determine a triangle. However, we can still say something. Here is what we can do:*

(1) *Put the known angle at one end of the line segment. (Note: In the picture below, it is at the left end of the line segment, opening downwards.)*

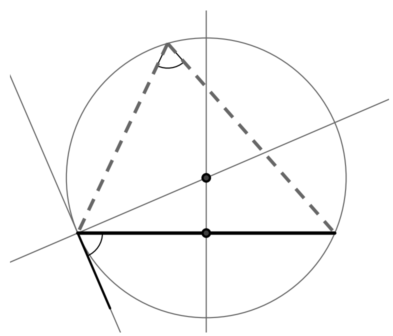
(2) *Construct the perpendicular bisector of the given segment.*

(3) *Construct a perpendicular to the other leg of the angle at its vertex.*

(4) *See where the bisector in step 2 intersects the perpendicular drawn in step 3.*

(5) *Draw circle centered at the point found in step 4 and touching the endpoints of the original segment.*

(6) *The segment cuts the circle into two arcs, one of which is opposite the angle placed in step 1. Every point on that arc is a valid choice for the vertex of the triangle.*



Question Why does the above method work?

?

Question You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays between the lighthouses as measured from the boat. How do you figure out where you are?

?

2.3.2 Problem Solving Strategies

The harder constructions discussed in this section can be difficult to do. There is no rote method to solve these problems, hence you must rely on your brain. Here are some hints that you may find helpful:

Construct what you can. You should start by constructing anything you can, even if you don't see how it will help you with your final construction. In doing so you are "chipping away" at the problem just as a rock-cutter chips away at a large boulder. Here are some guidelines that may help when constructing triangles:

(1) If a side is given, then you should draw it.

(2) If an angle is given and you know where to put it, draw it.

(3) If an altitude of length ℓ is given, then draw a line parallel to the side that the altitude is perpendicular to. This new line must be distance ℓ from the side.

(4) If a median is given, then bisect the segment it connects to and draw a circle centered around the bisector, whose radius is the length of the median.

(5) If you are working on a figure, construct any "mini-figures" inside the figure you are trying to construct. For example, many of the problems below ask you to construct a triangle. Some of these constructions have right-triangles inside of them, which are easier to construct than the final figure.

Sketch what you are trying to find. It is a good idea to try to sketch the figure that you are trying to construct. Sketch it accurately and label all pertinent parts. If there are special features in the figure, say two segments have the same length or there is a right-angle, make a note of it on your sketch. Also mark what is unknown in your sketch. We hope that doing this will help organize your thoughts and get your "brain juices" flowing.

Question Why are the above strategies good?

?

Problems for Section 2.3

(1) Construct a square. Explain the steps in your construction.

(2) Construct a regular hexagon. Explain the steps in your construction.

(3) Your friend Margy is building a clock. She needs to know how to align the twelve numbers on her clock so that they are equally spaced on a circle. Explain how to use a compass and straightedge construction to help her out. Illustrate your answer with a construction and explain the steps in your construction.

(4) Construct a triangle given two sides of a triangle and the angle between them. Explain the steps in your construction.

(5) State the SAS Theorem.

(6) Construct a triangle given three sides of a triangle. Explain the steps in your construction.

(7) State the SSS Theorem.

(8) Construct a triangle given a side and two angles where one of the angles does not touch the given side. Explain the steps in your construction.

(9) State the SAA Theorem.

(10) Construct a triangle given a side between two given angles. Explain the steps in your construction.

(11) State the ASA Theorem.

(12) Explain why when given an isosceles triangle, that two of its angles have equal measure. Hint: Use the SAS Theorem.

(13) Construct a figure showing that a triangle cannot always be uniquely determined when given an angle, a side adjacent to that angle, and the side opposite the angle. Explain the steps in your construction and explain how your figure shows what is desired. Explain what this says about the possibility of a SSA theorem. Hint: Draw many pictures to help yourself out.

(14) Give a construction showing that a triangle is uniquely determined if you are given a right-angle, a side touching that angle, and another side not touching the angle. Explain the steps in your construction and explain how your figure shows what is desired.

(15) Construct a triangle given two adjacent sides of a triangle and a median to one of the given sides. Explain the steps in your construction.

(16) Construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

(17) Construct a triangle given a side, the median to the side, and the angle opposite to the side. Explain the steps in your construction.

(18) Construct a triangle given an altitude, and two angles not touching the altitude. Explain the steps in your construction.

(19) Construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(20) Construct a triangle, given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(21) Construct a circle with a given radius tangent to two other given circles. Explain the steps in your construction.

(22) Does a given angle and a given opposite side uniquely determine a triangle? Explain your answer.

(23) You are on the bank of a river. There is a tree directly in front of you on the other side of the river. Directly left of you is a friend a known distance away. Your friend knows the angle starting with them, going to the tree, and ending with you. How wide is the river? Explain your work.

(24) You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays from the lighthouses. How do you figure out where you are? Explain your work.

(25) Construct a triangle given an angle, the length of a side adjacent to the given angle, and the length of the angle's bisector to the opposite side. Explain the steps in your construction.

(26) Construct a triangle given an angle, the length of the opposite side, and the length of the altitude of the given angle. Explain the steps in your construction.

(27) Construct a triangle given one side, the length of the altitude of the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.

(28) Construct a triangle given one side, the length of the altitude of an adjacent angle, and the radius of the circumcircle. Explain the steps in your construction.

(29) Construct a triangle given one side, the length of the median connecting that side to the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.

(30) Construct a triangle given one angle and the lengths of the altitudes to the two other angles. Explain the steps in your construction.

(31) Construct a circle with a given radius tangent to two given intersecting lines. Explain the steps in your construction.

(32) Given a circle and a line, construct another circle of a given radius that is tangent to both the original circle and line. Explain the steps in your construction.

(33) Construct a circle with three smaller circles of equal size inside such that each smaller circle is tangent to the other two and the larger outside circle. Explain the steps in your construction.

3 Folding and Tracing Constructions

> We don't even know if Foldspace introduces us to one universe or many. . .
>
> —Frank Herbert

3.1 Constructions

While origami as an art form is quite ancient, folding and tracing constructions in mathematics are relatively new. The earliest mathematical discussion of folding and tracing constructions that I know of appears in T. Sundara Row's book *Geometric Exercises in Paper Folding*,first published near the end of the Nineteenth Century. In the Twentieth Century it was shown that every construction that is possible with a compass and straightedge can be done with folding and tracing. Moreover, there are constructions that are possible via folding and tracing that are *impossible* with compass and straightedge alone. This may seem strange as you can draw a circle with a compass, yet this seems impossible to do via paper-folding. We will address this issue in due time. Let's get down to business—here are the rules of folding and tracing constructions:

Rules for Folding and Tracing Constructions

(1) You may only use folds, a marker, and semi-transparent paper.

(2) Points can only be placed in two ways:

  (a) As the intersection of two lines.

(b) By marking "through" folded paper onto a previously placed point. Think of this as when the ink from a permanent marker "bleeds" through the paper.

(3) Lines can only be obtained in three ways:

(a) By joining two points—either with a drawn line or a fold.

(b) As a crease created by a fold.

(c) By marking "through" folded paper onto a previously placed line.

(4) One can only fold the paper when:

(a) Matching up points with points.

(b) Matching up a line with a line.

(c) Matching up two points with two intersecting lines.

Now we are going to present several basic constructions. Compare these to the ones done with a compass and straightedge. We will proceed by the order of difficulty of the construction.

Construction (Transferring a Segment) Given a segment, we wish to move it so that it starts on a given point, on a given line.

Construction (Copying an Angle) Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.

Transferring segments and copying angles using folding and tracing without a "bleeding marker" can be tedious. Here is an easy way to do it:

Use 2 sheets of paper and a pen that will mark through multiple sheets.

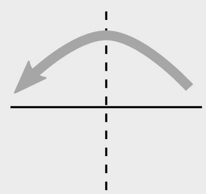
Question Can you find a way to do the above constructions without using a marker whose ink will pass through paper?

?

Construction (Bisecting a Segment) Given a segment, we wish to cut it in half.

(1) Fold the paper so that the endpoints of the segment meet.

(2) The crease will bisect the given segment.



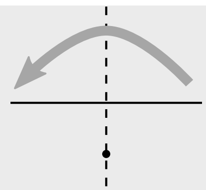
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Perpendicular through a Point) Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.

(1) Fold the given line onto itself so that the crease passes though the given point.

(2) The crease will be the perpendicular line.



Question Which rule for folding and tracing constructions are we using above?

?

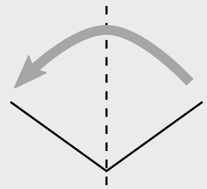
Question Does the construction work even when the point is on the line?

?

Construction (Bisecting an Angle) We wish to divide an angle in half.

(1) Fold a point on one leg of the angle to the other leg so that the crease passes though the vertex of the angle.

(2) The crease will bisect the angle.



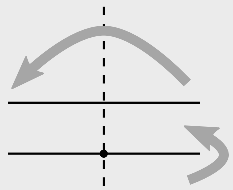
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Parallel through a Point) Given a line and a point not on the line, we wish to construct another line parallel to the first that passes through the given point.

(1) Fold a perpendicular line through the given point.

(2) Fold a line perpendicular to this new line through the given point.



Now there may be a pressing question in your head:

Question How the heck are we going to fold a circle?

First of all, remember the definition of a circle:

Definition A **circle** is the set of points that are a fixed distance from a given point.

Question Is the center of a circle part of the circle?

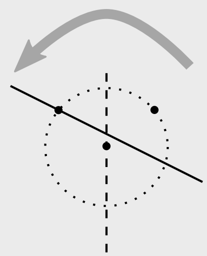
?

Secondly, remember that when doing compass and straightedge constructions we can **only** mark points that are intersections of lines and lines, lines and circles, and circles and circles. Thus while we technically draw circles, we can only actually mark certain points on circles. When it comes to folding and tracing constructions, drawing a circle amounts to marking points a given distance away from a given point—that is exactly what we can do with compass and straightedge constructions.

Construction (Intersection of a Line and a Circle) We wish to construct the points where a given line meets a given circle. Note: A circle is given by a point on the circle and the central point.

(1) Fold the point on the circle onto the given line so that the crease passes through the center of the circle.

(2) Mark this point though both sheets of paper onto the line.



Question Which rule for folding and tracing constructions are we using above?

?

Question How could you check that your folding and tracing construction is correct?

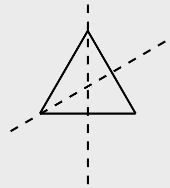
?

Construction (Equilateral Triangle) We wish to construct an equilateral triangle given the length of one side.

(1) Bisect the segment.

(2) Fold one end of the segment onto the bisector so that the crease passes through the other end of the segment. Mark this point onto the bisector.

(3) Connect the points.



Question Which rules for folding and tracing constructions are we using above?

?

Construction (Intersection of Two Circles) We wish to intersect two circles, each given by a center point and a point on the circle.

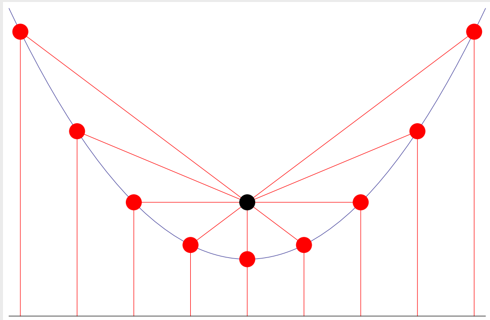
(1) Use four sheets of tracing paper. On the first sheet, mark the centers of both circles. On the next two sheets, mark the center and point on each of the circle—one circle per sheet.

(2) Simply move the two sheets with the centers and points on the circles, so that the centers are over the centers from the first sheet, and the points on the circles coincide. Now on the fourth sheet, mark all points.

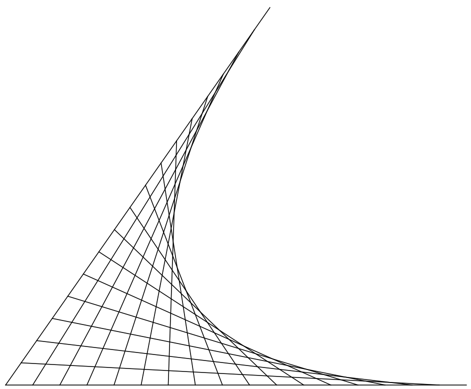
?

Think about the definition of a circle. In a similar fashion we can define other common geometric figures:

Definition Given a point and a line, a **parabola** is the set of points such that each of these points is the same distance from the given point as it is from the given line.



We can also form a parabola from an *envelope of tangents*:



Using a similar idea we can essentially obtain a parabola using folding and tracing.

Construction (Parabola) Given a point and a line we wish to construct a parabola.

(1) Make a series of equally spaced marks on your line.

(2) Fold the point onto the marks.

(3) Repeat the above step until an envelope of tangents forms.

Question Considering the definition of the parabola, can you explain why the above construction makes sense?

?

Question In the envelope of tangents, each line is tangent to the parabola. How do you find points that actually on the parabola?

?

Question Can you give a compass and straightedge construction of a parabola?

?

Our final basic folding and tracing construction is one that **cannot** be done with compass and straightedge alone.

This construction was discovered by S.T. Gormsen and verified by S.H. Kung.

Construction (Angle Trisection) We wish to divide an angle into thirds.

(1) Bisect the given angle.

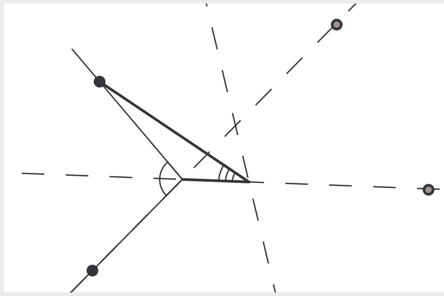
(2) Find two points (one on each leg of the angle) equidistant from the vertex of the angle.

(3) Fold the two points found above so that one of them lands on the extension (behind the angle) of the angle bisector and one lands on the line containing the other leg of the triangle—this will be behind the vertex. You are basically folding the angle back over itself.

(4) The crease from the last step will intersect the angle bisector at some point, mark it.

(5) The angle with the above mark as its vertex, the bisector found above as one of its legs, and the line to either of the points found in step 2 above will be

one third of the starting angle.



Problems for Section 3.1

(1) What are the rules for folding and tracing constructions?

(2) Use folding and tracing to bisect a given line segment. Explain the steps in your construction.

(3) Given a line segment with a point on it, use folding and tracing to construct a line perpendicular to the segment and passing through the given point. Explain the steps in your construction.

(4) Use folding and tracing to bisect a given angle. Explain the steps in your construction.

