EXPLICIT REALIZATION OF
A METAPLECTIC REPRESENTATION

By
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0. Let $F \not= \mathbb{C}$ be a local field with $\text{char } F \not= 2$. In [W] Weil explicitly constructed a model of a genuine unitary representation $\theta$ of the two-fold covering group $\hat{\text{Sp}}$ of the symplectic group $\text{Sp}$ over $F$. In particular, the existence of the covering group $\hat{\text{Sp}}$ was first proven in [W]. It is now known (see, e.g., [M]) how to construct $r$-fold covering groups of split semi-simple groups over a field $F \not= \mathbb{C}$ containing a primitive $r$th root of unity. In particular, when $r = 2$, such $F$ has $\text{char } F \not= 2$. In the case of $\text{GL}(n)$ the analogous genuine unitarizable representation $\Theta$ of a covering group is defined in [KP1] as a sub- or quotient of some induced representation. This $\Theta$ corresponds to the trivial representation of $\text{GL}(n)$ by the metaplectic correspondence (see [KP2], [FK1]). The purpose of this paper is to construct an explicit model of the representation $\Theta = \Theta_3$ of a two-fold covering group $G$ of $\text{GL}(3)$ over a local field $F \not= \mathbb{C}$ of characteristic $\not= 2$, analogous to the explicit model of the representation of Weil [W]. We also determine the unitary completion of the unitarizable $\Theta_3$. The unitary completion of our model coincides with the model of Torasso [T] when $F = \mathbb{R}$. The existence of our model has interesting applications in harmonic analysis. Some of these applications are discussed in detail in §3. In a sequel [F1] the techniques of this paper are generalized to construct an explicit model of $\Theta_n$ for any $n \geq 3$.

1. The representation

To state our Theorem and its Corollaries, we begin by specifying the representation $\Theta$ to be studied.

1.1. Let $F$ be a local field $\not= \mathbb{C}$ of characteristic $\not= 2$. For every integer $n > 1$ there exists (see [M]) a unique non-trivial topological central double covering group $p: S_n \to \text{SL}(n, F)$. Choose a section $s: \text{SL}(n, F) \to S_n$ corresponding to a choice of a two-cocycle $\beta'_n: S_n \times S_n \to \ker p$ which defines the group law on $S_n$. Embed $\hat{G}_n = \text{GL}(n, F)$ in $\text{SL}(n + 1, 1, F)$ by

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Denote by $G'_n$ the preimage $p^{-1}(t(G_n))$. Let $(\cdot, \cdot) : F^2 \times F^2 \to \{1, -1\}$ be the Hilbert symbol. Identify $\{1, -1\}$ with the kernel of $p$. Put $\beta(g, g') = \beta'(g, g')(\det g, \det g') (g, g' \in G_n)$. Let $s : G_n \to G'_n$ be the restriction of the section used in the definition of $S_{n+1}$. Denote by $G_n$ the group which is equal to $G'_n$ as a set, whose product rule is given by $s(g)z \cdot s(g')z' = s(gg')zz' \beta(g, g')$. Then $G_n$ is a non-trivial topological double covering group of $G_n$. Let $\mathcal{A}$ and $\mathcal{B}$ be the groups of diagonal and upper-triangular matrices in $\mathcal{G}_n$, and $\mathcal{A}$ and $\mathcal{B}$ their preimages in $G_n$. Note that $s$ is a homomorphism on the group $\mathcal{N}$ of upper-triangular unipotent matrices, and put $N = s(\mathcal{N})$. Let $\mathcal{Z}$ be the center of $G_n$ and $\mathcal{Z}$ the center of $G_n$.

**Lemma 1.** Let $\mathcal{A}$ be the group of squares in $\mathcal{A}$, and put $A^2 = p^{-1}(A^2)$. Then

(i) the group $ZA^2$ is the center of $A$,

(ii) if $n$ is even then $Z = A^2 \cap p^{-1}(\mathcal{Z})$,

(iii) if $n$ is odd then $Z = p^{-1}(\mathcal{Z})$, and $p$ defines an isomorphism

$$p : Z/(Z \cap A^2) \to \mathcal{Z}/\mathcal{Z}^2 \cong F^x/F^x.$$

**Proof.** See [KP1], Prop. 0.1.1.

Define a map $t = t_n : \mathcal{A} \to A^2$ by $t(h) = s(h)^2u(h)$, where

$$u(h) = \Pi_{1 \leq i < j \leq n} (h_i, h_j)$$

for a diagonal matrix $h = \text{diag}(h_i)$ with entries $h_i$ ($1 \leq i \leq n$). Note that $t$ is independent of the choice of the section $s$. Using the product rule in $G_n$ (see [KP1], p. 39), it is easy to check that our section $s$ satisfies $t(h) = s(h^2)$ for every $h$ in $\mathcal{A}$.

**Lemma 2.** The map $t$ is a group homomorphism.

**Proof.** This follows from the multiplication law on $A \subset G_n$.

**Definition.** Let $\delta = \delta_n : \mathcal{A} \to \mathbb{C}^\times$ be the character $\delta(\text{diag}(h_i)) = \Pi_{1 \leq i < j \leq n} |h_i|^{(2i-1-n)2}$. A character $\delta = \delta_n : ZA^2 \to \mathbb{C}^\times$ whose restriction to $\ker p$ is non-trivial is called exceptional if $\delta(t(h)) = \delta(h)$ for all $h$ in $\mathcal{A}$.

Note that $A^2 = t(A) \cdot \ker p$ is equal to $ZA^2$ if $n$ is even. If $n$ is odd then $ZA^2/A^2 \cong F^x/F^x$, hence it is possible to extend $\delta$ from $A^2$ to $ZA^2$, and there exist exceptional characters for all $n$.

**Lemma 3.** (i) For any exceptional character $\delta$ of $ZA^2$ there exists a unique (up to isomorphism) irreducible representation $\rho_\delta$ of $A$ whose restriction to $ZA^2$ is $\delta \cdot \text{Id}$.
(ii) Extend \( \rho_b \) to a representation of \( B \) trivial on \( N \). Let \( (\pi_\delta, \hat{\rho}_\delta) \) be the representation of \( G_n \), normalizedly (see \([BZ2], (1.8)\)) induced from \( \rho_b \). Then \( (\pi_\delta, \hat{\rho}_\delta) \) has a unique irreducible subrepresentation. When \( n = 2 \), \( (\pi_\delta, \hat{\rho}_\delta) \) has a unique proper non-zero subrepresentation.

(iii) The unique irreducible subrepresentation of \( (\pi_\delta, \hat{\rho}_\delta) \) is unitarizable.

**Proof.** See \([KP1], p. 72, \) for (i), (ii); and Theorem II.2.1, p. 118, for (iii).

**Definition.** By the *exceptional representation* \( (\pi_\delta, \hat{\rho}_\delta) \) of \( G_n \) we mean the unique irreducible subrepresentation of \( (r, \pi_\delta) \).

1.2. Lemma 1(ii) implies that for an even \( n \) the group \( G_n \) has a unique exceptional representation, denoted \( (\Theta, V) \) or \( (\Theta_n, V) \).

**Lemma 4.** Assume that \( n \) is odd. Then there exists a map \( v : \hat{Z} \to Z \) such that \( p \circ v = \text{Id} \) and \( v(z_1)v(z_2) = v(z_1z_2)z_1^{(a-1)/2} \). Moreover, such a map is unique up to a composition with an involution of \( G_n \).

**Proof.** First note that the section \( s \) satisfies the required properties. To prove the uniqueness, let \( v_1 \) and \( v_2 \) be two such maps. Then \( \chi = v_1/v_2 \) defines a homomorphism \( \chi : F^\times \approx \hat{Z} \to \ker p \). Let \( \hat{\chi} \) be the involution of \( G_n \) defined by \( \hat{\chi}(g) = \chi(\det p(g))g \). Then \( v_2 = \hat{\chi} \circ v_1 \), as required.

**Definition.** Fix a non-trivial additive character \( \psi : F \to \mathbb{C}^\times \) of \( F \). Denote by \( dx \) a Haar measure on \( F \). Define a function \( \gamma = \gamma_\psi : F^\times \to \mathbb{C}^\times \) by

\[
\gamma(a) = \frac{|a|^{1/2} \int_F \psi(ax^2)dx}{\int_F \psi(x^2)dx}.
\]

Clearly, we have \( \gamma(a^2) = 1 \). Moreover, we have

**Lemma 5.** For every \( a, b \) in \( F^\times \) the function \( \gamma \) satisfies \( \gamma(ab) = \gamma(a)\psi(b)(a, b) \).

