EXPLICIT REALIZATION OF A METAPLECTIC REPRESENTATION

By

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0. Let $F \neq \mathbf{C}$ be a local field with $char\, F \neq 2$. In [W] Weil explicitly constructed a model of a genuine unitary representation θ of the two-fold covering group $\tilde{\mathrm{S}}\mathrm{p}$ of the symplectic group Sp over F. In particular, the existence of the covering group $\tilde{\mathrm{S}}\mathrm{p}$ was first proven in [W]. It is now known (see, e.g., [M]) how to construct r-fold covering groups of split semi-simple groups over a field $F \neq \mathbf{C}$ containing a primitive rth root of unity. In particular, when $r = 2$, such F has $char\, F \neq 2$. In the case of GL(n) the analogous genuine unitarizable representation Θ of a covering group is defined in [KP1] as a sub- or quotient of some induced representation. This Θ corresponds to the trivial representation of GL(n) by the metaplectic correspondence (see [KP2], [FK1]). The purpose of this paper is to construct an explicit model of the representation $\Theta = \Theta_3$ of a two-fold covering group G of GL(3) over a local field $F \neq \mathbf{C}$ of characteristic $\neq 2$, analogous to the explicit model of the representation of Weil [W]. We also determine the unitary completion of the unitarizable Θ_3. The unitary completion of our model coincides with the model of Torasso [T] when $F = \mathbf{R}$. The existence of our model has interesting applications in harmonic analysis. Some of these applications are discussed in detail in §3. In a sequel [F1] the techniques of this paper are generalized to construct an explicit model of Θ_n for any $n \geqq 3$.

1. The representation

To state our Theorem and its Corollaries, we begin by specifying the representation Θ to be studied.

1.1. Let F be a local field $\neq \mathbf{C}$ of characteristic $\neq 2$. For every integer $n > 1$ there exists (see [M]) a unique non-trivial topological central double covering group $p: S_n \to \mathrm{SL}(n, F)$. Choose a section $\mathbf{s}: \mathrm{SL}(n, F) \to S_n$ corresponding to a choice of a two-cocycle $\beta_n': S_n \times S_n \to ker\ p$ which defines the group law on S_n. Embed $\bar{G}_n = \mathrm{GL}(n, F)$ in $\mathrm{SL}(n+1, F)$ by

† Partially supported by an NSF grant and a Seed grant. This author wishes to express his deep gratitude to IHES for its hospitality when the paper was written.

†† Partially supported by an NSF grant.

‡ Sloan Doctoral Fellow.

$$\imath : g \to \begin{pmatrix} g & 0 \\ 0 & det\, g^{-1} \end{pmatrix}.$$

Denote by G'_n the preimage $p^{-1}(\imath(\bar{G}_n))$. Let $(\cdot, \cdot) : \mathrm{F}^2 \times \mathrm{F}^2 \to \{1, -1\}$ be the Hilbert symbol. Identify $\{1, -1\}$ with the kernel of p. Put $\beta(g, g') = \beta'(g, g')(det\, g, det\, g')$ $(g, g'$ in $\bar{G}_n)$. Let $\mathbf{s} : \bar{G}_n \to G'_n$ be the restriction of the section used in the definition of S_{n+1}. Denote by G_n the group which is equal to G'_n as a set, whose product rule is given by $\mathbf{s}(g)\zeta \cdot \mathbf{s}(g')\zeta' = \mathbf{s}(gg')\zeta\zeta'\beta(g, g')$. Then G_n is a non-trivial topological double covering group of $\bar{G}_n$. Let $\bar{A}$ and $\bar{B}$ be the groups of diagonal and upper-triangular matrices in $\bar{G}_n$, and A and B their preimages in G_n. Note that $\mathbf{s}$ is a homomorphism on the group $\bar{N}$ of upper-triangular unipotent matrices, and put $N = \mathbf{s}(\bar{N})$. Let $\bar{Z}$ be the center of $\bar{G}_n$ and Z the center of G_n.

Lemma 1. *Let $\bar{A}^2$ be the group of squares in $\bar{A}$, and put $A^2 = p^{-1}(\bar{A}^2)$. Then*
(i) *the group ZA^2 is the center of A,*
(ii) *if n is even then $Z = A^2 \cap p^{-1}(\bar{Z})$,*
(iii) *if n is odd then $Z = p^{-1}(\bar{Z})$, and p defines an isomorphism*

$$p : Z/(Z \cap A^2) \to \bar{Z}/\bar{Z}^2 \cong F^\times/F^{\times 2}.$$

Proof. See [KP1], Prop. 0.1.1.

Define a map $t = t_n : \bar{A} \to A^2$ by $t(h) = \mathbf{s}(h)^2 u(h)$, where

$$u(h) = \Pi_{1 \leqq i < j \leqq n} (h_i, h_j)$$

for a diagonal matrix $h = diag(h_i)$ with entries h_i $(1 \leqq i \leqq n)$. Note that t is independent of the choice of the section $\mathbf{s}$. Using the product rule in G_n (see [KP1], p. 39), it is easy to check that our section $\mathbf{s}$ satisfies $t(h) = \mathbf{s}(h^2)$ for every h in $\bar{A}$.

Lemma 2. *The map t is a group homomorphism.*

Proof. This follows from the multiplication law on $A \subset G_n$.

Definition. Let $\bar{\delta} = \bar{\delta}_n : \bar{A} \to \mathbf{C}^\times$ be the character $\bar{\delta}(diag(h_i)) = \Pi_{i=1}^n |h_i|^{(2i-1-n)/2}$. A character $\delta = \delta_n : ZA^2 \to \mathbf{C}^\times$ whose restriction to $ker\, p$ is non-trivial is called *exceptional* if $\delta(t(h)) = \bar{\delta}(h)$ for all h in $\bar{A}$.

Note that $A^2 = t(A) \cdot ker\, p$ is equal to ZA^2 if n is even. If n is odd then $ZA^2/A^2 \cong F^\times/F^{\times 2}$, hence it is possible to extend δ from A^2 to ZA^2, and there exist exceptional characters for all n.

Lemma 3. (i) *For any exceptional character δ of ZA^2 there exists a unique (up to isomorphism) irreducible representation ρ_δ of A whose restriction to ZA^2 is* $\delta \cdot \mathrm{Id}$.

(ii) *Extend* ρ_δ *to a representation of* B *trivial on* N. *Let* $(\pi_\delta, \hat{V}_\delta)$ *be the representation of* G_n *normalizedly* (see [BZ2], (1.8)) *induced from* ρ_δ. *Then* $(\pi_\delta, \hat{V}_\delta)$ *has a unique irreducible subrepresentation. When* $n = 2$, $(\pi_\delta, \hat{V}_\delta)$ *has a unique proper non-zero subrepresentation.*

(iii) *The unique irreducible subrepresentation of* $(\pi_\delta, \hat{V}_\delta)$ *is unitarizable.*

Proof. See [KP1], p. 72, for (i), (ii); and Theorem II.2.1, p. 118, for (iii).

Definition. By the *exceptional* representation $(\pi_\delta, \hat{V}_\delta)$ of G_n we mean the unique irreducible subrepresentation of $(\pi_\delta, \hat{V}_\delta)$.

1.2. Lemma 1(ii) implies that for an even n the group G_n has a unique exceptional representation, denoted (Θ, V) or (Θ_n, V).

Lemma 4. *Assume that* n *is odd. Then there exists a map* $v : \bar{Z} \to Z$ *such that* $p \circ v = \mathrm{Id}$ *and* $v(z_1)v(z_2) = v(z_1z_2)(z_1, z_2)^{(n-1)/2}$. *Moreover, such a map is unique up to a composition with an involution of* G_n.

Proof. First note that the section s satisfies the required properties. To prove the uniqueness, let v_1 and v_2 be two such maps. Then $\chi = v_1/v_2$ defines a homomorphism $\chi : F^\times \cong \bar{Z} \to ker\ p$. Let $\hat{\chi}$ be the involution of G_n defined by $\hat{\chi}(g) = \chi(det\ p(g))g$. Then $v_2 = \hat{\chi} \circ v_1$, as required.

Definition. Fix a non-trivial additive character $\psi : F \to \mathbf{C}^\times$ of F. Denote by dx a Haar measure on F. Define a function $\gamma = \gamma_\psi : F^\times \to \mathbf{C}^\times$ by

$$\gamma(a) = \frac{|a|^{1/2} \int \psi(ax^2)dx}{\int \psi(x^2)dx}.$$

Clearly, we have $\gamma(a^2) = 1$. Moreover, we have

Lemma 5. *For every* a, b *in* $F^\times$ *the function* γ *satisfies* $\gamma(ab) = \gamma(a)\gamma(b)(a, b)$.