(5) Given a point and a line, use folding and tracing to construct a line perpendicular to the given line and passing through the given point. Explain the steps in your construction.

(6) Given a point and a line, use folding and tracing to construct a line parallel to the given line and passing through the given point. Explain the steps in your construction.

(7) Given a circle (a center and a point on the circle) and line, use folding and tracing to construct the intersection. Explain the steps in your construction.

(8) Given a line segment, use folding and tracing to construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction.

(9) Explain how to use folding and tracing to transfer a segment.

(10) Given an angle and some point, use folding and tracing to copy the angle so that the new angle has as its vertex the given point. Explain the steps in your construction.

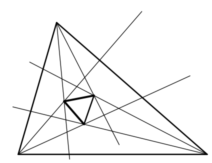
(11) Explain how to use folding and tracing to construct envelope of tangents for a parabola.

(12) Explain how to use folding and tracing to trisect a given angle.

(13) Use folding and tracing to construct a square. Explain the steps in your construction.

(14) Use folding and tracing to construct a regular hexagon. Explain the steps in your construction.

(15) Morley's Theorem states: If you trisect the angles of any triangle with lines, then those lines form a new equilateral triangle inside the original triangle.



Give a folding and tracing construction illustrating Morley's Theorem. Explain the steps in your construction.

(16) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction.

(17) Construct a 30-60-90 right triangle. Explain the steps in your construction.

(18) Given a length of 1, construct a triangle with a perimeter of $3 + \sqrt{5}$. Explain the steps in your construction.

3.2 Anatomy of Figures Redux

Remember, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for folding and tracing constructions.

Question In regards to folding and tracing constructions, what is a *point*?

?

Question In regards to folding and tracing constructions, what is a *line*?

?

Question In regards to folding and tracing constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them (I know we've already gone over this, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

Problems for Section 3.2

(1) In regards to folding and tracing constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle.

(2) Explain how a perpendicular bisector is different from an altitude. Use folding and tracing to illustrate the difference.

(3) Explain how a median different from an angle bisector. Use folding and tracing to illustrate the difference.

(4) Given a triangle, use folding and tracing to construct the circumcenter. Explain the steps in your construction.

(5) Given a triangle, use folding and tracing to construct the orthocenter. Explain the steps in your construction.

(6) Given a triangle, use folding and tracing to construct the incenter. Explain the steps in your construction.

(7) Given a triangle, use folding and tracing to construct the centroid. Explain the steps in your construction.

(8) Could the circumcenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(9) Could the orthocenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

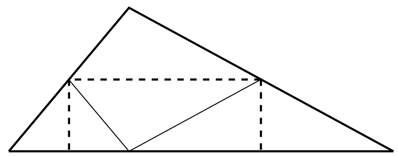
(10) Could the incenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(11) Could the centroid be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(12) Where is the circumcenter of a right triangle? Explain your reasoning and illustrate your ideas with folding and tracing.

(13) Where is the orthocenter of a right triangle? Explain your reasoning and illustrate your ideas with folding and tracing.

(14) The following picture shows a triangle that has been folded along the dotted lines:



Explain how the picture "proves" the following statements:

(a) The interior angles of a triangle sum to 180°.

(b) The area of a triangle is given by $bh/2$.

(15) Use folding and tracing to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(16) Use folding and tracing to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(17) Use folding and tracing to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(18) Use folding and tracing to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

4 Toward Congruence and Similarity

4.1 Transformations, Symmetry, and Congruence

In school mathematics, transformations and symmetry have typically been niche topics, separate from each other, separate from most of the rest of school mathematics, and receiving little curricular attention. Congruence, on the other hand, is a more prominent idea that begins informally in the elementary grades as "same shape, same size" and culminates in high school with theorems and proofs, sometimes based on explicit postulates.

In this section, we demonstrate how transformations can undergird both symmetry and congruence, thereby strengthening all three topics and also establishing groundwork for an analogous approach to similarity.

4.1.1 Transformations

Informally, a transformation of the plane is a "motion," such as a rotation or a stretch of the plane. More formally, a transformation is a function that takes points in the plane as inputs and gives points as outputs.[G-CO.2] In school mathematics, we consider only transformations that take lines to lines, so that key geometric features are "preserved." For example a triangle remains a triangle when it is rotated and even when it is stretched.

CCSS G-CO.2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

Transformations are often specified using a coordinate system, but coordinates are not necessary. For now, we will explore transformations without a coordinate system. Later, we will use coordinates, along with matrices and vectors, to describe transformations.

Definition Transformations that preserve distances and angles are called *isometries*, and the most important of these are *basic rigid motions*: translations, rotations, and reflections.

Question Is a transformation that stretches the plane an isometry? Explain.

?

Through exploration with transparencies, tracing paper, and software, it is not hard to see that the basic rigid motions have important properties.[8.G.1 8.G.1a 8.G.1b 8.G.1c]

Based on such explorations, we write careful definitions of translation, reflection, rotation, focusing what is required to specify each transformation.[G-CO.4]

Question What does it take to specify a translation? A reflection? A rotation?

?

Definition The *identity transformation*, sometimes called the "do nothing" transformation, doesn't move the plane at all. As a function, the identity transformation takes a point to itself: The output is identical to the input.

Question Is the identity transformation a translation, rotation, or reflection? Explain.

?

4.1.2 Symmetry

A *symmetry* of a figure is a transformation that takes the figure onto itself,[G-CO.3] so that the figure is "preserved" by the transformation. In everyday language, we may say a figure is "symmetrical," but mathematically we can be more precise by specifying the symmetry transformation(s) of the figure.

CCSS 8.G.1: Verify experimentally the properties of rotations, reflections, and translations:

CCSS 8.G.1a: Lines are taken to lines, and line segments to line segments of the same length.

CCSS 8.G.1b: Angles are taken to angles of the same measure.

CCSS 8.G.1c: Parallel lines are taken to parallel lines.

CCSS G-CO.4: Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

CCSS G-CO.3: Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

Question What are the symmetries of a rectangle? Be sure to specify the transformations.

?

4.1.3 Congruence

Congruence is sometimes described using angles and side lengths. But such a definition cannot apply to figures that are not polygons. A more inclusive definition is as follows:

Definition Two figures (in the plane) are said to be *congruent* to one another if there is a sequence of basic rigid motions that takes one figure onto the other.

The idea behind this definition is sometimes called the *principle of superposition*, which states that congruent figures can be placed exactly on top of one another. The above definition is more precise than superposition because it calls for an explicit sequence of basic rigid motions (e.g., translations, rotations, and reflections) rather than merely "movement" of one figure onto the other.

Question When we say that two polygons are congruent, why is the order of labeling the vertices important? For example, if we know $\triangle ABC \simeq \triangle XYZ$, does it follow that $\triangle ABC \simeq \triangle YXZ$? Explain. (Hint: Which angle of $\triangle XYZ$ corresponds to $\angle A$? Which side of $\triangle ABC$ corresponds to $\overline{XZ}$?)

?

The above definition of congruence helps us in two directions.[8.G.2] First, if we have a sequence of basic rigid motions that takes one figure onto another, then we know the two figures are congruent. Furthermore, the sequence of basic rigid motions sets up the correspondences between various parts of the figures. Conversely, if two figures are congruent, then we know it is possible to find a sequence of basic rigid motions that takes one figure onto the other. And the sequence of basic rigid motions often takes advantage of corresponding parts that are known to be congruent.

CCSS 8.G.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

For triangles, we still have the familiar congruence criteria, such as side-side-side (SSS), side-angle-side (SAS), and angle-side-angle (ASA). The key idea is that although triangles have six measures of sides and angles, most of the time (but not always) just three of these measures are sufficient to determine the triangle uniquely. Students can develop intuition about these criteria by drawing triangles from given conditions.[7.G.2] The next step is to show, first, that the above definition fits with traditional notions of triangle congruence[G-CO.7], and, second, to prove that the triangle congruence criteria follow from the properties of the basic rigid motions.[G-CO.8]

Then, because the triangle congruence criteria can be established from sequences of rigid motions, we can prove theorems using triangle congruence criteria, basic rigid motions, or a combination of the two approaches.

CCSS 7.G.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

CCSS G-CO.7: Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

CCSS G-CO.8: Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Problems for Section 4.1

(1) What is required to specify a translation?

(2) What is required to specify a rotation?

(3) What is required to specify a reflection?

(4) Write a careful definition of translation. Hint: Describe how to find the image P' of a point P.

(5) Write a careful definition of rotation. Hint: Describe how to find the image P' of a point P.

(6) Write a careful definition of reflection. Hint: Describe how to find the image P' of a point P.

Sometimes a sequence of transformations can be described as a single translation, rotation, or reflection.

(7) What kind of transformation is a translation followed by a translation? Explain. Be sure to consider any special cases.

(8) What kind of transformation is a rotation followed by a rotation? Explain. Be sure to consider any special cases.

(9) What kind of transformation is a reflection followed by another reflection? Explain. Be sure to consider any special cases.

(10) Will the letter F look like an F after a reflection? What about after a sequence of two reflections? What about after a sequence of 73 or 124 reflections? Explain your reasoning.

(11) How will your answer to the previous problem change if you use a capital D? Explain.

(12) Given a figure and its image after a translation, how do find the direction and distance of the translation? How many points and images do you need?

(13) Given a figure and its image after a reflection, how do you find the line of reflection? How many points and images do you need?

(14) Given a figure and its image after a rotation, how do you find the center and the angle of the rotation? How many points and images do you need?

(15) Categorize the capital letters of the alphabet by their symmetries.

(16) Write the words COKE and PEPSI in capital letters so that they read vertically. Use a mirror to look at a reflection of the words. What is different about the reflections of the two words? Explain.

(17) Describe all of the symmetries of the following figures:

(a) An equilateral triangle

(b) An isosceles triangle that is not equilateral

(c) A square

(d) A rectangle that is not a square

(e) A rhombus that is not a square

(f) A (non-special) parallelogram

(g) A regular n−gon

(18) What are the symmetries of a circle?

(19) How can you use the symmetries of a circle to determine whether a figure is indeed a circle?

(20) What are the symmetries of a line?

(a) Describe all translation symmetries.

(b) Describe all rotation symmetries.

(c) Describe two types of reflection symmetries.

(d) Given a line, describe a rotation symmetry and a reflection symmetry that have the same effect on a line. How do the corresponding transformations differ in what they do to the surrounding space?

(21) How can you use the symmetries of a line to determine whether a figure is indeed a line?

(22) Find some tessellations. For each tessellation, describe all of its symmetries.

4.2 Euclidean and non-Euclidean Geometries

The geometry of school mathematics is called *Euclidean Geometry* for it is the geometry organized and detailed by Euclid more than 2,000 years ago. To better understand the assumptions that underlie Euclidean geometry and the results that follow, it helps to be aware of non-Euclidean geometries. Perhaps the most accessible of these is spherical geometry, because we can make use of basketballs that we can hold in our hands, and we can take advantage of our experience traveling on our (approximately spherical) Earth, modeled by a globe.

Question Before we talk about spheres, what does it mean to say that a plane is two-dimensional and space is three-dimensional? What is "dimension"?

?

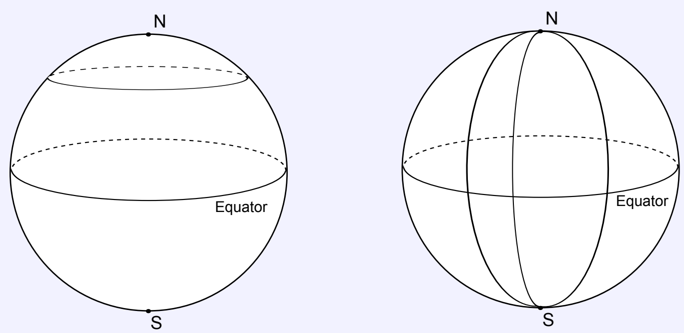
To think about spherical geometry, it helps to imagine a bug crawling on the surface of a sphere. From the bug's perspective, the surface of the sphere is very much the same as the surface of a Euclidean plane. Both surfaces are two-dimensional in the sense that the bug has two degrees of freedom: forward/backward and left/right. Any other movement can be expressed as a combination of these. (We are assuming the bug must stay *on the surface*: It can neither fly away from nor burrow underneath the surface.) Whereas the surface of a Euclidean plane is infinite and flat, the surface of a sphere is finite and curved. But if the sphere is reasonably large (compared to the bug), then even a very smart bug might have trouble determining whether she or he was walking on a sphere or on a flat plane.

Question Explain in your own words how to think about the surface of a sphere as two-dimensional.

?

Points in spherical geometry are taken to be points on the surface of the sphere. But "lines" present more of a challenge: We want lines to be "straight", but any path on the surface of a sphere curves with the surface. Suppose the bug travels forward along a path that is as straight as possible, being very careful to veer neither right nor left. Alternatively, because lines should determine "shortest paths" between two points, stretch a rubber band between two points on a basketball or on a globe to find the shortest path. (Try this!) In both cases, you will find that best answer is that a "line" in spherical geometry is a *great circle*, which is to say a circle that is as big as possible on the sphere. From a three-dimensional perspective, the center of a great circle is the same as the center of the sphere.

Question Consider the pictures below.

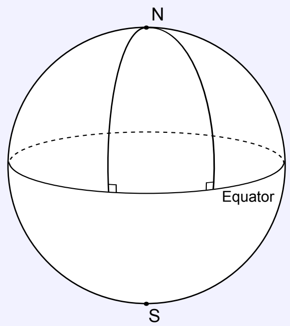


Are longitude lines on the earth "lines" in spherical geometry? What about latitude lines? Explain your reasoning.

?

In non-Euclidean geometries, many familiar results no longer hold. In spherical geometry, for example, there are no parallel lines because any two "lines" (i.e., great circles) intersect in two points, and the sum of the angles in a triangle is greater than 180°.

Question Use the following picture to explain that the sum of the angles in a triangle in spherical geometry can be greater than 180°.



?

Other non-Euclidean geometries are even stranger than spherical geometry! In hyperbolic geometry, for example, parallel lines are not a fixed distance apart, and the sum of the angles in a triangle is less than 180°.

The following statements characterize three different types of geometries:

- **Euclidean geometry**: Given a line and a point not on the line, there is *exactly one line* parallel to the given line.
- **Spherical geometry**: Given a line and a point not on the line, there are *no lines* through the point parallel to the given line.
- **Hyperbolic geometry**: Given a line and a point not on the line, there is *more than one line* parallel to the given line.

In this course, we explore neither spherical nor hyperbolic geometry in detail, but keep these contrasting ideas in mind as we continue to dig into Euclidean geometry.

Problems for Section 4.2

(1) From the above statements about angle sums in triangles, what can you conclude about angle sums in quadrilaterals in spherical and hyperbolic geometries?