**Proof.** Let \( \gamma_\psi \) be the \( \gamma \) defined in \([W]\) by

\[
|a|^{1/2} \int_F f(x)\psi(ax^2)dx = \gamma_\psi(ax^2) \int_F \hat{f}(x)\psi(-a^{-1}x^2)dx
\]

for integrable \( f \) and \( \hat{f} \); here \( \hat{f} \) is the \( \psi \)-Fourier transform with respect to the self-dual Haar measure. Since \( \gamma_\psi \) satisfies the relation

\[
\gamma_\psi(x^2 - ay^2 - bz^2 + abt^2) = (a, b)
\]

(see \([W], p. 176, \) bottom line), and \( \gamma(a) = \gamma_\psi(ax^2)/\gamma_\psi(x^2) \), the lemma follows.
**Definition.** Let $\delta_\nu$ be the function of $ZA^2$ defined by

$$
\delta_\nu(\zeta s(z)i(h)) = \zeta \gamma(z) \delta(h) \quad (\zeta \in \ker p, z \in \mathbb{Z} \cong F^\times, h \in \mathbb{A})
$$

if $n \equiv 3 \pmod{4}$; if $n \equiv 1 \pmod{4}$ define $\delta_\nu$ by $\delta_\nu(\zeta s(z)i(h)) = \zeta \delta(h)$.

It is clear that $\delta_\nu$ is an exceptional character of $ZA^2$. Denote by $(\Theta, V)$, or $(\Theta_n, V)$, the corresponding representation of $G = G_n$.

**1.3.** It is important for us to work with an extension of $\Theta$ to a semi-direct product $G^* = G \rtimes (\sigma)$, where $\sigma$ is an involution of $G$ which we proceed to define. Let $w_n$ be the anti-diagonal matrix $((-1)^{i+1} \delta_{i,-i-n+1})$ in $G_n$. Consider $w_n$ as an element of $SL(n + 1, F)$ via $j$. Denote by $\tilde{\sigma}$ the involution $\tilde{\sigma}(g) = w_n^{-1} \cdot g^{-1} \cdot w_n$ of $SL(n + 1, F)$. Since the Steinberg group $St(n + 1, F)$ is generated by elementary matrices (see [M], p. 39), $\tilde{\sigma}$ maps elementary matrices to elementary matrices, and $\tilde{\sigma}$ preserves the relations which define $St(n + 1, F)$, then $\tilde{\sigma}$ lifts to an involution of $St(n + 1, F)$, hence to an involution $\tilde{\sigma}$ of $G$.

Suppose that $n$ is odd. Then both $s$ and $s^\sigma = \tilde{\sigma} \circ s \circ \tilde{\sigma}$ satisfy the conditions of Lemma 4. Hence there exists a character $\tilde{\chi} : F^\times \rightarrow \{1, -1\}$ such that $s^\sigma = \tilde{\chi} \circ s$. Define $\sigma = \tilde{\chi} \circ \tilde{\sigma}$; it is an involution of $G$. Since $\sigma \circ s = \tilde{\chi} \circ \tilde{\sigma} \circ s = s \circ \sigma$ on $ZA^2$, we have

$$
\delta(\sigma(s(z)i(h^2))) = \delta(s(\sigma z)s(\sigma h^2)) \quad \text{for all } z \in \mathbb{Z}, \ h \in \mathbb{A};
$$

hence $\delta(\sigma(x)) = \delta(x)$ for all $x$ in $ZA^2$. By Lemma 3(i) we have $\rho_{t} \circ \sigma \cong \rho_t$, where $\rho_t$ is the unique extension of $\delta$ to $A$. Hence $\pi_t \circ \sigma \cong \pi_t$, and by Lemma 3(ii) we have $\Theta \circ \sigma \cong \Theta$. It follows that there exists a non-zero operator $I : V \rightarrow V$ such that $\Theta(g)I = I \Theta(\sigma(g))$ for all $g$ in $G$. Since $\Theta$ is irreducible, by Schur's lemma $I^2$ is a scalar, which we normalize to be 1. This determines $I$ uniquely up to a sign. The choice of $\Theta(\sigma) = I$ determines an extension of $\Theta$ to the semi-direct product $G^* = G \rtimes (\sigma)$.

**Remark.** (i) It is easy to check (consider first the case where $h_j = 1$ for all $j \neq i$) that

$$
\tilde{\sigma}(s(diag(h_i))) = s(diag(h_{n+1-i}^{-1})) \cdot \prod_{i=1}^{n} \left( h_i, \prod_{j=i+1}^{n} h_j \right).
$$

In particular

$$
\tilde{\sigma}(s(z)) = s(z^{-1}) \cdot (z, -1)^{n(n-1)/2} \quad \text{for } z \in F^\times \cong \mathbb{Z}.
$$

Consequently

$$
\sigma(g) = (-1, \det p(g))^{(n-1)/2} \tilde{\sigma}(g) \quad \text{and} \quad \chi(x) = (-1, x)^{(n-1)/2}.
$$

(ii) Since $(\det \tilde{\sigma}(g), \det \tilde{\sigma}(g')) = (\det g, \det g') (g, g' \in G)$, the formula in (i) for the involution $\sigma$ on $G$ defines also an involution $\sigma'$ on $G'$ which satisfies $p \circ \sigma' = \tilde{\sigma} \circ p$ on $G'$ and $\sigma \circ s = s \circ \tilde{\sigma}$ on $ZA^2$. 
1.4. An explicit model for $\Theta_2$ is easily obtained (see [F1], §1, Example, or [FM], and the proof of Proposition 1, §5, below) from that of the even Weil representation (see [F], p. 145). Indeed, this Weil representation is a representation of $S_2$, which extends to a representation of $s(\mathbb{Z})S_2$ (by the character $\gamma = \gamma_{\varphi}$ on $s(\mathbb{Z})$). The representation $\Theta_2$ is the $G_2$-module induced from this extension to $s(\mathbb{Z})S_2$.

In this paper we construct an explicit realization of the unitarizable $G_2$-module $\Theta_3$. When $F = \mathbb{R}$ the unitary completion of $\Theta_3$, or at least its restriction to $p^{-1}(\text{SL}(3, \mathbb{R}))$, coincides with the unitary $p^{-1}(\text{SL}(3, \mathbb{R}))$-module constructed by Torasso [T].

2. The realization

The representation $\Theta = \Theta_1$ will be realized in a space of functions on a two-fold covering space $X$ of the punctured affine plane $\mathring{X} = F \times F - \{(0, 0)\}$. Clearly $\mathring{X} = \Gamma \backslash \text{GL}(2, F)$, where

$$\Gamma = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}.$$

It is easy to see that the restriction of $s$ to $\Gamma$ is a homomorphism. Hence we can define the double cover $X$ of $\mathring{X}$ to be $s(\Gamma) \backslash G_2$. Then $X$ is a homogeneous space under the action of $G_2$. To be able to write explicit formulas for the action of $G_2$ on $X$, recall the explicit construction of $G_2$. Put

$$x \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c, & c \neq 0, \\ d, & c = 0, \end{cases}$$

and

$$\beta(g, g') = \left( \frac{x(gg')}{x(g)}, \frac{x(gg')}{x(g') \det g} \right).$$

Then $G_2$ is the group of pairs $(g, \zeta)$ ($g$ in $\text{GL}(2, F)$, $\zeta$ in $\text{ker } p$) with the multiplication law

$$(g, \zeta)(g', \zeta') = (gg', \zeta \zeta' \beta(g, g')).$$

Given $\tilde{z} = (x, y)$ in $\mathring{X}$, put $x(\tilde{z}) = x$ if $x \neq 0$ and $x(\tilde{z}) = y$ if $x = 0$. Identify $X$ with $\mathring{X} \times \text{ker } p$ by mapping the image in $X$ of the element $s(h)\zeta$ of $G$, where

$$h = \begin{pmatrix} z & t \\ x & y \end{pmatrix},$$

to the element $(x, y; \zeta(x(h), \det h))$ of $\mathring{X} \times \text{ker } p$. Then the action of $G_2$ on $\mathring{X} \times \text{ker } p$ implied by this identification is given by
Remark. Replacing $(\cdot, \cdot)$ by the $n$th Hilbert symbol, $(*)$ defines an $n$-fold covering of the punctured plane $\hat{X}$ as the homogeneous space $s(\Gamma) \setminus G_2$.

Definition. A function $f: X \to \mathbb{C}$ is called genuine if $f(\zeta z) = \zeta f(z)$ for $\zeta$ in $\ker p$, $z$ in $X$. It has bounded support if there is a compact subset of $F \times F$ which contains all $z$ in $\hat{X}$ with $f(z; \zeta) \neq 0$. It is called homogeneous if $f(t^2x, t^2y; \zeta) = |t|^{-1}f(x, y; \zeta)$ ($t$ in $F^\times$). Let $L^2(X)$ be the space of genuine, square-integrable, complex-valued functions on $X$. Let $C(X)$ be the space of smooth functions in $L^2(X)$. Denote by $C_b(X)$ the space of $f$ in $C(X)$ with bounded support. Denote by $C_h(X)$ the space of homogeneous $f$ in $C(X)$.