Proof. Let γ_W be the γ defined in [W] by

$$|a|^{1/2} \int_F f(x)\psi(ax^2)dx = \gamma_W(ax^2) \int_F \hat{f}(x)\psi(-a^{-1}x^2)dx$$

for integrable f and $\hat{f}$; here $\hat{f}$ is the ψ-Fourier transform with respect to the self-dual Haar measure. Since γ_W satisfies the relation

$$\gamma_W(x^2 - ay^2 - bz^2 + abt^2) = (a, b)$$

(see [W], p. 176, bottom line), and $\gamma(a) = \gamma_W(ax^2)/\gamma_W(x^2)$, the lemma follows.

Definition. Let δ_ψ be the function of ZA^2 defined by

$$\delta_\psi(\zeta \mathbf{s}(z)t(h)) = \zeta\gamma(z)\bar{\delta}(h) \qquad (\zeta \in ker\, p, z \in \bar{Z} \cong F^\times, h \in \bar{A})$$

if $n \equiv 3(mod\, 4)$; if $n \equiv 1(mod\, 4)$ define δ_ψ by $\delta_\psi(\zeta \mathbf{s}(z)t(h)) = \zeta\bar{\delta}(h)$.

It is clear that δ_ψ is an exceptional character of ZA^2. Denote by (Θ, V), or (Θ_n, V), the corresponding representation of $G = G_n$.

1.3. It is important for us to work with an extension of Θ to a semi-direct product $G^\# = G \rtimes \langle \sigma \rangle$, where σ is an involution of G which we proceed to define. Let w_n be the anti-diagonal matrix $((-1)^{i+1}\delta_{i,n+1-j})$ in $\bar{G}_n$. Consider w_n as an element of $\mathrm{SL}(n+1, F)$ via j. Denote by $\bar{\sigma}$ the involution $\bar{\sigma}(g) = w_n^{-1} \cdot {}^t g^{-1} \cdot w_n$ of $\mathrm{SL}(n+1, F)$. Since the Steinberg group $St(n+1, F)$ is generated by elementary matrices (see [M], p. 39), $\bar{\sigma}$ maps elementary matrices to elementary matrices, and $\bar{\sigma}$ preserves the relations which define $\mathrm{St}(n+1, F)$, then $\bar{\sigma}$ lifts to an involution of $\mathrm{St}(n+1, F)$, hence to an involution $\tilde{\sigma}$ of G.

Suppose that n is odd. Then both $\mathbf{s}$ and $\mathbf{s}^\sigma = \bar{\sigma} \circ \mathbf{s} \circ \bar{\sigma}$ satisfy the conditions of Lemma 4. Hence there exists a character $\chi : F^\times \to \{1, -1\}$ such that $\mathbf{s}^\sigma = \hat{\chi} \circ \mathbf{s}$. Define $\sigma = \hat{\chi} \circ \tilde{\sigma}$; it is an involution of G. Since $\sigma \circ \mathbf{s} = \hat{\chi} \circ \tilde{\sigma} \circ \mathbf{s} = \mathbf{s} \circ \bar{\sigma}$ on $\bar{Z}\bar{A}^2$, we have

$$\delta(\sigma(\mathbf{s}(z)\mathbf{s}(h^2))) = \delta(\mathbf{s}(\bar{\sigma}z)\mathbf{s}(\bar{\sigma}h^2)) \qquad \text{for all } z \in \bar{Z}, \quad h \in \bar{A};$$

hence $\delta(\sigma(x)) = \delta(x)$ for all x in ZA^2. By Lemma 3(i) we have $\rho_\delta \circ \sigma \cong \rho_\delta$, where ρ_δ is the unique extension of δ to A. Hence $\pi_\delta \circ \sigma \cong \pi_\delta$, and by Lemma 3(ii) we have $\Theta \circ \sigma \cong \Theta$. It follows that there exists a non-zero operator $I: V \to V$ such that $\Theta(g)I = I\Theta(\sigma(g))$ for all g in G. Since Θ is irreducible, by Schur's lemma I^2 is a scalar, which we normalize to be 1. This determines I uniquely up to a sign. The choice $\Theta(\sigma) = I$ determines an extension of Θ to the semi-direct product $G^\# = G \rtimes \langle \sigma \rangle$.

Remark. (i) It is easy to check (consider first the case where $h_j = 1$ for all $j \neq i$) that

$$\tilde{\sigma}(\mathbf{s}(diag(h_i))) = \mathbf{s}(diag(h_{n+1-i}^{-1})) \cdot \prod_{i=1}^{n-1} \left(h_i, \prod_{j=i+1}^{n} h_j\right).$$

In particular

$$\tilde{\sigma}(\mathbf{s}(z)) = \mathbf{s}(z^{-1}) \cdot (z, -1)^{n(n-1)/2} \qquad \text{for } z \in F^\times \cong \bar{Z}.$$

Consequently

$$\sigma(g) = (-1, det\, p(g))^{(n-1)/2}\tilde{\sigma}(g) \quad \text{and} \quad \chi(x) = (-1, x)^{(n-1)/2}.$$

(ii) Since $(det\, \bar{\sigma}(g), det\, \bar{\sigma}(g')) = (det\, g, det\, g')$ $(g, g' \in \bar{G})$, the formula in (i) for the involution σ on G defines also an involution σ' on G' which satisfies $p \circ \sigma' = \bar{\sigma} \circ p$ on G' and $\sigma \circ \mathbf{s} = \mathbf{s} \circ \bar{\sigma}$ on $\bar{Z}\bar{A}^2$.

1.4. An explicit model for Θ_2 is easily obtained (see [F1], §1, Example, or [FM], and the proof of Proposition 1, §5, below) from that of the even Weil representation (see [F], p. 145). Indeed, this Weil representation is a representation of S_2, which extends to a representation of $\mathbf{s}(\bar{Z})S_2$ (by the character $\gamma = \gamma_\psi$ on $\mathbf{s}(\bar{Z})$). The representation Θ_2 is the G_2-module induced from this extension to $\mathbf{s}(\bar{Z})S_2$.

In this paper we construct an explicit realization of the unitarizable G_3-module Θ_3. When $F = \mathbf{R}$ the unitary completion of Θ_3, or at least its restriction to $p^{-1}(\mathrm{SL}(3, \mathbf{R}))$, coincides with the unitary $p^{-1}(\mathrm{SL}(3, \mathbf{R}))$-module constructed by Torasso [T].

2. The realization

The representation $\Theta = \Theta_3$ will be realized in a space of functions on a two-fold covering space X of the punctured affine plane $\bar{X} = F \times F - \{(0, 0)\}$. Clearly $\bar{X} = \Gamma \setminus \mathrm{GL}(2, F)$, where

$$\Gamma = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}.$$

It is easy to see that the restriction of s to Γ is a homomorphism. Hence we can define the double cover X of $\bar{X}$ to be $\mathbf{s}(\Gamma) \setminus G_2$. Then X is a homogeneous space under the action of G_2. To be able to write explicit formulas for the action of G_2 on X, recall the explicit construction of G_2. Put

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c, & c \neq 0, \\ d, & c = 0, \end{cases}$$

and

$$\beta(g, g') = \left(\frac{x(gg')}{x(g)}, \frac{x(gg')}{x(g')\det g} \right).$$

Then G_2 is the group of pairs (g, ζ) (g in $\mathrm{GL}(2, F)$, ζ in $\ker p$) with the multiplication law

$$(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'\beta(g, g')).$$

Given $\bar{z} = (x, y)$ in $\bar{X}$, put $x(\bar{z}) = x$ if $x \neq 0$ and $x(\bar{z}) = y$ if $x = 0$. Identify X with $\bar{X} \times \ker p$ by mapping the image in X of the element $\mathbf{s}(h)\zeta$ of G, where

$$h = \begin{pmatrix} z & t \\ x & y \end{pmatrix},$$

to the element $(x, y; \zeta(x(h), \det h))$ of $\bar{X} \times \ker p$. Then the action of G_2 on $\bar{X} \times \ker p$ implied by this identification is given by

(*) $$(\bar{z}, \zeta)(g, \zeta') = \left(\bar{z}g, \zeta\zeta'\left(\frac{x(\bar{z}g)}{x(\bar{z})}, \frac{x(\bar{z}g)}{x(g)}\right)(x(\bar{z}g), det\, g)\right).$$

Remark. Replacing $(\cdot, \cdot)$ by the nth Hilbert symbol, (*) defines an n-fold covering of the punctured plane $\bar{X}$ as the homogeneous space $\mathbf{s}(\Gamma) \setminus G_2$.