(2) In Euclidean geometry, a rectangle is a quadrilateral with four right angles.

(a) What can you conclude about rectangles in spherical and hyperbolic geometries? Explain.

(b) What does this imply about the usefulness of familiar (Euclidean) area formulas in these other geometries? Explain your reasoning.

(3) In Euclidean geometry, when three distinct points A, B, and C, lie on a line, it is easy to tell which point is between the other two. Does this work in spherical geometry? Explain your reasoning.

(4) A bear goes traveling. She walks due south for one mile, turns left 90°, and walks due east for one mile. She again turns left 90°, and then walks due north for one mile, ending in the place where she started. What color is the bear? Explain your reasoning.

(5) When walking on a sphere, how could a bug check whether she or he was traveling straight.

(6) In Euclidean geometry, any two distinct points determine a unique line. This is sometimes (but not always) true in spherical geometry. What can you say about two distinct points that do not lie on a unique line in spherical geometry?

(7) In Euclidean geometry, given a line and a point, there is a unique perpendicular to the given line through the given point. Describe how this sometimes fails in spherical geometry.

(8) Can the Euclidean definition of a circle make sense on a sphere? Be sure that the center of the circle is a point on the sphere. How would you measure the radius of the circle?

4.3 Assumptions in Mathematics

Every area of mathematics is based on a set of assumptions, sometimes called axioms or postulates, which are merely statements that are accepted without proof. They serve as the foundation of the theory being developed, and all other facts are proven beginning with these assumptions. This approach is called the *axiomatic method*.

. . . Or at least that's how mathematics is imagined to work. In practice, because mathematics is so vast and interconnected, most mathematical reasoning and problem solving starts "in the middle" from a collection of accepted facts, with little worry about which statements were taken as assumptions and which were proven as theorems.

In classical mathematics, "axioms" were self-evident statements that were common to many areas of science (including mathematics), whereas "postulates" were common-sense facts drawn from experience in specific areas, such as geometry. In modern mathematics, this distinction is no longer seen as significant, and most assumptions are merely called axioms. In deference to Euclid's *Elements*, the word postulate is used almost exclusively to discuss key assumptions in geometry, as you will see below.

In this course, we started in the middle. In this section, we are examining the foundation.

Question In school mathematics we can "explain" the properties of whole or rational numbers by appealing to models and to meanings of the arithmetic operations. But in advanced mathematics courses, the real numbers are usually specified via axioms, some of which have names.

What are the names of the following axioms:

(1) $a + b = b + a$

(2) $a(bc) = (ab)c$

(3) $a(b + c) = ab + ac$

(4) If $a = b$ and $b = c$ then $a = c$

?

Chances are you used the word "property" or "law" rather than "axiom" in your responses. Some properties of arithmetic have important names, such as the *distributive property of multiplication over addition*. The fourth property above is called the transitive property of equality. But in school mathematics, it is neither necessary nor instructive to insist that every such property have a name that students are expected to recall.

4.3.1 Assumptions for School Geometry

We propose the following set of assumptions for school geometry:

(A1) Through two distinct points passes a unique line.

(A2) Given a line and a point not on the line, there is exactly one line passing through the point which is parallel to the given line (Parallel postulate).

(A3) The points on a line can be placed in one-to-one correspondence with the real numbers so that differences measure distances (Ruler postulate).

(A4) The rays with a common endpoint can be numbered so that differences measure angles and so that straight angles measure 180° (Protractor postulate).

(A5) Every basic rigid motion (rotation, reflection, or translation) has the following properties:

- (i) It maps a line to a line, a ray to a ray, and a segment to a segment.
- (ii) It preserves distance and angle measure.

(A6) Areas of geometric figures have the following properties:

- (i) Congruent figures enclose equal areas.
- (ii) Area is additive, i.e., the area of the union of two regions that overlap only at their boundaries is the sum of their areas.
- (iii) A rectangle with side-lengths a and b has area ab, where a and b can be any non-negative real numbers.

In addition to these geometric assumptions, we of course assume the properties of the algebra of real numbers.

These formal axioms, we should be clear, are intended not for students but for teachers. And even teachers need not memorize them. Instead, we suggest that teachers remember them informally in the following chunks:

- Points, lines, and parallel lines behave as they should (A1 and A2)
- Distance and angle measure behave as they should (A3 and A4)
- Basic rigid motions behave as they should (A5)
- Area behaves as it should (A6)

We are almost ready to use these axioms to prove some basic results. First, we need a crucial definition.

Definition In a plane, two distinct lines are said to be **parallel** if they have no point in common.

Most of the time, of course, two distinct lines will have exactly one point in common.

Question Can the two distinct lines have more than one point in common? Use the above axioms to explain your reasoning.

?

The ruler postulate gives us a definition of betweenness, which allows to to define line segment and ray.

Definition If points A, X, and B are on a line l, we say that X is *between* A and B if $AX + XB = AB$.

Question Use the concept of betweenness to define line segment $\overline{AB}$. Now use the concept of betweenness to define ray $\overrightarrow{AB}$.

?

Question Use the protractor postulate to provide a definition of adjacent angles, analogous to betweenness for distances.

?

Theorem 4.3.1 *Let l be a line and O be a point on l. Let R be the 180° rotation around O. Then R maps l to to itself.*

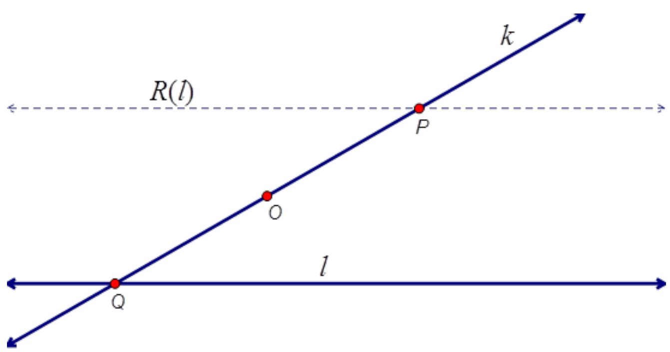
Question Can you prove this theorem? (Hint: Pick points P and Q on l so that O is between them, and consider the straight angle $\angle POQ$.)

?

Theorem 4.3.2 *Let l be a line and O be a point* not *lying on l. Let R be the 180° rotation around O. Then R maps l to a line parallel to itself.*

Note: The following proof uses function notation to describe the images under the rotation R. Thus $R(l)$ is the image of line l, and $R(Q)$ is the image of point Q.

Proof *Let P be an arbitrary point on $R(l)$, the rotated image of l. To show that $R(l)$ is parallel to l, it is sufficient to show that P cannot lie also on l.*



Because P is on $R(l)$, there is a point Q on l such that $P = R(Q)$. The rotated image of the ray OQ is the ray OP, and because $\angle QOP$ is 180°, it follows that Q, O, and P are collinear. Call that line k. We know line k is distinct from l because O is on k but not on l. Now, if P were on l, then points P and Q would be on two distinct lines, k and l, contradicting A1 (i.e., on two points there is a unique line). The theorem is proved.

Problems for Section 4.3

(1) Use adjacent angles to prove that vertical angles are equal.

(2) Now use rotations to prove that vertical angles are equal.

(3) Prove that alternate interior angles and corresponding angles of a transversal with respect to a pair of parallel lines are equal.

(4) Prove that the sum of the interior angles of a triangle is $180°$.

(5) Prove: If a pair of alternate interior angles or a pair of corresponding angles of a transversal with respect to two lines are equal, then the lines are parallel.

4.4 Dilations, Scaling, and Similarity

In a previous section, we saw how transformations can be used as a foundation for describing congruence and explaining the triangle congruence criteria. In this section, we show how transformations can be used to describe similarity. Because the basic rigid motions all preserve distances, we need a new kind of transformation: a dilation.

Definition Given a point O and a positive number r, a *dilation* about O by scale factor r, is a mapping that takes a point P to a point P' so that $OP' = r \cdot OP$.

With this definition, rubber bands are natural tools for exploring dilations. Through explorations with rubber bands and with geometry software, we observe that a dilation has the following properties:[G-SRT.1]

CCSS G-SRT.1: Verify experimentally the properties of dilations given by a center and a scale factor:

(i) It maps lines to lines, rays to rays, and segments to segments.

(ii) It changes distance by a factor of r, where r is the scale factor of the dilation.

(iii) It maps every line passing through the center of dilation to itself, and it maps every line not passing through the center of the dilation to a parallel line.

(iv) It preserves angle measure.

We could assume these properties, just as we have assumed the properties of the basic rigid motions. Instead, we use our assumptions about area to prove some of these properties. These are the Side-Splitter Theorems.

Now we are ready to define similarity.[8.G.4]

CCSS 8.G.4: Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

Definition A geometric figure is *similar* to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations.

4.4.1 Theorems for Similar Triangles

We need to show that this general definition of similarity fits with ideas about similar triangles that we may remember from school mathematics. Here is one way of thinking about similar triangles:

$$\triangle ABC \sim \triangle A'B'C' \quad \Leftrightarrow \quad \begin{array}{l} \angle A \simeq \angle A' \\ \angle B \simeq \angle B' \\ \angle C \simeq \angle C' \end{array}$$

Question What does this mean?

?

Here is another way of thinking about similar triangles:

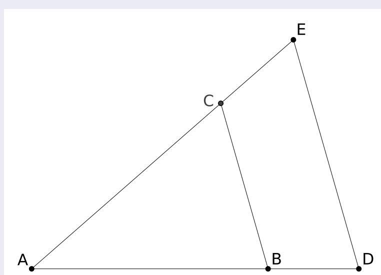
$$\triangle ABC \sim \triangle A'B'C' \quad \Leftrightarrow \quad \begin{array}{l} AB = k \cdot A'B' \\ BC = k \cdot B'C' \\ CA = k \cdot C'A' \end{array}$$

Question What does this mean?

?

Using merely the formula for the area of a triangle, we (meaning you) will explain why the following important theorem is true. Throughout this discussion we will use the convention that when we write AB we mean the *length* of the segment AB.

Theorem 4.4.1 (Parallel-Side) *Given:*



If side BC is parallel to side DE, then

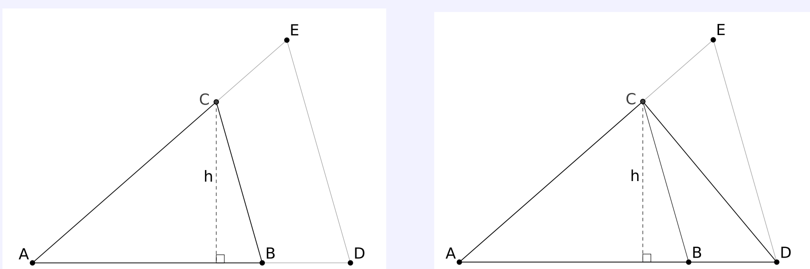
$$\frac{AB}{AD} = \frac{AC}{AE}.$$

Question Can you tell me in English what this theorem says? How does it relate to the definition of similarity in terms of rigid motions and dilations?

?

Now we (meaning you) are going to explore a bit. See if answering these questions sheds light on this.

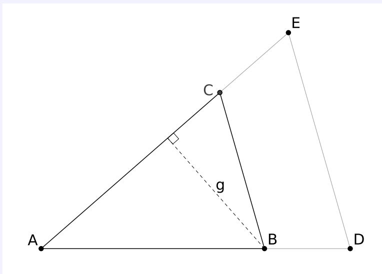
Question If h is the height of $\triangle ABC$, find formulas for the areas of $\triangle ABC$ and $\triangle ADC$.

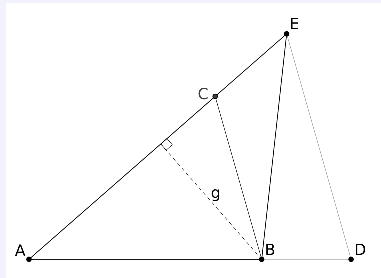


?

Question If g is the height of $\triangle ACB$, find formulas for the areas of $\triangle ACB$ and

$\triangle AEB$.





?

Question Explain why

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle ACB).$$

?

Question Explain why

$$\text{Area}(\triangle CBE) = \text{Area}(\triangle CBD).$$

Big hint: Use the fact that you have two parallel sides! Draw a picture to help clarify your explanation.

?

Question Explain why

$$\text{Area}(\triangle ADC) = \text{Area}(\triangle AEB).$$

?

Question Explain why

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} = \frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle AEB)}$$

?

Question Compute and simplify both of the following expressions:

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} \quad \text{and} \quad \frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle AEB)}$$

?

Question How can you conclude that:

$$\frac{AB}{AD} = \frac{AC}{AE}$$

?

Question Why is it important that line DE is parallel to line CB?

?

Question Can you sketch out (in words) how the questions above prove the Parallel-Side Theorem?

?

Now comes the moment of truth.

Question Can you use the Parallel-Side Theorem to explain why if you know that if you have two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with:

$$\angle A \simeq \angle A'$$
$$\angle B \simeq \angle B'$$
$$\angle C \simeq \angle C'$$

then we must have that

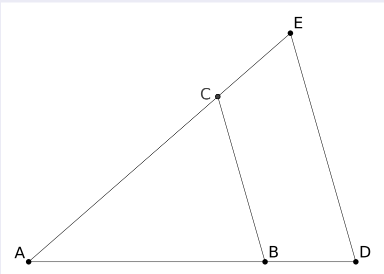
$$AB = k \cdot A'B'$$
$$BC = k \cdot B'C'$$
$$CA = k \cdot C'A'$$

These notes do not describe why side CA is also scaled by k. You address that question in the Side-Splitter Theorem activity.

?

The Converse The converse of the Parallel-Side Theorem states:

Theorem 4.4.2 (Split-Side) *Given:*



If side BC intersects (splits) the sides of $\triangle ADE$ so that

$$\frac{AB}{AD} = \frac{AC}{AE},$$

then side BC is parallel to side DE and in the same ratio.

Now we (meaning you) will answer questions in the hope that they will help us see why the above theorem is true.

Question Suppose that you **doubt** that side BC is parallel to side DE. Explain how to place a point C' on side AE so that side BC' is parallel to line DE. Be sure to sketch the situation(s).

?

Question You now have a triangle $\triangle ADE$ whose sides are split by a line BC' such that the line BC' is parallel to line DE. What does the Parallel-Side Theorem have to say about this?

?

Question What can you conclude about points C and C'?

?

Question What does this tell you about the Split-Side Theorem?

?

Let's see if you can put this all together:

Question Can you use the Split-Side Theorem to explain why you know that if you have two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with:

$$AB = k \cdot A'B'$$
$$BC = k \cdot B'C'$$
$$CA = k \cdot C'A'$$

then we must have that

$$\angle A \simeq \angle A'$$
$$\angle B \simeq \angle B'$$
$$\angle C \simeq \angle C'$$

?