Let $\bar{P} \subset \bar{B}$ be the standard maximal parabolic subgroup of type $(2, 1)$ of $G$, and consider the subgroup $P = p^{-1}(\bar{P})$ of $G$. Define the action of $P$ on $L^2(X)$ as follows (we denote the action by $\Theta$):

1. \[ \Theta \left( s \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) f(z) = |\det g|^{1/2} f(\zeta z) \quad (g \text{ in } \text{GL}(2, F)); \]

2. \[ \Theta \left( s \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right) f(z) = \psi(ux + vy) f(z) \quad (u, v \text{ in } F); \]

3. \[ \Theta \left( s \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) \zeta f(z) = \zeta \gamma(a) f(z) \quad (a \text{ in } F^\times). \]

Under the action (1) the space $C_h(X)$ is a $G_2$-module; it has a unique proper non-zero $G_2$-submodule $C_h(X)^0$, isomorphic to $\Theta_2 \otimes |\det|^{1/4}$ (see [F], p. 145). Indeed, the space

\[ I(s) = \left\{ \varphi: G_2 \to \mathbb{C}; \varphi \left( s \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \right) \zeta \right\} \]

\[ = \zeta |a/b|^{1/2}\varphi(g), \quad a \in F^\times, \quad b \in F^\times, \quad \zeta \in \ker p \]
is a $G_2$-module under the action \( \rho(g)p(h) = |\det g|^{1/4}p(hg) \). At \( s = -\frac{1}{4} \) it is reducible, of length two. Its unique proper non-zero submodule is \( \Theta_2 \otimes |\det|^{1/4} \).

The map \( \varphi \to f, f(0, 1)g = |\det g|^{-1/2} \varphi(g) \), establishes a \( G_2 \)-module isomorphism from \( I(s) \) to the space

\[
J(s) = \{ f: X \to C; f(b(x, y); \zeta) = \zeta |b|^{-1-2}f(x, y; 1), b \in F^\times \} ,
\]

with the \( G_2 \)-action \( \rho(g)f(z) = |\det p(g)|^{1/4}f(zg) (z \in X) \).

**Definition.** Denote by \( C_b(X)^0 \) the space of \( f \) in \( C_b(X) \) for which there exists \( f_0 \) in \( C_b(X)^0 \) and \( A_f > 0 \) such that \( f(z) = f_0(z) \) for all \( z = (x, y; \zeta) \) with \( \max(|x|, |y|) \leq A_f \).

In particular, for every \( f \) in \( C_b(X)^0 \) there is \( A_f > 0 \) such that

\[
f(t^2x, t^2y; \zeta) = |t|^{-1}f(x, y; \zeta) \quad \text{if} \quad \max(|x|, |y|) \leq A_f \quad \text{and} \quad |t| \leq 1.
\]

**Theorem.** (i) The genuine representation \( \Theta \) of \( G^* = G \rtimes \langle \sigma \rangle \) can be realized in the space \( C_b(X)^0 \) by the operators (1), (2), (3) and

\[
(4) \quad (\Theta(\sigma)f)(x, y; \zeta) = \gamma(-1)^{1/2}y(x)^{-1}|x|^{-1/2} \int f(-x, u; \zeta)\psi(uy/x)du.
\]

(ii) The space \( C_b(X)^0 \) is contained in \( L^2(X) \). There is a unique (up to a scalar multiple) Hermitian scalar product on the unitarizable representation \( (\Theta, C_b(X)^0) \). It is given by the \( L^2 \)-product.

**Remark.** (i) Since \( G^* \) is generated by \( P \) and \( \sigma \), the action of \( G^* \) is completely defined by (1)–(4).

(ii) It follows from (ii) in the Theorem that the unitary completion of \( (\Theta, C_b(X)^0) \) is \( (\Theta, L^2(X)) \). As noted in (1.4), when \( F = \mathbb{R} \) the restriction to \( p^{-1}(\text{SL}(3, \mathbb{R})) \) of this realization of the unitary completion of \( \Theta \) coincides with the model constructed by Torasso [T].

(iii) Erasing the symbols \( s \) in (1), (2), (3), \( \zeta \) in (3), (4), \( \gamma(a) \) in (3), and \( \gamma(-1)^{1/2}y(x)^{-1} \) in (4), the (modified) operators (1)–(4) define an explicit realization of the representation \( I(1_F; \text{GL}(3, F), \bar{\rho}) \) of \( \text{GL}(3, F) \) normalizedly induced from the trivial representation \( 1_F \) of a maximal parabolic subgroup \( \bar{\rho} \). This model is isomorphic to the model \( (\tau_0, V_0) \) in [FK2], middle of p. 497, by the map \( (\tau_0, V_0) \exists \phi \to f, f(x, y) = \int x \phi(x, y, z)\psi(z)dz \).

### 3. Corollaries

The Theorem is proven in §§5–6. In this section we deduce three Corollaries, assuming the Theorem.
3.1. Let $F$ be a local field as in (1.1), and $\psi$ an additive character as in (1.2). The function
\[
g(x) = \gamma_\psi(x)\psi(-1/x)|x|^{-1/2}
\]
is locally integrable on $F$. Let
\[
\hat{g}(x) = \int_F g(y)\psi(-xy)dy
\]
be its Fourier transform. Put
\[
K(z, z') = (x, -x') \cdot \hat{g} \left( - \det \begin{pmatrix} x' & y' \\ x & y \end{pmatrix} \right) \cdot \zeta \zeta'
\]
if $z = (x, y; \zeta), \ z' = (x', y'; \zeta'),$
and for every $f \in L^2(X)$ write
\[
f^\vee(z) = \int_X f(z')K(z, z')dz'.
\]
Denote the action of $S_2$ on $L^2(X)$ by $\rho$; thus $(\rho(s)f)(z) = f(\rho(s)z)$ for $f \in L^2(X), s \in S_2, z \in X$.

**Corollary 1.** The map $f \mapsto f^\vee$ takes $L^2(X)$ to $L^2(X)$ and $C_b(X)^0$ to $C_b(X)^0$. Moreover, we have (i) $(f^\vee)^\vee = \gamma(-1)^{-1/2}f$, and (ii) $(\rho(s)f)^\vee = \rho(s)f^\vee$ for all $s$ in $S_2$.

**Proof.** Put
\[
\alpha = s \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
and $F = \Theta(\sigma)\Theta(\alpha)\Theta(\sigma)\Theta(\alpha)\Theta(\sigma)$.

Using (4) and (1), we have $Ff = \gamma(-1)^{1/2}f^\vee$. Assuming the Theorem it is easy to check that $F^2 = \rho(-1)$, and that $F$ commutes with $\rho(s)$ for every $s$ in $S_2$, as required.

**Remark.** The transform $f \mapsto f^\vee$ is analogous to the Fourier transform
\[
\hat{f}^\vee(x, y) = \int \int \hat{f}(x', y')\psi \left( \det \begin{pmatrix} x' & y' \\ x' & y' \end{pmatrix} \right) dx'dy'
\]
on $L^2(X)$, which satisfies $(\hat{f}^\vee)^\vee = \hat{f}$ and $(\rho(s)f)^\vee = \rho(s)\hat{f}^\vee$ for every $s$ in $\text{SL}(2, F)$; here we put $(\rho(s)f)(z) = \hat{f}(\rho(s)z)$. 
3.2. Let $F$ be a local field as in (1.1), and $\psi$, $g$ and $\hat{g}$ as in (3.1).

**Corollary 2.** The support of $\hat{g}$ is contained in the set $F^2$ of squares of $F$.

**Proof.** Corollary 1(ii) with $s = s(\alpha)$, $\alpha = \binom{1}{0}$, asserts that

$$K(zs(\alpha), z's(\alpha)) = K(z, z') \quad \text{for all } z = (x, y; \xi) \quad \text{and} \quad z' = (x', y'; \xi').$$

Hence for all $z, z'$ we have

\begin{equation}
\ast \quad g^\ast \left(-\det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}\right) [1 - (y', -x)(y', -x')(y, -y')(x, -x')] = 0.
\end{equation}

Since $(a + b, -b/a) = (a, b)$, we have

$$(xy', -x'y) = \left(-\det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}, xx'y'y'\right).$$

Put

$$a = -\det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}, \quad b = x'y.$$

Then (\ast) implies that

$$g^\ast(a)[1 - (a, b(a + b))] = 0$$

for all $a, b$ in $F$ with $ab(a + b) \neq 0$. Note that $1 + b/a \in F^\times$ if $|b|$ is sufficiently smaller than $|a|$. If follows that if $a \neq 0$ and $g^\ast(a) \neq 0$, then $a \in F^\times$, as required.

