COMPUTATION OF A TWISTED CHARACTER OF A SMALL REPRESENTATION OF GL(3, E)

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Summary. Let E/F be a quadratic extension of p-adic fields, $p \neq 2$. Let $x \mapsto \overline{x}$ be the involution of E over F. The representation π of $\mathrm{GL}(3, E)$ normalizedly induced from the trivial representation of the maximal parabolic subgroup is invariant under the involution $\sigma(g) = J^t\overline{g}^{-1}J$. We compute – by purely local means – the σ-twisted character χ_π^σ of π. We show that it is σ-unstable, namely its value at one σ-regular-elliptic conjugacy class within a stable such class is equal to negative its value at the other such conjugacy class within the stable class, or zero when the σ-regular-elliptic stable conjugacy class consists of a single such conjugacy class. Further, we relate this twisted character to the twisted endoscopic lifting from the trivial representation of the "unstable" twisted endoscopic group $\mathrm{U}(2,E/F)$ of $\mathrm{GL}(3, E)$. In particular π is σ-elliptic, that is, χ_π^σ is not identically zero on the σ-elliptic set.

1. Introduction

Let F be a finite field extension of $\mathbb{Q}_p$, $\mathbf{G}'$ a reductive connected linear algebraic group over F, and σ an involution of $\mathbf{G}'$ over F. Put $G' = \mathbf{G}'(F)$ for the group of F-rational points on $\mathbf{G}'$. Denote by σ also the induced involution on G'. Let π be an admissible ([BZ], [B]) irreducible representation of G' in a complex vector space. It is called *σ-invariant* if it is equivalent to the representation ${}^\sigma\pi$, defined by ${}^\sigma\pi(g) = \pi(\sigma g)$, $g \in G'$. In this case there is an intertwining operator A on the space of π with $\pi(g)A = A\pi(\sigma g)$ for all $g \in G'$. Since $\sigma^2 = 1$ we have $\pi(g)A^2 = A^2\pi(g)$ for all $g \in G'$. As π is irreducible A^2 is a scalar by Schur's lemma ([BZ]). Multiplying by a scalar we may choose A with $A^2 = 1$. This determines A up to a sign. Extend π to $G' \rtimes \langle\sigma\rangle$ by $\pi(\sigma) = A$.

Let K' be a maximal compact subgroup of G' with $\sigma K' = K'$. A representation π is called *unramified* if the space of π contains a nonzero K'-fixed vector. The dimension of the space of K'-fixed vectors is bounded by one if π is irreducible. If π is σ-invariant, irreducible and unramified, and $v_0 \neq 0$ is a K'-fixed vector in the space of π, then Av_0 is also K'-fixed, hence is a multiple of v_0. Thus $Av_0 = cv_0$, with $c = \pm 1$. Replace A by cA to have $Av_0 = v_0$.

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Since π is irreducible, by Schur's lemma there is a character ω' of the center Z' of G' with $\pi(zg) = \omega'(z)\pi(g)$ for all $z \in Z'$, $g \in G'$. Fix a Haar measure dg' on G'/Z' ([BZ]). Denote by $\mathbb{H}'$ the convolution algebra of complex valued locally constant measures $\phi dg'$ on G' which are compactly supported on G' modulo Z' and which transform under Z' by ω'^{-1}.

For any admissible representation π and any $\phi dg' \in \mathbb{H}'$ the convolution operator $\pi(\phi dg')$ $= \int_{G'/Z'} \phi(g)\pi(g)dg'$ has finite rank. If π is σ-invariant denote by $\operatorname{tr}\pi(\phi dg' \times \sigma)$ the trace of

$$\pi(\phi dg' \times \sigma) = \pi(\phi dg') \times \pi(\sigma) = \int_{G'/Z'} \phi(g')\pi(g')\pi(\sigma)dg'.$$

The distribution $\phi \mapsto \operatorname{tr}\pi(\phi dg' \times \sigma)$ is represented by a function χ_π^σ, named the *σ-twisted character* of π. Thus χ_π^σ is a locally constant complex valued function on the σ-regular set of G' (where $g\sigma(g)$ is regular) which transforms under the center Z' via ω' and is σ-conjugacy invariant (thus $\chi_\pi^\sigma(h^{-1}g\sigma(h)) = \chi_\pi^\sigma(g)$ for all $h \in G'$), which satisfies $\operatorname{tr}(\phi dg' \times \sigma) = \int_{G'/Z'} \chi_\pi^\sigma(g')\phi(g')dg'$ for all $\phi \in \mathbb{H}'$ supported on the σ-regular set (in fact for all $\phi \in \mathbb{H}'$ as χ_π^σ is locally integrable on G', see [H], [K], [C], hence it is uniquely determined by its restriction to the σ-regular set).

It is a basic question in representation theory to compute χ_π^σ. Methods to carry out such computations are scarce. The question is related to the theory of liftings, and can be studied by global techniques of comparison of trace formulae. We study here one particular interesting example by a purely local technique, based on the usage of a convenient model for the representation in question. This extends a technique first introduced in [FK] and then extended by us in [FZ1], [FZ2], [FZ3] in more involved cases, the case considered in this paper being the most advanced of them all. It will be interesting to develop a local proof of our theorems to cover the archimedean case as well. Naturally our purely local approach is useful to compare with the global approach (of, e.g., [F]).

Our example concerns $G' = \mathrm{GL}(3,E)$, and the involution $\sigma(g) = J^t\overline{g}^{-1}J$, where E/F is a quadratic extension of p-adic fields, $x \mapsto \overline{x}$ denotes the nontrivial automorphism of E over F, $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$, and $\overline{g} = (\overline{g}_{ij})$ if $g = (g_{ij})$. Then $G' = \mathbf{G}'(F)$ where $\mathbf{G}'$ is the F-group $\mathrm{R}_{E/F}\,\mathbf{G}$ obtained from the unitary group $\mathbf{G} = \mathbf{U}(3, E/F)$ (regarded as an E-group) of the form J, upon restricting scalars from E to F. Thus $G = \{g \in G'; \sigma(g) = g\}$. Note that ${}^\sigma\pi \simeq \pi$ means that $\overline{\pi} \simeq \check{\pi}$, where $\check{\pi}$ is the contragredient ([BZ]) of π and $\overline{\pi}(g) = \pi(\overline{g})$.

A Levi subgroup M' of a maximal parabolic subgroup P' of $\mathrm{GL}(3,E)$ is isomorphic to $\mathrm{GL}(2,E) \times E^\times$. A representation π_1 of $\mathrm{GL}(2,E)$ extends to P' to have a trivial central character and be 1 on the unipotent radical N' of P'. Let δ denote the character of P' which is trivial on N' and whose value at $p = mn$ is $|\det(a^{-1}h)|$ if m corresponds to (h,a) in $\mathrm{GL}(2,E) \times E^\times$. Explicitly, if P' is the upper triangular parabolic subgroup of type (2,1), and m in M' is represented in $\mathrm{GL}(3,E)$ by $\begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$, then $\delta(m) = |(\det m')/m''^2|$ (m' lies in $\mathrm{GL}(2,E)$, m'' in $\mathrm{GL}(1,E)$). Denote by $I(\pi_1) = \mathrm{Ind}(\delta^{1/2}\pi_1; P', G')$ the representation of G' normalizedly induced from π_1 on P' to G'. It is clear from [BZ] that when $I(\pi_1)$ is irreducible it is σ-invariant iff $\overline{\pi}_1 \simeq \check{\pi}_1$, and $I(\pi_1)$ is unramified if and only if π_1 is unramified. Here $K'_{M'} = \mathrm{GL}(2, R_E)$ and $K' = \mathrm{PGL}(3, R_E)$, where R_E denotes the ring of integers of E.

We regard the twisted character χ_π^σ as a function on the set of regular σ-conjugacy classes in G'. These are naturally packed in stable σ-conjugacy classes. We explain the classification of these later, but note here that the norm map, defined using $g \mapsto g\sigma(g)$, defines a bijection from the set of regular stable σ-conjugacy classes in G' to the set of regular stable conjugacy classes in $G = \mathrm{U}(3, E/F) = \{g \in G'; \sigma(g) = g\}$. The centralizers of such regular classes in G are tori of 4 types. Put $E^1 = \{x \in E^\times; x\overline{x} = 1\}$.
(0) The diagonal torus (up to conjugacy) $T^* = \{\mathrm{diag}(a, b, \overline{a}^{-1}); a \in E^\times, b \in E^1\}$.
(1) Elliptic torus $\simeq (E^1)^3$.
(2) Elliptic torus which splits over the biquadratic extension EL of F. It is isomorphic to $(EL/K)^1 \times E^1$, where $(EL/K)^1$ is the kernel of the norm from $(EL)^\times$ to $K^\times$, where K is the quadratic extension of F other than E, L.
(3) Elliptic torus which splits over a cubic extension of E.

Let h be an element of the subgroup $\tilde{H} = \{(\alpha_{ij}); \alpha_{ij} = 0 \text{ if } i + j \text{ is odd}\} \simeq H \times E^1$ of $G = \mathrm{U}(3, E/F)$. Its eigenvalues are α, $\beta = \alpha_{22}$, γ. We view $H = \mathrm{U}(2, E/F)$, the unitary group of the form $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, as the subgroup $H \times 1$ ($\alpha_{22} = 1$) of G. Then $\tilde{H} = HZ$, where $Z \simeq E^1$ is the center of G. An element, conjugacy class, or stable conjugacy class in $\tilde{H}$ defines one in G. But note that the 3 distinct stable conjugacy classes in $\tilde{H}$ with eigenvalues α, β, γ in E^1 define the same stable conjugacy class in G.

Fix a character κ of $E^\times$ which is trivial on $NE^\times$, but nontrivial on $F^\times$. Put $\kappa_1(h) = \kappa(-(1 - \alpha/\beta)(1 - \gamma/\beta))$. If $h = \mathrm{diag}(\alpha, \beta, \gamma)$ then $\gamma = \overline{\alpha}^{-1}$, and $\kappa_1(h) = \kappa(\alpha/\beta)$. A σ-conjugacy class of type (0), (1) or (2) has a representative $g = (g_1, b)$ in $Z_{G'}(\mathrm{diag}(1, -1, 1)) \simeq \mathrm{GL}(2, E) \times E^\times$, which is unique up to σ-conjugacy in $Z_{G'}(\mathrm{diag}(1, -1, 1))$. Put $\det_1 g$ for $\det(g_1/b)$. Then $\kappa(\det_1 g) = \kappa(\det_1(h' g \sigma(h')^{-1}))$ ($h' \in Z_{G'}(\mathrm{diag}(1, -1, 1))$) is well-defined. Note that $|x|_E = |Nx|_F = |x|_F^2$ defines $|x|_F$ for x in E. Here N is $N_{E/F}$, the norm from E to F. Then $|\varepsilon|_F^2$ is $|N\varepsilon|_F$. Write $\mathrm{Ad}(u)X = uXu^{-1}$. For $g \in G$, $h \in \tilde{H}$, put

$$\Delta(g) = |1 - \det(\mathrm{Ad}(g))|\,\mathrm{Lie}(G/Z_G(g))|_F^{1/2}, \qquad \Delta_1(h) = |1 - \det(\mathrm{Ad}(h))|\,\mathrm{Lie}(H/Z_H(h))|_F^{1/2}.$$

Then $\Delta(g) = |(\alpha - \beta)^2(\gamma - \beta)^2(\alpha - \gamma)^2/(\alpha\beta\gamma)|_F^{1/2}$ if the eigenvalues of g are α, β, γ, and $\Delta_1(h) = |(\alpha - \gamma)^2/(\alpha\gamma)|_F^{1/2}$, as we may work with G over the algebraic closure.

Our result is the following. Suppose that E is unramified over F and $p \neq 2$ (throughout this paper, unless otherwise specified).

Theorem 1. *If π_1 is the trivial representation of $\mathrm{GL}(2, E)$, $\pi = I(\pi_1)$, and g is an element of $\mathrm{GL}(3, E)$ with elliptic regular norm Ng of type (1) or (2), then $\Delta(Ng)\chi_\pi^\sigma(g) = \sum_w \Delta_1(Ng^w)\kappa_1(Ng^w)/\kappa(\det_1 g^w)$. The sum is trivial except in case (1), where it ranges over the Weyl group S_3 of permutations of the eigenvalues of g, quotient $(S_2\backslash S_3)$ by the analogous Weyl group S_2 of $\mathrm{GL}(2, E)$ where g is viewed in $Z_{G'}(\mathrm{diag}(1, -1, 1))$. If g is an element of $\mathrm{GL}(3, E)$ of type (3) then $\chi_\pi^\sigma(g) = 0$.*

In this paper we deal only with case (1), as our main interest is in showing that the transfer factor is $\kappa_1(Ng)/\kappa(\det_1 g)$ (and the corresponding representation of $H = \mathrm{U}(2, E/F)$ is the trivial representation), so that it can be compared with the transfer factors formulated by Langlands, Kottwitz-Shelstad, Waldspurger, in general settings. We checked the Theorem in a case of case (2), but a full treatment of case (2) would require another paper.

The Theorem was found by global techniques, using the trace formula, as a case of the unstable twisted endoscopic lifting from $H = \mathrm{U}(2, E/F)$ to $G' = \mathrm{GL}(3, E)$, see [F], pp. 207-208 (note that in [F], p. 208, line 25, e is meant to be $b \circ e$) which is the counterpart of the endoscopic lifting from $H = \mathrm{U}(2, E/F)$ to $G = \mathrm{U}(3, E/F)$ and the basechange lifting from G to $G' = \mathrm{GL}(3, E)$; see [F], Prop. II.5.1.1, p. 340. The proof in [F] uses the Weyl integration formula and its twisted form. Our work provides an alternative, purely local and elementary, verification of this highly complicated relation. The works [FK], [FZ1] deal with an analogous computation related to (the unstable counterpart of) the symmetric square lifting from SL(2) to PGL(3), while [FZ2], [FZ3] deal with a computation related to the lifting from the (semisimple) rank two symplectic group GSp(2), to GL(4).