Putting all of our work above together, we may now say the following:

Theorem 4.4.3 *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are* ***similar*** *if either equivalent condition holds:*

$$\begin{matrix} \angle A \simeq \angle A' & & AB = k \cdot A'B' \\ \angle B \simeq \angle B' & \text{or} & BC = k \cdot B'C' \\ \angle C \simeq \angle C' & & CA = k \cdot C'A' \end{matrix}$$

Question How does this theorem connect back to the definition of similarity in terms of rigid motions and dilations?

?

4.4.2 A New Meaning of Multiplication

School mathematics makes sense when concepts have *meaning*.

Question What can multiplication mean? Can you give multiplication meaning involving groups of groups or something of the sort?

?

Question Can you give multiplication meaning involving areas or something of the sort?

?

Question Can you somehow give meaning to multiplication using similarity? Use “scale factor” or “scaling” in your explanation.

?

4.4.3 Problem Solving with Similarity

We now have several ways of thinking more deeply about the naïve "same shape" notion of similarity, imagined as zooming in and out. In this section, we defined similarity in terms of basic rigid motions and dilations, and we used calculations involving area to show that these ideas are consistent with triangle similarity described as "same angles" or as "proportional sides."

Again, the advantage of defining similarity in terms of basic rigid motions and dilations, is that the approach applies not just to polygons but to figures of any shape. And the key is identifying the scale factor.

Here are some key ideas that arise in the activities and homework problems:

(1) Many real-world problems can be solved using similar triangles or other similar figures. For example, you can use shadows to compute the height of a flagpole. Maps, scale drawings, and scale models all involve similarity.

(2) A crticial issue is being able to distinguish situations in which figures are similar from those in which they are not.

(3) When using proportional relationships between corresponding parts of similar figures, it helps distinguish "within figure" ratios from "across figure" ratios, the latter being a scale factor.[G-SRT.2] When figures overlap, one challenge is being consistent about part-part versus part-whole ratios.

(4) You may use the definition of similarity to show that any two circles are similar.[G-C.1] You can also see the more surprising result that any two parabolas are similar.

(5) Similarity turns out to be very useful in right triangles. First, the altitude to the hypotenuse creates two triangles similar to the first. Second, among right triangles, similarity requires specifying only one non-right angle, which leads to right triangle trigonometry.

CCSS G-SRT.2: Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

CCSS G-C.1: Prove that all circles are similar.

Problems for Section 4.4

(1) Compare and contrast the ideas of *equal triangles*, *congruent triangles*, and *similar triangles*.

(2) Explain why all equilateral triangles are similar to each other.

(3) Explain why all isosceles right triangles are similar to each other.

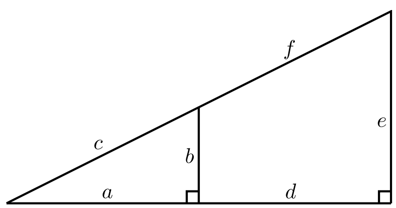
(4) Explain why when given a right triangle, the altitude of the right angle divides the triangle into two smaller triangles each similar to the original right triangle.

(5) The following sets contain lengths of sides of similar triangles. Solve for all unknowns—give all solutions. In each case explain your reasoning.

- (a) $\{3, 4, 5\}$, $\{6, 8, x\}$
- (b) $\{3, 3, 5\}$, $\{9, 9, x\}$
- (c) $\{5, 5, x\}$, $\{10, 4, y\}$
- (d) $\{5, 5, x\}$, $\{10, 8, y\}$
- (e) $\{3, 4, x\}$, $\{4, 5, y\}$

(6) A *Pythagorean Triple* is a set of three positive integers $\{a, b, c\}$ such that $a^2 + b^2 = c^2$. Write down an infinite list of Pythagorean Triples. Explain your reasoning and justify all claims.

(7) Here is a right triangle. Note that it is **not** drawn to scale:



Solve for all unknowns in the following cases.

- (a) $a = 3$, $b = ?$, $c = ?$, $d = 12$, $e = 5$, $f = ?$
- (b) $a = ?$, $b = 3$, $c = ?$, $d = 8$, $e = 13$, $f = ?$
- (c) $a = 7$, $b = 4$, $c = ?$, $d = ?$, $e = 11$, $f = ?$
- (d) $a = 5$, $b = 2$, $c = ?$, $d = 6$, $e = ?$, $f = ?$

In each case explain your reasoning.

(8) Suppose you have two similar triangles. What can you say about the area of one in terms of the area of the other? Be specific and explain your reasoning.

(9) During a solar eclipse we see that the apparent diameter of the Sun and Moon are nearly equal. If the Moon is around 240000 miles from Earth, the Moon's diameter is about 2000 miles, and the Sun's diameter is about 865000 miles how far is the Sun from the Earth?

- (a) Draw a relevant (and helpful) picture showing the important points of this problem.
- (b) Solve this problem, be sure to explain your reasoning.

(10) When jets fly above 8000 meters in the air they form a vapor trail. Cruising altitude for a commercial airliner is around 10000 meters. One day I reached my arm into the sky and measured the length of the vapor trail with my hand—my hand could just span the entire trail. If my hand spans 9 inches and my arm extends 25 inches from my eye, how long is the vapor trail? Explain your reasoning.

- (a) Draw a relevant (and helpful) picture showing the important points of this problem.
- (b) Solve this problem, be sure to explain your reasoning.

(11) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Dave sits comfortably with his dog Fritz at a distance of 10 feet. How far must Michael stand from his TV to have the "same" viewing experience? Explain your reasoning.

- (a) Draw a relevant (and helpful) picture showing the important points of this problem.
- (b) Solve this problem, be sure to explain your reasoning.

(12) You love IMAX movies. While the typical IMAX screen is 72 feet by 53 feet, your TV is only a 32 inch screen—it has a 32 inch diagonal. How close do you have to sit to your screen to simulate the IMAX format? Explain your reasoning.

- (a) Draw a relevant (and helpful) picture showing the important points of this problem.
- (b) Solve this problem, be sure to explain your reasoning.

(13) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Michael stands and watches his TV at a distance of 2 feet. Dave sits comfortably with his dog Fritz at a distance of 10 feet. Whose TV appears bigger to the respective viewer? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(14) Here is a personal problem: Suppose you are out somewhere and you see that when you stretch out your arm, the width of your thumb is the same apparent size as a distant object. How far away is the object if you know the object is:

(a) 6' long (as tall as a person).

(b) 16' long (as long as a car).

(c) 40' long (as long as a school bus).

(d) 220' long (as long as a large passenger airplane).

(e) 340' long (as long as an aircraft carrier).

Explain your reasoning.

(15) I was walking down Woody Hayes Drive, standing in front of St. John Arena when a car pulled up and the driver asked, "Where is Ohio Stadium?" At this point I was a bit perplexed, but nevertheless I answered, "Do you see the enormous concrete building on the other side of the street that looks like the Roman Colosseum? That's it."

The person in the car then asked, "Where are the Twin-Towers then?" Looking up, I realized that the towers were in fact just covered by top of Ohio Stadium. I told the driver to just drive around the stadium until they found two enormous identical towers—that would be them. They thanked me and I suppose they met their destiny.

I am about 2 meters tall, I was standing about 100 meters from the Ohio Stadium and Ohio Stadium is about 40 meters tall. If the Towers are around 500 meters from the rotunda (the front entrance of the stadium), how tall could they be and still be obscured by the stadium? Explain your reasoning—for the record, the towers are about 80 meters tall.

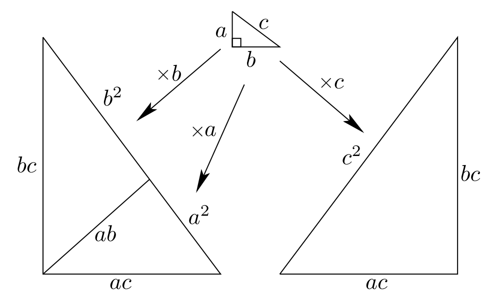
(16) Explain how to use the notion of similar triangles to multiply numbers with your answer expressed as a segment of the appropriate length.

(17) Explain how to use the notion of similar triangles to divide numbers with your answer expressed as a segment of the appropriate length.

(18) Consider the following combinations of S's and A's. Which of them produce a *Congruence Theorem*? Which of them produce a *Similarity Theorem*? Explain your reasoning.

SSS, SSA, SAS, SAA, ASA, AAA

(19) Explain how the following picture "proves" the Pythagorean Theorem.



4.5 Length, Area, and Volume

To be written.

5 Coordinate Constructions

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> —Joseph Louis Lagrange

5.1 Constructions

One of the deepest and powerful aspects of mathematics is that it allows one to see connections between disparate areas. So far we have used different physical techniques (compass and straightedge constructions along with origami constructions) to solve similar problems. Take a minute and reflect upon that—isn't it cool that similar problems can be solved by such different methods? You back? OK—so let's see if we can solidify these connections through abstraction and in the process, make a third connection. We are going to see the algebra behind the geometry we've done. Making these connections isn't easy and can be scary. Thankfully, you are a fearless (yet gentle) reader.

Rules for Coordinate Constructions

(1) A point is an ordered pair (x, y) of real numbers x and y. Points can only be placed as the intersection of lines and/or circles.

(2) Lines are defined as all points (x, y) that are solutions to equations of the form

$$ax + by = c \qquad \text{for given } a, b, c.$$

(3) Circles centered at (a, b) of radius c are defined as all solutions to equations of the form

$$(x - a)^2 + (y - b)^2 = c^2 \qquad \text{for given } a, b, c.$$

(4) The distance between two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$ is given by

$$d(A, B) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}.$$

Just as we have done before, we will present several basic constructions. Compare these to the ones done with a compass and straightedge and the ones done by folding and tracing. We will proceed by the order of difficulty of the construction.

Construction (Bisecting a Segment) Given a segment, we wish to cut it in half.

(1) Let (x_1, y_1) and (x_2, y_2) be the endpoints of your segment.

(2) We claim the midpoint is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Question Can you explain why this works?

?

Construction (Parallel through a Point) Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.

(1) Let $ax + by = c$ be the line and let (x_0, y_0) be the point.

(2) Set $c_0 = ax_0 + by_0$.

(3) The line $ax + by = c_0$ is the desired parallel line.

Question Can you explain why this works?

?

Construction (Perpendicular through a Point) Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.

(1) Let (x_0, y_0) be the given point and let $ax + by = c$ be the given line.

(2) Find $c_0 = bx_0 - ay_0$.

(3) The desired line is $bx + (-a)y = c_0$.

Question Can you explain why this works? Can you give some examples of it in action?

?

Construction (Line between two Points) Given two points, we wish to give the line connecting them.

(1) Call the two points (x_1, y_1) and (x_2, y_2).

(2) Write

$$
\begin{aligned}
ax_1 + by_1 &= c, \\
ax_2 + by_2 &= c.
\end{aligned}
$$

(3) Solve for $-a/b$ and c.

Example 5.1.1) Suppose you want to find the line between the points $(3, 1)$ and $(2, 5)$. Write

$$a \cdot 3 + b \cdot 1 = c,$$
$$a \cdot 2 + b \cdot 5 = c,$$

and subtract these equations to get:

$$a - b \cdot 4 = 0$$

Now we see

$$-b \cdot 4 = -a,$$
$$-4 = -a/b.$$

Now we can take **any** values of a and b that make the equation above true, and plug them back in to $a \cdot 3 + b = c$ to obtain c. **You should explain why this works!** I choose $a = 4$ and $b = 1$. From this I see that $c = 13$ so the line we desire is:

$$4x + y = 13$$

Construction (Intersection of a Line and a Circle) We wish to find the points where a given line meets a given circle.

(1) Let $ax + by = c$ be the given line.

(2) Let $(x - x_0)^2 + (y - y_0)^2 = r^2$ be the given circle.

(3) Solve for x and y.

Question Can you give an example and draw a picture of this construction?

?

Construction (Bisecting an Angle) We wish to divide an angle in half.

(1) Find two points on the angle equidistant from the vertex.

(2) Bisect the segment connecting the point above.

(3) Find the line connecting the vertex to the bisector above.

Question Can you give an example and draw a picture of this construction?

?

Construction (Intersection of Two Circles) Given two circles, we wish to find the points where they meet.

(1) Let $(x - a_1)^2 + (y - b_1)^2 = c_1^2$ be the first circle.

(2) Let $(x - a_2)^2 + (y - b_2)^2 = c_2^2$ be the second circle.

(3) Solve for x and y.

Question Can you give an example and draw a picture of this construction? How many examples should you give for "completeness" sake?

?

Question We wish to construct an equilateral triangle given the length of one side. Can you do this?

?

Problems for Section 5.1

(1) What are the rules for coordinate constructions?

(2) Explain how to transfer a segment using coordinate constructions.

(3) Explain how to copy an angle using coordinate constructions (but don't actually do it!)

(4) Given two points, use coordinate constructions to construct a line between both points. Explain the steps in your construction.

(5) Given segment, use coordinate constructions to bisect the segment. Explain the steps in your construction.

(6) Given a point and line, use coordinate constructions to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

(7) Given a point and line, use coordinate constructions to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(8) Given a line and a circle, use coordinate constructions to construct the intersection of these figures. Explain the steps in your construction.

(9) Use coordinate constructions to bisect a given angle. Explain the steps in your construction.

(10) Given two circles, use coordinate constructions to construct the intersection of these figures. Explain the steps in your construction.

(11) Use algebra to help explain why lines intersect in zero, one, or infinitely many points.

(12) Use algebra to help explain why circles and lines intersect in zero, one, or two points.

(13) Use algebra to help explain why circles intersect in zero, one, two, or infinitely many points.

(14) Use coordinate constructions to construct an equilateral triangle. Explain the steps in your construction.

(15) Use coordinate constructions to construct a square. Explain the steps in your construction.

(16) Use coordinate constructions to construct a regular hexagon. Explain the steps in your construction.

5.2 Brave New Anatomy of Figures

Once more, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for coordinate constructions.

Question In regards to coordinate constructions, what is a *point*?

?

Question In regards to coordinate constructions, what is a *line*?

?

Question In regards to coordinate constructions, what is a *circle*?

?

Now I'm going to quiz you about them (I know we've already gone over this *twice*, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

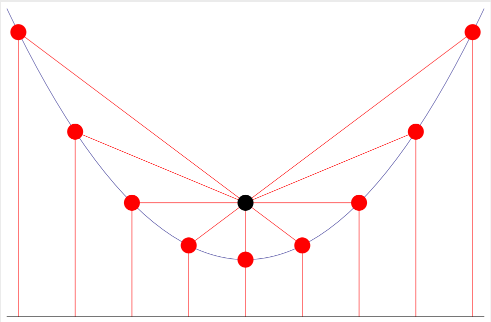
Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

5.2.1 Parabolas

Recall the definition of a *parabola*:

Definition Given a point and a line, a **parabola** is the set of points such that each of these points is the same distance from the given point as it is from the given line.