**Scholium.** The following is a sketch of an alternative, elementary proof of Corollary 2, communicated to us by J.L. Waldspurger. Recall that $F$ is a local non-archimedean field with $\text{char } F \neq 2$, $\psi : F \to \mathbb{C}^\times$ is a non-trivial continuous character, and $g : F \to \mathbb{C}$ is defined almost everywhere by $g(x) = \psi(-1/x)\alpha(x)/\alpha(1)$, where $\alpha(x) = \int_F \psi(xy)dy$. The Fourier transform $f^\ast$ is defined by $f^\ast(x) = \int_F \psi(-xy)f(y)dy$, and we claim that $g^\ast$ is supported on $F^2$.

Note that $g(x) = \alpha(1)^{-1}\int_F \psi(xy^2 - x^{-1})dy$. Making the change $y \mapsto y + x^{-1}$, we get

$$g(x) = \alpha(1)^{-1}\int_F \psi(xy^2 + 2y)dy.$$

For a function $f : F \to \mathbb{C}$ supported on $F^2$, the change $z = y^2$ of variables yields the identity
\[
\int_{F} f(z) |z|^{-1/2} dz = \frac{|2|}{2} \int_{F} f(y^2) dy.
\]

For a fixed \( x \in F \), consider the function

\[
f(z) = \begin{cases} 
\sum_{(y, y' = z)} \psi(xz + 2y), & z \in F^2, \\
0, & z \notin F^2.
\end{cases}
\]

Then

\[
\int_{F} f(z) |z|^{-1/2} dz = \frac{|2|}{2} \int_{F} \psi(xy^2) [\psi(2y) + \psi(-2y)] dy
\]

\[
= |2| \int_{F} \psi(xy^2 + 2y) dy.
\]

Hence

\[
g(x) = (|2| \alpha(1))^{-1} \int_{F} f(z) |z|^{-1/2} dz.
\]

Now put

\[
h(z) = \begin{cases} 
(|2| \alpha(1))^{-1} |z|^{-1/2} \sum_{(y, y' = z)} \psi(2y), & z \in F^2, \\
0, & z \notin F^2.
\end{cases}
\]

Then \( g(x) = \int_{F} \psi(xz) h(z) dz \), namely \( g(x) = h^\vee(-x) \). The Fourier inversion formula \( (h^\vee)^\vee(x) = h(-x) \) implies that \( g^\vee(x) = h(x) \). Hence \( g^\vee \) is supported on \( F^2 \) as required.

**Remark.** (i) Since \( \text{SL}(2, F) \) is generated by

\[
u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in F^\times,
\]

and \( a \), and since \( Kuzu, z'u) = K(z, z') \) is trivially true, Corollary 2 is equivalent to (ii) of Corollary 1.

(ii) Denote by \( a^{1/2} \) the non-negative square-root of \( a \geq 0 \), and by \( i \) the square root of \(-1\) in the upper half-plane in \( \mathbb{C} \). Define a function \( \sqrt{x} \) or \( \mathbb{R} \) by

\[
\sqrt{x} = \begin{cases} 
|x|^{1/2}, & \text{if } x \geq 0, \\
i|x|^{1/2}, & \text{if } x \leq 0.
\end{cases}
\]
Corollary 2 implies that: The Fourier transform $g_\mathcal{R}(x) = \int_{\mathbb{R}} g_\mathcal{R}(y)e^{-iyx}dy$ of the locally integrable function $g_\mathcal{R}(x) = e^{-it\sqrt{x}}$ on $\mathbb{R}$ is supported on the set of non-negative real numbers. Indeed, this is the special case where $F = \mathbb{R}$ and $\psi(x) = e^{ix}$; then $\gamma_\psi(x) = 1$ if $x > 0$ and $\gamma_\psi(x) = 1/i$ if $x < 0$ by [W], top of p. 174. Hence $\gamma_\psi(x)|x|^{-1/2} = 1/\sqrt{x}$, and $g_\mathcal{R}(x)$ is $g(x)$ of Corollary 2. However, it is easy to see directly that $g_\mathcal{R}$ is supported on $\mathbb{R}_{\leq 0}$ since $g_\mathcal{R}(x)$ extends to a function $g_\mathcal{C}(x)$ analytic in the upper half-plane and vanishing at infinity, and our assertion then follows from the Paley-Wiener theorem.

(iii) In fact the Theorem can be reduced to Corollary 2. This observation is due to Torasso [T]. He proved first that $g_\mathcal{R}$ is supported on $\mathbb{R}_{\leq 0}$ and this is the basis of his proof of the Theorem when $F = \mathbb{R}$.

(iv) Corollary 2 suggests the existence of a theory of “analytic” complex-valued functions on a local field $F$, in which the space of “analytic functions on the upper half-plane” is replaced by the space $R_\psi$ of functions $f$ on $F$ such that the support of $\hat{f}$ lies in the set of squares. However $R_\psi$ is not a ring, and we do not know how to develop the theory of such “analytic” functions on $F$.

3.3. Suppose that $F$ is non-archimedean, denote by $R$ its ring of integers, and fix a generator $\pi$ of the maximal ideal of $R$. Denote by $\text{val}$ the additive, integer-valued function on $F^\times$ normalized by $\text{val}(\pi) = 1$. Put $h(x) = |x|^{-1/2}$ if $\text{val}(x)$ is even and non-negative, and $h(x) = 0$ otherwise. Suppose that the residual characteristic of $F$ is odd. There exists a unique group-theoretic section of $p: p^{-1}(\text{SL}(4, R)) \rightarrow \text{SL}(4, R)$, denoted by $\kappa^*$; see [KP1], p. 43. Then $K = \text{GL}(3, R)$ embeds as a subgroup of $G_3$ via $\kappa^*$. An irreducible genuine $G$-module is called unramified if it has a (necessarily unique up to a scalar multiple) non-zero $K$-fixed vector.

**Corollary 3.** If the residual characteristic of $F$ is odd, then the $G$-module $\Theta$ is unramified. If $\psi$ is trivial on $R$ but not on $\pi^{-1}R$, then the $K$-fixed vector in $\Theta$ is a multiple of the vector

$$\phi(x, y; \zeta) = \begin{cases} \zeta h(x), & \text{if } |y| \leq |x|, \\ (x, y)\zeta h(y), & \text{if } 0 < |x| < |y|, \\ \zeta h(y), & \text{if } x = 0. \end{cases}$$

**Proof.** The group $K$ is generated by its upper-triangular matrices, by

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
and \( \sigma_\omega \). The sections \( \kappa^* \) and \( s \) coincide on these matrices (see [KP1], Prop. 0.1.3). Using the Theorem it is easy to check that \( \phi \) is invariant under the image of these matrices. Hence the corollary follows.

**Remark.** Note that the function \( \phi \) of Corollary 3 is locally constant at \((0, y, \zeta), \ y \in F^\times\), since the limit of \((x, y; (-x, y))\) as \( x \to 0 \) \((x, y \neq 0)\) is \((0, y; 1)\).

### 4. Preliminaries

Here we collect various facts used in the proof of the Theorem. Since the Theorem is already proven in [T] when \( F = R \), we restrict our attention to the case when \( F \) is non-archimedean.

#### 4.1. Given a group \( H \) and a smooth \( H \)-module \( V = V(H) \), let \( V'(H) \) be the Hermitian dual of \( V \), namely the smooth \( H \)-module obtained on conjugating the complex structure of the smooth dual of \( V \). We write \( V' \) for \( V'(H) \) when the group \( H \) is specified. Note that in general \( V'(H) \neq V'(H') \) when \( V \) is both \( H \)- and \( H' \)-module. Observe that an \( H \)-invariant Hermitian form on \( V \) is equivalent to an \( H \)-invariant map from \( V \) to \( V' \) (= \( V'(H) \)). Note that if \( \alpha \in V' \), \( \nu \in V \) and \( h \in H \), then \((h \cdot \alpha)(\nu) = \alpha(h^{-1} \cdot \nu)\).

#### 4.2. Let \( Q = SR \) be the semi-direct product of a group \( S \) and an abelian normal subgroup \( R \). The group \( Q \) acts on \( R \) by \( q: r \to q \cdot rq^{-1} \), hence also on the group \( \hat{R} \) of characters \( \psi_R \) on \( R \) by \( \psi_R^R(r) = \psi_R(q^{-1}rq) \). For any character \( \psi_R \) of \( R \) we denote by \( \text{Stab}_Q(\psi_R) \) the stabilizer of \( \psi_R \) in \( Q \), and put \( \text{Stab}_S(\psi_R) = S \cap \text{Stab}_Q(\psi_R) \). For any irreducible representation \( \tau \) of \( \text{Stab}_Q(\psi_R) \) the tensor product \( \tau \otimes \psi_R \) defines a representation of \( \text{Stab}_Q(\psi_R) = \text{Stab}_S(\psi_R) \). Denote by \( \pi(\tau \otimes \psi_R) \) the \( Q \)-module \( \text{ind}(\tau \otimes \psi_R; Q, \text{Stab}_Q(\psi_R)) \), where, as in [BZ1], (2.21) and (2.22), \( \text{Ind} \) indicates the functor of (unnormalized) induction, and \( \text{ind} \) the functor of induction with compact supports (we do not normalize these functors as in [BZ2], p. 444). As in [BZ2], top of p. 444, define the positive-valued character \( \Delta_Q: Q \to R^\times_0 \) by \( d(g^{-1}gg) = \Delta_Q(g)dg \) \((g \in Q)\), where \( dq \) is a Haar measure on \( Q \).