Definition. A function $f: X \to \mathbf{C}$ is called *genuine* if $f(z\zeta) = \zeta f(z)$ for ζ in $ker\ p$, z in X. It has *bounded support* if there is a compact subset of $F \times F$ which contains all $\bar{z}$ in $\bar{X}$ with $f(\bar{z}; \zeta) \neq 0$. It is called *homogeneous* if $f(t^2x, t^2y; \zeta) = |t|^{-1}f(x, y; \zeta)$ (t in $F^\times$). Let $L^2(X)$ be the space of genuine, square-integrable, complex-valued functions on X. Let $C(X)$ be the space of smooth functions f in $L^2(X)$. Denote by $C_b(X)$ the space of f in $C(X)$ with bounded support. Denote by $C_h(X)$ the space of homogeneous f in $C(X)$.

Let $\bar{P}$ ($\supset \bar{B}$) be the standard maximal parabolic subgroup of type (2, 1) of G, and consider the subgroup $P = p^{-1}(\bar{P})$ of G. Define the action of P on $L^2(X)$ as follows (we denote the action by Θ):

(1) $$\left[\Theta\left(\mathbf{s}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)f\right](z) = |det\, g|^{1/2} f(z\mathbf{s}(g)) \qquad (g \text{ in } GL(2, F));$$

(2) $$\left[\Theta\left[\mathbf{s}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}\right]f\right](z) = \psi(ux + vy)f(z) \qquad (u, v \text{ in } F);$$

(3) $$\left[\Theta\left[\mathbf{s}\begin{bmatrix} a & & 0 \\ & a & \\ 0 & & a \end{bmatrix}\zeta\right]\right](z) = \zeta\gamma(a)f(z) \qquad (a \text{ in } F^\times).$$

Under the action (1) the space $C_h(\mathrm{X})$ is a G_2-module; it has a unique proper non-zero G_2-submodule $C_h(X)^0$, isomorphic to $\Theta_2 \otimes |det|^{1/4}$ (see [F], p. 145). Indeed, the space

$$I(s) = \left\{\varphi: G_2 \to \mathbf{C};\ \varphi\left(\mathbf{s}\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\zeta\right)\right.$$
$$\left. = \zeta\, |a/b|^{1/2+s}\varphi(g),\ a \in F^\times,\ b \in F^{\times 2},\ \zeta \in ker\ p\right\}$$

is a G_2-module under the action $\rho(g)\varphi(h) = |\det g|^{1/4}\varphi(hg)$. At $s = -\frac{1}{4}$ it is reducible, of length two. Its unique proper non-zero submodule is $\Theta_2 \otimes |\det|^{1/4}$. The map $\varphi \to f$, $f((0, 1)g) = |\det g|^{-1/2-s}\varphi(g)$, establishes a G_2-module isomorphism from $I(s)$ to the space

$$J(s) = \{ f\colon X \to \mathbf{C};\, f(b(x, y);\zeta) = \zeta\, |b|^{-1-2s} f(x, y; 1),\, b \in F^{\times 2}\},$$

with the G_2-action $\rho(g)f(z) = |\det p(g)|^{3/4+s} f(zg)$ $(z \in X)$.

Definition. Denote by $C_b(X)^0$ the space of f in $C_b(X)$ for which there exists f_0 in $C_h(X)^0$ and $A_f > 0$ such that $f(z) = f_0(z)$ for all $z = (x, y; \zeta)$ with $max(|x|, |y|) \leqq A_f$.

In particular, for every f in $C_b(X)^0$ there is $A_f > 0$ such that

$$f(t^2x, t^2y; \zeta) = |t|^{-1} f(x, y; \zeta) \qquad \text{if } max(|x|, |y|) \leqq A_f \text{ and } \quad |t| \leqq 1.$$

Theorem. (i) *The genuine representation* Θ *of* $G^{\#} = G \rtimes \langle\sigma\rangle$ *can be realized in the space* $C_b(X)^0$ *by the operators* (1), (2), (3) *and*

$$(4) \quad (\Theta(\sigma)f)(x, y; \zeta) = \gamma(-1)^{1/2}\gamma(x)^{-1}|x|^{-1/2} \int_F f(-x, u; \zeta)\psi(uy/x)du.$$

(ii) *The space* $C_b(X)^0$ *is contained in* $L^2(X)$. *There is a unique* (*up to a scalar multiple*) *Hermitian scalar product on the unitarizable representation* $(\Theta, C_b(X)^0)$. *It is given by the* L^2*-product.*

Remark. (i) Since $G^{\#}$ is generated by P and σ, the action of $G^{\#}$ is completely defined by (1)–(4).

(ii) It follows from (ii) in the Theorem that the unitary completion of $(\Theta, C_b(X)^0)$ is $(\Theta, L^2(X))$. As noted in (1.4), when $F = \mathbf{R}$ the restriction to $p^{-1}(\mathrm{SL}(3, \mathbf{R}))$ of this realization of the unitary completion of Θ coincides with the model constructed by Torasso [T].

(iii) Erasing the symbols s in (1), (2), (3), ζ in (3), (4), $\gamma(a)$ in (3), and $\gamma(-1)^{1/2}\gamma(x)^{-1}$ in (4), the (modified) operators (1)–(4) define an explicit realization of the representation $I(\mathbf{1}_{\bar{P}}; \mathrm{GL}(3, F), \bar{P})$ of $\mathrm{GL}(3, F)$ normalizedly induced from the trivial representation $\mathbf{1}_{\bar{P}}$ of a maximal parabolic subgroup $\bar{P}$. This model is isomorphic to the model (τ_0, V_0) in [FK2], middle of p. 497, by the map $(\tau_0, V_0) \ni \phi \to f$, $f(x, y) = \int_F \phi(x, y, z)\psi(z)dz$.

3. Corollaries

The Theorem is proven in §§5–6. In this section we deduce three Corollaries, assuming the Theorem.

3.1. Let F be a local field as in (1.1), and ψ an additive character as in (1.2). The function

$$g(x) = \gamma_\psi(x)\psi(-1/x)|x|^{-1/2}$$

is locally integrable on F. Let

$$\check{g}(x) = \int_F g(y)\psi(-xy)dy$$

be its Fourier transform. Put

$$K(z, z') = (x, -x')\cdot\check{g}\left(-det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}\right)\cdot\zeta\zeta'$$

$$\text{if } z = (x, y; \zeta), \quad z' = (x', y'; \zeta'),$$

and for every f in $L^2(X)$ write

$$f^\vee(z) = \int_{\check{X}} f(z')K(z, z')dz'.$$

Denote the action of S_2 on $L^2(X)$ by ρ; thus $(\rho(s)f)(z) = f(zs)$ for f in $L^2(X)$, s in S_2, z in X.

Corollary 1. *The map $f \to f^\vee$ takes $L^2(X)$ to $L^2(X)$ and $C_b(X)^0$ to $C_b(X)^0$. Moreover, we have* (i) $(f^\vee)^\vee = \gamma(-1)^{-1}\rho(-1)f$, *and* (ii) $(\rho(s)f^\vee = \rho(s)f^\vee$ *for all s in S_2.*

Proof. Put

$$\mathbf{\alpha} = \mathbf{s}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \Theta(\sigma)\Theta(\mathbf{\alpha})\Theta(\sigma)\Theta(\mathbf{\alpha})\Theta(\sigma).$$

Using (4) and (1), we have $\mathbf{F}f = \gamma(-1)^{1/2}f^\vee$. Assuming the Theorem it is easy to check that $\mathbf{F}^2 = \rho(-1)$, and that $\mathbf{F}$ commutes with $\rho(s)$ for every s in S_2, as required.

Remark. The transform $f \to f^\vee$ is analogous to the Fourier transorm

$$\bar{f}^\vee(x, y) = \int\int \bar{f}(x', y')\psi\left(det\begin{pmatrix} x & y \\ x' & y' \end{pmatrix}\right)dx'dy'$$

on $L^2(X)$, which satisfies $(\bar{f}^\vee)^\vee = \bar{f}$ and $(\bar{\rho}(s)\bar{f})^\vee = \bar{\rho}(s)\bar{f}^\vee$ for every s in SL(2, F); here we put $(\bar{\rho}(s)\bar{f})(\bar{z}) = \bar{f}(\bar{z}s)$.