Fancy folks call the point the **focus** and they call the line the **directrix**.

However I know that you—being rather cosmopolitan in your knowledge and experience—know that from a coordinate geometry point of view that the formula for a parabola should be *something* like:

$$y = ax^2 + bx + c$$

Question How do you rectify these two different notions of a parabola?

I'm feeling chatty, so let me take this one. What would be really nice is if we could extract the focus and directrix from any formula of the form $y = ax^2 + bx + c$. I think we'll work it for a specific example. Consider:

$$y = 3x^2 + 6x - 7$$

Step 1 Complete the square. Write:

$$\begin{aligned} y &= 3x^2 + 6x - 7 \\ &= 3(x^2 + 2x) - 7 \\ &= 3(x^2 + 2x + 1 - 1) - 7 \\ &= 3(x^2 + 2x + 1) - 3 - 7 \\ &= 3(x + 1)^2 - 10 \end{aligned}$$

Step 2 Compare with the following basic form:

$$y = u(x - v)^2 + w$$

Given a parabola in the form above, we have that

$$\text{focus} : \left(v, w + \frac{1}{4u}\right) \qquad \text{and} \qquad \text{directrix} : y = w - \frac{1}{4u}.$$

So in our case the focus is at

$$\left(-1, -10 + \frac{1}{12}\right)$$

and our directrix is the line

$$y = -10 - \frac{1}{12}.$$

Question Can you use the distance formula to show that every point on the parabola is the same distance from focus as it is from the directrix?

?

Problems for Section 5.2

(1) In regards to coordinate constructions, what is a *point*? Compare and contrast this to a naive notion of a point.

(2) In regards to coordinate constructions, what is a *line*? Compare and contrast this to a naive notion of a line.

(3) In regards to coordinate constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle. In particular, explain how the formula for the circle arises.

(4) Explain what is meant by the *focus* of a parabola.

(5) Explain what is meant by the *directrix* of a parabola.

(6) Will the following formula

$$y = ax^2 + bx + c$$

really plot *any* parabola in the plane? If so why? If not, can you give a formula that will? Explain your reasoning.

(7) For each parabola given, find the focus and directrix:

- (a) $y = x^2$
- (b) $y = 7x^2$
- (c) $y = -2x^2$
- (d) $y = x^2 - 4x$
- (e) $y = x^2 - 12$
- (f) $y = x^2 - x + 1$
- (g) $y = x^2 + 2x - 5$
- (h) $y = 2x^2 - 3x - 7$
- (i) $y = -17x^2 + 42x - 3$
- (j) $x = y^2 - 5y$
- (k) $x = 3y^2 - 23y + 17$

In each case explain your reasoning.

(8) Explain in general terms (without appealing to an example) how to find the focus and directrix of a parabola $y = ax^2 + bx + c$.

(9) Use coordinate constructions to construct the circle that passes through the points:

$$A = (0, 0), \qquad B = (3, 3), \qquad C = (4, 0).$$

Sketch this situation and explain your reasoning.

(10) Consider the points

$$A = (1, 1) \qquad \text{and} \qquad B = (5, 3).$$

- (a) Find the midpoint between A and B.
- (b) Find the line the connects A and B. Use algebra to show that the midpoint found above is actually on this line.
- (c) Use algebra to show that this midpoint is equidistant from both A and B.

Sketch this situation and explain your reasoning in each step above.

(11) Consider the parabola $y = x^2/4 + x + 2$.

- (a) Find the focus and directrix of this parabola.
- (b) Sketch the parabola by plotting points.
- (c) Use folding and tracing to fold the envelope of tangents of the parabola.

Present the above items simultaneously on a single graph. Explain the steps in your work.

(12) Consider the following line and circle:

$$x - y = -1 \qquad \text{and} \qquad (x-1)^2 + (y-1)^2 = 5$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(13) Consider the following two circles:

$$x^2 + y^2 = 5 \qquad \text{and} \qquad (x-1)^2 + (y-1)^2 = 5$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(14) Consider the following two circles:

$$(x+1)^2 + (y-1)^2 = 9 \qquad \text{and} \qquad (x-3)^2 + (y-2)^2 = 4$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(15) Explain how to find the minimum or maximum of a parabola of the form:

$$y = ax^2 + bx + c$$

(16) Given a triangle, use coordinate constructions to construct the circumcenter. Explain the steps in your construction.

(17) Given a triangle, use coordinate constructions to construct the orthocenter. Explain the steps in your construction.

(18) Given a triangle, use coordinate constructions to construct the incenter. Explain the steps in your construction.

(19) Given a triangle, use coordinate constructions to construct the centroid. Explain the steps in your construction.

(20) Use coordinate constructions to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(21) Use coordinate constructions to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(22) Use coordinate constructions to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(23) Use coordinate constructions to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

5.3 Constructible Numbers

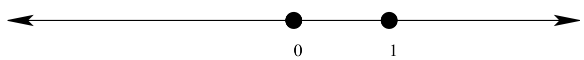
We've now practiced three types of constructions:

(1) Compass and straightedge constructions.

(2) Folding and Tracing constructions.

(3) Coordinate constructions.

You may be wondering what is meant by the words "constructible numbers." Imagine a line with two points on it:



Label the left point 0 and the right point 1. If we think of this as a starting point for a number line, then a **constructible number** is nothing more than a point we can obtain on the above number line using one of the construction techniques above starting with the points 0 and 1.

(1) Denote the set of numbers constructible by compass and straightedge with $\mathbb{C}$. We'll call $\mathbb{C}$ the set of *constructible numbers*.

(2) Denote the set of numbers constructible by folding and tracing with $\mathcal{F}$. We'll call $\mathcal{F}$ the set of *folding and tracing numbers*.

(3) Denote the set of numbers constructible by coordinate constructions with $\mathcal{D}$. We'll call $\mathcal{D}$ the set of *Descartes numbers*.

Be warned, this notion of so-called "Descartes numbers" is unique to these pages.

Mostly in this chapter we'll be talking about $\mathbb{C}$. You'll have to deal with $\mathcal{F}$ and $\mathcal{D}$ yourself.

Question Exactly what numbers are in $\mathbb{C}$?

How do we attack this question? Well first let's get a bit of notation. Recall that we use the symbol "∈" to mean *is in*. So we know that 0 and 1 are *in* the set of constructible numbers. So we write

$$0 \in \mathbb{C} \qquad \text{and} \qquad 1 \in \mathbb{C}.$$

Question Is this true for $\mathcal{F}$, the set of folding and tracing numbers? What about $\mathcal{D}$, the set of Descartes numbers?

?

If we could use constructions to make the operations $+$, $-$, $\cdot$, and $\div$, then we would be able to say a lot more. In fact we will do just this.

Question How does one add and subtract using a compass and straightedge?

?

Question Starting with 0 and 1, what numbers could we add to our number line by simply adding and subtracting?

At this point we have all the positive whole numbers, zero, and the negative whole numbers. We have a special name for this set, we call it the **integers** and denote it by the letter $\mathbb{Z}$:

$$\mathbb{Z} = \{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}.$$

Question Are the integers contained in $\mathcal{F}$, the set of folding and tracing numbers? Are the integers contained in $\mathcal{D}$, the set of Descartes numbers?

?

We still have some more operations:

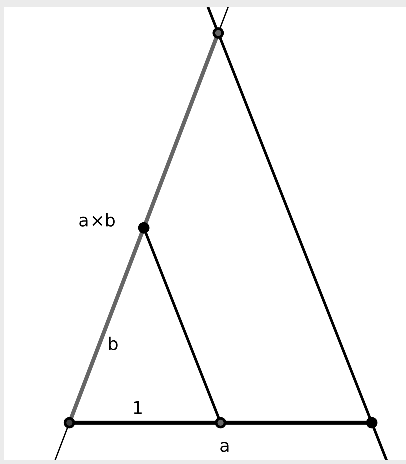
Construction (Multiplication) This construction is based on the idea of similar triangles. Start with given segments of length a, b, and 1:

(1) Make a small triangle with the segment of length 1 and segment of length b.

(2) Now place the segment of length a on top of the unit segment with one end at the vertex.

(3) Draw a line parallel to the segment connecting the unit to the segment of length b starting at the other end of segment of length a.

(4) The length from the vertex to the point that the line containing b intersects the line drawn in step 3 is of length $a \cdot b$.



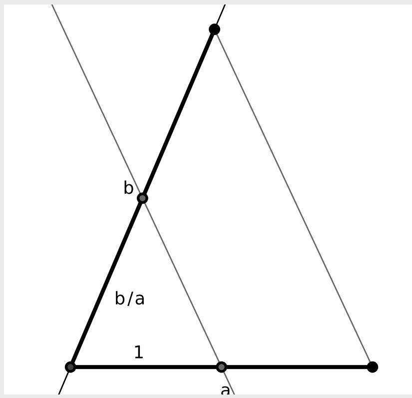
Construction (Division) This construction is also based on the idea of similar triangles. Again, you start with given segments of length a, b, and 1:

(1) Make a triangle with the segment of length a and the segment of length b.

(2) Put the unit along the segment of length a starting at the vertex where the segment of length a and the segment of length b meet.

(3) Make a line parallel to the third side of the triangle containing the segment of length a and the segment of length b starting at the end of the unit.

(4) The distance from where the line drawn in step 3 meets the segment of length b to the vertex is of length b/a.



Question What does our number line look like at this point?

Currently we have $\mathbb{Z}$, the integers, and all of the fractions. In other words:

$$\mathbb{Q} = \left\{\frac{a}{b} \text{ such that } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0\right\}$$

Fancy folks will replace the words *such that* with a colon ":" to get:

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0\right\}$$

We call this set the **rational numbers**. The letter $\mathbb{Q}$ stands for the word *quotient*, which should remind us of fractions.

In mathematics we study sets of numbers. In any field of science, the first step to understanding something is to classify it. One sort of classification that we have is the notion of a *field*.

Definition A **field** is a set of numbers, which we will call F, that is closed under two associative and commutative operations + and · such that:

(1) (a) There exists an additive identity $0 \in F$ such that for all $x \in F$,

$$x + 0 = x.$$

(b) For all $x \in F$, there is an additive inverse $-x \in F$ such that

$$x + (-x) = 0.$$

(2) (a) There exists a multiplicative identity $1 \in F$ such that for all $x \in F$,

$$x \cdot 1 = x.$$

(b) For all $x \in F$ where $x \neq 0$, there is a multiplicative inverse x^{-1} such that

$$x \cdot x^{-1} = 1.$$

(3) Multiplication distributes over addition. That is, for all $x, y, z \in F$

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Now, a word is in order about three tricky words I threw in above: *closed*, *associative*, and *commutative*:

Definition A set F is **closed** under an operation * if for all $x, y \in F$, $x * y \in F$.

Example 5.3.1) The set of integers, $\mathbb{Z}$, is closed under addition, but is not closed under division.

Definition An operation * is **associative** if for all x, y, and z

$$x * (y * z) = (x * y) * z.$$

Definition An operation * is **commutative** if for all x, y

$$x * y = y * x.$$

Question Is $\mathbb{Z}$ a field? Is $\mathbb{Q}$ a field? Can you think of other fields? What about the set of constructible numbers $\mathbb{C}$? What about the folding and tracing numbers $\mathcal{F}$? What about the Descartes numbers $\mathcal{D}$?

?

From all the constructions above we see that the set of constructible numbers $\mathbb{C}$ is a field. However, which field is it? In fact, the set of constructible numbers is bigger than $\mathbb{Q}$!

Construction (Square-Roots) Start with given segments of length a and 1:

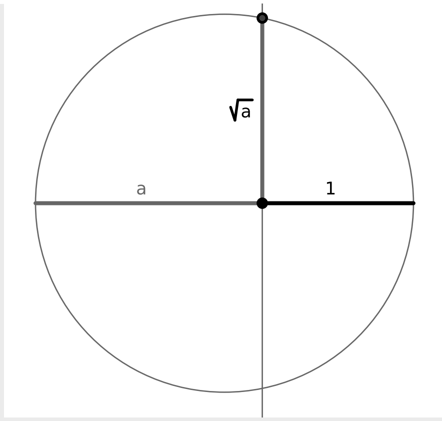
(1) Put the segment of length a immediately to the left of the unit segment on a line.

(2) Bisect the segment of length $a + 1$.

(3) Draw an arc centered at the bisector that starts at one end of the line segment of length $a + 1$ and ends at the other end.

(4) Construct the perpendicular at the point where the segment of length a meets the unit.

(5) The line segment connecting the meeting point of the segment of length a and the unit to the arc drawn in step 3 is of length $\sqrt{a}$.



This tells us that square-roots are constructible. In particular, the square-root of two is constructible. But the square-root of two is not rational! That is, there is no fraction

$$\frac{a}{b} = \sqrt{2} \qquad \text{such that } a, b \in \mathbb{Z}.$$

Question Can you remind me, how do we know that $\sqrt{2}$ is not rational?

?

Question Are square-roots found in $\mathcal{F}$, the set of folding and tracing numbers? What about $\mathcal{D}$, the set of Descartes numbers?

?

OK, so how do we talk about a field that contains both $\mathbb{Q}$ and $\sqrt{2}$? Simple, use this notation:

$$\mathbb{Q}(\sqrt{2}) = \{\text{the smallest field containing both } \mathbb{Q} \text{ and } \sqrt{2}\}$$

So the set of constructible numbers contains all of $\mathbb{Q}(\sqrt{2})$. Does the set of constructible numbers contain even more numbers? Yes! In fact the $\sqrt{3}$ is also not rational, but is constructible. So here is our situation:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{C}$$

So all the numbers in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ are also in $\mathbb{C}$. But is this all of $\mathbb{C}$? Hardly! We could keep on going, adding more and more square-roots 'til the cows come home, and we still will not have our hands on all of the constructible numbers. But all is not lost. We can still say something:

Theorem 5.3.2 *The use of compass and straightedge alone on a field F can at most produce numbers in a field $F(\sqrt{\alpha})$ where $\alpha \in F$.*

Question Can you explain why the above theorem is true? Big hint: What is the relationship between $\mathbb{C}$ and $\mathcal{D}$?

?

The upshot of the above theorem is that the only numbers that are constructible are expressible as a combination of rational numbers and the symbols:

$$+ \quad - \quad \cdot \quad \div \quad \sqrt{}$$

So what are examples of numbers that are not constructible? Well to start $\sqrt[3]{2}$ is not constructible. Also π is not constructible. While both of these facts can be carefully explained, we will spare you gentle reader—for now.