**Mackey’s Theorem.** (i) The \( Q \)-module \( \pi(\tau \otimes \psi_R) \) is irreducible.

(ii) We have \( \pi(\tau \otimes \psi_R) \cong \pi(\tau^* \otimes \psi_R^\circ) \) if and only if there is \( s \) in \( S \) such that \( \psi_R^s = \psi_R^\circ \) and \( \tau^* = \tau^s \).

(iii) Every irreducible \( Q \)-module is equivalent to \( \pi(\tau \otimes \psi_R) \) for some \( \tau \) and \( \psi_R \).

(iv) The \( Q \)-module \( \pi(\tau \otimes \psi_R) \) (see (4.1)) is equivalent to \( \text{Ind}((\Delta_Q/\Delta_S)\tau^* \otimes \psi_R^s; Q, S) \), where \( S = \text{Stab}_Q(\psi_R) \).
Proof. See [BZ1], (2.23) and (5.10), for (i)-(iii), and [BZ1], (2.25), for (iv); when \( F = \mathbb{R} \) see [K], §13.3, Theorem 1.

4.3. Let \( Q \) be a parabolic subgroup of \( G \), \( R \) its unipotent radical, \( M = Q/R \) its Levi component, and \( \psi_R \) a character of \( R \). For any \( Q \)-module \( V \), let \( V_{R, \psi} \) be the \( \text{Stab}_M(\psi_R) \)-module of \((R, \psi_R)\)-coinvariants in \( V \) (see [BZ1], (2.30)). Put \( V_{\psi} \) for \( V_{R, \psi} \) when \( \psi_R \) is trivial. In this paper the functor of coinvariants is not normalized (as in [BZ1], in contrast with [BZ2], p. 444). For the reader's convenience, we record

Frobenius Reciprocity ([BZ2], (1.9(b)), p. 445). For any smooth \( Q \)-module \( V \), and any smooth \( \text{Stab}_M(\psi_R) \)-module \( W \), we have

\[
\text{Hom}_{\text{Stab}_M(\psi_R)}(V_{R, \psi}, W) = \text{Hom}_Q(V, \text{Ind}(W \otimes \psi_R; Q, \text{Stab}_Q(\psi_R))).
\]

4.4. We use below the Geometric Lemma (2.12) of [BZ2], which we now record (in the notations of [BZ2]). Let \( G \) be a covering group of a reductive connected group \( \bar{G} \) over a local field \( F \), fix a minimal parabolic subgroup \( P_0 \) and a Levi subgroup thereof, and denote by \( M \), \( N \) standard Levi subgroups of \( G \) (notations: \( \bar{M}, N < \bar{G} \)). Denote by \( W_G, W_M, W_N \) the Weyl groups of \( G, M, N \) (note that \( W_G = W_G, \ldots \) ). Each double coset \( W_N \setminus W_G / W_M \) has a unique representative of minimal length. The set of these representatives will be denoted by \( W_G^{N, M} \). For each \( w \) in \( W_G^{N, M} \) put

\[
M_w = M \cap w^{-1}(N) < M, \quad N_w = w(M_n) = w(M) \cap N < N.
\]

Denote by \( \text{Alg} M \) the category of smooth (= algebraic in [BZ2]) \( M \)-modules. Let \( P \) be the parabolic subgroup of \( G \) which contains \( P_0 \) and whose Levi component is \( M \). Put \( \delta_P(p) \) for \( \Delta_P(p)^{-1} \), for \( p \) in \( P \). Put \( i_{GM} \) for \( \text{Ind}(\delta_P^{-1}; V; G, M) \) and \( r_{NG} \) for \( \delta_P^{-1/2} \otimes V_N; i_{GM} \) and \( r_{NG} \) are the functors of normalized (as in [BZ2]) induction and coinvariants.

Composition Theorem. The functor \( F = r_{NG} \circ i_{GM} : \text{Alg} M \to \text{Alg} N \) is glued from the functors \( F_w = i_{N, M_w} \circ w \circ r_{M, M} \) for \( w \) in \( W_G^{N, M} \). More precisely, choose an ordering \( \{w_1, \ldots, w_l\} \) of \( W_G^{N, M} \) such that \( w_j < w_i \) implies \( i < j \) (\( < \) is the standard partial order on \( W_G \)). Then \( F \) has a canonical filtration \( 0 = F_0 \subset F_1 \subset \cdots \subset F_r = F \) such that \( F_i / F_{i-1} \) is canonically isomorphic to \( F_{w_i} \).

Proof. This is the Geometric Lemma (2.12) of [BZ2], which is stated there only for the algebraic group \( \bar{G} \), but its proof is valid also in the context of the covering group \( G \).

4.5. In this subsection we summarize properties of \( \Theta \) used in the proof of the Theorem in §§5–6 below.
The $G_n$-module $(\Theta_n, V_n)$ is defined in §1 as the unique irreducible submodule of the induced $G_n$-module $(\pi_n, V_n)$. Its character $\chi_{\Theta_n}$ is computed in [KP2], Theorem 6.1, at least when $n = 2, 3$ (the computation for a general $n$ is reduced to a certain conjecture about orbital integrals). This character computation implies that $\Theta_n$ corresponds to the trivial $GL(n, F)$-module $1_n$ by the metaplectic correspondence ([KP2], Conjecture, p. 208, and Prop. 5.6, p. 213; or [FK1], (26.1)). We shall record here two applications of this character computation, to be used below.

For any diagonal matrix $h = \text{diag}(h_i)$ in $A$ put

$$\Delta(h) = \left| \prod_{i<j} (h_i - h_j)^2 / h_i h_j \right|^{1/2},$$

and for $\hat{h}$ in $A$ put $\Delta(\hat{h}) = \Delta(p(\hat{h}))$. The character computation implies that there is a $\beta > 0$ (explicitly given in [KP2]) such that

$$\Delta(t(h))\chi_{\Theta_n}(t(h)) = \beta \Delta(h)$$

for every $h$ in $\tilde{A}$ with $|h_i| \neq |h_j|$ for all $i \neq j$. In particular, when $n = 3$ and $h = \text{diag}(a, b, c)$ with $|a| < |b| < |c|$, we have $\Delta(h) = |c/a|$, hence

$$\Delta(\chi_{\Theta_n})(t(h)) = \beta |c/a|.$$ (5)

To state the second application, denote by $\psi_N$ a non-degenerate character of the unipotent upper-triangular subgroup $N$ of $G_n$. A Whittaker model of a $G$-module $(\pi, V)$ is an injection $I: V \rightarrow \text{Ind}_{G_n} G_n, N)$. The space of Whittaker functionals $I$ is then dual to the space

$$V_{N, \psi_n} = V/(\pi(n)v - \psi_n(n)v; v \in V, n \in N).$$

Corollary 6.2 of [KP2] asserts that (at least for $n = 2, 3$) we have

$$\dim V_{N, \psi_n} = \frac{a}{r!n} \sum_{h \in A, x' = 1} \Delta(h), \quad a = \frac{n}{(n, r - 1)} \left| \frac{n, r - 1}{n^r} \right|^{1/2}.$$

In our case $r = 2$. Consequently we have the following

**Lemma 6.** (i) When $n = 2$, $\dim(\Theta_2, V_2)_{N, \psi_n} = 1$, and $\Theta_2$ has a unique (up to a scalar multiple) Whittaker functional. (ii) When $n \geq 3$, $\dim(\Theta_n, V_n)_{N, \psi_n} = 0$, and $\Theta_n$ has no Whittaker model.

**Remark.** The proof of the character relation [KR2], Theorem 6.1, is based on the (global) trace formula. Hence the proof of (ii) is presently complete only for $n = 3$. For $F$ with $|2| = 1$ a purely local proof of Lemma 6 is given [KP1], Theorem 1.3.5.