3.2. Let F be a local field as in (1.1), and ψ, g and $\check{g}$ as in (3.1).

Corollary 2. *The support of $\check{g}$ is contained in the set F^2 of squares of F.*

Proof. Corollary 1(ii) with $s = \mathbf{s}(\alpha)$, $\alpha = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, asserts that

$$K(z\mathbf{s}(\alpha), z'\mathbf{s}(\alpha)) = K(z, z') \qquad \text{for all } z = (x, y; \zeta) \text{ and } z' = (x', y'; \zeta').$$

Hence for all z, z' we have

$$(*) \qquad g^{\vee}\left(-\det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}\right)[1 - (y, -x)(y', -x')(y, -y')(x, -x')] = 0.$$

Since $(a + b, -b/a) = (a, b)$, we have

$$(xy', -x'y) = \left(-\det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}, xx'yy'\right).$$

Put

$$a = -\det\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}, \qquad b = x'y.$$

Then $(*)$ implies that

$$g^{\vee}(a)[1 - (a, b(a + b))] = 0$$

for all a, b in F with $ab(a + b) \neq 0$. Note that $1 + b/a \in F^{\times 2}$ if $|b|$ is sufficiently smaller than $|a|$. If follows that if $a \neq 0$ and $g^{\vee}(a) \neq 0$, then $a \in F^{\times 2}$, as required.

Scholium. The following is a sketch of an alternative, elementary proof of Corollary 2, communicated to us by J.L. Waldspurger. Recall that F is a local non-archimedean field with $\mathit{char}\, F \neq 2$, $\psi: F \to \mathbf{C}^{\times}$ is a non-trivial continuous character, and $g: F \to \mathbf{C}$ is defined almost everywhere by $g(x) = \psi(-1/x)\alpha(x)/\alpha(1)$, where $\alpha(x) = \int_F \psi(xy^2)dy$. The Fourier transform $f^{\vee}$ is defined by $f^{\vee}(x) = \int_F \psi(-xy)f(y)dy$, and we claim that $g^{\vee}$ is supported on F^2.

Note that $g(x) = \alpha(1)^{-1}\int_F \psi(xy^2 - x^{-1})dy$. Making the change $y \mapsto y + x^{-1}$, we get

$$g(x) = \alpha(1)^{-1} \int_F \psi(xy^2 + 2y)dy.$$

For a function $f: F \to \mathbf{C}$ supported on F^2, the change $z = y^2$ of variables yields the identity

$$\int_F f(z)|z|^{-1/2}dz = \frac{|2|}{2} \int_F f(y^2)dy.$$

For a fixed $x \in F$, consider the function

$$f(z) = \begin{cases} \sum_{\{y;\, y^2 = z\}} \psi(xz + 2y), & z \in F^2, \\ 0, & z \notin F^2. \end{cases}$$

Then

$$\int_F f(z)|z|^{-1/2}dz = \frac{|2|}{2} \int_F \psi(xy^2)[\psi(2y) + \psi(-2y)]dy$$

$$= |2| \int_F \psi(xy^2 + 2y)dy.$$

Hence

$$g(x) = (|2|\alpha(1))^{-1} \int_F f(z)|z|^{-1/2}dz.$$

Now put

$$h(z) = \begin{cases} (|2|\alpha(1))^{-1}|z|^{-1/2} \sum_{\{y:\, y^2 = z\}} \psi(2y), & z \in F^2, \\ 0, & z \notin F^2. \end{cases}$$

Then $g(x) = \int_F \psi(xz)h(z)dz$, namely $g(x) = h^\vee(-x)$. The Fourier inversion formula $(h^\vee)^\vee(x) = h(-x)$ implies that $g^\vee(x) = h(x)$. Hence $g^\vee$ is supported on F^2 as required.

Remark. (i) Since $\mathrm{SL}(2, F)$ is generated by

$$u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad b \in F^\times,$$

and α, and since $K(zu, z'u) = K(z, z')$ is trivially true, Corollary 2 is equivalent to (ii) of Corollary 1.

(ii) Denote by $a^{1/2}$ the non-negative square-root of $a \geqq 0$, and by i the square root of -1 in the upper half-plane in $\mathbf{C}$. Define a function $\sqrt{x}$ or $\mathbf{R}$ by

$$\sqrt{x} = \begin{cases} |x|^{1/2}, & \text{if } x \geqq 0, \\ i|x|^{1/2}, & \text{if } x \leqq 0. \end{cases}$$

Corollary 2 implies that: *The Fourier transform $g_{\mathbf{R}}^{\vee}(x)=\int_{\mathbf{R}} g_{\mathbf{R}}(y)e^{-ixy}dy$ of the locally integrable function $g_{\mathbf{R}}(x)=e^{-i/x}/\sqrt{x}$ on* $\mathbf{R}$ *is supported on the set of non-negative real numbers*. Indeed, this is the special case where $F=\mathbf{R}$ and $\psi(x)=e^{ix}$; then $\gamma_\psi(x)=1$ if $x>0$ and $\gamma_\psi(x)=1/i$ if $x<0$ by [W], top of p. 174. Hence $\gamma_\psi(x)|x|^{-1/2}=1/\sqrt{x}$, and $g_{\mathbf{R}}(x)$ is $g(x)$ of Corollary 2. However, it is easy to see directly that $\check{g}_{\mathbf{R}}$ is supported on $\mathbf{R}_{\geqq 0}$ since $g_{\mathbf{R}}(x)$ extends to a function $g_{\mathbf{C}}(z)$ analytic in the upper half-plane and vanishing at infinity, and our assertion then follows from the Paley-Wiener theorem.

(iii) In fact the Theorem can be reduced to Corollary 2. This observation is due to Torasso [T]. He proved first that $g_{\mathbf{R}}^{\vee}$ is supported on $\mathbf{R}_{\geqq 0}$ and this is the basis of his proof of the Theorem when $F=\mathbf{R}$.

(iv) Corollary 2 suggests the existence of a theory of "analytic" complex-valued functions on a local field F, in which the space of "analytic functions on the upper half-plan" is replaced by the space R_ψ of functions f on F such that the support of $\check{f}$ lies in the set of squares. However R_ψ is not a ring, and we do not know how to develop the theory of such "analytic" functions on F.

3.3. Suppose that F is non-archimedean, denote by R its ring of integers, and fix a generator π of the maximal ideal of R. Denote by *val* the additive, integer-valued function on $F^\times$ normalized by $val(\pi)=1$. Put $h(x)=|x|^{-1/2}$ if $val(x)$ is even and non-negative, and $h(x)=0$ otherwise. Suppose that the residual characteristic of F is odd. There exists a unique group-theoretic section of $p: p^{-1}(\mathrm{SL}(4,R))\to \mathrm{SL}(4,R)$, denoted by κ^*; see [KP1], p. 43. Then $K=\mathrm{GL}(3,R)$ embeds as a subgroup of G_3 via κ^*. An irreducible genuine G-module is called *unramified* if it has a (necessarily unique up to a scalar multiple) non-zero K-fixed vector.

Corollary 3. *If the residual characteristic of F is odd, then the G-module Θ is unramified. If ψ is trivial on R but not on $\pi^{-1}R$, then the K-fixed vector in Θ is a multiple of the vector*

$$\phi(x,y;\zeta)=\begin{cases}\zeta h(x), & \text{if } |y|\leqq|x|,\\ (x,y)\zeta h(y), & \text{if } 0<|x|<|y|,\\ \zeta h(y), & \text{if } x=0.\end{cases}$$

Proof. The group K is generated by its upper-triangular matrices, by

$$\boldsymbol{\alpha}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

and $\sigma\alpha\sigma$. The sections κ^* and s coincide on these matrices (see [KP1], Prop. 0.1.3). Using the Theorem it is easy to check that ϕ is invariant under the image of these matrices. Hence the corollary follows.

Remark. Note that the function ϕ of Corollary 3 is locally constant at $(0, y, \zeta)$, $y \in F^\times$, since the limit of $(x, y; (-x, y))$ as $x \to 0$ $(x, y \neq 0)$ is $(0, y; 1)$.

4. Preliminaries

Here we collect various facts used in the proof of the Theorem. Since the Theorem is already proven in [T] when $F = \mathbf{R}$, we restrict our attention to the case when F is non-archimedean.