Question Which of the following numbers are constructible?

$$3.1415926, \qquad \sqrt[16]{5}, \qquad \sqrt[3]{27}, \qquad \sqrt[6]{27}.$$

?

Problems for Section 5.3

(1) Explain what the set denoted by $\mathbb{Z}$ is.

(2) Explain what the set denoted by $\mathbb{Q}$ is.

(3) Explain what the set $\mathbb{C}$ of constructible numbers is.

(4) Given two line segments a and b, construct $a+b$. Explain the steps in your construction.

(5) Given two line segments a and b, construct $a-b$. Explain the steps in your construction.

(6) Given three line segments 1, a, and b, construct $a \cdot b$. Explain the steps in your construction.

(7) Given three line segments 1, a, and b, construct a/b. Explain the steps in your construction.

(8) Given a unit, construct $4/3$. Explain the steps in your construction.

(9) Given a unit, construct $3/4$. Explain the steps in your construction.

(10) Use the construction for multiplication to explain why when multiplying two numbers between 0 and 1, the product is always still between 0 and 1.

(11) Explain why the construction for multiplication works.

(12) Use the construction for division to explain why when dividing a positive number by a number between 0 and 1, the quotient is always larger than the initial positive number.

(13) Explain why the construction for division works.

(14) Given a unit, construct $\sqrt{2}$. Explain the steps in your construction.

(15) Use algebra to help explain why the construction for square-roots works.

(16) Give relevant and revealing examples of numbers in the set $\mathbb{Z}$.

(17) Give relevant and revealing examples of numbers in the set $\mathbb{Q}$.

(18) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2})$.

(19) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(20) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

(21) Which of the following are constructible numbers? Explain your answers.

(a) 3.141

(b) $\sqrt[3]{5}$

(c) $\sqrt{3+\sqrt{17}}$

(d) $\sqrt[8]{5}$

(e) $\sqrt[10]{37}$

(f) $\sqrt[16]{37}$

(g) $\sqrt[3]{28}$

(h) $\sqrt[3]{27}$

(i) $\sqrt{13+\sqrt[3]{2}+\sqrt{11}}$

(j) $3+\sqrt[5]{4}$

(k) $\sqrt{3+\sqrt{19}+\sqrt{10}}$

(22) Is $\sqrt{7}$ a rational number? Is it a constructible number? Explain your reasoning.

(23) Is $\sqrt{8}$ a rational number? Is it a constructible number? Explain your reasoning.

(24) Is $\sqrt{9}$ a rational number? Is it a constructible number? Explain your reasoning.

(25) Is $\sqrt[3]{7}$ a rational number? Is it a constructible number? Explain your reasoning.

(26) Is $\sqrt[3]{8}$ a rational number? Is it a constructible number? Explain your reasoning.

(27) Is $\sqrt[3]{9}$ a rational number? Is it a constructible number? Explain your reasoning.

5.4 Impossibilities

Oddly enough, the importance of compass and straightedge constructions is not so much what we can construct, but what we cannot construct. It turns out that classifying what we cannot construct is an interesting question. There are three classic problems which are impossible to solve with a compass and straightedge alone:

(1) Doubling the cube.

(2) Squaring the circle.

(3) Trisecting the angle.

5.4.1 Doubling the Cube

The goal of this problem is to double the volume of a given cube. This boils down to trying to construct roots to the equation:

$$x^3 - 2 = 0$$

But we can see that the only root of the above equation is $\sqrt[3]{2}$ and we already know that this number is not constructible.

Question Why does doubling the cube boil down to constructing a solution to the equation $x^3 - 2 = 0$?

?

5.4.2 Squaring the Circle

Given a circle of radius r, we wish to construct a square that has the same area. Why would someone want to do such a thing? Well to answer this question you must ask yourself:

Question What is area?

?

So what is the deal with this problem? Well suppose you have a circle of radius 1. Its area is now π square units. How long should the edge of a square be if it has the same area? Well the square should have sides of length $\sqrt{\pi}$ units. In 1882, it was proved that π is not the root of any polynomial equation, and hence $\sqrt{\pi}$ is not constructible. Therefore, it is impossible to square the circle.

5.4.3 Trisecting the Angle

This might sound like the easiest to understand, but it's a bit subtle. Given any angle, the goal is to trisect that angle. It can be shown that this cannot be done using a compass and straightedge. In particular, it is impossible to trisect a 60 degree angle with compass and straightedge alone. However, we are not saying that you cannot trisect some angles with compass and straightedge alone, in fact there are *special* angles which can be trisected using a compass and straightedge. However the methods used to trisect those special angles will fail miserably in nearly all other cases.

Question Can you think of any angles that can be trisected using a compass and straightedge?

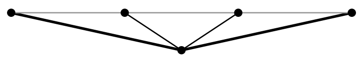
?

Just because it is impossible to trisect an arbitrary angle with compass and straightedge alone does not stop people from trying.

Question If you did not know that it was impossible to trisect an arbitrary angle with a compass and straightedge alone, how might you try to do it?

?

One common way that people try to trisect angles is to take an angle, make an isosceles triangle using the angle, and divide the line segment opposite the angle into three equal parts. While you can divide the opposite side into three equal parts, it in fact **never** trisects the angle. When you do this procedure to acute angles, it *seems* to work, though it doesn't really. You can see that it doesn't by looking at an obtuse angle:



Trisecting the line segment opposite the angle clearly leaves the middle angle much larger than the outer two angles. This happens regardless of the measure of the angle. This mistake is common among people who think that they can trisect an angle with compass and straightedge alone.

5.4.4 Folding and Tracing's Time to Shine

We know that:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{C} = \mathcal{D}$$

Where does the set of folding and tracing numbers $\mathcal{F}$ fit into the parade? I'll tell you, if you promise not to tell anybody that I did. . . $\mathcal{F}$ is the leader of the pack! We already know that you can trisect angles using folding and tracing constructions. In fact you can even solve cubic equations! We'll show you how to do this.

Construction (Solving Cubic Equations) We wish to solve equations of the form:

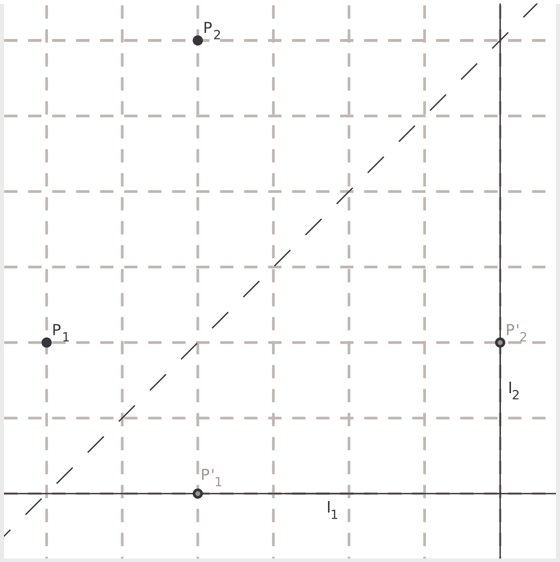
$$x^3 + ax^2 + bx + c = 0$$

(1) Plot the points: $P_1 = (a, 1)$ and $P_2 = (c, b)$.

(2) Plot the lines: $\ell_1 : y = -1$ and $\ell_2 : x = -c$.

(3) With a single fold, place P_1 onto ℓ_1 and P_2 onto ℓ_2.

(4) The slope of the crease is a solution to $x^3 + ax^2 + bx + c = 0$.



Question How do we get the “solution” from the slope?

?

Since folding and tracing constructions can duplicate every compass and straightedge construction and more, we have that $\mathbb{C} \subseteq \mathcal{F}$.

Problems for Section 5.4

(1) Explain the three classic problems that cannot be solved with a compass and straightedge alone.

(2) Use a compass and straightedge construction to trisect an angle of 90°. Explain the steps in your construction.

(3) Use a compass and straightedge construction to trisect an angle of 135°. Explain the steps in your construction.

(4) Use a compass and straightedge construction to trisect an angle of 45°. Explain the steps in your construction.

(5) Use a compass and straightedge construction to trisect an angle of 67.5°. Explain the steps in your construction.

(6) Use folding and tracing to construct an angle of 20°. Explain the steps in your construction.

(7) Use folding and tracing to construct an angle of 10°. Explain the steps in your construction.

(8) Is it possible to use compass and straightedge constructions to construct an angle of 10°? Why or why not?

(9) We have seen that:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathbb{C} \subseteq \mathcal{F}$$

Give explicit examples showing that the set inclusions above are strict—none of them are set equality. Explain your reasoning.

(10) Use folding and tracing to find a solution to the following cubic equations:

(a) $x^3 - x^2 - x + 1 = 0$

(b) $x^3 - 2x^2 - x + 2 = 0$

(c) $x^3 - 3x - 2 = 0$

(d) $x^3 - 4x^2 + 5x - 2 = 0$

(e) $x^3 - 2x^2 - 5x + 6 = 0$

Explain the steps in your constructions.

5.5 Functions and More Functions

To be written.

6 City Geometry

> I always like a good math solution to any love problem.
>
> —Carrie Bradshaw

6.1 Welcome to the City

One day I was walking through the city—that's right, New York City. I had the most terrible feeling that I was lost. I had just passed a *Starbucks Coffee* on my left and a *Sbarro Pizza* on my right, when what did I see? Another *Starbucks Coffee* and *Sbarro Pizza*! Three options occurred to me:

(1) I was walking in circles.

(2) I was at the nexus of the universe.

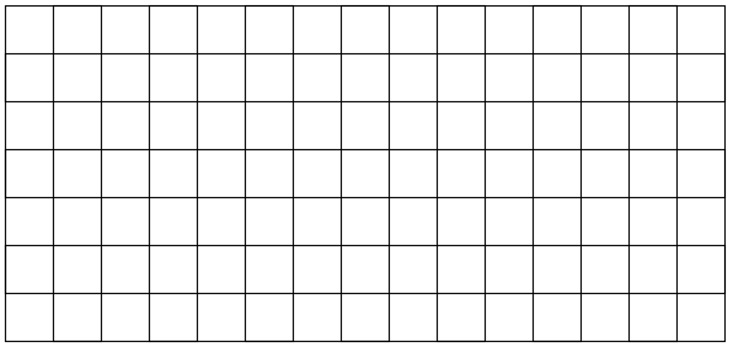
(3) New York City had way too many *Starbucks* and *Sbarro Pizzas*!

Regardless, I was lost. My buddy Joe came to my rescue. He pointed out that the city is organized like a grid.

"Ah! city geometry!" I exclaimed. At this point all Joe could say was "Huh?"

Question What the heck was I talking about?

Let me tell you: *Euclidean geometry* is regular old plane (not plain!) geometry. It is the geometry that we've been exploring thus far in our journey. In *city geometry* we have points and lines, just like in Euclidean geometry. However, most cities can be viewed as a grid of city blocks

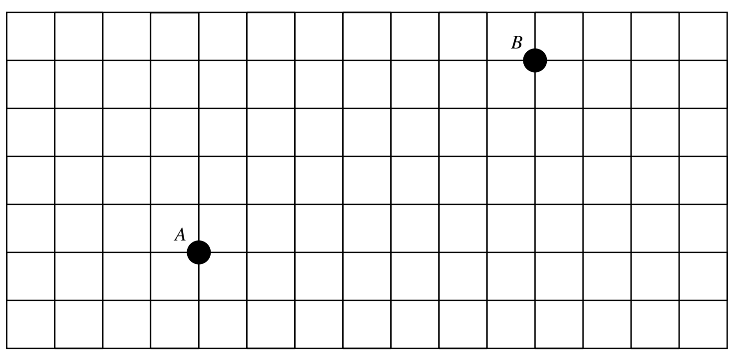


and when we travel in a city, we can only travel on the streets—we can't cut through the blocks. This means that we don't measure distance as the crow flies. Instead we use the *taxicab distance*:

Definition Given two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$, we define the **taxicab distance** as:

$$d_T(A, B) = |a_x - b_x| + |a_y - b_y|$$

Example 6.1.1) Consider the following points:



Let $A = (0, 0)$. Now we see that $B = (7, 4)$. Hence

$$\begin{aligned} d_T(A, B) &= |0 - 7| + |0 - 4| \\ &= 7 + 4 \\ &= 11. \end{aligned}$$

Of course in real life, you would want to add in the appropriate units to your final answer.

Question How do you compute the distance between A and B as the crow flies?

?

Definition The geometry where points and lines are those from Euclidean geometry but distance is measured via taxicab distance is called **city geometry**.

Question Compare and contrast the notion of a line in Euclidean geometry and in city geometry. In either geometry is a line the unique shortest path between any two points?

?

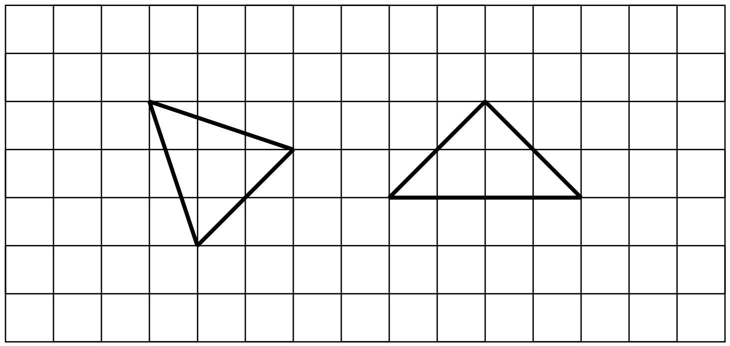
6.1.1 (Un)Common Structures

How different is life in city geometry from life in Euclidean geometry? Let's find out!

Triangles If we think back to Euclidean geometry, we may recall some lengthy discussions on triangles. Yet so far, we have not really discussed triangles in city geometry.

Question What does a triangle look like in city geometry and how do you measure its angles?

I'll take this one. Triangles look the same in city geometry as they do in Euclidean geometry. Also, you measure angles in exactly the same way. However, there is one minor hiccup. Consider these two triangles in city geometry:



Question In city geometry, what are the lengths of the sides of each of these triangles? Why is this odd?

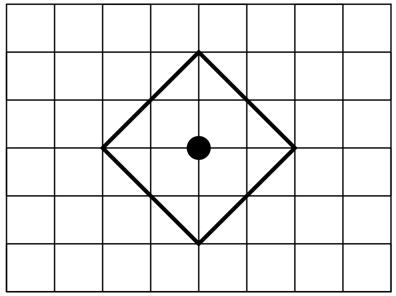
?

Hence we see that triangles are a bit funny in city geometry.

Circles Circles are also discussed in many geometry courses and this course is no different. However, in city geometry the circles are a little less round. The first question we must answer is the following:

Question What is a circle?