**4.6.** In (5.1) below we use a special case of the Theorem of [C], which we record here in a form useful for (5.1), in the notations of (4.5).
**Theorem** ([C]). Let \( \pi \) be an admissible \( G_\pi \)-module, and \( h \) the matrix \( \text{diag}(h_i) \), with \( |h_i| < |h_{i+1}| \) (1 \( \leq i < n \)). Then \( (\Delta\chi_{\pi})(t(h)) = \chi_{r_{\pi}A}(t(h)). \)

Here \( r_{\pi,A} \) is an \( A \)-module (see (4.4)). The center of \( A \) is \( ZA^2 \); it is of finite index in \( A \). The irreducible constituents of the restriction of \( r_{\pi,A} \) to \( ZA^2 \) are characters. We use this Theorem in two cases. First, the Theorem, together with (5), implies

**Lemma 7.** When \( n = 3 \) and \( \pi = \Theta \), the restriction of \( r_{\pi,A} \Theta \) to \( t(A) \) is a multiple of the character which maps \( t(h), h = \text{diag}(a, b, c), \) to \( |c/a| \).

Note that a genuine character of \( ZA^2 \) which transforms on \( s(Z) \) according to \( \gamma \) is uniquely determined by its values on \( t(A) \).

**Remark.** Lemma 7 can be proven also using [KP1], Theorem 1.2.9(e), instead of using [C] and the character relation (5).

The second application concerns the case \( n = 2 \). Let \( \mu_i : F^x \rightarrow C^x \) (\( i = 1, 2 \)) be two characters of \( F^x \). Extend the character \( (\mu_1, \mu_2) : t(\text{diag}(a, b)) \rightarrow \mu_1(a)\mu_2(b) \) to a genuine character \( \mu \) of a maximal abelian subgroup \( A_\mu \) of \( A_2 \). Extend \( \mu \) to \( A_\mu N \) (trivially on \( N \)), and induce (normalized) to a \( G_2 \)-module \( \pi(\mu_1, \mu_2) \). The character of \( \pi = \pi(\mu_1, \mu_2) \) is computed in [F], p. 141: on \( t(A_2) \) we have that \( \Delta\chi_{\pi} \) is equal to a scalar multiple of \( (\mu_1, \mu_2)(\mu_2, \mu_1) \). Theorem [C] then implies

**Lemma 8.** Each irreducible constituent of the restriction of \( r_{\pi,A}[\pi(\mu_1, \mu_2)] \) to \( t(A_2) \) is isomorphic to the character \( (\mu_1, \mu_2) \) or \( (\mu_2, \mu_1) \).

### 5. Restriction to \( P \)

Denote by \( P \) and \( P^+ \) (\( \supset B \)) the preimages in \( G \) of the standard maximal parabolic subgroups of type (2, 1) and (1, 2) in \( GL(3, F) \), and by \( U \) and \( U^+ \) (\( \subset N \)) their unipotent radicals. Our construction of the explicit realization of \( \Theta \) is accomplished in two steps. In this section we study the restriction of \( \Theta \) to \( P \). In the next section we construct the action of \( \sigma \). Since \( P \) and \( \sigma \) generates \( G^* = G \rtimes \langle \sigma \rangle \) we thus obtain the required explicit realization.

#### 5.1. Let \( \psi : F \rightarrow C^x \) be a character as in (1.2), and define a character \( \psi_U \) of \( N \) by \( \psi_U(n) = \psi(n_{2,3}) \). The restriction of \( \psi_U \) to the subgroup \( U \) of \( N \) will again be denoted by \( \psi_U \). Since \( \psi_U \) is trivial on \( U^+ \) it defines a character of \( N^+ = N/U^+ \), denoted again by \( \psi_U \).

Embed \( G_2 \) in \( P \) by

\[
g \rightarrow \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.
\]
Put \( G_2 = p^{-1}(G_2) \subset P \). Since \( P = ZG_2U \), we identify below a \( P \)-module which transforms trivially under \( U \) and by \( \gamma \) under \( s(Z) \), with a \( G_2 \)-module. The analogous convention is applied to \( P^+ \)-modules. Let \( V_U \) be the \( P \)-module of \( U \)-coinvariants of \( V \) (see (4.3)).

**Proposition 1.** (i) As a \( G_2 \)-module, \( V_U \) is isomorphic to \( \Theta_3 \otimes |\text{det}|^{1/4} \). In particular, \( \mathbf{s}^{(k, 0)}_{\mathbf{k}} \) acts as multiplication by \(|h|\).

(ii) As a \( G_2 \)-module, \( V_{U^*} \) is isomorphic to \( \Theta_3 \otimes |\text{det}|^{-1/4} \).

(iii) The element \( \mathbf{s}^{(k, k)}_{\mathbf{k}} \) acts on any Whittaker function on \( V_U \) as multiplication by \(|h|^{1/2} \gamma(h)^{-1}\).

**Proof.** (i) By definition (see Lemma 3(ii) of §1), \( \Theta = \Theta_3 \) is the unique irreducible submodule of the induced \( G_3 \)-module \((\pi_{\delta}, \tilde{V}_{\delta})\). Since the functor \( r \) of coinvariants is exact (see [BZ1], Prop. 2.35), the \( P \)-module \( r_{M, \Theta} \) is a submodule of \( r_{M, \Theta}(\pi_{\delta}, \tilde{V}_{\delta}) \), where \( M \) is the standard Levi subgroup of \( P \). The Composition Theorem (4.4) applies to \( r_{M, \Theta}(\pi_{\delta}, \tilde{V}_{\delta}) \) with \( M = B \) and \( N = P \), and \( W^B_{\Theta} \) consists of the elements \( w_3 = \text{id} \), \( w_2 = (23) \) and \( w_i = (12)(13) = (132) \) of \( W_G \). It asserts that there is a composition series \( 0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \tilde{V}_3 = (\tilde{V}_{\delta})_U \) of \( P \)-modules (i.e. \( G_2 \)-modules), where \( \tilde{V}_i/\tilde{V}_{i-1} \cong i_{\rho, \Theta}(w_i \otimes \rho_{\delta}) \rho_{\delta} \) is defined by Lemma 3(i); \( w_i \otimes \rho_{\delta} \) is the \( B \)-module extended trivially on \( N \) from \( A \). Now it follows from Lemma 8 that each irreducible constituent of the normalized \( A \)-module of \( N \)-coinvariants \( r_{A, \Theta} \otimes i_{\rho, \Theta}(w_i \otimes \rho_{\delta}) \) \((i = 1, 2, 3)\) is acted upon by the element \( t(h) \) of the center of \( A \), where \( h = \text{diag}(a, b, c) \in A \), according to the characters: \(|c/a|\) or \(|c/b|\) if \( w_i = \text{id} \) \((i = 3)\), \(|b/a|\) or \(|b/c|\) if \( w_i = (23) \) \((i = 2)\), \(|a/b|\) or \(|a/c|\) if \( w_i = (12)(13) \) \((i = 1)\). On the other hand, Lemma 7 implies that \( t(h) \) acts according to the character \(|c/a|\) on each irreducible constituent of the \( A \)-module \( r_{A, \Theta} = r_{A, \Theta}(\pi_{\delta}, \tilde{V}_{\delta}) \). Since the functor of coinvariants is exact, we thus obtain that \( \text{Hom}_P(r_{M, \Theta}, \tilde{V}_3) = 0 \), and that the submodule \( r_{M, \Theta} \tilde{V}_3 \) of \( \tilde{V}_3 \) is a proper non-zero \( P \)-submodule of the quotient \( \tilde{V}_3/\tilde{V}_2 \cong i_{\rho, \Theta}(\rho_{\delta}) = \pi_{\delta} \otimes |\text{det}|^{-1/4} \) as a \( G_2 \)-module. However, Lemma 3(ii) asserts that the \( G_2 \)-module \( \pi_{\delta} \) has a unique proper non-zero submodule, which is \( \Theta_2 \). Hence \( r_{M, \Theta} \tilde{V}_3 = \Theta_2 \otimes |\text{det}|^{-1/4} \), and

\[
\Theta_U = \delta^{1/2}_P \otimes r_{M, \Theta} \Theta = \Theta_2 \otimes |\text{det}|^{1/4} \quad \left( \text{since } \delta_P \left( \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right) \right) = |\text{det} g|,
\]

as required.

For the last claim in (i), note that \( \mathbf{s}^{(k, k)}_{\mathbf{k}} \) acts trivially on \( \Theta_2 \) by definition of \( \Theta_2 \).

Part (ii) is of course analogous to (i).