Although the representation π of the Theorem is (properly) parabolically induced ([BZ], [B]), we study in the Theorem the character of its extension to $\mathrm{GL}(3, E) \rtimes \langle\sigma\rangle$. This extension is not induced from any proper σ-invariant parabolic subgroup of $\mathrm{GL}(3, E)$. Hence its twisted character is nonzero on the regular σ-elliptic set. Elliptic representations (ones whose character is nonzero on the regular elliptic set) of a connected reductive p-adic group include the square integrable representations, in particular the cuspidal representations. Cuspidal representations of connected reductive groups can be constructed by compact induction (see a recent survey by [Kim], with references to previous works by Bushnell-Kutzko, Moy-Prasad, Yu and others). This construction can be used to compute their characters. It will be interesting to extend this theory to nonconnected groups, as in our example. Computations of twisted characters are useful to compare with character identities obtained by global techniques, as in [F]. Our example of an unramified nontempered representation is of course another extreme in a direction away from cuspidal representations.

For transfer to local fields of positive characteristic see [K1].

To compute the character of π we shall express π as an integral operator in a convenient model, and integrate the kernel over the diagonal. Denote by μ_s the character $\mu_s(x) = |x|_E^{(s+1)/2}$ of $E^\times$. It defines a character $\mu_{s,P}$ of P', trivial on N', by $\mu_{s,P}(p) = \mu_s((\det m')/m''^2)$ if $p = mn$ and $m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$ with m' in $\mathrm{GL}(2, E)$, m'' in $\mathrm{GL}(1, E)$. If $s = 0$, then $\mu_{s,P} = \delta^{1/2}$. Let W_s be the space of complex-valued smooth functions ψ on G' with $\psi(pg) = \mu_{s,P}(p)\psi(g)$ for all p in P' and g in G'. The group G' acts on W_s by right translation: $(\pi_s(g)\psi)(h) = \psi(hg)$. By definition, $I(\pi_1)$ is the G'-module W_s with $s = 0$. The parameter s is introduced for purposes of analytic continuation.

We prefer to work in another model V_s of the G'-module W_s. Let V denote the space of column 3-vectors over E. Let V_s be the space of smooth complex-valued functions ϕ on $V - \{0\}$ with $\phi(\lambda v) = \mu_s(\lambda)^{-3}\phi(v)$. The expression $\mu_s(\det g)\phi({}^t gv)$, which is initially defined for g in $\mathrm{GL}(3, E)$, depends only on the image of g in $\mathrm{PGL}(3, E)$. The group $\mathrm{PGL}(3, E)$ acts on V_s by $(\tau_s(g)\phi)(v) = \mu_s(\det g)\phi({}^t gv)$. Let $v_0 \neq 0$ be a vector of V such that the line $\{\lambda v_0; \lambda \text{ in } E\}$ is fixed under the action of ${}^tP'$. Explicitly, we take $v_0 = {}^t(0, 0, 1)$. It is clear that the map $V_s \to W_s$, $\phi \mapsto \psi = \psi_\phi$, where $\psi(g) = (\tau_s(g)\phi)(v_0) = \mu_s(\det g)\phi({}^t gv_0)$, is a PGL$(3,E)$-module isomorphism, with inverse $\psi \mapsto \phi = \phi_\psi$, $\phi(v) = \mu_s(\det g)^{-1}\psi(g)$ if $v = {}^t g v_0$ (PGL$(3,E)$ acts transitively on $V - \{0\}$).

In this section (only), $|.|$ denotes $|.|_E$. For $v = {}^t(x,y,z)$ in V put $\|v\| = \max(|x|,|y|,|z|)$. Let V^0 be the quotient of the set of v in V with $\|v\| = 1$ by the equivalence relation $v \sim \alpha v$ if α is a unit in R_E. Denote by $\mathbb{P}V$ the projective space of lines in $V - \{0\}$. If Φ is a function on $V - \{0\}$ with $\Phi(\lambda v) = |\lambda|^{-3}\Phi(v)$ and $dv = dx\ dy\ dz$, then $\Phi(v)dv$ is homogeneous of degree zero. Define

$$\int_{\mathbb{P}V} \Phi(v)dv \quad \text{to be} \quad \int_{V^0} \Phi(v)dv.$$

Clearly we have

$$\int_{\mathbb{P}V} \Phi(v)dv = \int_{\mathbb{P}V} \Phi(gv)d(gv) = |\det g| \int_{\mathbb{P}V} \Phi(gv)dv.$$

Put $\nu(x) = |x|$ and $m = 3(s-1)/2$. Note that $\nu/\mu_s = \mu_{-s}$. Put $\langle v,w\rangle = {}^t v J \overline{w}$. Then $\langle gv, \sigma(g)w\rangle = \langle v,w\rangle$.

Lemma 1.1. *The operator $T_s : V_s \to V_{-s}$,*

$$(T_s\phi)(v) = \int_{\mathbb{P}V} \phi(w)|\langle w,v\rangle|^m dw,$$

converges when $\operatorname{Re} s > 1/3$ *and satisfies $T_s\tau_s(g) = \tau_{-s}(\sigma g)T_s$ for all g in* PGL$(3,E)$ *where it converges.*

Proof. We have

$$\begin{aligned}
(T_s(\tau_s(g)\phi))(v) &= \textstyle\int (\tau_s(g)\phi)(w)|{}^t wJ\overline{v}|^m dw = \mu_s(\det g) \int \phi({}^t gw)|{}^t wJ\overline{v}|^m dw \\
&= |\det g|^{-1}\mu_s(\det g) \textstyle\int \phi(w)|{}^t({}^t g^{-1}w)J\overline{v}|^m dw \\
&= (\mu_s/\nu)(\det g) \textstyle\int \phi(w)|{}^t wJ \cdot Jg^{-1}J\overline{v}|^m dw \\
&= (\mu_s/\nu)(\det g) \textstyle\int \phi(w)|\langle w, \sigma({}^t g)v\rangle|^m dw = (\nu/\mu_s)(\det \sigma g)\cdot (T_s\phi)(\sigma({}^t g)v) \\
&= [(\tau_{-s}(\sigma g))(T_s\phi)](v).
\end{aligned}$$

Further, $\int_{|w|\le 1} |w|^m dw = (1-q^{-1})\int_{|w|\le 1}|w|^{m+1}d^\times w = (1-q^{-1})\sum_{n\ge 0} q^{-(m+1)n}$ converges iff $\operatorname{Re}(m)+1>0$. □

The spaces V_s are isomorphic to the space W of locally-constant complex-valued functions on V^0, and T_s is equivalent to an operator T_s^0 on W. The proof of Lemma 1.1 implies also

Corollary 1.2. *The operator $T_s^0 \circ \tau_s(g^{-1})$ is an integral operator with kernel*

$$(\mu_s/\nu)(\det \sigma g)|\langle w, \sigma({}^t g^{-1})v\rangle|^m \qquad (v, w \ \textit{in}\ V^0)$$

and trace

$$\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})] = (\nu/\mu_s)(\det g) \int_{V^0} |{}^t v g J \overline{v}|^m dv.$$

Remark. (1) In the domain where the integral converges, it is clear that $\mathrm{tr}[T_s^0 \circ \tau_s(g^{-1})]$ depends only on the σ-conjugacy class of g if (and only if) $s = 0$. (2) We evaluate below this integral at $s = 0$ in a case where it converges for all s, and no analytic difficulties occur. However, we claim that to compute the trace of the analytic continuation of $T_s^0 \circ \tau_s(g^{-1})$ it suffices to compute this trace for s in the domain of convergence, and then evaluate the resulting expression at the desired s. Indeed, for each compact open σ-invariant subgroup K of $\mathrm{PGL}(3, E)$ the space W_K of K-biinvariant functions on W is finite dimensional. Denote by $p_K : W \to W_K$ the natural projection. Then $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ acts on W_K, and the trace of the analytic continuation of $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ is the analytic continuation of the trace of $p_K \circ T_s^0 \circ \tau_s(g^{-1})$. Since K can be taken to be arbitrarily small the claim follows.

Next we normalize the operator T_s so that it acts trivially on the one-dimensional space of K'-fixed vectors in V_s. This space is spanned by the function ϕ_0 in V_s with $\phi_0(v) = 1$ for all v in V^0. Fix a local uniformizer $\boldsymbol{\pi}_E$ in R_E. Let $q = q_E$ be the cardinality of the quotient field of R_E. Normalize the valuation $|\cdot|$ by $|\boldsymbol{\pi}_E| = q^{-1}$. Normalize the measure dx by $\int_{|x|\le 1} dx = 1$, so that $\int_{|x|=1} dx = 1 - q^{-1}$. In particular, the volume of V^0 is

$$(1 - q^{-3})/(1 - q^{-1}) = 1 + q^{-1} + q^{-2}.$$

Lemma 1.3. *We have* $(T_s\phi_0)(v_0) = (1 - q^{-3(s+1)/2})(1 - q^{(1-3s/2})^{-1}\phi_0(v_0)$.

When $s = 0$ *the constant is* $-q^{-1/2}(1 + q^{-1/2} + q^{-1}) = -q^{-3/2}(1 + q^{1/2} + q)$.

Proof. We have

$$\int \phi_0(v)|{}^t vJ\overline{v}_0|^m dv = \int_{V^0} |x|^m dx\ dy\ dz = (1 - q^{-3(s+1)/2}) \int_{|x|\le 1} |x|^m dx / \int_{|x|=1} dx,$$

as required. □

To prove the theorem we have to compute $\mathrm{tr}[T \circ \tau_s(g^{-1})]$ where $T = -\frac{q^{3/2}}{1+q^{1/2}+q} T_s^0$.

For this purpose, in section 2, we describe the σ-stable conjugacy classes and find explicit representatives for the σ-conjugacy classes within the stable classes. We begin with a description of conjugacy and stable conjugacy, after which will come the study of the twisted analogue. In section 3, we state the main theorem of this paper. In particular we introduce the parameters n_1 and n_2, and observe that it suffices to deal with three cases, where (n_1, n_2) is (0,0), (1,0), and (1,1). The proof requires numerous and lengthy computations that took us many years to complete. To improve the readability and the clarity of this exposition, we put the computations later in the paper. They are put into the corresponding Appendices. Thus, in section 4, we deal with the case $n_1 = 0$, $n_2 = 0$, postponing the needed computations to appendix B. In section 5, we deal with the case $n_1 = 1$, $n_2 = 0$, postponing the needed computations to appendix C. Finally, in section 6, we deal with the case $n_1 = 1$, $n_2 = 1$, postponing the needed computations to appendix D.

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2. Conjugacy classes

Let $\mathbf{G}$ be a connected reductive group defined over a local or global field F. Fix an algebraic closure $\overline{F}$. Denote by $\overline{G} = \mathbf{G}(\overline{F})$ the group of $\overline{F}$-points on the variety $\mathbf{G}$. Now $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{G}$. The group $\mathbf{G}(F)$ of fixed points is denoted by G. An *F-torus* $\mathbf{T}$ in $\mathbf{G}$ is a maximal F-subgroup $\overline{F}$-isomorphic to a power of $\mathbb{G}_m$. Its group T of F-points is also called a torus. An element t of G is *regular* if the centralizer $Z_{\mathbf{G}}(t)$ of t in $\mathbf{G}$ is a maximal F-torus $\mathbf{T}$ (it is often named "strongly regular"). The elements t, t' of G are *conjugate* if there is g in G with $t' = gtg^{-1}$. They are *stably conjugate* if there is such a g in $\overline{G}$. Tori T and T' are *stably conjugate* if there is g in $\overline{G}$ with $T' = gTg^{-1}$, so that the map $\mathrm{Int}(g) : \mathbf{T} \to \mathbf{T}'$, $\mathrm{Int}(g)(t) = gtg^{-1}$, is defined over F. Then $g_\tau = g^{-1}\tau(g)$ centralizes T for all τ in $\mathrm{Gal}(\overline{F}/F)$, hence lies in $\overline{T}$, since $\mathbf{G}$ is connected and reductive.

Of course the notion of stable conjugacy can be defined by $t' = g^{-1}tg$, which will lead to the definition of the cocycle as $g_\tau = g\tau(g^{-1})$. The change from g to g^{-1} should lead to no confusion, and we use both conventions.

We shall now *list all stable conjugacy classes of tori in* G. Let $\mathbf{T}^*$ be a fixed F-torus, $\mathbf{N}$ its normalizer in $\mathbf{G}$, and $\mathbf{W} = \mathbf{T}^*\backslash\mathbf{N} = \mathbf{N}/\mathbf{T}^*$ the absolute Weyl group. For each $\mathbf{T}$ there is g in $\mathbf{G}(\overline{F})$ with $\mathbf{T} = g\mathbf{T}^*g^{-1}$. Since $\mathbf{T}$ is defined over F, g_τ normalizes $\mathbf{T}^*$, and the cocycle $\tau \mapsto g_\tau$ defines a class in the first cohomology group $H^1(F, \mathbf{N})$ of $\mathrm{Gal}(\overline{F}/F)$ with coefficients in $\mathbf{N}(\overline{F})$. Denote by $\{g'_\tau\}$ the image of $\{g_\tau\}$ under the natural map $H^1(F, \mathbf{N}) \to H^1(F, \mathbf{W})$, obtained from $\mathbf{N} \to \mathbf{W}$.

The stable conjugacy classes are determined by means of the following.