Well, a circle is the collection of all points equidistant from a given point. So in city geometry, we must conclude that a circle of radius 2 would look like:



Question What sort of shape should a city geometry compass draw?

?

Question How many points are there at the intersection of two circles in Euclidean geometry? How many points are there at the intersection of two circles in city geometry?

?

Problems for Section 6.1

(1) Given two points A and B in city geometry, does $d_T(A, B) = d_T(B, A)$? Explain your reasoning.

(2) It was once believed that Euclid's five postulates

(a) A line can be drawn from a point to any other point.

(b) A finite line can be extended indefinitely.

(c) A circle can be drawn, given a center and a radius.

(d) All right angles are ninety degrees.

(e) If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line is less than the sum of two right angles, then the lines meet on that side and not on the other side.

were sufficient to completely describe plane geometry. Explain how city geometry shows that Euclid's five postulates are **not** enough to determine all of the familiar properties of the plane.

(3) In Euclidean geometry are all equilateral triangles congruent assuming they have the same side length? Is this true in city geometry? Explain your reasoning.

(4) How many points are there at the intersection of two circles in Euclidean geometry? How many points are there at the intersection of two circles in city geometry? Explain your reasoning.

(5) What sort of shape should a city geometry compass draw? Explain your reasoning.

(6) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for an equilateral triangle using a city geometry compass.

(7) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for bisecting a segment using a city geometry compass.

(8) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for a perpendicular through a point using a city geometry compass.

(9) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for bisecting an angle using a city geometry compass.

(10) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for copying an angle using a city geometry compass.

(11) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for a parallel through a point using a city geometry compass.

6.2 Anatomy of Figures and the City

When we study geometry, what do we seek? That's right—we wish to discover the points that can be obtained given a set of rules. With city geometry, the major rule involved is the taxicab distance. Let's answer these questions!

Question In regards to city geometry, what is a *point*?

?

Question In regards to city geometry, what is a *line*?

?

Question In regards to city geometry, what is a *circle*?

?

Now I'm going to quiz you about them (I know we've already gone over this *twice*, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a city geometry circle containing these points? If no, why not? If yes, how do you know?

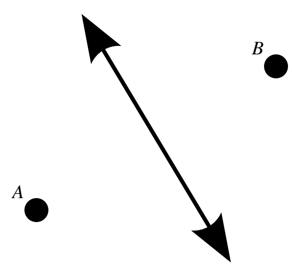
?

Midsets

Definition Given two points A and B, their **midset** is the set of points that are an equal distance away from both A and B.

Question How do we find the midset of two points in Euclidean geometry? How do we find the midset of two points in city geometry?

In Euclidean geometry, we just take the the following line:



If we had no idea what the midset should look like in Euclidean geometry, we could start as follows:

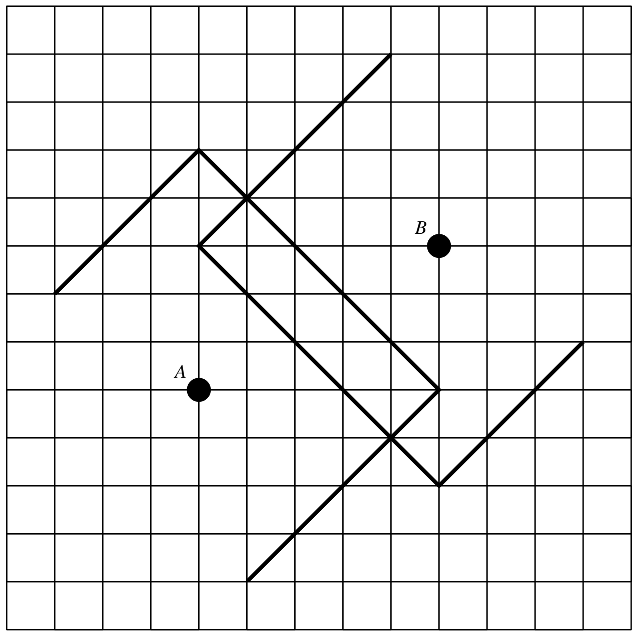
- Draw circles of radius r_1 centered at both A and B. If these circles intersect, then their points of intersection will be in our midset. (Why?)
- Draw circles of radius r_2 centered at both A and B. If these circles intersect, then their points of intersection will be in our midset.
- We continue in this fashion until we have a clear idea of what the midset looks like. It is now easy to check that the line in our picture is indeed the midset.

How do we do it in city geometry? We do it basically the same way.

Example 6.2.1) Suppose you wished to find the midset of two points in city geometry.

We start by fixing coordinate axes. Considering the diagram below, if $A = (0, 0)$, then $B = (5, 3)$. We now use the same idea as in Euclidean geometry. Drawing circles of radius 3 centered at A and B respectively, we see that there are no points 3 points away from both A and B. Since $d_T(A, B) = 8$, this is to be expected. We will need to draw larger taxicab circles before we will find points in the midset. Drawing taxicab circles of radius 5, we see that the points $(1, 4)$

and $(4, -1)$ are both in our midset.

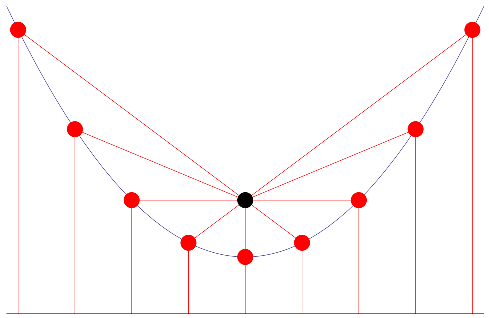


Now it is time to sing along. You draw circles of radius 6, to get two more points $(1, 5)$ and $(4, -2)$. Drawing circles with larger radii yields more and more points "due north" of $(1, 5)$ and "due south" of $(4, -2)$. However, if we draw circles of radius 4 centered at A and B respectively, their intersection is the line segment between $(1, 3)$ and $(4, 0)$. Unlike Euclidean circles, distinct city geometry circles can intersect in more than two points and city geometry midsets can be more complicated than their Euclidean counterparts.

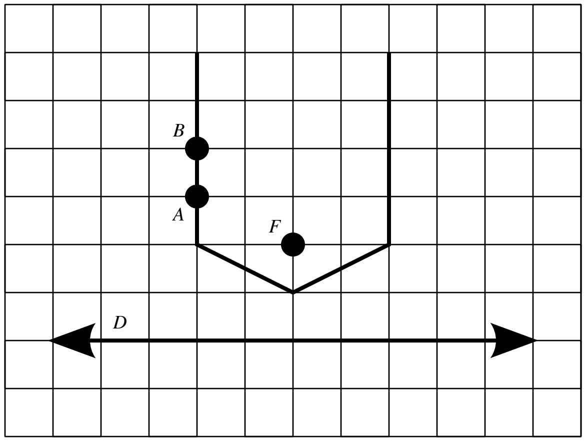
Question How do you draw the city geometry midset of A and B? What could the midsets look like?

?

Parabolas Recall that a parabola is a set of points such that each of those points is the same distance from a given point, F, as it is from a given line, D.



This definition still makes sense when we work with taxicab distance instead of Euclidean distance. To start, choose a value r and draw a line parallel to D at taxicab distance r away from D. Now draw a City circle of radius r centered at F. The points of intersection of this line and this circle will be r away from D and r away from F and so will be points on our City parabola. Repeat this process for different values of r.



Unlike the Euclidean case, the City parabola need not grow broader and broader as the distance from the line increases. In the picture above, as we go from A to B

on the parabola, both the taxicab and Euclidean distances to the line D increase by 1. The taxicab distance from the point F also increases by 1 as we go from A to B but the Euclidean distance increases by less than 1. For the Euclidean distance from F to the parabola to keep increasing at the same rate as the distance to the line D, the Euclidean parabola has to keep spreading to the sides.

Question How do you draw city geometry parabolas? What do different parabolas look like?

?

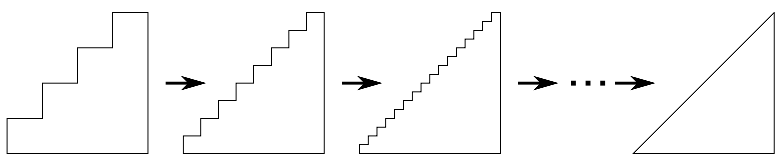
A Paradox To be completely clear on what a paradox is, here is the definition we will be using:

Definition A **paradox** is a statement that seems to be contradictory. This means it seems both true and false at the same time.

There are many paradoxes in mathematics. By studying them we gain insight—and also practice tying our brain into knots! Here is a paradox:

Paradox $\sqrt{2} = 2$.

Proof (False-Proof) *Consider the following sequence of diagrams:*



On the far right-hand side, we see a right-triangle. Suppose that the lengths of the legs of the right-triangle are one. Now by the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

However, we see that the triangles coming from the left converge to the triangle *on the right. In every case on the left, the stair-step side has length* 2*. Hence when our sequence of stair-step triangles converges, we see that the hypotenuse of the right-triangle will have length* 2*. Thus* $\sqrt{2} = 2$.

Question What is wrong with the proof above?

?

Problems for Section 6.2

(1) Suppose that you have two triangles $\triangle ABC$ and $\triangle DEF$ in city geometry such that

(a) $d_T(A, B) = d_T(D, E)$.

(b) $d_T(B, C) = d_T(E, F)$.

(c) $d_T(C, A) = d_T(F, D)$.

Is it necessarily true that $\triangle ABC \equiv \triangle DEF$? Explain your reasoning.

(2) In city geometry, if all the angles of $\triangle ABC$ are 60°, is $\triangle ABC$ necessarily an equilateral triangle? Explain your reasoning.

(3) In city geometry, if two right triangles have legs of the same length, is it true that their hypotenuses will be the same length? Explain your reasoning.

(4) Considering that π is the ratio of the circumference of a circle to its diameter, what is the value of π in city geometry? Explain your reasoning.

(5) Considering that the area of a circle of radius r is given by πr^2, what is the value of π in city geometry? Explain your reasoning.

(6) When is the Euclidean midset of two points equal to their city geometry midset? Explain your reasoning.

(7) Find the city geometry midset of $(-2, 2)$ and $(3, 2)$.

(8) Find the city geometry midset of $(-2, 2)$ and $(4, -1)$.

(9) Find the city geometry midset of $(-2, 2)$ and $(2, 2)$.

(10) Draw the city geometry parabola determined by the point $(0, 2)$ and the line $y = 0$.

(11) Draw the city geometry parabola determined by the point $(3, 0)$ and the line $x = 0$.

(12) Draw the city geometry parabola determined by the point $(2, 0)$ and the line $y = x$.

(13) Find the distance in city geometry from the point $(3, 4)$ to the line $y = -1/3x$. Explain your reasoning.

(14) Draw the city geometry parabola determined by the point $(0, 4)$ and the line $y = x/3$. Explain your reasoning.

(15) Draw the city geometry parabola determined by the point $(0, 6)$ and the line $y = x/2$. Explain your reasoning.

(16) Draw the city geometry parabola determined by the point $(1, 4)$ and the line $y = 2x/3$. Explain your reasoning.

(17) Draw the city geometry parabola determined by the point $(3, 3)$ and the line $y = x/2$. Explain your reasoning.

(18) Find all points P such that $d_T(P, A) + d_T(P, B) = 8$. Explain your work. (In Euclidean geometry, this condition determines an *ellipse*. The solution to this problem could be called the *city geometry ellipse*.)

(19) True/False: Three noncollinear points lie on a unique Euclidean circle. Explain your reasoning.

(20) True/False: Three noncollinear points lie on a unique city geometry circle. Explain your reasoning.

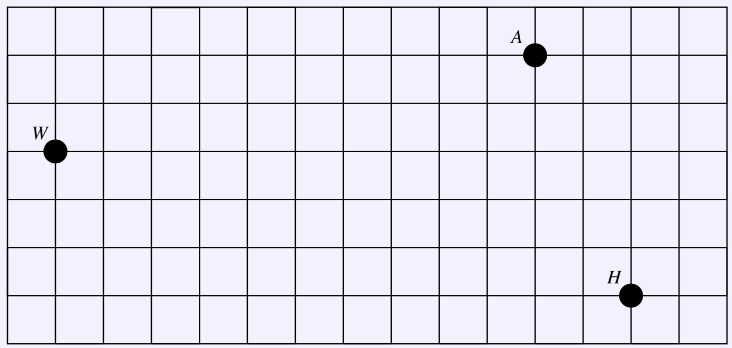
(21) Explain why no Euclidean circle can contain three collinear points. Can a city geometry circle contain three collinear points? Explain your conclusion.

(22) Can you find a false-proof showing that $\pi = 2$?

6.3 Getting Work Done

If you are interested in *real-world* types of problems, then maybe city geometry is the geometry for you. The concepts that arise in city geometry are directly applicable to everyday life.

Question Will just bought himself a brand new gorilla suit. He wants to show it off at three parties this Saturday night. The parties are being held at his friends' houses: the Antidisestablishment (A), Hausdorff (H), and the Wookie Loveshack (W). If he travels from party A to party H to party W, how far does he travel this Saturday night?



Proof (Solution) *We need to compute*

$$d_T(A, H) + d_T(H, W)$$

Let's start by fixing a coordinate system and making A *the origin. Then* H *is* $(2, -5)$ *and* W *is* $(-10, -2)$. *Then*

$$\begin{aligned} d_T(A, H) &= |0 - 2| + |0 - (-5)| \\ &= 2 + 5 \\ &= 7 \end{aligned}$$

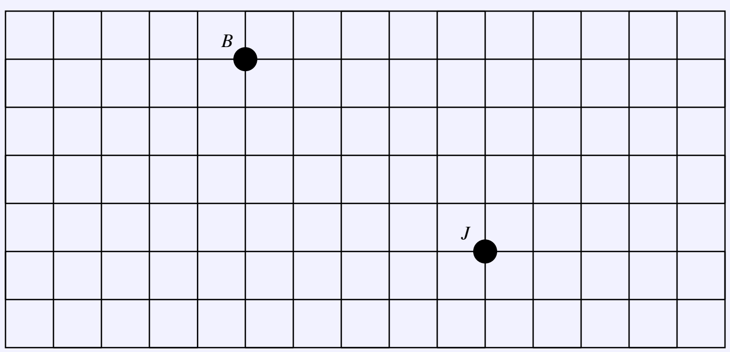
and

$$\begin{aligned} d_T(H,W) &= |2-(-10)| + |-5-(-2)| \\ &= 12+3 \\ &= 15. \end{aligned}$$

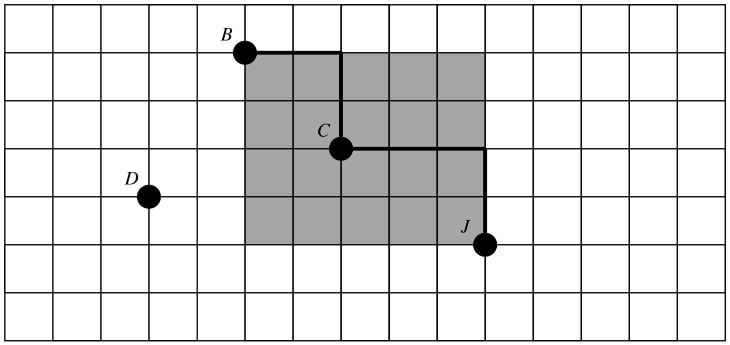
Will must trudge 7 + 15 = 22 *blocks in his gorilla suit.*

Okay, that's enough monkey business—I feel like pizza and a movie.