4.1. Given a group H and a smooth H-module $V = V(H)$, let $V'(H)$ be the Hermitian dual of V, namely the smooth H-module obtained on conjugating the complex structure of the smooth dual of V. We write V' for $V'(H)$ when the group H is specified. Note that in general $V'(H) \neq V'(H')$ when V is both H- and H'-module. Observe that an H-invariant Hermitian form on V is equivalent to an H-invariant map from V to V' $(= V'(H))$. Note that if $\alpha \in V'$, $v \in V$ and $h \in H$, then $(h \cdot \alpha)(v) = \alpha(h^{-1} \cdot v)$.

4.2. Let $Q = SR$ be the semi-direct product of a group S and an abelian normal subgroup R. The group Q acts on R by $q : r \to q r q^{-1}$, hence also on the group $\hat{R}$ of characters ψ_R on R by $\psi_R^q(r) = \psi_R(q^{-1} r q)$. For any character ψ_R of R we denote by $Stab_Q(\psi_R)$ the stabilizer of ψ_R in Q, and put $Stab_S(\psi_R) = S \cap Stab_Q(\psi_R)$. For any irreducible representation τ of $Stab_S(\psi_R)$ the tensor product $\tau \otimes \psi_R$ defines a representation of $Stab_Q(\psi_R) = Stab_S(\psi_R)R$. Denote by $\pi(\tau \otimes \psi_R)$ the Q-module $ind(\tau \otimes \psi_R; Q, Stab_Q(\psi_R))$, where, as in [BZ1], (2.21) and (2.22), *Ind* indicates the functor of (*un*normalized) induction, and *ind* the functor of induction with compact supports (we do not normalize these functors as in [BZ2], p. 444). As in [BZ2], top of p. 444, define the positive-valued character $\Delta_{\mathbf{Q}} : \mathbf{Q} \to \mathbf{R}_{>0}^\times$ by $d(g^{-1} q g) = \Delta_{\mathbf{Q}}(g) dq$ $(g \in \mathbf{Q})$, where dq is a Haar measure on $\mathbf{Q}$.

Mackey's Theorem. (i) *The Q-module $\pi(\tau \otimes \psi_R)$ is irreducible.*

(ii) *We have $\pi(\tau \otimes \psi_R) \cong \pi(\tau^\# \otimes \psi_R^\#)$ if and only if there is s in S such that $\psi_R^s = \psi_R^\#$ and $\tau^s \cong \tau^\#$.*

(iii) *Every irreducible Q-module is equivalent to $\pi(\tau \otimes \psi_R)$ for some τ and ψ_R.*

(iv) *The Q-module $\pi(\tau \otimes \psi_R)'$ (see (4.1)) is equivalent to*

$$Ind((\Delta_Q/\Delta_S)\tau' \otimes \psi_R; Q, \bar{S}), \qquad \text{where } \bar{S} = Stab_Q(\psi_R).$$

Proof. See [BZ1], (2.23) and (5.10), for (i)–(iii), and [BZ1], (2.25), for (iv); when $F = \mathbf{R}$ see [K], §13.3, Theorem 1.

4.3. Let Q be a parabolic subgroup of G, R its unipotent radical, $M = Q/R$ its Levi component, and ψ_R a character of R. For any Q-module V, let V_{R,ψ_R} be the $Stab_M(\psi_R)$-module of (R, ψ_R)-coinvariants in V (see [BZ1], (2.30)). Put V_R for V_{R,ψ_R} when ψ_R is trivial. In this paper the functor of coinvariants is not normalized (as in [BZ1], in contrast with [BZ2], p. 444). For the reader's convenience, we record

Frobenius Reciprocity ([BZ2], (1.9(b)), p. 445). *For any smooth Q-module V, and any smooth $Stab_M(\psi_R)$-module W, we have*

$$Hom_{Stab_M(\psi_R)}(V_{R,\psi_R}, W) = Hom_Q(V, Ind(W \otimes \psi_R; Q, Stab_Q(\psi_R))).$$

4.4. We use below the Geometric Lemma (2.12) of [BZ2], which we now record (in the notations of [BZ2]). Let G be a covering group of a reductive connected group $\bar{G}$ over a local field F, fix a minimal parabolic subgroup P_0 and a Levi subgroup thereof, and denote by M, N standard Levi subgroups of G (notations: $M, N < G$). Denote by W_G, W_M, W_N the Weyl groups of G, M, N (note that $W_G = W_{\bar{G}}, \dots$). Each double coset $W_N \backslash W_G / W_M$ has a unique representative of minimal length. The set of these representatives will be denoted by $W_G^{N,M}$. For each w in $W_G^{N,M}$ put

$$M_w = M \cap w^{-1}(N) < M, \qquad N_w = w(M_w) = w(M) \cap N < N.$$

Denote by $Alg\, M$ the category of smooth (= algebraic in [BZ2]) M-modules. Let P be the parabolic subgroup of G which contains P_0 and whose Levi component is M. Put $\delta_P(p)$ for $\Delta_P(p)^{-1}$, for p in P. Put $i_{GM}V$ for $ind(\delta_P^{1/2} \otimes V; G, M)$ and $r_{NG}V$ for $\delta_P^{-1/2} \otimes V_N$; i_{GM} and r_{NG} are the functors of normalized (as in [BZ2]) induction and coinvariants.

Composition Theorem. *The functor* $\mathbf{F} = r_{NG} \circ i_{GM} : Alg\, M \to Alg\, N$ *is glued from the functors* $\mathbf{F}_w = i_{N,N_w} \circ w \circ r_{M_w,M}$ *for w in $W_G^{N,M}$. More precisely, choose an ordering $\{w_1, \dots, w_r\}$ of $W_G^{N,M}$ such that $w_j < w_i$ implies $i < j$ ($<$ is the standard partial order on W_G). Then* $\mathbf{F}$ *has a canonical filtration* $0 = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \cdots \subset \mathbf{F}_r = \mathbf{F}$ *such that* $\mathbf{F}_i/\mathbf{F}_{i-1}$ *is canonically isomorphic to* $\mathbf{F}_{w_i}$.

Proof. This is the Geometric Lemma (2.12) of [BZ2], which is stated there only for the algebraic group $\bar{G}$, but its proof is valid also in the context of the covering group G.

4.5. In this subsection we summarize properties of Θ used in the proof of the Theorem in §§5–6 below.

The G_n-module (Θ_n, V_n) is defined in §1 as the unique irreducible submodule of the induced G_n-module $(\pi_{\delta_n}, \hat{V}_n)$. Its character χ_{Θ_n} is computed in [KP2], Theorem 6.1, at least when $n = 2, 3$ (the computation for a general n is reduced to a certain conjecture about orbital integrals). This character computation implies that Θ_n corresponds to the trivial $\mathrm{GL}(n, F)$-module $\mathbf{1}_n$ by the metaplectic correspondence ([KP2], Conjecture, p. 208, and Prop. 5.6, p. 213; or [FK1], (26.1)). We shall record here two applications of this character computation, to be used below.

For any diagonal matrix $h = diag(h_i)$ in $\bar{A}$ put

$$\Delta(h) = \left| \prod_{i<j} (h_i - h_j)^2/h_ih_j \right|^{1/2},$$

and for $\tilde{h}$ in A put $\Delta(\tilde{h}) = \Delta(p(\tilde{h}))$. The character computation implies that there is a $\beta > 0$ (explicitly given in [KP2]) such that

$$\Delta(t(h))\chi_{\Theta_n}(t(h)) = \beta\Delta(h)$$

for every h in $\bar{A}$ with $|h_i| \neq |h_j|$ for all $i \neq j$. In particular, when $n = 3$ and $h = diag(a, b, c)$ with $|a| < |b| < |c|$, we have $\Delta(h) = |c/a|$, hence

$$(\Delta\chi_\Theta)(t(h)) = \beta\,|c/a|. \tag{5}$$

To state the second application, denote by ψ_N a non-degenerate character of the unipotent upper-triangular subgroup N of G_n. A Whittaker model of a G-module (π, V) is an injection $l: V \to Ind(\psi_N; G_n, N)$. The space of Whittaker functionals l is then dual to the space

$$V_{N,\psi_N} = V/\langle \pi(n)v - \psi_N(n)v;\ v \text{ in } V, n \text{ in } N\rangle.$$

Corollary 6.2 of [KP2] asserts that (at least for $n = 2, 3$) we have

$$dim\, V_{N,\psi_N} = \frac{a}{r!n} \sum_{h \in \bar{A}, h^r = 1} \Delta(h), \qquad a = \frac{n}{(n, r-1)} \left| \frac{(n, r-1)}{n^r} \right|_F^{1/2}.$$

In our case $r = 2$. Consequently we have the following

Lemma 6. (i) *When $n = 2$, $dim(\Theta_2, V_2)_{N,\psi_N} = 1$, and Θ_2 has a unique (up to a scalar multiple) Whittaker functional.* (ii) *When $n \geqq 3$, $dim(\Theta_n, V_n)_{N,\psi_N} = 0$, and Θ_n has no Whittaker model.*

Remark. The proof of the character relation [KR2], Theorem 6.1, is based on the (global) trace formula. Hence the proof of (ii) is presently complete only for $n = 3$. For F with $|2| = 1$ a purely local proof of Lemma 6 is given [KP1], Theorem I.3.5.