Proposition 2.1. *The map $T \mapsto \{g'_\tau\}$ injects the set of stable conjugacy classes of tori in G into the image in $H^1(F, \mathbf{W})$ of* $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{G})]$. *This map is also surjective when $\mathbf{G}$ is quasisplit.*

Proof. If $\mathbf{T} = g\mathbf{T}^*g^{-1}$ and $\mathbf{T}'$ are stably conjugate, then there is x in $\overline{G}$ with $\mathbf{T}' = x\mathbf{T}x^{-1} = xg\mathbf{T}^*(xg)^{-1}$, and $(xg)_\tau = g^{-1}x_\tau g \cdot g_\tau$ has the image g'_τ in $H^1(F, \mathbf{W})$, since $g^{-1}x_\tau g$ lies in $\overline{T}^*$ (x_τ in $\overline{T}$). Hence the map of the proposition is welldefined.

Conversely, if $\mathbf{T} = g\mathbf{T}^*g^{-1}$, $\mathbf{T}' = g'\mathbf{T}^*g'^{-1}$, and $g_\tau = a(\tau)g'_\tau$ with $a(\tau)$ in $\overline{T}^*$, then $a(\tau) = g'^{-1}x(\tau)g'$ with $x(\tau)$ in $\overline{T}'$, and the map $t \mapsto gg'^{-1}t(gg'^{-1})^{-1}$ [t in $\overline{T}'$] is defined over F. Hence the map of the proposition is injective.

For the second claim, if $\{g_\tau\}$ lies in $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{G})]$, then it defines a new $\mathrm{Gal}(\overline{F}/F)$-action by $\widehat{\tau}(h) = g_\tau^{-1}\tau(h)g_\tau$ ($h = t^*$ in $\overline{T}^*$). If h is a fixed $\widehat{\tau}$-invariant regular element, then $\tau(h) = g_\tau h g_\tau^{-1}$, and the conjugacy class of h in $\overline{G}$ is defined over F. When $\mathbf{G}$ is quasisplit, a theorem of Steinberg and Kottwitz [Ko] implies the existence of h' in G which is conjugate to h in $\overline{G}$, since the field F is perfect. The centralizer of h' in G is a torus whose stable conjugacy class corresponds to $\{g_\tau\}$. Hence the map is surjective. □

Remark. Implicit in the proof is a description – used below – of the action of the Galois group on the torus. Let us make this explicit. All tori are conjugate in $\overline{G}$, thus $\overline{T} = g^{-1}\overline{T}^*g$ for some g in $\overline{G}$. For any t in $\overline{T}$ there is t^* in $\overline{T}^*$ with $t = g^{-1}t^*g$. For t in T, we have

$$\sigma g^{-1}\sigma t^*\sigma g = \sigma t = t = g^{-1}t^*g,$$

hence $\sigma t^* = g_\sigma^{-1} t^* g_\sigma \in \overline{T}^*$. Taking regular t (and t^*), $g_\sigma \in \overline{N}$ is uniquely determined modulo $\overline{T}^*$, namely in $\overline{W}$. For any t^* in $\overline{T}^*$ we then have

$$\sigma(g^{-1}t^*g) = g^{-1}(g\sigma(g^{-1}))\sigma(t^*)(\sigma(g)g^{-1})g,$$

hence the induced action on $\overline{T}^*$ is given by

$$\sigma^*(t^*) = g_\sigma \sigma(t^*) g_\sigma^{-1}.$$

The cocycle $\rho = \rho(T) \colon \mathrm{Gal}(\overline{F}/F) \to \overline{W}$, given by $\rho(\sigma) = g_\sigma \bmod \overline{T}^*$, determines $\mathbf{T}$ up to stable conjugacy.

Let $A(\mathbf{T}/F)$ be the pointed set of g in $\mathbf{G}(\overline{F})$ so that $\mathbf{T}' = {}^g\mathbf{T} = g\mathbf{T}g^{-1}$ is defined over F. Then the set

$$B(\mathbf{T}/F) = G\backslash A(\mathbf{T}/F)/\mathbf{T}(\overline{F})$$

parametrizes the morphisms of T into G over F, up to inner automorphisms by elements of G. If T is the centralizer of x in G then $B(\mathbf{T}/F)$ *parametrizes the set of conjugacy classes within the stable conjugacy class of* x in G. The map

$$g \mapsto \{\tau \mapsto g_\tau = g^{-1}\tau(g); \tau \in \mathrm{Gal}(\overline{F}/F)\}$$

defines a bijection

$$B(\mathbf{T}/F) \simeq \ker[H^1(F,\mathbf{T}) \to H^1(F,\mathbf{G})].$$

Let $p : \mathbf{G}^{\mathrm{sc}} \twoheadrightarrow \mathbf{G}^{\mathrm{der}}$ denote the simply connected covering group of the derived group $\mathbf{G}^{\mathrm{der}}$ of $\mathbf{G}$. If $\mathbf{T}$ is an F-torus in $\mathbf{G}$, let $\mathbf{T}^{\mathrm{sc}} = p^{-1}(\mathbf{T}^{\mathrm{der}})$ of $\mathbf{T}^{\mathrm{der}} = \mathbf{T} \cap \mathbf{G}^{\mathrm{der}}$. Then $\mathbf{G} = \mathbf{T}\mathbf{G}^{\mathrm{der}}$ and $\mathbf{G}/p(\mathbf{G}^{\mathrm{sc}}) = \mathbf{T}/p(\mathbf{T}^{\mathrm{sc}})$. Then the pointed set $B(\mathbf{T}/F)$ is a subset of the group $C(\mathbf{T}/F)$, defined to be the image of $H^1(F,\mathbf{T}^{\mathrm{sc}})$ in $H^1(F,\mathbf{T})$. If $H^1(F,\mathbf{G}^{\mathrm{sc}}) = \{0\}$, for example when F is a nonarchimedean local field, then $B(\mathbf{T}/F) = C(\mathbf{T}/F)$. If F is a global field with a ring $\mathbb{A}$ of adèles, then we put $C(\mathbf{T}/\mathbb{A}) = \oplus_v C(\mathbf{T}/F_v)$, $B(\mathbf{T}/\mathbb{A}) = \oplus_v B(\mathbf{T}/F_v)$. The sums are pointed. They range over all places v of F.

Let K be a finite Galois extension of F over which $\mathbf{T}$ splits. Denote $H^{-1}(\mathrm{Gal}(K/F), X)$ by $H^{-1}(X)$ and $\mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ by $X_*(\mathbf{T})$. In the local case the Tate-Nakayama duality (see [KS]) identifies $C(\mathbf{T}/F)$ with the image of $H^{-1}(X_*(\mathbf{T}^{\mathrm{sc}}))$ in $H^{-1}(X_*(\mathbf{T}))$. In the global case it yields an exact sequence

$$C(\mathbf{T}/F) \to C(\mathbf{T}/\mathbb{A}) \to \mathrm{Im}[H^{-1}(X_*(\mathbf{T}^{\mathrm{sc}})) \to H^{-1}(X_*(\mathbf{T}))].$$

The last term here is the quotient of the $\mathbb{Z}$-module of μ in $X_*(\mathbf{T}^{\mathrm{sc}})$ with $\sum_\tau \tau\mu = 0$ (sum over τ in $\mathrm{Gal}(K/F)$), by the submodule spanned by $\mu - \tau\mu$, where μ ranges over $X_*(\mathbf{T})$ and τ over $\mathrm{Gal}(K/F)$.

We denote by $W(T)$ the Weyl group of T in G, by $\mathbf{W} = S_3$ the Weyl group of (the algebraic group) $\mathbf{T}^*$ in $\mathbf{G}$, and by $W'(T)$ the Weyl group of T in $A(\mathbf{T}/F)$. We write σ for the non trivial element in $\mathrm{Gal}(E/F)$.

We shall now discuss *the above definitions in our case* where $\mathbf{G} = \mathbf{U}(3, E/F)$. The centralizer E' of T in the algebra $M(3, E)$ of $3{\times}3$ matrices over E, is a maximal commutative semisimple subalgebra. Hence it is isomorphic to a direct sum of field extensions of E.

There are three possibilities.
(1) $E' = E \oplus E \oplus E$.
(2) $E' = E'' \oplus E$, $[E'' : E] = 2$.
(3) E' is a cubic extension of E.

The absolute Weyl group $\mathbf{W}$ is the symmetric group on three letters, generated by the reflections (12), (23), (13). Note that $\sigma(12)$=(23), $\sigma(13)$=(13). In view of Proposition 1, the stable conjugacy classes of G are determined by $H^1(F, \mathbf{W})$. We also note that if the eigenvalues of g in G are α, β, γ in K, then τ in $\mathrm{Gal}(K/F)$ whose restriction to E is nontrivial, maps α, β, γ to $\tau\alpha^{-1}$, $\tau\beta^{-1}$, $\tau\gamma^{-1}$. The lattice $X_*(\mathbf{T})$ is the group of $\mu = (x, y, z)$ in $\mathbb{Z}^3$, and $X_*(\mathbf{T}^{\mathrm{sc}})$ is the subgroup of μ with $x + y + z = 0$. Indeed, $\mathbf{G}^{\mathrm{sc}} = \mathbf{SU}(3)$. If $\tau|E \neq 1$ it maps the set $\{x, y, z\}$ to the set $\{-x, -y, -z\}$.

Proposition 2.2. (1) *There are two stable conjugacy classes of F-tori in $\mathbf{G} = \mathbf{U}(3, E/F)$ which split over E. One, named of type* (0)*, consists of a single conjugacy class, represented by the torus $\mathbf{T}^*$ with*

$$T^* = \{\mathrm{diag}(a, b, \sigma a^{-1});\ a \in E^\times,\ b \in E^1 = \{x \in E^\times; x\sigma x = 1\}\}.$$

We have $W'(T^*) = W(T^*) = \mathbb{Z}/2$.

The other stable conjugacy class, named of type (1)*, consists of tori $\mathbf{T}$ with $T = (E^1)^3$, and $C(\mathbf{T}/F) = \{(a, b, c) \in (F^\times/NE^\times)^3;\ abc = 1\}$. We have $W'(T) = S_3$. This group acts transitively on the nontrivial elements in* (*and characters of*) $C(\mathbf{T}/F)$.
(2) *The stable conjugacy classes of F-tori in $\mathbf{G}$ whose splitting fields are quadratic extensions of E, named of type* (2)*, split over biquadratic extensions EL of F. Then $\mathrm{Gal}(EL/F) = \mathbb{Z}/2\times\mathbb{Z}/2$ is generated by σ which fixes L and τ which fixes E; put $K = (EL)^{\sigma\tau}$. Each such torus is $T \simeq \{(a, b, \sigma a^{-1}); a \in (EL/K)^1, b \in E^1\}$. Here $(EL/K)^1 = \{a \in EL; a\sigma\tau a = 1\}$. Further, $C(\mathbf{T}/F) = K^\times/N_{EL/K}(EL)^\times = \mathbb{Z}/2$ and $W'(T) = \mathbb{Z}/2$.*
(3) *The stable conjugacy classes of F-tori in $\mathbf{G}$ whose splitting fields are cubic extensions of E, named of type* (3)*, are split over cubic extensions ME of E, where M is a cubic extension of F. Each stable class consists of a single conjugacy class. If EM/F is not Galois then $W'(T)$ is trivial. If $\mathrm{Gal}(EM/F) = S_3$ or $\mathbb{Z}/3 \times \mathbb{Z}/2$ then $W'(T)$ is $\mathbb{Z}/3$.*

Proof. A cocycle in $H^1(\mathrm{Gal}(E/F), \mathbf{W})$ is determined by w_σ in $\mathbf{W} = S_3$ with $1 = w_{\sigma^2} = w_\sigma\sigma(w_\sigma)$. Thus w_σ is 1 or (13), or (12)(23) or (23)(12). As

$$\sigma((23))[(12)(23)](23) = 1 = \sigma((12))[(23)(12)](12),$$

the last two are cohomologous to 1. The cocycle $w_\sigma = 1$ defines the action $\sigma^*(t^*) = \sigma(t^*)$ on $\overline{T}^*$. To determine $C(\mathbf{T}^*/F)$, note that $H^1(F, \mathbf{T}^*) = H^1(\mathrm{Gal}(E/F), \mathbf{T}^*(E))$ is the quotient of the cocycles $t_\sigma = \mathrm{diag}(a, b, c) \in \mathbf{T}^*(E) = E^{\times 3}$, $t_\sigma\sigma(t_\sigma) = t_{\sigma^2} = 1$, thus $t_\sigma = \mathrm{diag}(a, b, \sigma a)$, $a \in E^\times$, $b \in F^\times$, by the coboundaries $t_\sigma\sigma(t_\sigma^{-1}) = \mathrm{diag}(a\sigma c, b\sigma b, c\sigma a)$. Since $\mathbf{G}^{\mathrm{sc}}$ is the subgroup of $\mathbf{G}$ of elements of determinant 1, the cocycles which come from

$H^1(F,\mathbf{T}^{*\mathrm{sc}})$ have the form $t_\sigma = \mathrm{diag}(a, 1/a\sigma a, \sigma a)$. These are coboundaries: $u_\sigma\sigma(u_\sigma^{-1})$, with $u_\sigma = (a, 1/a, 1)$, hence $C(\mathbf{T}^*/F)$ is trivial.

The cocycle $w_\sigma = (13)$ defines the action $\sigma^*(\mathrm{diag}(a,b,c)) = (\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1})$ on $\overline{T}^*$. Then $\mathbf{T} = g^{-1}\mathbf{T}^* g$ for some g in $\overline{G}$ with $g\sigma(g^{-1}) = J \pmod{\overline{T}^*}$, and $T = \mathbf{T}(F) = g^{-1}(E^1)^3 g$. A cocycle $t_\sigma = \mathrm{diag}(a,b,c) \in (E^\times)^3$ of $\mathrm{Gal}(E/F)$ in $\mathbf{T}^*(E)$ satisfies $1 = t_{\sigma^2} = t_\sigma\sigma^*(t_\sigma) = \mathrm{diag}(a/\sigma a, b/\sigma b, c/\sigma c)$, thus $a,b,c \in F^\times$ and it comes from $\mathbf{T}^{*\mathrm{sc}}(E)$ if $abc = 1$. The coboundaries take the form $t_\sigma\sigma^*(t_\sigma)^{-1} = \mathrm{diag}(a\sigma a, b\sigma b, c\sigma c)$, hence $C(\mathbf{T}/F) = \{(a,b,c) \in (F^\times/NE^\times)^3;\, abc = 1\}$.