For (iii), note that the \( G_2 \)-module \( \Theta_2 \) has the following realization (see, e.g., [FM] or [F1], Sect. 1, Example). Its space \( V_2 \) consists of all locally constant functions \( f : F^X \to \mathbb{C} \) whose support is compact in \( F \), for which there is \( A(f) > 0 \).
and $f' : F^x \to C^x$ satisfying $f'(xa^2) = |a|^{-1/2}f'(x)(x, a \text{ in } F^x)$ with $f(x) = f'(x)$ for $|x| \leq A(f)$. On this space the group $G_2$ acts by

$$\Theta_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) f(x) = |a|^{1/2}f(ax), \quad \Theta_2 \left( \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \right) f(x) = (x, z)\gamma(z)^{-1}f(x),$$

$$\Theta_2 \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) f(x) = \psi(bx)f(x), \quad \Theta_2 \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) f(x) = c\gamma(x)^{-1}|x|^{1/2} \int_f |y|^{1/2}f(xy)\psi(2xy)dy,$$

for some $c$ in $C^x$. By definition, a Whittaker functional on $(\Theta_2, V_2)$ is a linear form $L : V_2 \to C$ which satisfies

$$L\left( \Theta_2 \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) f - \psi(b)f \right) = 0 \quad \text{for all } b \text{ in } F \text{ and } f \text{ in } V_2.$$  

By Lemma 6(i) this functional is unique up to a scalar. Hence it is a multiple of $L(f) = f(1)$, which is clearly a Whittaker functional. Now

$$\Theta_2 \left( \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right) f = L\left( \Theta_2 \left( \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right) f \right) = L(\gamma(h)^{-1}(x, h)f(x))$$

$$= \gamma(h)^{-1}f(1) = \gamma(h)^{-1}L(f)$$

for every $f$ in $V_2$ and $h$ in $F^x$; this implies (iii) by virtue of (i).

**Remark.** Lemma 7 implies that $(\Theta_U, V_U)$ is a multiple of $\Theta_2 \otimes |\det|^{|1/4|}$. To show that this multiple is one, we use in the proof above the Composition Theorem (4.4). Alternatively, this can be proven on comparing the exact value of the character of $\Theta_U$ with that of $\Theta_2$ on the $h$ which appear in (5). In the proof above this comparison is done only up to a scalar multiple.

**5.2.** Let $V_0$ be the kernel of the natural surjection of $V$ on $V_U$. Put $P' = \text{Stab}_U(\psi_U)$. Then $V_0 = \text{ind}(V_U, \varphi_U \otimes \psi_U; P, P'U)$ by [BZ1], Prop. 5.12(d), or [BZ2], (3.5). Note that

$$\delta_p \left( \begin{pmatrix} g & * \\ 0 & b \end{pmatrix} \right) = |(\det g)/b^2| \quad (g \in \text{GL}(2, F)) \quad \text{and} \quad \delta_p \left( \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \right) = |a/b|.$$  

In particular
\[
\delta_p = \delta_{\rho} \quad \text{on} \quad \begin{pmatrix}
  a & \ast \\
  b & \\
  0 & b
\end{pmatrix}.
\]

Hence

(6) \quad V_0 = \delta_p^{1/2} \otimes \text{ind}(V'_1; P, P'U), \quad \text{where} \quad V_1 = \delta_p^{1/2} \otimes (V_{U, \psi} \otimes \psi_U).

**Proposition 2.**

(i) The \( P \)-module \( V_0 \) is irreducible.

(ii) The \( P'U \)-module \( V_1 \) is one-dimensional and unitary.

**Proof.**

(i) It suffices to prove that \( V_{U, \psi} \) is one-dimensional, for then it is irreducible and the proposition follows from Mackey's theorem (4.2(i)) and (6).

To prove the one-dimensionality, note that \( V_{N, \psi} = 0 \), where \( \psi_N(n) = \psi(n_{1,2} + n_{2,3}) \), by Lemma 6(ii). Hence \( U^+ \) acts trivially on \( V_{U, \psi} \), and so \( V_{U, \psi} = V_{N, \psi} \). By the transitivity property of the functor of coinvariants, we have \( V_{N, \psi} = (V_{U'}{N'}, \psi_U) \), where \( N^+ = N/U^+ \). By Proposition 1(ii), \( V_{U'} \) is the Weil representation of \( G_2 \) (up to a twist). Hence Lemma 6(i) implies that \( \dim V_{N, \psi} = 1 \), as required.

(ii) The one-dimensionality is proven in (i). Since \( N \) acts on \( V_1 \) via \( \psi_U \), it suffices to show that the element

\[
\begin{pmatrix}
  a & 0 \\
  b & \\
  0 & b
\end{pmatrix}
\]

acts on \( V_1 \) as multiplication by \( \gamma(b) \). By Proposition 1(iii),

\[
 s = \begin{pmatrix}
  1 & 0 \\
  b/a & \\
  0 & b/a
\end{pmatrix}
\]

acts on \( V_{U, \psi} = (V_{U'}{N'}, \psi_U) \), as \( |a/b|^{1/2} \gamma(b/a) \). Since \( \delta_p^{1/2}(s) = |a/b|^{1/2} \), and the central character of \( \Theta \) is \( \gamma \), the claim follows from

\[
\begin{pmatrix}
  a & 0 \\
  b & \\
  0 & b
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 0 \\
  b/a & \\
  0 & b/a
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & 0 \\
  b/a & \\
  0 & b/a
\end{pmatrix} \cdot (a, b/a).
\]
5.3. Let $V' = V'(P)$ be the $P$-module defined in (4.1) using the $P$-module $V$, and $V'_0$ the $P$-module obtained from $V_0$. Mackey's theorem (4.2(iv)) implies that $\text{ind}(V'_1) = \text{Ind}((\Delta_p/\Delta_p U)V'_1)$. By Proposition 5.2(ii) we have $V'_1 = V_1$. Since $\Delta_p/\Delta_p U = \Delta_p^{-1} = \delta_p = \delta_p'$ on $P'$, we have $\text{ind}(V'_1) = \delta_p' \otimes \text{Ind}(V_1)$. Hence

(7) $V'_0 = \delta_p'^{1/2} \otimes \text{Ind}(V_1; P, P'U)$.

As noted in (4.1), the unitary structure of the $P$-module $(\Theta, V)$ yields the following sequence of $P$-module morphisms:

$V_0 \to V \to V' \to V'_0$.

Denote by $\varphi$ the composite morphism from $V$ to $V'_0$.

**Proposition 3.** (i) The map $\varphi$ is an injection.

(ii) We have $\dim \text{Hom}_P(V_0, V'_0) = 1$. In particular, the restriction of $\varphi$ to $V_0$ is a multiple of the natural inclusion $\delta_p^{1/2} \otimes \text{ind}(V_1) \to \delta_p'^{1/2} \otimes \text{Ind}(V_1)$.

**Proof.** (i) The subspace $\ker \varphi$ is $U$-invariant since it is the orthogonal complement of $V_0$, and $V_0$ is spanned by the vectors $v - \Theta(u)v$, $v$ in $V$, $u$ in $U$. Hence the claim follows from

**Theorem (Howe–Moore [HM], Prop. 5.5, p. 85).** Let $G$ be a covering group of a simple reductive group, and $V$ a non-trivial irreducible unitarizable $G$-module. Then no one-parameter subgroup of $G$ fixes a non-zero vector in $V$.

(ii) By (7) and Frobenius reciprocity (see (4.3)), we have

$$\text{Hom}_P(V_0, V'_0) = \text{Hom}_P((V_0)_{U, \psi_U}, \delta_p^{1/2} \otimes V_1).$$

Since the functor of coinvariants is exact we have $(V_0)_{U, \psi_U} = V_{U, \psi_U}$. Note that $\delta_p^{1/2} \otimes V_1 = V_{U, \psi_U}$. Hence $\text{Hom}_P(V_0, V'_0) = 0$ and $\varphi : V_0 \to V'_0$ is a multiple of the natural inclusion.

**Proposition 4.** (i) The $P$-module $V'_0$ is isomorphic to the space of genuine functions on $X$ smooth with respect to the action of $P$ defined by (1), (2), (3) in §2.

(ii) The $P$-module $V_0$ can be realized by (1), (2), (3) on the space of smooth, genuine, compactly-supported functions $f$ on $X$.

**Proof.** This follows at once from (6) and (7) and the isomorphism of $X = s(\Gamma) \setminus G_2$ with $P'U \setminus P$.

6. Restriction to $B$

It remains to determine $V$ as a subspace of $V'_0$, and to extend the action of $P$ to an action of $G^* = G \rtimes \langle \sigma \rangle$ on $V$. 

Since

\[ P = B \cup P'UaB \quad \text{and} \quad a = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

it follows that the action of \( B \) on \( X = P'U \setminus P \) has two orbits, \( Y = \{z \in X; x \neq 0\} \), and \( X - Y = \{z \in X; x = 0\} \). Let \( W \) be the space of smooth, genuine, compactly-supported, complex-valued functions on \( Y \). It is a \( B \)-submodule of \( V_0 \). In fact \( W \) is an irreducible \( B \)-module, by Mackey's theorem (4.2(i)), since

\[ W = \delta_{\mu}^{1/2} \otimes \text{ind}(V_0^a; B, a \cdot P'U \cdot a^{-1} \cap B) \]

and \( V_1 \) is irreducible (see Proposition 2(ii)).