4.6. In (5.1) below we use a special case of the Theorem of [C], which we record here in a form useful for (5.1), in the notations of (4.5).

Theorem ([C]). *Let π be an admissible G_n-module, and h the matrix diag(h_i), with $|h_i| < |h_{i+1}|$ $(1 \leqq i < n)$. Then $(\Delta\chi_\pi)(t(h)) = \chi_{r_{A,G}\pi}(t(h))$.*

Here $r_{A,G}\pi$ is an A-module (see (4.4)). The center of A is ZA^2; it is of finite index in A. The irreducible constituents of the restriction of $r_{A,G}\pi$ to ZA^2 are characters. We use this Theorem in two cases. First, the Theorem, together with (5), implies

Lemma 7. *When $n = 3$ and $\pi = \Theta$, the restriction of $r_{A,G}\Theta$ to $t(\bar{A})$ is a multiple of the character which maps $t(h)$, $h = \mathrm{diag}(a, b, c)$, to $|c/a|$.*

Note that a genuine character of ZA^2 which transforms on $s(\bar{Z})$ according to γ is uniquely determined by its values on $t(\bar{A})$.

Remark. Lemma 7 can be proven also using [KP1], Theorem I.2.9(e), instead of using [C] and the character relation (5).

The second application concerns the case $n = 2$. Let $\mu_i : F^\times \to \mathbf{C}^\times$ $(i = 1, 2)$ be two characters of $F^\times$. Extend the character $(\mu_1, \mu_2) : t\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \to \mu_1(a)\mu_2(b)$ to a genuine character μ of a maximal abelian subgroup A_* of A_2. Extend μ to A_*N (trivially on N), and induce (normalizedly) to a G_2-module $\pi(\mu_1, \mu_2)$. The character of $\pi = \pi(\mu_1, \mu_2)$ is computed in [F], p. 141: on $t(\bar{A}_2)$ we have that $\Delta\chi_\pi$ is equal to a scalar multiple of $(\mu_1, \mu_2) + (\mu_2, \mu_1)$. Theorem [C] then implies

Lemma 8. *Each irreducible constituent of the restriction of $r_{A_2, G_2}[\pi(\mu_1, \mu_2)]$ to $t(\bar{A}_2)$ is isomorphic to the character (μ_1, μ_2) or (μ_2, μ_1).*

5. Restriction to P

Denote by P and P^+ $(\supset B)$ the preimages in G of the standard maximal parabolic subgroups of type $(2, 1)$ and $(1, 2)$ in GL$(3, F)$, and by U and U^+ $(\subset N)$ their unipotent radicals. Our construction of the explicit realization of Θ is accomplished in two steps. In this section we study the restriction of Θ to P. In the next section we construct the action of σ. Since P and σ generates $G^* = G \rtimes \langle \sigma \rangle$ we thus obtain the required explicit realization.

5.1. Let $\psi : F \to \mathbf{C}^\times$ be a character as in (1.2), and define a character ψ_U of N by $\psi_U(n) = \psi(n_{2,3})$. The restriction of ψ_U to the subgroup U of N will again be denoted by ψ_U. Since ψ_U is trivial on U^+ it defines a character of $N^+ = N/U^+$, denoted again by ψ_U.

Embed $\bar{G}_2$ in $\bar{P}$ by

$$g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Put $G_2 = p^{-1}(\bar{G}_2) \subset P$. Since $P = ZG_2U$, we identify below a P-module which transforms trivially under U and by γ under $s(\bar{Z})$, with a G_2-module. The analogous convention is applied to P^+-modules. Let V_U be the P-module of U-coinvariants of V (see (4.3)).

Proposition 1. (i) *As a G_2-module, V_U is isomorphic to $\Theta_2 \otimes |det|^{1/4}$. In particular, $s\left(\begin{smallmatrix} h^2 & 0 \\ 0 & h^2 \end{smallmatrix}\right)$ acts as multiplication by $|h|$.*

(ii) *As a G_2-module, V_{U^+} is isomorphic to $\Theta_2 \otimes |det|^{-1/4}$.*

(iii) *The element $s\left(\begin{smallmatrix} h & 0 \\ 0 & h \end{smallmatrix}\right)$ acts on any Whittaker functional on V_U as multiplication by $|h|^{1/2}\gamma(h)^{-1}$.*

Proof. (i) By definition (see Lemma 3(ii) of §1), $\Theta = \Theta_3$ is the unique irreducible submodule of the induced G_3-module $(\pi_{\delta_3}, \hat{V}_{\delta_3})$. Since the functor r of coinvariants is exact (see [BZ1], Prop. 2.35), the P-module $r_{M,G}\Theta$ is a submodule of $r_{M,G}(\pi_{\delta_3}, \hat{V}_{\delta_3})$, where M is the standard Levi subgroup of P. The Composition Theorem (4.4) applies to $r_{M,G}\pi_{\delta_3}$ with $M = B$ and $N = P$, and $W_G^{P,B}$ consists of the elements $w_3 = \text{id}$, $w_2 = (23)$ and $w_1 = (12)(13) = (132)$ of W_G. It asserts that there is a composition series $0 \subset \hat{V}_1 \subset \hat{V}_2 \subset \hat{V}_3 = (\hat{V}_{\delta_3})_U$ of P-modules (i.e. G_2-modules), where $\hat{V}_i / \hat{V}_{i-1} \cong i_{P,B}(w_i \circ \rho_{\delta_3})$ (ρ_{δ_3} is defined by Lemma 3(i); $w_i \circ \rho_{\delta_3}$ is the B-module extended trivially on N from A). Now it follows from Lemma 8 that each irreducible constituent of the normalized A-module of N-coinvariants $r_{A,M} \circ i_{P,B}(w_i \circ \rho_{\delta_3})$ ($i = 1, 2, 3$) is acted upon by the element $t(h)$ of the center of A, where $h = diag(a, b, c) \in \tilde{A}$, according to the characters: $|c/a|$ or $|c/b|$ if $w_i = \text{id}$ ($i = 3$), $|b/a|$ or $|b/c|$ if $w_i = (23)$ ($i = 2$), $|a/b|$ or $|a/c|$ if $w_i = (12)\,(13)$ ($i = 1$). On the other hand, Lemma 7 implies that $t(h)$ acts according to the character $|c/a|$ on each irreducible constituent of the A-module $r_{A,G}\Theta = r_{A,M}(r_{M,G}\Theta)$. Since the functor of coinvariants is exact, we thus obtain that $Hom_P(r_{M,G}\Theta, \hat{V}_2) = 0$, and that the submodule $r_{M,G}\Theta$ of $\hat{V}_3$ is a proper non-zero P-submodule of the quotient $\hat{V}_3 / \hat{V}_2 \cong i_{P,B}(\rho_{\delta_3})$ ($\cong \pi_{\delta_2} \otimes |det|^{-1/4}$ as a G_2-module). However, Lemma 3(ii) asserts that the G_2-module π_{δ_2} has a unique proper non-zero submodule, which is Θ_2. Hence $r_{M,G}\Theta = \Theta_2 \otimes |det|^{-1/4}$, and

$$\Theta_U = \delta_P^{1/2} \otimes r_{M,G}\Theta = \Theta_2 \otimes |det|^{1/4} \qquad \left(\text{since } \delta_P\left(s\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) = |det\, g|\right),$$

as required.

For the last claim in (i), note that $s\left(\begin{smallmatrix} h^2 & 0 \\ 0 & h^2 \end{smallmatrix}\right)$ acts trivially on Θ_2 by definition of Θ_2.

Part (ii) is of course analogous to (i).