Consider next an F-torus $\mathbf{T}$ in $\mathbf{G}$ which splits over a quadratic extension L_1 of E, but not over E. We claim that L_1/F is Galois. Indeed, the involution $\iota(x) = J\,{}^t\overline{x}J$ stabilizes $T = \mathbf{T}(F)$, and its centralizer $L_1^\times \times E^\times$ in $\mathrm{GL}(3,E)$. It induces on L_1 an automorphism whose restriction to E generates $\mathrm{Gal}(E/F)$. Hence L_1/F is Galois.

We claim that the Galois group of L_1/F is not $\mathbb{Z}/4$. Indeed, if $\mathrm{Gal}(L_1/F) = \mathbb{Z}/4$ were generated by τ, then τ^2 be trivial on E, $(w_{\tau^2})^2 = w_{\tau^4} = 1$ implies $w_{\tau^2} = 1$ or (13) up to coboundaries. But $(13) = w_{\tau^2} = w_\tau\tau(w_\tau) = w_\tau(13)w_\tau(13)$ implies $w_\tau^2 = (13)$, which has no solutions, and $w_{\tau^2} = 1$ implies that T splits over E. Then $\mathrm{Gal}(L_1/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$, and L_1 is the compositum of E and a quadratic extension L of F, not isomorphic to E. There are two such L (up to isomorphism), both ramified if E/F is unramified.

The Galois group $\mathrm{Gal}(LE/F)$ is generated by σ whose restriction to L is trivial, and τ whose restriction to E is trivial. Up to coboundaries, w_τ is 1 or (13). If $w_\sigma = (13)$, then $w_\tau \neq 1$ is of order 2. Up to coboundary which does not change w_σ, we have $w_\tau = (13)$, and replacing σ by $\sigma\tau$ (thus changing L) we may assume $w_\sigma = 1$. If $w_\sigma = 1$, $w_\tau w_\sigma = w_{\tau\sigma} = w_{\sigma\tau} = w_\sigma\sigma(w_\tau) = w_\sigma(13)w_\tau(13)$ implies that w_τ $(\neq 1)$ commutes with (13), hence $w_\tau = (13)$. Up to isomorphism, T consists of $(a,b,c) \in (LE)^{\times 3}$ which are fixed by $\sigma^*(a,b,c) = (\sigma c^{-1}, \sigma b^{-1}, \sigma a^{-1})$ and $\tau^*(a,b,c) = (\tau c, \tau b, \tau a)$. Thus $b = \tau b = \sigma b^{-1}$ lies in E^1, and $c = \sigma a^{-1} = \tau a$, namely $T \simeq \{(a, b, \sigma a^{-1}); a \in (EL/K)^1, b \in E^1\}$, where $(EL/K)^1 = \{a \in EL; a\sigma\tau a = 1\}$.

It is simplest to compute $C(\mathbf{T}/F)$ using Tate-Nakayama duality. Locally, the image of

$$\hat{H}^{-1}(F, X_*(\mathbf{T}^{\mathrm{sc}})) = \{X = (x,y,z) \in \mathbb{Z}^3; x+y+z = 0\}/\langle X - \sigma X, X - \tau X\rangle$$

in

$$\hat{H}^{-1}(F, X_*(\mathbf{T})) = \mathbb{Z}^3/\langle X - \tau\sigma X = (2x, 2y, 2z), \quad X - \tau X = (x-z, 0, z-x)\rangle$$

is $\mathbb{Z}/2$.

Here is an explicit computation of $H^1(\mathrm{Gal}(LE/F), \mathbf{T}(LE))$. We replace $\mathbf{T}$ by $\mathbf{T}^*$ if $\rho \in \mathrm{Gal}(LE/F)$ acts by ρ^*. To compute note that a cocycle in $H^1(\mathrm{Gal}(LE/F), \mathbf{T}^*(LE))$ is defined by $\{t_\sigma, t_\tau, t_{\sigma\tau}\}$ satisfying the cocycle relations. Thus $t_\tau = (a,b,c) \in (EL)^{\times 3}$ satisfies $1 = t_{\tau^2} = t_\tau\tau^*(t_\tau) = (a,b,c)(\tau c, \tau b, \tau a)$. So $b = b'/\tau b'$ and if $g = (a, b', 1)$, replacing our cocycle $\{t_\rho\}$ by its product $\{t_\rho g^{-1}\rho^*(g)\}$ with a coboundary, we may assume that $t_\tau = 1$. If $t_{\tau\sigma} = (u,v,w)$ then

$$1 = t_{(\tau\sigma)^2} = t_{\tau\sigma}(\sigma\tau)^*(t_{\tau\sigma}) = (u,v,w)(\tau\sigma u^{-1}, \tau\sigma v^{-1}, \tau\sigma w^{-1}).$$

Hence $(u,v,w) \in K^{\times 3}$. Here K is the fixed field of $\tau\sigma$ in LE. Further, $t_{\tau\sigma}(\tau\sigma)^*(t_\tau) = t_\sigma = t_\tau \tau^*(t_{\tau\sigma})$. Hence $t_{\tau\sigma} = (u,v,w) = (\tau w, \tau v, \tau u) = (u, v, \tau u)$, $u \in K^\times$, $v \in F^\times$. We can still multiply our cocycle t_ρ by a coboundary $g^{-1}\rho^*(g)$ with $g = \tau(g)$ (to preserve $t_\tau = 1$). Thus $g = (x, y, \tau x)$, $y = \tau y \in E^\times$. Then $g^{-1}(\tau\sigma)^*(g) = (1/u, 1/y\sigma(y), 1/\tau(u))$, $u = x\tau\sigma(x)$. Now $H^1(\mathrm{Gal}(LE/F), \mathbf{T}^{\mathrm{sc}}(LE))$ is spanned by the $t_{\tau\sigma} = (u, v, \tau u)$, $u \in K^\times / N_{EL/K}(EL)^\times$, $vu\tau u = 1$. Then $\mathrm{Im}[H^1(F, \mathbf{T}^{\mathrm{sc}}) \to H^1(F, \mathbf{T})]$ is represented by

$$(u, 1/u\tau u, \tau u), \qquad u \in K^\times / N_{EL/K}(EL)^\times \simeq \mathbb{Z}/2.$$

Consider next an F-torus $\mathbf{T}$ in $\mathbf{G}$ which splits over a cubic extension M_1 of E, but not over E. The involution $\iota(x) = J {}^t\overline{x} J$ stabilizes $T = \mathbf{T}(F)$, and its centralizer $M_1^\times$ in $\mathrm{GL}(3, E)$. It induces on the field M_1 an automorphism, denoted σ, whose restriction to E generates $\mathrm{Gal}(E/F)$. Define M to be the subfield of M_1 whose elements are fixed by σ. It is a cubic extension of F, $M_1 = ME$, and M_1/F is Galois precisely when M/F is. If M' is a Galois closure of M_1/F, then there is τ in $\mathrm{Gal}(M'/F)$ with $\tau(x,y,z) = (z,x,y)$ (up to order). But $X - \tau X = (x, y, -x-y)$ if $X = (x, x+y, 0)$. Hence $C(\mathbf{T}/F)$ is $\{0\}$.

There are two possible actions of the Galois group of the Galois closure of M_1 over F. In both cases we may assume that $\tau^*(x,y,z) = (\tau z, \tau x, \tau y)$.

If $\sigma^*(x,y,z) = (\sigma z^{-1}, \sigma y^{-1}, \sigma x^{-1})$ then $\tau\sigma = \sigma\tau^2$, the Galois group is S_3, and T^* consists of $(x, \tau x, \tau^2 x)$, $x \in M_1$ with $x\tau\sigma x = 1$.

If $\sigma^*(x,y,z) = (\sigma x^{-1}, \sigma y^{-1}, \sigma z^{-1})$ then $\tau\sigma = \sigma\tau$, the Galois group is $\mathbb{Z}/3 \times \mathbb{Z}/2$, and T^* consists of $(x, \tau x, \tau^2 x)$, $x \in M_1$ with $x\sigma x = 1$. □

In the case of $\mathbf{H} = \mathrm{U}(2)$, each torus $\mathbf{T}$ splits over a biquadratic extension of F, and $C(\mathbf{T}/F)$ is trivial, unless $\mathbf{T}$ splits over E and σ acts by $\sigma(x,y) = (-x,-y)$, where $C(\mathbf{T}/F)$ is $\mathbb{Z}/2$ in the local case.

We also need a *a twisted analogue of the above discussion*. Let $\mathbf{G}' = \mathrm{R}_{E/F}\,\mathbf{G}$ be the group obtained from $\mathbf{G} = \mathbf{U}(3, E/F)$ upon restricting scalars from E to F. It is defined over F. In fact, $\mathbf{G}'(\overline{F}) = \mathbf{G}(\overline{F}) \times \mathbf{G}(\overline{F})$, and $\mathrm{Gal}(\overline{F}/F)$ acts on $\mathbf{G}'(\overline{F})$ by $\tau(x,y) = (\tau x, \tau y)$ if $\tau|E = 1$, or by $\tau(x,y) = \iota(\tau x, \tau y)$ if $\tau|E \neq 1$. Here $\iota(x,y) = (y,x)$. Further, we have $\mathbf{G}'(E) = \mathbf{G}(E) \times \mathbf{G}(E)$, and $G' = \mathbf{G}'(F)$ consists of all $(x, \sigma x)$, x in $\mathbf{G}(E) = \mathrm{GL}(3, E)$. The group G embeds in G' as the diagonal.

Denote by $Z_{\mathbf{G}'}(x\iota)$ the ι-centralizer of $x = (x', x'')$ in $\mathbf{G}'$. It consists of the $y = (y', y'')$ in $\mathbf{G}'$ with $(y', y'')(x', x'') = (x', x'')\iota(y', y'')$. These y satisfy $y'x'x'' = x'x''y'$, $y'' = x'^{-1}y'x'$. If $x = (x', \sigma(x'))$ lies in G', $\mathbf{T} = Z_{\mathbf{G}'}(x\iota)$ is defined over F, since ι is. The group T of F-rational points consists of such y with $y'' = \sigma y'$. The ι-centralizer $\mathbf{T}$ is isomorphic to the σ-centralizer of x' in $\mathbf{G}$.

The elements x and x^1 in G' are called (*stably*) *σ-conjugate* if there is y in G' (resp. $\mathbf{G}'(\overline{F})$) so that $yx = x^1\iota(y)$. In this case $\tau x = x$ for all τ in $\mathrm{Gal}(\overline{F}/F)$, and $\tau(y)x = x^1\iota(\tau y)$. Hence the σ-conjugacy classes within the stable σ-conjugacy class of x are parametrized by the elements $\{\tau \mapsto y_\tau = y^{-1}\tau(y)\}$ of the kernel $B''(\mathbf{T}/F)$ of the natural map from $H^1(F, \mathbf{T})$ to $H^1(F, \mathbf{G}')$. Here $\mathbf{T}$ denotes the ι-centralizer of $x = (x', x'')$ in $\mathbf{G}'$.

The conjugacy class in $\mathbf{G}(\overline{F})$ of $x'x'' = x'\sigma(x')$ is defined over F. Hence it contains a member Nx of G by [Ko]. The element Nx is determined only up to stable conjugacy. The group T is isomorphic to the centralizer of Nx in G, over F, by the map $y = (y', y'') \mapsto y'$. Recall that y satisfies $y'x'x'' = x'x''y'$, $y'' = x'^{-1}y'x'$. The pointed set $H^1(F, \mathbf{G}')$ is trivial. Hence $B''(\mathbf{T}/F) = H^1(F, \mathbf{T})$.

We introduce the notion of (stable) σ-conjugacy since we shall use below orbital integrals $\int \phi(gx\sigma(g)^{-1})dg/dt$ over $G'/Z_{G'}(x)$ of functions ϕ which transform under the center $Z' = E^\times$ of $G' = \mathrm{GL}(3, E)$ via a character $\omega'(z) = \omega(z/\overline{z})$ of $z \in E^\times$. In particular ϕ transforms trivially on $F^\times$. Hence the actual notion of stable σ-conjugacy that we need is $yx\iota(y)^{-1} = zx$, for z in $F^\times$, viewed as $(z, \sigma(z) = z^{-1})$ in G'.

The map $z \mapsto \{z_\tau = (z, 1)\tau(z, 1)^{-1}\}$ embeds $F^\times$ in $B''(\mathbf{T}/F)$. Here z_τ acts on x in $\mathbf{G}'$ by $(z, 1)x\iota(z, 1)^{-1} = zx$ $(= (zx', \sigma(zx'))$ if $x = (x', \sigma x'))$. Thus z maps the member $\{y_\tau = y^{-1}\tau(y)\}$ of $B''(\mathbf{T}/F)$ to $\{(zy)_\tau\}$, which sends x to $[(z, 1)y]x\iota[(z, 1)y]^{-1} = (z, z^{-1})yx\iota(y^{-1})$. The quotient of $B''(\mathbf{T}/F)$ under this action of $F^\times$ is denoted by $B'(\mathbf{T}/F)$. Put $B'(\mathbf{T}/\mathbb{A}) = \oplus_v B'(\mathbf{T}/F_v)$ (pointed sum) if F is global.