Let \( W' = W'(B) \) be the Hermitian dual (4.1) of the \( B \)-module \( W \). By Mackey's theorem (4.2(iv)), \( W' \) is the space of genuine functions on \( Y \) smooth under the action of \( B \) defined by (1), (2), (3); in particular, the support of any \( f \) in \( W' \) is bounded in the \( y \)-direction. We have the following inclusions of \( B \)-modules:

\[ W \subseteq V_0 \subseteq V \subseteq V_0' = V_0(P) \subseteq W' = W'(B). \]

Fix a square root \( \gamma(-1)^{1/2} \) of \( \gamma(-1) \). For any \( f \) in \( W' \) define \( Jf(x, y; \zeta) \) by the integral

\[ (8) \quad \gamma(-1)^{1/2} \gamma(x)^{-1} |x|^{-1/2} \int_{F} f(-x, u; \zeta) \psi(uy/x) \, du. \]

It is clear that this integral converges, that \( J^2 = \text{Id} \), and that \( f \to Jf \) maps \( W \) to \( W \) and \( W' \) to \( W' \).

As noted in (1.3), since \( \Theta \) is \( \sigma \)-invariant there is an isomorphism \( I : V \to V \) such that \( f(\Theta(g)) = \Theta(\sigma g)I \) and \( I^2 = \text{Id} \). It is unique up to a sign. We claim that \( I \) is given on \( V \) by the integral (8). More precisely, we have

Proposition 5. (i) The operator \( J \) maps \( V \) to \( V \). (ii) There is a choice of \( I : V \to V \) such that the restriction \( J \big| V \) of \( J \) to \( V \) is equal to \( I \).

Proof. The \( B \)-module \( W' \) consists of functions on \( Y = \{z \in X; x \neq 0\} \). The subgroup \( N_{1,3} = U \cap U^+ \) of \( N \) acts on \( W' \) according to (2). Hence the only vector in \( W' \) fixed by \( N_{1,3} \) is the zero vector. On the other hand, for every \( u \) in \( F \), we have that \( \psi(u x) \) is 1 for a sufficiently small \( |x| \). Hence \( f \in W' \) and

\[ \Theta \begin{bmatrix} 1 & u \\ 1 & f \\ 0 & 1 \end{bmatrix} \]
are equal on a sufficiently small neighborhood of $X - Y = \{ z \in X; x = 0 \}$. Consequently

$$
\Theta \begin{pmatrix}
1 & u \\
1 & f - f \in W \\
0 & 1
\end{pmatrix}
$$

We conclude that $N_{i,i}$ acts trivially on $W'/W$. In particular, since $(W \subset V \subset W')$, we have

$$
\text{Hom}_B(V/W, W') = 0, \quad \text{Hom}_B((V/W)', W') = 0.
$$

Since for any $H$-modules $A, B$ we have $\text{Hom}_B(A, B) \rightarrow \text{Hom}_B(B', A')$, we also have that the submodule $\text{Hom}_B(W, V/W)$ of the zero-module $\text{Hom}_B((V/W)', W')$ is zero.

It follows that $I$ maps $W$ to $W$. Indeed, had this been false, the map $I$ would induce a non-trivial map $W \rightarrow V/W$, contradicting the fact that $\text{Hom}_B(V/W, W') = 0$.

We claim that the restrictions $I \mid W$ and $J \mid W$ of $I$ and $J$ to $W$ coincide. We have $(I \mid W)^2 = \text{Id}$, and $(I \mid W)\Theta(b) = \Theta(sb)(I \mid W)$ for all $b \in B$. By (1.3) we have

$$
\tilde{\sigma} \left( \begin{pmatrix}
a & 0 \\
b & c \\
0 & c
\end{pmatrix} \right) = s \begin{pmatrix}
c^{-1} & 0 \\
b^{-1} & 0 \\
0 & a^{-1}
\end{pmatrix} \cdot (a, bc)(b, c),
$$

and \( \sigma(g) = (-1, \det p(g))\tilde{\sigma}(g) \quad (g \in G) \).

Consequently, it is easy to check that $(J \mid W)\Theta(b) = \Theta(sb)(J \mid W)$ for all $b$ in $B$, and that $J^2 = \text{Id}$. Since $W$ is an irreducible $B$-module, we have $I \mid W = J \mid W$, up to a sign. Hence we can choose $I$ such that $I \mid W = J \mid W$, as claimed.

It now follows that $J \mid V - I$ defines a morphism $V/W \rightarrow W'$, necessarily zero since $\text{Hom}_B(W, V/W) = 0$, and the proposition follows.

Finally we prove the
Theorem. (i) The space \( V \) is isomorphic to \( C_b(X)^0 \). The \( G^* \)-module \((\Theta, V)\) is equivalent to the \( G^* \)-module defined by the operators (1)-(4) on the space \( C_b(X)^0 \).

(ii) There is a unique (up to scalar) Hermitian scalar product on the unitarizable \( G \)-module \((\Theta, C_b(X)^0)\). It is given by the \( L^2 \)-product.

Proof. (i) The space \( V \) is realized in Proposition 3(i) as a subspace of \( V_0 \). Moreover, we have the inclusions \( V_0 \hookrightarrow V \hookrightarrow V_0 \). By Proposition 4(i), \( V_0 \) is the space of genuine, smooth, complex-valued functions with bounded support on \( X \). The subspace \( V_0 \) of \( V \) consists, by Proposition 4(ii), of the compactly-supported \( f \) in \( V_0 \). By definition (in (5.2)) of \( V_0 \) as kernel \( (V \rightarrow V_V) \), the space \( V \) consists of the \( f \) in \( V_0 \) such that \( f = f \mod V_0 \) lies in \( V_V \). Proposition 1(i) asserts that \( V_V \equiv \Theta_2 \otimes |det|^{1/4} \).

In particular, for every \( f \) in \( V \) and \( t \) in \( F^X \), the vector

\[ \langle t \rangle^{-1} \Theta \begin{bmatrix} t^2 & 0 \\ t^2 & 1 \\ 0 & 1 \end{bmatrix} f - \langle t \rangle^{-1} \tilde{f} \]

is zero in \( V/V_0 \equiv \Theta_2 |det|^{1/4} \).

Hence for every \( f \) in \( V \) there is \( A_f > 0 \), and \( c \ (0 < c < 1) \), such that \( |t| f(t^2 x, t^2 y; \zeta) = f(x, y; \zeta) \) for \( max(|x|, |y|) \leq A_f \) and \( c \leq |t| \leq 1 \) (note that this domain of \( t \) is compact, and \( f \) is locally constant). But then this relation holds for all \( t \) with \( 0 < |t| \leq 1 \). Define \( f_0 \) on \( X \) by \( f_0(x, y; \zeta) = |t| f(t^2 x, t^2 y; \zeta) \) for \( t \) such that \( |t|^2 \max(|x|, |y|) \leq A_f \). Then \( f_0 \) lies in \( C_b(X) \).

We conclude so far that, for every \( f \) in \( V \), there is \( f_0 \in C_b(X) \) and \( A_f > 0 \) such that \( f(x, y; \zeta) = f_0(x, y; \zeta) \) for \( max(|x|, |y|) \leq A_f \). Proposition 1(i) then implies that the function \( f_0 \) lies in the unique irreducible \( G \)-submodule \( C_b(X)^0 \) \( (\equiv \Theta_2 \otimes |det|^{1/4}) \) of \( C_b(X) \). This determines the space \( V \) of \( \Theta \) to be \( C_b(X)^0 \), as asserted. The action of \( P \) is described by Proposition 4(i), and that of \( \sigma \) by Proposition 5. Since \( P \) and \( \sigma \) generate \( G^* \), (i) follows.

(ii) By Proposition 3(ii), we have \( dim \text{Hom}_p(V_0, V_0') = 1 \). Since \( V' \hookrightarrow V_0' \), the space \( \text{Hom}_p(V_0, V') \) is a subspace of \( \text{Hom}_p(V_0, V_0) \), necessarily one-dimensional. Consider the map \( \text{Hom}_p(V, V') \to \text{Hom}_p(V_0, V') \), obtained by restriction from \( V \) to \( V_0 \). Its kernel is \( \text{Hom}_p(V/V_0, V') \). Now \( V/V_0 \cong V_V \), and \( U \) acts trivially on \( V_V \).

On the other hand, the only vector in \( W' \), and in particular in \( V' \subset W' \), which is fixed by \( U \), is the zero vector. Hence \( \text{Hom}_p(V, V') \) injects in \( \text{Hom}_p(V_0, V') \), and it is one-dimensional. The \( L^2 \)-product on \( V \) yields a \( P \)-invariant Hermitian form on \( V \), hence a non-zero \( P \)-module morphism \( i : V \to V' \). The unitary structure on \( V \) yields a non-zero morphism \( j : V \to V' \) of \( G \)-modules. In particular \( j \) is a \( P \)-module morphism. Since \( dim \text{Hom}_p(V, V') = 1 \), the morphism \( j \) is a multiple of the morphism \( i \), as required.
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