For (iii), note that the G_2-module Θ_2 has the following realization (see, e.g., [FM] or [F1], Sect. 1, Example). Its space V_2 consists of all locally constant functions $f: F^\times \to \mathbf{C}$ whose support is compact in F, for which there is $A(f) > 0$

and $f': F^\times \to \mathbf{C}^\times$ satisfying $f'(xa^2) = |a|^{-1/2} f'(x)(x, a$ in $F^\times)$ with $f(x) = f'(x)$ for $|x| \leqq A(f)$. On this space the group G_2 acts by

$$\Theta_2\left(\mathbf{s}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) f(x) = |a|^{1/2} f(ax), \quad \Theta_2\left(\mathbf{s}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) f(x) = (x, z)\gamma(z)^{-1} f(x),$$

$$\Theta_2\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(x) = \psi(bx) f(x), \qquad \Theta_2\left(\mathbf{s}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) f(x)$$

$$= c\gamma(x)^{-1} |x|^{1/2} \int_F |y|^{1/2} f(xy^2)\psi(2xy)dy,$$

for some c in $\mathbf{C}^\times$. By definition, a Whittaker functional on (Θ_2, V_2) is a linear form $L: V_2 \to \mathbf{C}$ which satisfies

$$L\left(\Theta_2\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f - \psi(b) f\right) = 0 \qquad \text{for all } b \text{ in } F \text{ and } f \text{ in } V_2.$$

By Lemma 6(i) this functional is unique up to a scalar. Hence it is a multiple of $L(f) = f(1)$, which is clearly a Whittaker functional. Now

$$\mathbf{s}\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} L(f) = L\left(\mathbf{s}\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} f\right) = L(\gamma(h)^{-1}(x, h) f(x))$$

$$= \gamma(h)^{-1} f(1) = \gamma(h)^{-1} L(f)$$

for every f in V_2 and h in $F^\times$; this implies (iii) by virtue of (i).

Remark. Lemma 7 implies that (Θ_U, V_U) is a multiple of $\Theta_2 \otimes |det|^{1/4}$. To show that this multiple is one, we use in the proof above the Composition Theorem (4.4). Alternatively, this can be proven on comparing the exact value of the character of Θ_U with that of Θ_2 on the h which appear in (5). In the proof above this comparison is done only up to a scalar multiple.

5.2. Let V_0 be the kernel of the natural surjection of V on V_U. Put $P' = Stab_M(\psi_U)$. Then $V_0 = ind(V_{U,\psi_U} \otimes \psi_U; P, P'U)$ by [BZ1], Prop. 5.12(d), or [BZ2], (3.5). Note that

$$\delta_P\begin{pmatrix} g & * \\ 0 & b \end{pmatrix} = |(det\, g)/b^2| \quad (g \in \mathrm{GL}(2, F)) \quad \text{and} \quad \delta_{P'}\begin{bmatrix} a & & * \\ & b & \\ 0 & & b \end{bmatrix} = |a/b|.$$

In particular

$$\delta_P = \delta_{P'} \quad \text{on} \quad \begin{bmatrix} a & & * \\ & b & \\ 0 & & b \end{bmatrix}.$$

Hence

$$V_0 = \delta_P^{1/2} \otimes ind(V_1; P, P'U), \qquad \text{where } V_1 = \delta_{P'}^{1/2} \otimes (V_{U,\psi_u} \otimes \psi_U). \tag{6}$$

Proposition 2. (i) *The P-module V_0 is irreducible.*
(ii) *The $P'U$-module V_1 is one-dimensional and unitary.*

Proof. (i) It suffices to prove that V_{U,ψ_U} is one-dimensional, for then it is irreducible and the proposition follows from Mackey's theorem (4.2(i)) and (6). To prove the one-dimensionality, note that $V_{N,\psi_N} = 0$, where $\psi_N(n) = \psi(n_{1,2} + n_{2,3})$, by Lemma 6(ii). Hence U^+ acts trivially on V_{U,ψ_U}, and so $V_{U,\psi_U} = V_{N,\psi_U}$. By the transitivity property of the functor of coinvariants, we have $V_{N,\psi_U} = (V_{U^+})_{N^+,\psi_U}$, where $N^+ = N/U^+$. By Proposition 1(ii), V_{U^+} is the Weil representation of G_2 (up to a twist). Hence Lemma 6(i) implies that $dim\ V_{N,\psi_U} = 1$, as required.

(ii) The one-dimensionality is proven in (i). Since N acts on V_1 via ψ_U, it suffices to show that the element

$$\mathbf{s}\begin{bmatrix} a & & 0 \\ & b & \\ 0 & & b \end{bmatrix}$$

acts on V_1 as multiplication by $\gamma(b)$. By Proposition 1(iii),

$$s = \mathbf{s}\begin{bmatrix} 1 & & 0 \\ & b/a & \\ 0 & & b/a \end{bmatrix}$$

acts on $V_{U,\psi_U} = (V_{U^+})_{N^+,\psi_U}$, as $|a/b|^{1/2}\gamma(b/a)$. Since $\delta_{P'}^{1/2}(s) = |a/b|^{1/2}$, and the central character of Θ is γ, the claim follows from

$$\mathbf{s}\begin{bmatrix} a & & 0 \\ & b & \\ 0 & & b \end{bmatrix} = \mathbf{s}\begin{bmatrix} a & & 0 \\ & a & \\ 0 & & a \end{bmatrix} \mathbf{s}\begin{bmatrix} 1 & & 0 \\ & b/a & \\ 0 & & b/a \end{bmatrix} \cdot (a, b/a).$$

5.3. Let $V' = V'(P)$ be the P-module defined in (4.1) using the P-module V, and V_0' the P-module obtained from V_0. Mackey's theorem (4.2(iv)) implies that $ind(V_1)' = Ind((\Delta_P/\Delta_{P'U})V_1')$. By Proposition 5.2(ii) we have $V_1' = V_1$. Since $\Delta_P/\Delta_{P'U} = \Delta_{P'}^{-1} = \delta_{P'} = \delta_P$ on P', we have $ind(V_1)' = \delta_P \otimes Ind(V_1)$. Hence

$$V_0' = \delta_P^{1/2} \otimes Ind(V_1; P, P'U). \tag{7}$$

As noted in (4.1), the unitary structure of the P-module (Θ, V) yields the following sequence of P-module morphisms:

$$V_0 \to V \to V' \to V_0'.$$

Denote by φ the composite morphism from V to V_0'.

Proposition 3. (i) *The map φ is an injection.*

(ii) *We have* $\dim Hom_P(V_0, V_0') = 1$. *In particular, the restriction of φ to V_0 is a multiple of the natural inclusion* $\delta_P^{1/2} \otimes ind(V_1) \hookrightarrow \delta_P^{1/2} \otimes Ind(V_1)$.

Proof. (i) The subspace $\ker\varphi$ is U-invariant since it is the orthogonal complement of V_0, and V_0 is spanned by the vectors $v - \Theta(u)v$, v in V, u in U. Hence the claim follows from

Theorem (Howe–Moore [HM], Prop. 5.5, p. 85). *Let G be a covering group of a simple reductive group, and V a non-trivial irreducible unitarizable G-module. Then no one-parameter subgroup of G fixes a non-zero vector in V.*

(ii) By (7) and Frobenius reciprocity (see (4.3)), we have

$$Hom_P(V_0, V_0') = Hom_{P'}((V_0)_{U,\psi_U}, \delta_{P'}^{1/2} \otimes V_1).$$

Since the functor of coinvariants is exact we have $(V_0)_{U,\psi_U} = V_{U,\psi_U}$. Note that $\delta_{P'}^{1/2} \otimes V_1 = V_{U,\psi_U}$. Hence $Hom_P(V_0, V_0') = \mathbf{C}$ and $\varphi: V_0 \to V_0'$ is a multiple of the natural inclusion.

Proposition 4. (i) *The P-module V_0' is isomorphic to the space of genuine functions on X smooth with respect to the action of P defined by* (1), (2), (3) *in* §2.

(ii) *The P-module V_0 can be realized by* (1), (2), (3) *on the space of smooth, genuine, compactly-supported functions f on X.*

Proof. This follows at once from (6) and (7) and the isomorphism of $X = \mathrm{s}(\Gamma) \setminus G_2$ with $P'U \setminus P$.

6. Restriction to B

It remains to determine V as a subspace of V_0', and to extend the action of P to an action of $G^* = G \rtimes \langle\sigma\rangle$ on V.