The Tate-Nakayama theory implies that $B'(\mathbf{T}/F)$ (in the local case) or $B'(\mathbf{T}/\mathbb{A})$/Image $B'(\mathbf{T}/F)$ (in the global case), is the quotient of the $\mathbb{Z}$-module of the X in $X_*(\mathbf{T})$ modulo $\mathbf{Z}$ with $\sum_\tau \tau X = 0$ (τ in $\mathrm{Gal}(K/F)$), by the span of $X - \tau X$ for all X in $X_*(\mathbf{T})$ and τ in $\mathrm{Gal}(K/F)$, where K is a Galois extension of F over which $\mathbf{T}$ splits.

The map $x \mapsto Nx$ gives a bijection from the set of stable σ-conjugacy classes in G' (parametrized by $B'(\mathbf{T}/F)$), to the set of stable conjugacy classes in G. In fact, for our present work it suffices to consider regular x in G (x with distinct eigenvalues), and σ-regular x in G' (Nx is regular). Hence there are three types of stable σ-conjugacy classes of σ-regular elements in G', denoted by (1), (2), (3) as in the nontwisted case. Using the Tate-Nakayama theory we see (in the local case) that $B'(\mathbf{T}/F)$ is trivial if $\mathbf{T}$ is $\mathbf{T}^*$ (the diagonal torus), and in case (3); it is $\mathbb{Z}/2$ in case (2); it is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if $\mathbf{T}$ splits over E but $\mathbf{T}$ is not (stably) conjugate to $\mathbf{T}^*$.

To compute orbital integrals, we need explicit representatives.

Lemma 2.3. *If $\mathbf{T}$ splits over E but is not $\mathbf{T}^*$, $H^1(F, \mathbf{T})/F^\times$ is $F^{\times 3}/F^\times NE^{\times 3}$, where $F^\times/NE^\times$ embeds diagonally into $F^{\times 3}/NE^{\times 3}$. If $\mathbf{T}$ splits over a biquadratic extension LE of F, $\mathrm{Gal}(LE/F) = \langle \tau, \sigma \rangle$, $L = (LE)^\sigma$, $E = (LE)^\tau$, $K = (LE)^{\sigma\tau}$ are the quadratic extensions of F in EL, then $H^1(F, \mathbf{T})/F^\times$ is $K^\times/N_{LE/K}(LE)^\times$, which is $\mathbb{Z}/2\mathbb{Z}$.*

Proof. If $\mathbf{T}$ splits over E but is not $\mathbf{T}^*$, a cocycle $t_\sigma = (a, b, c)$ in $H^1(E, \mathbf{T}(E))$ satisfies

$$1 = t_{\sigma^2} = t_\sigma \sigma^*(t_\sigma) = (a, b, c)(\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1}).$$

Thus (a, b, c) lies in $F^{\times 3}$. A coboundary has the form $t_\sigma \sigma^*(t_\sigma)^{-1} = (a, b, c)(\sigma a, \sigma b, \sigma c)$. Hence we get $NE^{\times 3}$, and $H^1(F, \mathbf{T})/F^\times$ is $F^{\times 3}/F^\times NE^{\times 3}$.

If $\mathbf{T}$ splits over a biquadratic extension LE of F, the group $H^1(\mathrm{Gal}(LE/F), \mathbf{T}(LE))$ is computed in the proof of Proposition 2. Then $H^1(\mathrm{Gal}(LE/F), \mathbf{T}(LE))/F^\times$ is represented by $t_{\tau\sigma} = (u, 1, \tau u)$, $u \in K^\times/N_{LE/K}(LE)^\times$. □

We also need an explicit realizations of the twisted stable conjugacy classes in the cases that they contain several twisted conjugacy classes, namely the cases corresponding to the tori $T=(E^1)^3$ and $T=(EL/K)^1\times E^1$. This is useful in computations of twisted orbital integrals and twisted characters. We consider only classes whose norm is regular (has distinct eigenvalues) in $G=\mathrm{U}(3,E/F)$. We name such classes twisted-regular or σ-regular.

Proposition 2.4. (*i*) *A set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of a twisted regular x in* $\mathrm{GL}(3,E)$ *with norm in an anisotropic torus which splits over E, thus $Nx=h^{-1}\operatorname{diag}(a/\overline{a},b/\overline{b},c/\overline{c})h$ in a torus*

$$T_1=h^{-1}\mathbf{T}^*(E^1)h,\qquad h=\begin{pmatrix}1&&1\\&1&\\-1&&1\end{pmatrix},$$

distinct $a/\overline{a}$, $b/\overline{b}$, $c/\overline{c}$, is given by $x_1=h^{-1}\operatorname{diag}(a,b,c)h$, $x_2=h^{-1}\operatorname{diag}(a,b\rho,c)h$, $x_3=h^{-1}\operatorname{diag}(a\rho,b,c)h$, $x_4=h^{-1}\operatorname{diag}(a,b,c\rho)h$, where a, b, c lie in $E^\times$, $\rho\in F-NE$.

If $y=h^{-1}\operatorname{diag}(\alpha,\beta,\gamma)h$ then $yx_1\sigma y^{-1}=h^{-1}\operatorname{diag}(a\alpha\overline{\alpha},b\beta\overline{\beta},c\gamma\overline{\gamma})h$.

(*ii*) *Let $L=F(\sqrt{A})=(EL)^\sigma$, $E=(EL)^\tau=F(\sqrt{D})$, and $K=(EL)^{\sigma\tau}=F(\sqrt{DA})$ be the distinct quadratic extensions of F. Here $\{A,D,AD\}=\{\boldsymbol{\pi},u,u\boldsymbol{\pi}\}$ and $u\in R_E-R_E^2$.*

A set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of a σ-regular x in $\mathrm{GL}(3,E)$ *with norm in a torus which splits over a biquadratic extension EL of F can be realized by*

$$t=t_\alpha=h^{-1}\begin{pmatrix}(a+b\sqrt{A})\alpha&&\\&c&\\&&(a-b\sqrt{A})\tau(\alpha)\end{pmatrix}h,\qquad h=\begin{pmatrix}1&&\sqrt{A/D}\\&1&\\-\frac{\sqrt{D/A}}{2}&&\frac{1}{2}\end{pmatrix},$$

where a, b, $c\in E^\times$ and $\alpha\in K^\times/N_{EL/K}(EL)^\times$. Then ($Nt$ is regular) and

$$Nt=t\sigma(t)=h^{-1}\operatorname{diag}((a+b\sqrt{A})/(\overline{a}-\overline{b}\sqrt{A}),c/\overline{c},(a-b\sqrt{A})/(\overline{a}+\overline{b}\sqrt{A}))h.$$

The norm map is surjective.

Proof. First note that $x_1=h^{-1}\operatorname{diag}(a,b,c)h$ satisfies

$$Nx_1=x_1\sigma(x_1)=h^{-1}\operatorname{diag}(a,b,c)h\cdot\sigma(h^{-1})\operatorname{diag}(1/\overline{c},1/\overline{b},1/\overline{a})\sigma(h).$$

Since $\sigma(h^{-1})=h$ and $h^2=\operatorname{diag}(2,-1,-2)J$, this is

$$=h^{-1}\operatorname{diag}(a/\overline{a},b/\overline{b},c/\overline{c})\operatorname{diag}(2,-1,-2)^t h^{-1}J.$$

But $\operatorname{diag}(2,-1,-2)^t h^{-1}J=h$. In particular the norm N is onto the torus $T\simeq(E^1)^3$, which we realize as $T_1=h^{-1}T^*h$.

The stable ι-conjugates of x_1 are given by $y'x_1y''^{-1}$ where $y_\sigma=y'^{-1}\sigma(y'')$ lies in $H^1(F,\mathbf{T}_1)/F^\times$, $\mathbf{T}_1=h^{-1}\mathbf{T}^*h$, where $\mathbf{T}^*$ denotes the diagonal torus, as discussed in the paragraph where (stable) σ-conjugacy is defined, prior to Proposition 2.3. A set of representatives for the stable ι-conjugates of x_1 up to ι-conjugacy is given as y_σ ranges over $h^{-1}th$, where t ranges over $\mathbf{T}^*(F)/\mathbf{Z}(F)N\mathbf{T}^*(E)$; $\mathbf{Z}$ is the diagonal. Choose $\rho\in F-NE$.

Thus we may take t to be 1, $\mathrm{diag}(1,\rho,1)$, $\mathrm{diag}(\rho,1,1)$, $\mathrm{diag}(1,1,\rho)$. Taking y'' to be 1, we choose $y' = h^{-1}th$, to get x_i $(1 \le i \le 4)$ of the proposition.

In the case of the torus splitting over EL and isomorphic to $\ker N_{EL/K} \times E^1$, note that $\sigma(h) = h$, and that $\sigma^*(a,b,c) = (\sigma c^{-1}, \sigma b^{-1}, \sigma a^{-1})$. We write $\sigma a = \overline{a}$, and σ fixes $\sqrt{A}$. The σ-conjugacy classes within the stable σ-conjugacy class are parametrized in Lemma 3. □

Remark. (1) Define a', b' in E by $a' + b'\sqrt{A} = (a + b\sqrt{A})\alpha$ in $EL = E(\sqrt{A})$. Then

$$t_\alpha = \begin{pmatrix} a' & & b'A/\sqrt{D} \\ & c & \\ b'\sqrt{D} & & a' \end{pmatrix}.$$

(2) If E/F is unramified then K/F and L/F are ramified. We may take D to be a nonsquare unit in F. Hence we may choose the $\alpha \neq 1$ to be $\sqrt{AD}$, which is in K but not in $N_{EL/K}(EL)$.

3. Elements in G with norm in the torus $(E^1)^3$

In the case of an element whose norm in G lies in the torus $(E^1)^3$, the expression $\Delta(Ng)\chi_\pi^\sigma(g)$ of Theorem 1 is the value at $s = 0$ of the expression (a, b, c in $E^\times$, distinct $a/\overline{a}$, $b/\overline{b}$, $c/\overline{c}$ in E^1; take x_1 in Proposition 2.4, use Corollary 1.2 and Lemma 1.3; $|.| = |.|_E$ and $|\boldsymbol{\pi}_E|_E = q_E^{-1}$ there; $|x|_F = |N_{E/F}x|_F^{1/2} = |x|_E^{1/2}$)

$$\xi_s(a,b,c) = -\frac{q_E^{3/2}}{q_E + q_E^{1/2} + 1}\left|\left(\frac{a}{\overline{a}} - \frac{b}{\overline{b}}\right)\left(\frac{c}{\overline{c}} - \frac{b}{\overline{b}}\right)\left(\frac{a}{\overline{a}} - \frac{c}{\overline{c}}\right)\right|_F |abc|_E^{\frac{1-s}{2}}$$

times

$$\int_{V^0}\left|-by\overline{y} + \frac{a}{2}(x+z)(\overline{x}+\overline{z}) - \frac{c}{2}(x-z)(\overline{x}-\overline{z})\right|_E^m dxdydz, \qquad m = \frac{3}{2}(s-1).$$

In this and the following sections, q is q_F, $|.|$ is $|.|_F$, R is R_F.

Since the σ-character depends only on the σ-conjugacy class, we may replace any of a, b, c by its multiple with an element of $NE^\times$. Suppose that E/F is unramified and that the residual characteristic p is odd ($\neq 2$). Then we can take $\rho \in F^\times - NE^\times$ to be a generator $\boldsymbol{\pi}$ of the maximal ideal in the ring $R = R_F$ of integers in F, and $E = F(\sqrt{D})$, $D \in R^\times - R^{\times 2}$. We may assume that $|b| \ge |a|$, $|c|$, and that a/b and c/b lie in $R_E^\times$ or in $\boldsymbol{\pi}R_E^\times$. Since the central character of the representation π is trivial, thus $\chi_\pi^\sigma(bg) = \chi_\pi^\sigma(g)$ for $b \in E^\times$, we may replace a, c by a/b, c/b, hence assume that $b = 1$, and that a, c are in $R_E^\times$ or in $\boldsymbol{\pi}R_E^\times$. Fix $\iota \in E$ with $N\iota = 2$. After the change of variables: $x' = (x+z)/\iota$, $z' = (x-z)/\iota$, renaming (x', z') back to (x, z), and noting that $-1 \in NE^\times$, the integral becomes

$$\int_{V^0}|aN(x) + N(y) + cN(z)|_E^m \, dxdydz.$$

Let $a = a_1 + \sqrt{D}a_2$ and $c = c_1 + \sqrt{D}c_2$, where a_1, a_2, c_1, $c_2 \in R$. Replace (x, y) by (y, x). Set

$$Q_1(x,y,z) = N(x) + a_1 N(y) + c_1 N(z), \qquad Q_2(x,y,z) = a_2 N(y) + c_2 N(z).$$

Then our integral can be written as

$$I_s(a,c) \;=\; \int_{V^0} \left| Q_1(x,y,z) + \sqrt{D} Q_2(x,y,z) \right|_E^m dxdydz.$$

By definition,

$$|Q_1 + \sqrt{D}Q_2|_E \;=\; |Q_1^2 - DQ_2^2| \;=\; \max\{|Q_1|^2, |Q_2|^2\}.$$

Theorem 2. *If $a = a_1 + \sqrt{D}a_2$, $c = c_1 + \sqrt{D}c_2$ lie in $R_E^\times$ and $a/\overline{a} \neq c/\overline{c} \neq 1 \neq a/\overline{a}$, thus $a_2 \neq 1 \neq c_2$ and $u = a_1c_2/a_2c_1 \neq 1$, at $s = 0$, $\xi_s(a\boldsymbol{\pi}^{n_1}, 1, c\boldsymbol{\pi}^{n_2}) I_s(a\boldsymbol{\pi}^{n_1}, c\boldsymbol{\pi}^{n_2})$ is equal to*

$$(-1)^{n_1+n_2}\kappa(a_2c_2)|a_2c_1 - a_1c_2| + (-1)^{n_2}\kappa(a_2(a_2c_1 - a_1c_2))|c_2| + (-1)^{n_1}\kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$

As noted above, it suffices to show this for n_1, $n_2 \in \{0, 1\}$. Obviously, the expression of Theorem 2 does not change if $(a\boldsymbol{\pi}^{n_1}, c\boldsymbol{\pi}^{n_2})$ is replaced by $(c\boldsymbol{\pi}^{n_2}, a\boldsymbol{\pi}^{n_1})$.