Question Brad and Melissa are going to downtown Champaign, Illinois. Brad wants to go to *Jupiter's* for pizza (J) while Melissa goes to *Boardman's Art Theater* (B) to watch a movie. Where should they park to minimize the total distance walked by both?



Proof (Solution) *Again, let's set up a coordinate system so that we can say* *what points we are talking about. If J is $(0, 0)$, then B is $(-5, 4)$.*



No matter where they park, Brad and Melissa's two paths joined together must make a path from B to J. This combined path has to be at least 9 blocks long since $d_T(B, J) = 9$. They should look for a parking spot in the rectangle formed by the points $(0, 0), (0, 4), (-5, 0)$, and $(-5, 4)$.

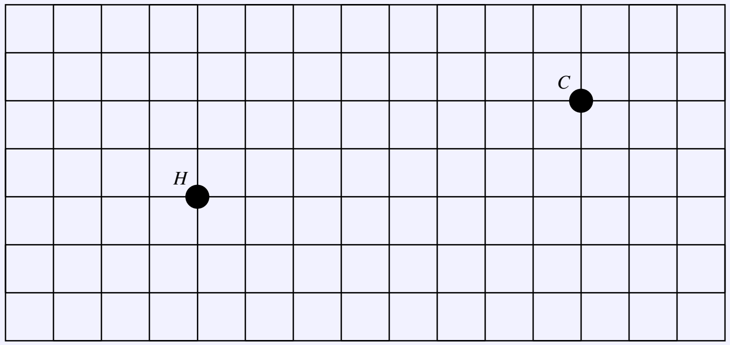
Suppose they park within this rectangle and call this point C. Melissa now walks 4 blocks from C to B and Brad walks 5 blocks from C to J. The two paths joined together form a path from B to J of length 9.

If they park outside the rectangle described above, for example at point D, then the corresponding path from B to J will be longer than 9 blocks. Any path from B to J going through D goes a block too far west and then has to backtrack a block to the east making it longer than 9 blocks.

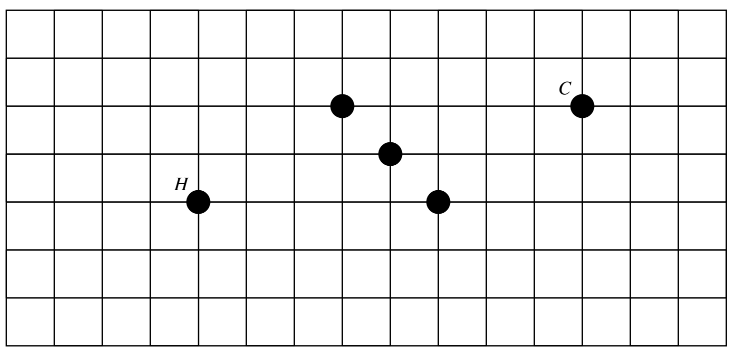
Question If we consider the same question in Euclidean geometry, what is the answer?

?

Question Tom is looking for an apartment that is close to Altgeld Hall (H) but is also close to his favorite restaurant, *Crane Alley* (C). Where should Tom live?



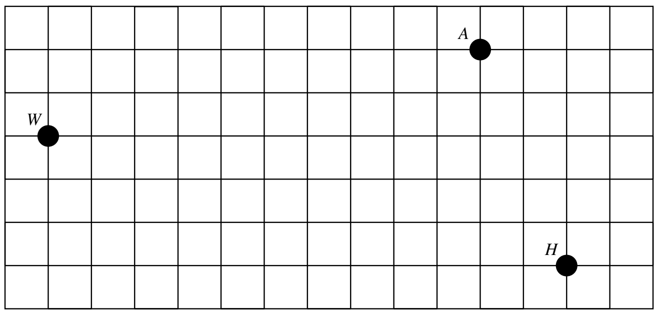
Proof (Solution) *If we fix a coordinate system with its origin at Altgeld Hall, H, then C is at $(8, 2)$. We see that $d_T(H, C) = 10$. If Tom wants to live as close as possible to both of these, he should look for an apartment, A, such that $d_T(A, H) = d_T(A, C) = 5$. He would then be living halfway along one of the shortest paths from Altgeld to the restaurant. Mark all the points 5 blocks away from H. Now mark all the points 5 blocks away from C.*



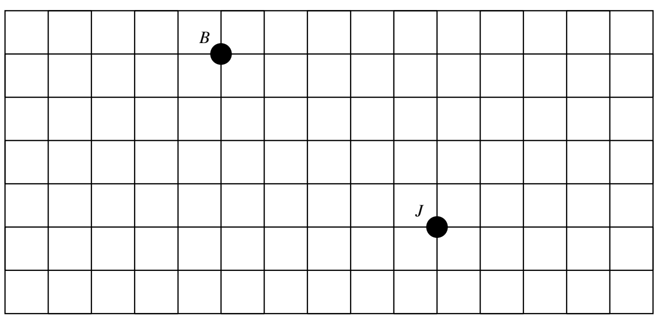
We now see that Tom should check out the apartments near $(5, 0)$, $(4, 1)$, and $(3, 2)$.

Problems for Section 6.3

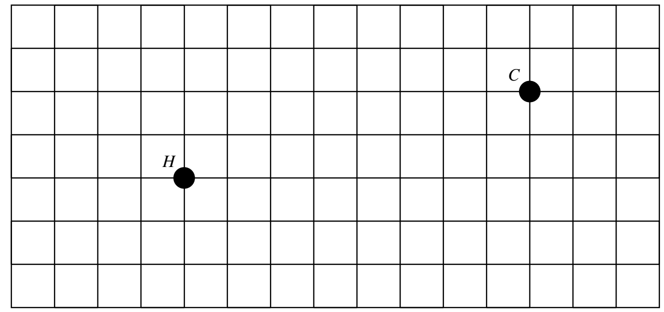
(1) Will just bought himself a brand new gorilla suit. He wants to show it off at three parties this Saturday night. The parties are being held at his friends' houses: the Antidisestablishment (A), Hausdorff (H), and the Wookie Loveshack (W). If he travels from party A to party H to party W, how far does he travel this Saturday night? Explain your reasoning.



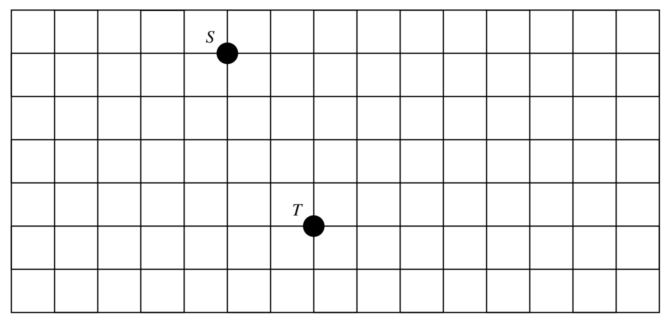
(2) Brad and Melissa are going to downtown Champaign, Illinois. Brad wants to go to *Jupiter's* for pizza (J) while Melissa goes to *Boardman's Art Theater* (B) to watch a movie. Where should they park to minimize the total distance walked by both? Explain your reasoning.



(3) Tom is looking for an apartment that is close to Altgeld Hall (H) but is also close to his favorite restaurant, *Crane Alley* (C). Where should Tom live? Explain your reasoning.

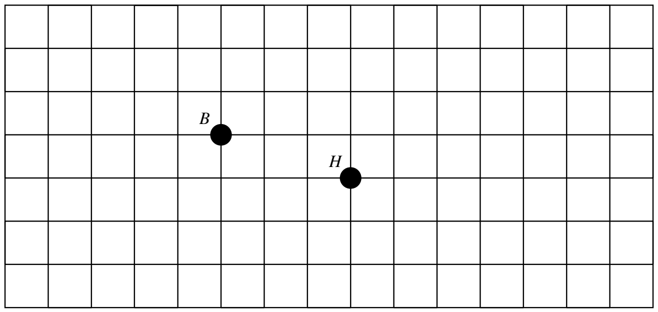


(4) Johann and Amber are going to German Village. Johann wants to go to *Schmidt's* (S) for a cream-puff while Amber goes to the *Thurman Cafe* (T) for some spicy wings. Where should they park to minimize the total distance walked by both if Amber insists that Johann should not have to walk a longer distance than her? Explain your reasoning.



(5) Han and Tom are going to downtown Clintonville. Han wants to go to get a haircut (H) and Tom wants to look at the bookstore (B). Where should they park to keep the total distance walked by both

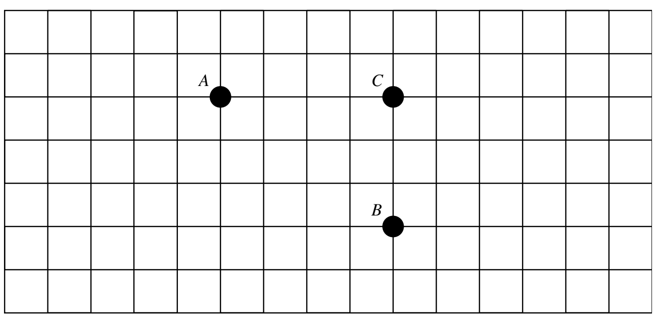
less than 8 blocks? Explain your reasoning.



(6) The university is installing emergency phones across campus. Where should they place them so that their students are never more than a block away from an emergency phone? Explain your reasoning.

(7) Tom and Ben have devised a ingenious *Puzzle-Stroll* (aka a *scavenger-hunt*). Here is one of the puzzles:

> To find what you seek, you must be one with the city—using it's distance, the treasure is 4 blocks from (A), 3 blocks from (B), and 2 blocks from (C).

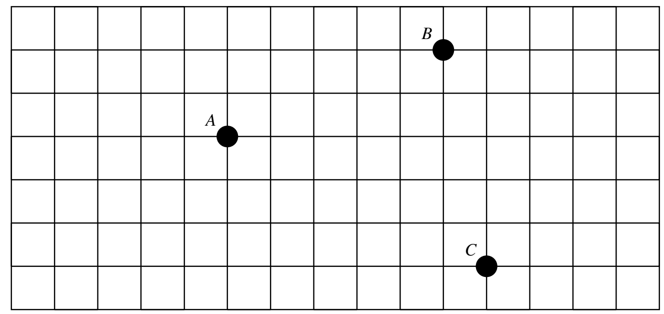


Where's the treasure? Explain your reasoning.

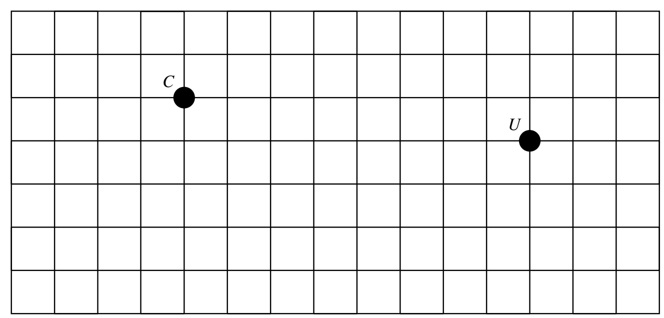
(8) Johann is starting up a new business, *Cafe Battle Royale*. He knows mathematicians drink a lot of coffee so he wants it to be near Altgeld Hall. Balancing this against how expensive rent is near campus, he decides the cafe should be 3 blocks from Altgeld Hall. Where should his cafe be located? Explain your reasoning.

(9) *Cafe Battle Royale, Inc.* is expanding. Johann wants his potential customers to always be within 4 blocks of one of his cafes. Where should his cafes be located? Explain your reasoning.

(10) There are hospitals located at A, B, and C. Ambulances should be sent to medical emergencies from whichever hospital is closest. Divide the city into regions in a way that will help the dispatcher decide which ambulance to send. Explain your reasoning.

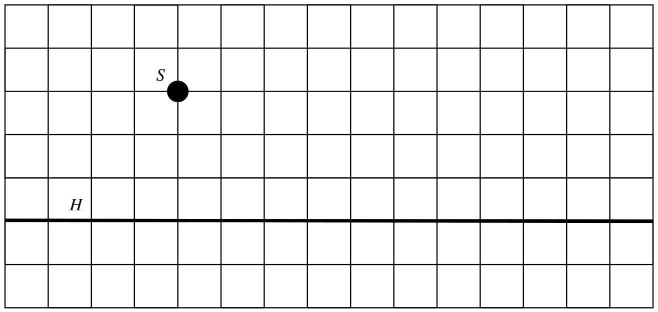


(11) Sylvia is going to open a new restaurant called *Grillvia's* where customers make their own food and then she grills it for them. She wants her restaurant to be equidistant from the heart of Champaign (C) and the heart of Urbana (U). Where should she put her restaurant? Explain your reasoning.

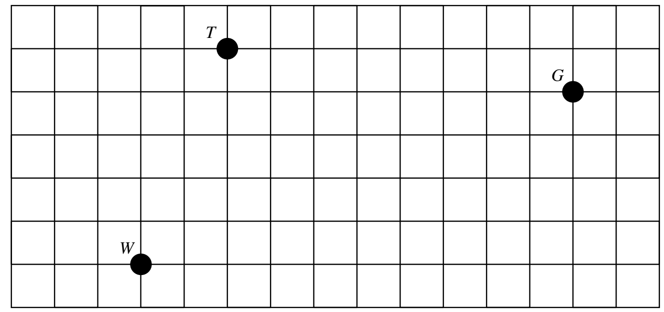


(12) Chris wants to live an equal distance from his favorite hangout *Studio 35* (S) and High Street (H) where he can catch the Number

2 bus. Where should he live? Explain your reasoning.



(13) Lisa just bought a 3-wheeled zebra-striped electric car and its range is limited. Suppose that each day Lisa likes to go to work (W), and then to the tea shop (T) **or** the garden shop (G) but not both, and then back home (H). Where should Lisa live? Give several options depending on how efficient her zebra-striped car is. Explain your reasoning.



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