Since

$$P = B \cup P'U\boldsymbol{\alpha}B \quad \text{and } \boldsymbol{\alpha} = \mathbf{s}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that the action of B on $X = P'U \backslash P$ has two orbits, $Y = \{z \text{ in } X; x \neq 0\}$, and $X - Y = \{z \text{ in } X; x = 0\}$. Let W be the space of smooth, genuine, compactly-supported, complex-valued functions on Y. It is a B-submodule of V_0. In fact W is an irreducible B-module, by Mackey's theorem (4.2(i)), since

$$W = \delta_P^{1/2} \otimes ind(V_1^{\boldsymbol{\alpha}}; B, \boldsymbol{\alpha} \cdot P'U \cdot \boldsymbol{\alpha}^{-1} \cap B)$$

and V_1 is irreducible (see Proposition 2(ii)).

Let $W' = W'(B)$ be the Hermitian dual (4.1) of the B-module W. By Mackey's theorem (4.2(iv)), W' is the space of genuine functions on Y smooth under the action of B defined by (1), (2), (3); in particular, the support of any f in W' is bounded in the y-direction. We have the following inclusions of B-modules:

$$W \subset V_0 \subset V \subset V_0' = V_0'(P) \subset W' = W'(B).$$

Fix a square root $\gamma(-1)^{1/2}$ of $\gamma(-1)$. For any f in W' define $Jf(x, y; \zeta)$ by the integral

$$\gamma(-1)^{1/2}\gamma(x)^{-1}|x|^{-1/2} \int_F f(-x, u; \zeta)\psi(uy/x)du. \tag{8}$$

It is clear that this integral converges, that $J^2 = \mathrm{Id}$, and that $f \rightarrow Jf$ maps W to W and W' to W'.

As noted in (1.3), since Θ is σ-invariant there is an isomorphism $I: V \rightarrow V$ such that $I\Theta(g) = \Theta(\sigma g)I$ and $I^2 = \mathrm{Id}$. It is unique up to a sign. We claim that I is given on V by the integral (8). More precisely, we have

Proposition 5. (i) *The operator J maps V to V.* (ii) *There is a choice of $I: V \rightarrow V$ such that the restriction $J \mid V$ of J to V is equal to I.*

Proof. The B-module W' consists of functions on $Y = \{z \in X; x \neq 0\}$. The subgroup $N_{1,3} = U \cap U^+$ of N acts on W' according to (2). Hence the only vector in W' fixed by $N_{1,3}$ is the zero vector. On the other hand, for every u in F, we have that $\psi(ux)$ is 1 for a sufficiently small $|x|$. Hence $f \in W'$ and

$$\Theta\begin{bmatrix} 1 & & u \\ & 1 & \\ 0 & & 1 \end{bmatrix} f$$

are equal on a sufficiently small neighborhood of $X - Y = \{z \in X; x = 0\}$. Consequently

$$\Theta\begin{bmatrix} 1 & & u \\ & 1 & \\ 0 & & 1 \end{bmatrix} f - f \in W.$$

We conclude that $N_{1,3}$ acts trivially on W'/W. In particular, since $(W \subset) V \subset W'$, we have

$$Hom_B(V/W, W') = 0, \qquad Hom_B((V/W)', W') = 0.$$

Since for any H-modules A, B we have $Hom_H(A, B) \hookrightarrow Hom_H(B', A')$, we also have that the submodule $Hom_B(W, V/W)$ of the zero-module $Hom_B((V/W)', W')$ is zero.

It follows that I maps W to W. Indeed, had this been false, the map I would induce a non-trivial map $W \to V/W$, contradicting the fact that $Hom_B(V/W, W') = 0$.

We claim that the restrictions $I \mid W$ and $J \mid W$ of I and J to W coincide. We have $(I \mid W)^2 = \mathrm{Id}$, and $(I \mid W)\Theta(b) = \Theta(\sigma b)(I \mid W)$ for all $b \in B$. By (1.3) we have

$$\tilde{\sigma}\left[\mathbf{s}\begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \end{bmatrix}\right] = \mathbf{s}\begin{bmatrix} c^{-1} & & 0 \\ & b^{-1} & \\ 0 & & a^{-1} \end{bmatrix} \cdot (a, bc)(b, c),$$

$$\text{and} \quad \sigma(g) = (-1, det\ p(g))\tilde{\sigma}(g) \quad (g \in G).$$

Consequently, it is easy to check that $(J \mid W)\Theta(b) = \Theta(\sigma b)(J \mid W)$ for all b in B, and that $J^2 = \mathrm{Id}$. Since W is an irreducible B-module, we have $I \mid W = J \mid W$, up to a sign. Hence we can choose I such that $I \mid W = J \mid W$, as claimed.

It now follows that $J \mid V - I$ defines a morphism $V/W \to W'$, necessarily zero since $Hom_B(W, V/W) = 0$, and the proposition follows.

Finally we prove the

Theorem. (i) *The space V is isomorphic to $C_b(X)^0$. The G^*-module (Θ, V) is equivalent to the G^*-module defined by the operators* (1)–(4) *on the space $C_b(X)^0$.*

(ii) *There is a unique (up to scalar) Hermitian scalar product on the unitarizable G-module $(\Theta, C_b(X)^0)$. It is given by the L^2-product.*

Proof. (i) The space V is realized in Proposition 3(i) as a subspace of V_0'. Moreover, we have the inclusions $V_0 \hookrightarrow V \hookrightarrow V_0'$. By Proposition 4(i), V_0' is the space of genuine, smooth, complex-valued functions with bounded support on X. The subspace V_0 of V consists, by Proposition 4(ii), of the compactly-supported f in V_0'. By definition (in (5.2)) of V_0 as $ker(V \to V_U)$, the space V consists of the f in V_0' such that $\bar{f} = f \bmod V_0$ lies in V_U. Proposition 1(i) asserts that $V_U \cong \Theta_2 \otimes |det|^{1/4}$. In particular, for every f in V and t in $F^\times$, the vector

$$|t|^{-1}\Theta\left(\mathbf{s}\begin{bmatrix} t^2 & & 0 \\ & t^2 & \\ 0 & 0 & 1 \end{bmatrix}\right)\bar{f} - |t|^{-1}\bar{f}$$

is zero in $V/V_0 \cong \Theta_2 |det|^{1/4}$.

Hence for every f in V there is $A_f > 0$, and c $(0 < c < \frac{1}{2})$, such that $|t| f(t^2x, t^2y; \zeta) = f(x, y; \zeta)$ for $max(|x|, |y|) \leqq A_f$ and $c \leqq |t| \leqq 1$ (note that this domain of t is compact, and f is locally constant). But then this relation holds for all t with $0 < |t| \leqq 1$. Define f_0 on X by $f_0(x, y; \zeta) = |t| f(t^2x, t^2y; \zeta)$ for t such that $|t|^2 max(|x|, |y|) \leqq A_f$. Then f_0 lies in $C_h(X)$.

We conclude so far that, for every f in V, there is f_0 in $C_h(X)$ and $A_f > 0$ such that $f(x, y; \zeta) = f_0(x, y; \zeta)$ for $max(|x|, |y|) \leqq A_f$. Proposition 1(i) then implies that the function f_0 lies in the unique irreducible G_2-submodule $C_h(X)^0$ $(\cong \Theta_2 \otimes |det|^{1/4})$ of $C_h(X)$. This determines the space V of Θ to be $C_b(X)^0$, as asserted. The action of P is described by Proposition 4(i), and that of σ by Proposition 5. Since P and σ generate G^*, (i) follows.

(ii) By Proposition 3(ii), we have $dim\, Hom_P(V_0, V_0') = 1$. Since $V' \hookrightarrow V_0'$, the space $Hom_P(V_0, V')$ is a subspace of $Hom_P(V_0, V_0')$, necessarily one-dimensional. Consider the map $Hom_P(V, V') \to Hom_P(V_0, V')$, obtained by restriction from V to V_0. Its kernel is $Hom_P(V/V_0, V')$. Now $V/V_0 \cong V_U$, and U acts trivially on V_U. On the other hand, the only vector in W', and in particular in V' $(\subset W')$, which is fixed by U, is the zero vector. Hence $Hom_P(V, V')$ injects in $Hom_P(V_0, V')$, and it is one-dimensional. The L^2-product on V yields a P-invariant Hermitian form on V, hence a non-zero P-module morphism $i: V \to V'$. The unitary structure on V yields a non-zero morphism $j: V \to V'$ of G-modules. In particular j is a P-module morphism. Since $dim\, Hom_P(V, V') = 1$, the morphism j is a multiple of the morphism i, as required.

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(Received October 31, 1988)