For each $n \geq 0$ we define the set

$$V_n(a,c) = \{v = (x,y,z) \in E^3;\; ||(x,y,z)||_E = 1, \max\{|Q_1|, |Q_2|\} = q^{-n}\},$$

where $Q_1 = Q_1(x,y,z)$, $Q_2 = Q_2(x,y,z)$. The space $V^0 = V/\sim$, where $V = \{v \in R_E^3; ||v||_E = 1\}$ and $\sim$ is the equivalence relation $v \sim \alpha v$ for $\alpha \in R_E^\times$, is the disjoint union of the subspaces

$$V_n^0 = V_n^0(a,c) = V_n(a,c)/\sim, \qquad n \geq 0.$$

Thus for s with $\mathrm{Re}(s)$ sufficiently large to assure convergence we have

$$I_s(a,c) = \sum_{n=0}^{\infty} q^{-2nm}\, \mathrm{vol}(V_n^0(a,c)).$$

Note that $\mathrm{vol}(R_E^\times) = 1 - q^{-2}$. The problem is to compute the volumes

$$\mathrm{vol}(V_n^0(a,c)) = \mathrm{vol}(V_n(a,c))/(1 - q^{-2}) \qquad (n \geq 0).$$

Notations. Write $a = a_1 + a_2\sqrt{D}$, $c = c_1 + c_2\sqrt{D}$ with a_1, a_2, c_1, c_2 in F. Put $u = \frac{a_1c_2}{a_2c_1}$. The condition $a/\overline{a} \neq c/\overline{c}$ means that $u \neq 1$.

The proof of Theorem 2 consists of three cases, the case of $n_1 = 0$, $n_2 = 0$ is dealt with in section 4, that of $n_1 = 1$, $n_2 = 0$ in section 5, and that of $n_1 = 1$, $n_2 = 1$ in section 6. Each of these cases consists of several subcases. To clarify the structure of the proof, the computations of $\mathrm{vol}(V_n^0)$, $n \geq 0$, used in each of these sections are stated and proved in the appendices to these sections.

4. Case of $n_1 = 0$, $n_2 = 0$

In this section $|.|$ is $|.|_F$, q is q_F, R is R_F. In section 3 we defined the quadratic forms

$$Q_1(x,y,z) = N(x) + a_1N(y) + c_1N(z),\ Q_2(x,y,z) = a_2N(y) + c_2N(z),$$

where $\max\{|a_1|,|a_2|\} = \max\{|c_1|,|c_2|\} = 1$ and the set

$$V_n = \{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, \max\{|Q_1(x,y,z)|, |Q_2(x,y,z)|\} = q^{-n}\}.$$

Note that if $|a_1| < 1$ then $|a_2| = 1$ and we can replace Q_1 with $Q_1 + Q_2$. Since in this case $|a_1 + a_2| = 1$, without loss of generality we can assume that $|a_1| = 1$, i.e. $a_1 \in R^\times$ is a unit in R. Recall that $u = a_1c_2/(a_2c_1)$. Then, we have the following possible cases.
(4.1) c_1, $a_2/c_2 \in R^\times$ and $|1-u| = 1$;
(4.2) c_1, $a_2/c_2 \in R^\times$ and $|1-u| < 1$;
(4.3) $|a_2| < 1$, $|a_2| \neq |c_2|$, $\max\{|c_1|,|c_2|\} = 1$;
(4.4) c_2, $a_2 \in R^\times$, $|c_1| < 1$.
The case a_1, a_2, $c_1 \in R^\times$, $|c_2| < 1$ can be reduced to that of 4.3 on interchanging a, c.

4.1 We proceed to prove the theorem when $n_1 = 0 = n_2$, a_1, c_1, u, $u-1$ are units in $R^\times$. When $|a_2| = |c_2| = q^{-d}$, to emphasize we denote V_n by $V_{n,d}$. Similarly we denote V_n^0 by $V_{n,d}^0$, thus $\mathrm{vol}(V_n^0)$ is $\mathrm{vol}(V_n)/(1-q^{-2})$. These volumes are computed in Lemma B.4.1.

Proof of theorem 2. Denote $I_s(a,c)$ by $I_{s,d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm}\,\mathrm{vol}(V_{n,d}^0).$$

When $d = 0$, by Lemma B.4.1, this is equal to

$$1 - 2q^{-3} + (1-q^{-2})(1+q^{-1})^2 \sum_{n=1}^{\infty} \alpha^n,$$

where $\alpha = q^{-2n(m+1)}$. This sum is equal to

$$1 - 2q^{-3} + (1-q^{-2})(1+q^{-1})^2\alpha(1-\alpha)^{-1}.$$

In our case, when $s = 0$, thus $m = -3/2$ and $\alpha = q$, this equals

$$= 1 - 2q^{-3} - (1+q^{-1})^3 = -3q^{-1}(1+q^{-1}+q^{-2}).$$

When $d \geq 1$, the sum

$$\sum_{n=0}^{d} q^{3n} \operatorname{vol}(V^0_{n,d})$$

by Lemma B.4.1 is equal to

$$1 - q^{-1} + q^{-2} + (1 - q^{-2})(1 - q^{-1} + q^{-2}) \sum_{n=1}^{d-1} q^{2n} + (1 - q^{-2} - q^{-3} - q^{-4}) q^{2d},$$

which is

$$= (1 - 2q^{-3}) q^{2d} = q^{2d} \operatorname{vol}(V^0_{0,0}).$$

Note that, by Lemma B.4.1, $\operatorname{vol}(V^0_{n,d}) = q^{-d} \operatorname{vol}(V^0_{n-d,0})$ $(n > d)$. Thus

$$I_{s,d} = \sum_{n=0}^{\infty} q^{-2mn} \operatorname{vol}(V^0_{n,d}) = \sum_{n=0}^{d} q^{-2mn} \operatorname{vol}(V^0_{n,d}) + \sum_{n=d+1}^{\infty} q^{-2nm} \operatorname{vol}(V^0_{n,d})$$

$$= \sum_{n=0}^{d} q^{-2mn} \operatorname{vol}(V^0_{n,d}) + q^{-d(1+2m)} \sum_{j=0}^{\infty} q^{-2jm} \operatorname{vol}(V^0_{j,0}) - q^{-d(1+2m)} \operatorname{vol}(V^0_{0,0}).$$

The sum in the middle of last line is equal to $q^{-d(1+2m)} I_{s,0}$. At $s = 0$, thus $m = -3/2$, the other two terms cancel each other by the equality found in the last paragraph, and we obtain

$$I_{0,d} = q^{2d} I_{0,0} = -3q^{2d} \times q^{-3}(1 + q + q^2).$$

In our case $|ac|_E = 1$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2|$ and $|a/\overline{a} - c/\overline{c}| = |a_2 c_1 - a_1 c_2| = |a_2 c_1||1-u| = |a_2|$. We put $|a| = |N_{E/F} a|^{1/2} = |a|_E^{1/2}$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1+q+q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, b, c)$. Hence $\xi_0(a, 1, c) I_0(a, c) = 3|a_2|$.

The H-side, namely the displayed formula in Theorem 2, is equal to

$$\kappa(a_2 c_2)|a_2 c_1 - a_1 c_2| + \kappa(a_2(a_2 c_1 - a_1 c_2))|c_2| + \kappa(c_2(a_2 c_1 - a_1 c_2))|a_2|.$$

Note that $\kappa(a_2) = \kappa(c_2) = \kappa(a_2 c_1 - a_1 c_2) = 1$. Since $|a_2 c_1 - a_1 c_2| = |c_2| = |a_2|$, each of the three summands is equal to $|a_2|$. Theorem 2 follows for $n_1 = n_2 = 0$, a_1, c_1, $a_2/c_2 \in R^\times$, $|u - 1| = 1$. □

4.2 Consider the case of $n_1 = 0$, $n_2 = 0$, a_1, c_1, $u \in R^\times$, and $|u - 1| < 1$. When $|a_2| = |c_2| = q^{-d}$, as in case 4.1, we denote V_n by $V_{n,d}$. Similarly we denote V_n^0 by $V^0_{n,d}$, thus $\operatorname{vol}(V_n^0)$ is $\operatorname{vol}(V_n)/(1 - q^{-2})$. The $\operatorname{vol}(V^0_{n,d})$ are computed in Lemma B.4.2.

Proof of theorem 2. Denote $I_s(a,c)$ by $I_{s,d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm}\,\mathrm{vol}(V^0_{n,d}).$$

We split this sum into two: a finite sum over n, which we evaluate at once at $s=0$ ($m=-3/2$), and an infinite sum, in which we first take big m, to have convergence, then add up the sum and only then evaluate the result at $m=-3/2$.

Thus when $|1-u|=q^{-2k}$ and $d=0$, by Lemma B.4.2, we have the sum of

$$(1+q^{-2}-q^{-3})+q^3(q^{-3}-q^{-5})+(1-q^{-2})\left(\sum_{n=3}^{2k-1} q^{n-1}+(1+q^{-1}+q^{-2})\sum_{n=2}^{2k-2} q^n\right)$$

and

$$(1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{2k}+(1-q^{-2})(1+q^{-1})^2\sum_{n=2k+1}^{\infty}\alpha^n,$$

where the first sum is taken over the odd n ($3\le n\le 2k-1$), the second over the even n ($2\le n\le 2k-2$), and the third over all $n\ge 2k+1$, and $\alpha=q^{-2n(m+1)}$. We have

$$\sum_{n=3}^{2k-1} q^{n-1}=\sum_{n=2}^{2k-2} q^n=(1-q^{-2})^{-1}(q^{2k-2}-1),$$

and

$$\sum_{n=2k+1}^{\infty}\alpha^n=-(1-\alpha^{-1})^{-1}\alpha^{2k}.$$

At $m=-3/2$, thus $\alpha=q$, the above expression equals

$$2-q^{-3}+(2+q^{-1}+q^{-2})(q^{2k-2}-1)+(1+q^{-1}-q^{-2}-2q^{-3}-q^{-4}-(1+q^{-1})^3)q^{2k}.$$

The coefficient of q^{2k},

$$2q^{-2}+q^{-3}+q^{-4}+1+q^{-1}-q^{-2}-2q^{-3}-q^{-4}-1-3q^{-1}-3q^{-2}-q^{-3},$$

is equal to $-2q^{-1}(1+q^{-1}+q^{-2})$. Hence we obtain

$$-q^{-3}(1+q+q^2)(1+2q^{2k}).$$

When $|1-u|=q^{-2k-1}$ and $d=0$, by Lemma B.4.2, we have the sum

$$2-q^{-3}+(1-q^{-2})\left(\sum_{n=3}^{2k-1} q^{n-1}+(1+q^{-1}+q^{-2})\sum_{n=2}^{2k} q^n\right)+(1+q^{-1})q^{2k},$$

where the first sum is taken over the odd n and second over the even n. To make both sums of the same length, we split the term $n = 2k$ from the second sum. Once the sums are computed, our expression is equal to

$$2 - q^{-3} + (1 + q^{-1} - q^{-3} - q^{-4})q^{2k} + (2 + q^{-1} + q^{-2})(q^{2k-2} - 1) + (1 + q^{-1})q^{2k}$$
$$= -q^{-1}(1 + q^{-1} + q^{-2})(1 - 2q^{2k+1}).$$

When $|1 - u| = q^{-1}$, by Lemma B.4.2, we have the sum

$$1 + q^{-2} - q^{-3} + q^3(q^{-3} + q^{-4}) = -q^{-1}(1 + q^{-1} + q^{-2}) + 2(1 + q^{-1} + q^{-2}),$$

which coincides to the odd case when $k = 0$. Thus we see that, if $d = 0$, then $I_{0,0}$ is equal to

$$-(1 + 2\kappa(1 - u)|1 - u|^{-1})q^{-3}(1 + q + q^2),$$

where $\kappa(1 - u)$ is 1 when $|1 - u| = q^{-2k}$ and is -1 when $|1 - u| = q^{-2k-1}$.

When $d \geq 1$, the sum

$$\sum_{n=0}^{d} q^{3n} \operatorname{vol}(V^0_{n,d})$$

by Lemma B.4.2, is equal to

$$1 - q^{-1} + q^{-2} + (1 - q^{-2})(1 - q^{-1} + q^{-2}) \sum_{n=1}^{d-1} q^{2n} + (1 - q^{-4})q^{2d},$$

which is

$$(1 + q^{-2} - q^{-3})q^{2d} = q^{2d} \operatorname{vol}(V^0_{0,0}).$$

Moreover, since, by Lemma B.4.2, $\operatorname{vol}(V^0_{n,d}) = q^{-d} \operatorname{vol}(V^0_{n-d,0})$ for $n > d$, we have the same equality as in Lemma B.4.1:

$$I_{0,d} = q^{2d} I_{0,0}.$$

In our case $|ac|_E = 1$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2|$, and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2| = |a_2c_1||1 - u|$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, b, c)$. Hence $\xi_0(a, 1, c)I_0(a, c)$ is equal to

$$|a_2c_1 - a_1c_2| + 2\kappa(1 - u)|a_2|.$$

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1 - a_1c_2| + \kappa(a_2(a_2c_1 - a_1c_2))|c_2| + \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$

Note that $\kappa(a_2c_2)=1$, and $\kappa(a_2(a_2c_1-a_1c_2))=\kappa(1-u)$. Thus the above expression is equal to $|a_2c_1-a_1c_2|+2\kappa(1-u)|a_2|$. Theorem 2 follows when $n_1=0$, $n_2=0$, a_1, c_1, $a_2/c_2\in R^\times$, $|1-u|<1$. □

4.3 Consider the case of $n_1=0$, $n_2=0$, $a_1\in R^\times$, $\max\{|c_1|,|c_2|\}=1$, $|a_2|<1$, $|a_2|\neq|c_2|$. Recall that we are dealing with the quadratic forms

$$Q_1(x,y,z)=N(x)+a_1N(y)+c_1N(z),\ Q_2(x,y,z)=a_2N(y)+c_2N(z).$$

Thus we can interchange (a,y) with (c,z) to reduce the case of $|a_2|=1$, $|c_1|<1$ to the case $|a_2|<1$, $|c_1|=1$ of Lemma B.4.3. When $|c_2|=q^{-d}$, to emphasize we denote V_n by $V_{n,d}$. Similarly, as in 4.1, we denote V_n^0 by $V_{n,d}^0$, thus $\mathrm{vol}(V_n^0)$ is $\mathrm{vol}(V_n)/(1-q^{-2})$. The $\mathrm{vol}(V_{n,d}^0)$ are computed in Lemma B.4.3.

Proof of theorem 2. Denote $I_s(a,c)$ by $I_{s,d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm}\,\mathrm{vol}(V_{n,d}^0).$$

Note that $\mathrm{vol}(V_{n,d}^0)$ ($d>0$ and $0\le n\le d$) of Lemma B.4.3 are equal to $\mathrm{vol}(V_{n,d}^0)$ of Lemma B.4.1, thus when $m=-3/2$, we have

$$\sum_{n=0}^{d} q^{3n}\,\mathrm{vol}(V_{n,d}^0)=q^{2d}\,\mathrm{vol}(V_{0,0}^0).$$

Moreover $\mathrm{vol}(V_{n,0}^0)$ ($n\ge 0$) of Lemma B.4.3 are related to those of Lemma B.4.2. Namely, when $|a_2|$ of Lemma B.4.3 is equal to $|1-u|$ of Lemma B.4.2, then $\mathrm{vol}(V_{n,0}^0)$ of Lemma B.4.3 is equal to $\mathrm{vol}(V_{n,0}^0)$ of Lemma B.4.2. Since $\mathrm{vol}(V_{n,d}^0)=q^{-d}\,\mathrm{vol}(V_{n-d,0}^0)$ for $n>d$, we have the same equality as in Lemma B.4.2:

$$I_{0,d}=q^{2d}I_{0,0}=-|c_2|^{-2}(1+2\kappa(a_2)|a_2|^{-1})q^{-3}(1+q+q^2),$$

where $\kappa(a_2)$ is equal to 1 when $|a_2|=q^{-2k}$ and to -1 when $|a_2|=q^{-2k-1}$.

In our case $|ac|_E=1$, $|a/\overline{a}-1|=|a_2|$, $|c/\overline{c}-1|=|c_2|$, and $|a/\overline{a}-c/\overline{c}|=|a_2c_1-a_1c_2|=|c_2|$ (since $|a_2c_1|<1$). Here q is q_F, q_E is q^2, so $-q^{-3}(1+q+q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E+q_E^{1/2}+1)^{-1}$ in $\xi_0(a,b,c)$. Hence $\xi_0(a,1,c)I_0(a,c)$ is equal to $|a_2|+2\kappa(a_2)$.

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1-a_1c_2|+\kappa(a_2(a_2c_1-a_1c_2))|c_2|+\kappa(c_2(a_2c_1-a_1c_2))|a_2|.$$

Note that $\kappa(c_2(a_2c_1 - a_1c_2)) = 1$, and since $|c_2| = 1$, then $\kappa(a_2c_2) = \kappa(a_2(a_2c_1 - a_1c_2)) = \kappa(a_2)$. Thus the above expression is equal to $|a_2| + 2\kappa(a_2)$. Theorem 2 follows when $n_1 = 0$, $n_2 = 0$, $a_1 \in R^\times$, $|a_2| < 1$, $|a_2| \neq |c_2|$ $\max\{|c_1|, |c_2|\} = 1$. □

4.4 Consider the case of $n_1 = 0$, $n_2 = 0$, a_1, a_2, $c_2 \in R^\times$, $|c_1| < 1$. As in the previous sections, the computation of $\mathrm{vol}(V^0_{n,d})$ is done in Lemma B.4.4.

Proof of theorem 2. Note that the $\mathrm{vol}(V^0_n)$ of Lemma B.4.4 is the same as the $\mathrm{vol}(V^0_{n,0})$ of Lemma B.4.1. Thus, $I_0(a, c)$ is equal to $-3q^{-3}(1 + q + q^2)$.

In our case $|ac|_E = 1$, $|a/\overline{a} - 1| = |a_2| = 1$, $|c/\overline{c} - 1| = |c_2| = 1$ and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2| = 1$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, b, c)$. Hence $\xi_0(a, 1, c)I_0(a, c) = 3$.

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1 - a_1c_2| + \kappa(a_2(a_2c_1 - a_1c_2))|c_2| + \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$

As each of the three terms on the right, namely the H-side, of the identity of the theorem, is equal to 1 (as $|a_2| = |c_2| = |a_2c_1 - a_1c_2| = 1$), Theorem 2 follows when $n_1 = n_2 = 0$, a_1, a_2, $c_2 \in R^\times$, and $|c_1| < 1$ □

5. Case of $n_1 = 1$, $n_2 = 0$

In this section $|.|$ is $|.|_F$, q is q_F, R is R_F, and the quadratic forms are

$$Q_1(x, y, z) = N(x) + a_1\boldsymbol{\pi}N(y) + c_1N(z),\ Q_2(x, y, z) = a_2\boldsymbol{\pi}N(y) + c_2N(z),$$

where $\max\{|a_1|, |a_2|\} = \max\{|c_1|, |c_2|\} = 1$. As in the case of $n_1 = n_2 = 0$, without loss of generality we assume that $|a_1| = 1$. Then, we have the following possible cases.

(5.1) $c_2 \in R^\times$;

(5.2) $|c_2| < 1$, $c_1 \in R^\times$, $|a_2\boldsymbol{\pi}| > |c_2|$;

(5.3) $|c_2| < 1$, $c_1 \in R^\times$, $|a_2\boldsymbol{\pi}| \leq |c_2|$.

5.1 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 0$, $c_2 \in R^\times$. The computation of $\mathrm{vol}(V^0_n)$ is done in Lemma C.5.1.

Proof of theorem 2. The integral $I_s(\boldsymbol{\pi}a, c)$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm}\,\mathrm{vol}(V^0_n).$$

By Lemma C.5.1, the sum is $\mathrm{vol}(V_0^0) + \frac{1}{q^{2m}}\mathrm{vol}(V_1^0)$. When $s = 0$, thus $m = -3/2$, this equals

$$1 + q^{-2} + q^{-4} \cdot q^3 = 1 + q^{-1} + q^{-2} = q \times q^{-3}(1 + q + q^2).$$

Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1+q+q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(\boldsymbol{\pi} a, 1, c)$. Moreover $|\boldsymbol{\pi} ac|_E^{-1/2} = q_E^{-1/2} = q^{-1}$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2| = 1$, and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2|$. Hence $\xi_0(\boldsymbol{\pi} a, 1, c)I_0(\boldsymbol{\pi} a, c)$ is equal to $-|a_2c_1 - a_1c_2||a_2|$. Note that this is equal to $-|a_2|$ when $|a_2c_1| < 1$ and to $-|a_2c_1 - a_1c_2|$ when $|a_2c_1| = 1$.

The H-side is equal to

$$-\kappa(a_2c_2)|a_2c_1 - a_1c_2| + \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$

When $|a_2c_1| < 1$ thus $|a_2c_1 - a_1c_2| = 1$, $\kappa(a_2(a_2c_1 - a_1c_2)) = \kappa(a_2)$, and $\kappa(c_2(a_2c_1 - a_1c_2)) = 1$, so the H-side is equal to

$$-\kappa(a_2c_2) + \kappa(a_2) - |a_2| = -|a_2|.$$

When $|a_2c_1| = 1$ we have $\kappa(a_2c_2) = 1$, and $-\kappa(a_2(a_2c_1 - a_1c_2)) + \kappa(c_2(a_2c_1 - a_1c_2)) = 0$, so the H-side is equal to

$$-|a_2c_1 - a_1c_2|.$$

This establishes the theorem for $n_1 = 1$, $n_2 = 0$, $c_2 \in R^\times$. □

5.2 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 0$, a_1, $c_1 \in R^\times$ and $|c_2| < |a_2\boldsymbol{\pi}|$. When $|a_2| = q^{-d}$, to emphasize we denote V_n by $V_{n,d}$. Similarly we denote V_n^0 by $V_{n,d}^0$. The computation of $\mathrm{vol}(V_{n,d}^0)$ is done in Lemma C.5.2.

Proof of theorem 2. Denote $I_s(\boldsymbol{\pi} a, c)$ by $I_{s,d}$, where $|a_2| = q^{-d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm}\,\mathrm{vol}(V_{n,d}^0).$$

We split (as in Lemma B.4.2) this sum into two: a finite sum over n, which we evaluate at once at $s = 0$ ($m = -3/2$), and an infinite sum, in which we first take big m, to have convergence, then add up the sum and only then evaluate the result at $m = -3/2$.

Thus, when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k}$ and $d = 0$, by Lemma C.5.2, we have the sum of

$$(1 - q^{-1}) + q^3(q^{-1} + q^{-2} - q^{-5}) + (1 - q^{-2})\left((1 + q^{-1} + q^{-2})\sum_{n=3}^{2k-1} q^{n+1} + \sum_{n=2}^{2k} q^n\right)$$

and

$$(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2k+2} + (1 - q^{-2})(1 + q^{-1})^2 \sum_{n=2k+2}^{\infty} \alpha^{n+1}.$$

The first sum is taken over the odd n ($3 \le n \le 2k-1$), the second over the even n ($2 \le n \le 2k$), and the third over all $n \ge 2k+2$, and $\alpha = q^{-2n(m+1)}$. We have

$$\sum_{n=3}^{2k-1} q^{n+1} = (1-q^{-2})^{-1}q^2(q^{2k-2}-1), \qquad \sum_{n=2}^{2k} q^n = (1-q^{-2})^{-1}(q^{2k}-1),$$

and

$$\sum_{n=2k+2}^{\infty} \alpha^{n+1} = -(1-\alpha^{-1})^{-1}\alpha^{2k+2}.$$

At $m=-3/2$, thus $\alpha = q$, the above expression equals to the sum of

$$1-q^{-1}+q^2+q-q^{-2}+(q^{2k}-1)+(1+q^{-1}+q^{-2})(q^{2k}-q^2)$$

and

$$((1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})-(1+q^{-1})^3)q^{2k+2}$$

Once simplified, this is equal to

$$-q^{-2}(1+q+q^2)(1+2q^{2k+1}).$$

When $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k-1}$ and $d=0$, by Lemma C.5.2, we have the sum of $(1-q^{-1}) + q^3(q^{-1}+q^{-2}-q^{-5})$ and

$$(1-q^{-2})(1+q^{-1}+q^{-2})\sum_{n=3}^{2k+1} q^{n+1} + (1-q^{-2})\sum_{n=2}^{2k} q^n + (1+q^{-1})q^{2k+2},$$

where the first sum is taken over the odd n ($3 \le n \le 2k+1$), the second over the even n ($2 \le n \le 2k$), and the third over all $n \ge 2k+2$. Once simplified, this is equal to

$$-q^{-2}(1+q+q^2)(1-2q^{2k}).$$

Thus we see that, when $d=0$, $I_s(a,c)$ is equal to

$$-q^{-2}(1+q+q^2)(1+2\kappa(c_2/(a_2\boldsymbol{\pi}))q^{2k+1}),$$

where $\kappa(c_2/(a_2\boldsymbol{\pi}))$ is 1 when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k}$ ($k \ge 1$) and -1 when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k-1}$ ($k \ge 0$).

When $|a_2| = q^{-d}$, $d \ge 1$, the sum

$$\sum_{n=0}^{d+1} q^{3n}\,\mathrm{vol}(V^0_{n,d})$$

by Lemma C.5.2, is equal to

$$1 - q^{-1} + q^3(q^{-1} - q^{-3} + q^{-4}) + (1 - q^{-2}) \sum_{n=2}^{d} q^{2n} + (1 - q^{-2})(1 + q^{-1} + q^{-2})q^{2d+2},$$

which is $q^{2d} + (1 - q^{-2})(1 + q + q^2)q^{2d}$

$$= (q^2 + q + 1 - q^{-1} - q^{-2})q^{2d} = q^{2d}(\operatorname{vol}(V^0_{0,0}) + q^3 \operatorname{vol}(V^0_{1,0})).$$

Moreover, since $\operatorname{vol}(V^0_{n,d}) = q^{-d} \operatorname{vol}(V^0_{n-d,0})$ for $n \geq d + 2$, we have

$$I_{s,d} = \sum_{n=0}^{\infty} q^{3n} \operatorname{vol}(V^0_{n,d}) = \sum_{n=0}^{d+1} q^{3n} \operatorname{vol}(V^0_{n,d}) + \sum_{n=d+2}^{\infty} q^{-2nm} \operatorname{vol}(V^0_{n,d})$$

$$= \sum_{n=0}^{d+1} q^{3n} \operatorname{vol}(V^0_{n,d}) + q^{-d(1+2m)} \sum_{j=0}^{\infty} q^{-2jm} \operatorname{vol}(V^0_{j,0}) - q^{-d(1+2m)}(\operatorname{vol}(V^0_{0,0}) + q^3 \operatorname{vol}(V^0_{1,0})).$$

The sum in the middle is $q^{-d(1+2m)} I_{s,0}$. At $s = 0$, thus $m = -3/2$, the other two terms cancel each other by the equality above, and we obtain that $I_{0,d} = q^{2d} I_{0,0}$, is equal to

$$-q^{-2}|a_2|^{-2}(1 + q + q^2)(1 + 2\kappa(c_2/(a_2\boldsymbol{\pi}))q^{2k+1})$$
$$= -q^{-2}(1 + q + q^2)|a_2|^{-1}(|a_2|^{-1} + 2\kappa(c_2/(a_2\boldsymbol{\pi}))|c_2|^{-1}).$$

In our case $|\boldsymbol{\pi} ac|_E^{1/2} = q^{-1}$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2|$, and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2| = |a_2|$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, b, c)$. Hence $\xi_0(\boldsymbol{\pi} a, 1, c)I_0(\boldsymbol{\pi} a, c)$ is equal to

$$|c_2| - 2\kappa(c_2/a_2)|a_2|,$$

where $\kappa(c_2/(a_2\boldsymbol{\pi})) = -\kappa(c_2/a_2)$.

The H-side is equal to

$$-\kappa(a_2c_2)|a_2c_1 - a_1c_2| + \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|$$
$$= -2\kappa(a_2c_2)|a_2| + \kappa(a_2^2)|c_2| = |c_2| - 2\kappa(a_2c_2)|a_2|.$$

Note that $\kappa(c_2/a_2) = \kappa(a_2c_2)$. Theorem 2 follows for $n_1 = 1$, $n_2 = 0$, a_1, $c_1 \in R^\times$, $|c_2| < 1$. □

5.3 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 0$, a_1, $c_1 \in R^\times$, $|c_2| < 1$ and $|\boldsymbol{\pi} a_2| \leq |c_2|$.

Proof of theorem 2. Denote $I_s(\boldsymbol{\pi} a, c)$ by $I_{s,d}$, where $|c_2| = q^{-d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm} \operatorname{vol}(V^0_{n,d}).$$

Applying Lemma C.5.3, when $|a_2\boldsymbol{\pi}| < |c_2|$ and $d = 1$, we have

$$(1 - q^{-1}) + q^3(q^{-1} + q^{-2} + q^{-4}) = q^2(1 + q^{-1} + q^{-2}).$$

Applying Lemma C.5.3, when $d \geq 2$, we have

$$1 - q^{-1} + q^3(q^{-1} - q^{-3} + q^{-4}) + (1 - q^{-2}) \sum_{n=2}^{d-1} q^{2n} + (1 + q^{-1})q^{2d}.$$

This is equal to

$$1 - q^{-1} + q^2 - 1 + q^{-1} + (q^{2d-2} - q^2) + (1 + q^{-1})q^{2d} = (1 + q^{-1} + q^{-2})q^{2d}.$$

When $|a_2\boldsymbol{\pi}| = |c_2|$, we split this sum into two: a finite sum over n, which we evaluate at once at $s = 0$ ($m = -3/2$), and an infinite sum, in which we first take big m, to have convergence, then add up the sum and only then evaluate the result at $m = -3/2$. Set $\alpha = q^{-2n(m+1)}$, thus when $m = -3/2$ we have $\alpha = q$. When $d = 1$, we have

$$(1 - q^{-1}) + q^3(q^{-1} + q^{-2} - q^{-3} - q^{-4} - q^{-5}) + (1 - q^{-2})(1 + q^{-1})^2 \sum_{n=2}^{\infty} \alpha^{1+n}.$$

$$= 1 - q^{-1} + q^2 + q^1 - 1 - q^{-1} - q^{-2} - q^2(1 + q^{-1})^3 = -(1 + q^{-1} + q^{-2})(1 + 2q).$$

When $d \geq 2$, we have

$$1 - q^{-1} + q^3(q^{-1} - q^{-3} + q^{-4}) + (1 - q^{-2}) \sum_{n=2}^{d-1} q^{2n} + (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2d}$$

$$+(1 - q^{-2})(1 + q^{-1})^2 q^d \sum_{n=d+1}^{\infty} \alpha^n.$$

This is equal to

$$q^2 + (q^{2d-2} - q^2) + (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2d} - (1 + q^{-1})^3 q^{2d}$$

$$= -(1 + q^{-1} + q^{-2})(1 + 2q)q^{2d-2}.$$

Thus we see that, when $|c_2| = q^{-d}$, $d \geq 0$, the integral $I_{0,d}$ is equal to

$$q^{-2}(1 + q + q^2)q^{2d} = q^{-2}(1 + q + q^2)|c_2|^{-2}$$

when $|a_2\boldsymbol{\pi}| < |c_2|$ and to

$$-q^{-2}(1+q+q^2)(1+2q)q^{2d-2} = -q^{-2}(1+q+q^2)(1+2|a_2||c_2|^{-1})|a_2|^{-2}$$

when $|a_2\boldsymbol{\pi}| = |c_2|$.

In our case $|\boldsymbol{\pi} ac|_E^{1/2} = q^{-1}$, $|a/\overline{a}-1| = |a_2|$, $|c/\overline{c}-1| = |c_2|$, and $|a/\overline{a}-c/\overline{c}| = |a_2c_1-a_1c_2|$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1+q+q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E+q_E^{1/2}+1)^{-1}$ in $\xi_0(a,b,c)$. Hence $\xi_0(\boldsymbol{\pi} a,1,c)I_0(\boldsymbol{\pi} a,c)$ is equal to

$$-|a_2||c_2|^{-1}|a_2c_1-a_1c_2| = -|a_2|,$$

when $|a_2| < |c_2|$ (thus $|a_2c_1-a_1c_2| = |c_2|$), to

$$-|a_2||c_2|^{-1}|a_2c_1-a_1c_2| = -|a_2c_1-a_1c_2|,$$

when $|a_2| = |c_2|$ and to

$$|c_2|+2|a_2|,$$

when $|a_2\boldsymbol{\pi}| = |c_2|$ (thus $|a_2c_1-a_1c_2| = |a_2|$).

The H-side is equal to

$$-\kappa(a_2c_2)|a_2c_1-a_1c_2| + \kappa(a_2(a_2c_1-a_1c_2))|c_2| - \kappa(c_2(a_2c_1-a_1c_2))|a_2|.$$

When $|a_2| < |c_2|$, thus $\kappa(a_2(a_2c_1-a_1c_2)) = \kappa(a_2c_2)$, and $\kappa(c_2(a_2c_1-a_1c_2)) = \kappa(c_2^2) = 1$, the H-side is equal to

$$-\kappa(a_2c_2)|c_2| + \kappa(a_2c_2)|c_2| - \kappa(c_2^2)|a_2| = -|a_2|.$$

When $|a_2| = |c_2|$, thus $\kappa(a_2(a_2c_1-a_1c_2))|c_2| - \kappa(c_2(a_2c_1-a_1c_2))|a_2| = 0$, and $\kappa(a_2c_2) = 1$, the H-side is equal to

$$-|a_2c_1-a_1c_2|.$$

When $|\boldsymbol{\pi} a_2| = |c_2|$ we have that $\kappa(a_2c_2) = -1$, $\kappa(a_2(a_2c_1-a_1c_2)) = \kappa(a_2^2) = 1$, and $\kappa(c_2(a_2c_1-a_1c_2)) = \kappa(c_2a_2) = -1$. Then the H-side is equal to

$$|a_2c_1-a_1c_2| + |c_2| + |a_2| = |c_2| + 2|a_2|.$$

Theorem 2 follows for $n_1 = 1$, $n_2 = 0$, a_1, $c_1 \in R^\times$, $|c_2| < 1$, and $|\boldsymbol{\pi} a_2| \le |c_2|$. $\square$

6. Case of $n_1 = 1$, $n_2 = 1$

In this section $|.|$ is $|.|_F$, q is q_F, R is R_F. The quadratic forms are

$$Q_1(x,y,z) = N(x) + a_1\boldsymbol{\pi}N(y) + c_1\boldsymbol{\pi}N(z),\ Q_2(x,y,z) = a_2\boldsymbol{\pi}N(y) + c_2\boldsymbol{\pi}N(z),$$

where without loss of generality $|a_1| = 1$, $\max\{|c_1|, |c_2|\} = 1$ and, as in 3,

$$V_n = \{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, \max\{|Q_1(x,y,z)|, |Q_2(x,y,z)|\} = q^{-n}\}.$$

Recall that $u = a_1c_2/(a_2c_1)$. Then, we have the following possible cases.
(6.1) c_1, a_2/c_2, $u \in R^\times$, and $|1-u| = 1$;
(6.2) c_1, a_2/c_2, $u \in R^\times$, and $|1-u| < 1$;
(6.3) $c_2 \in R^\times$, $|a_2| < 1$;
(6.4) $c_1 \in R^\times$, $|a_2| < 1$, $|c_2| < 1$ and $|a_2| \neq |c_2|$;
(6.5) c_2, $a_2 \in R^\times$, $|c_1| < 1$.
The case a_1, a_2, $c_1 \in R^\times$, $|c_2| < 1$ can be reduced to that of 6.3 on interchanging a, c.

6.1 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 1$, a_1, $c_1 \in R^\times$, $|u-1| = 1$, where $u = a_1c_2/a_2c_1$.

Proof of theorem 2. Denote $I_s(\boldsymbol{\pi}a, \boldsymbol{\pi}c)$ by $I_{s,d}$, where $|a_2| = |c_2| = q^{-d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm}\,\mathrm{vol}(V_{n,d}^0).$$

By Lemma D.6.1, this sum is finite, so we may evaluate at $s = 0$ already now, thus we take $m = -3/2$. When $d = 0$, we have

$$\mathrm{vol}(V_0^0) + q^3\,\mathrm{vol}(V_1^0) = 1 + q^3(q^{-2} + q^{-4}) = q^{-1}(1 + q + q^2).$$

By Lemma D.6.1, when $d \geq 1$, we have

$$1 + q^3(q^{-2} - q^{-3}) + (1 - q^{-2})\sum_{n=2}^{d} q^{2n} + (1 + q^{-1})q^{2d+1}.$$

This is equal to

$$q + q^{-1}(q^{2d} - q^2) + (q+1)q^{2d} = q^{-1}(1 + q + q^2)|a_2|^{-1}|c_2|^{-1}.$$

In our case $|\boldsymbol{\pi}^2ac|_E^{1/2} = q^{-2}$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2|$, and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2| = |a_2|$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1+q+q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a,1,c)$. Hence $\xi_0(\boldsymbol{\pi}a, 1, \boldsymbol{\pi}c)I_0(\boldsymbol{\pi}a, \boldsymbol{\pi}c)$ is equal to $-|a_2|$.

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1 - a_1c_2| - \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$

Since $|a_2| = |c_2|$, we have $\kappa(a_2c_2) = \kappa(a_2(a_2c_1 - a_1c_2)) = 1$, so the above expression is equal to $-|a_2|$. This establishes the theorem for $n_1 = 1$, $n_2 = 1$, a_1, c_1, a_2/c_2, $u - 1 \in R^\times$. □

6.2 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 1$, a_1, $c_1 \in R^\times$ and $|u-1| = q^{-2k-\varepsilon}$ ($k \geq 1$, $\varepsilon = 0, 1$) and $u = a_1c_2/a_2c_1$. When $|a_2| = |c_2| = q^{-d} \leq 1$ ($d \geq 0$) to emphasize as usual we denote V_n by $V_{n,d}$. Similarly we denote V_n^0 by $V_{n,d}^0$, thus $\mathrm{vol}(V_n^0)$ is $\mathrm{vol}(V_n^0)/(1-q^{-2})$. It is computed in Lemma D.6.2.

Proof of theorem 2. Denote $I_s(\boldsymbol{\pi} a, \boldsymbol{\pi} c)$ by $I_{s,d}$, where $|a_2| = |c_2| = q^{-d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm} \,\mathrm{vol}(V_{n,d}^0).$$

We split this sum into two: a finite sum over n, which we evaluate at once at $s = 0$ ($m = -3/2$), and an infinite sum, in which we first take big m, to have convergence, then add up the sum and only then evaluate the result at $m = -3/2$.

Applying Lemma D.6.2, when $|1-u| = q^{-2k}$ and $d = 0$, we have

$$1 + q^3(q^{-2} - q^{-3}) + (1-q^{-2})\left((1+q^{-1}+q^{-2})\sum_{n=3}^{2k-1} q^n + \sum_{n=2}^{2k} q^{n+1}\right) + (1+q^{-1})q^{2k+1}.$$

where the first sum is taken over the odd n ($3 \leq n \leq 2k-1$), the second over the even n ($2 \leq n \leq 2k$), and the third over all $n \geq 2k+2$. The above expression equals to

$$q + (1+q^{-1}+q^{-2})(q^{2k+1} - q) + (q^{2k-1} - q) + (1+q^{-1})q^{2k+1}$$

Once simplified, this is equal to

$$-q^{-3}(1+q+q^2)(1-2q^{2k}).$$

Applying Lemma D.6.2, when $|1-u| = q^{-2k-1}$ and $d = 0$, we have the sum of

$$1 + q^3(q^{-2} - q^{-3}) + (1-q^{-2})\left((1+q^{-1}+q^{-2})\sum_{n=3}^{2k+1} q^n + \sum_{n=2}^{2k} q^{n+1}\right) + (1+q^{-1})q^{2k+1}.$$

and

$$(1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{2k+3} + (1-q^{-2})(1+q^{-1})^2 q \sum_{n=2k+3}^{\infty} q^{-2(m+1)},$$

where the first sum is taken over the odd n ($3 \le n \le 2k+1$), the second over the even n ($2 \le n \le 2k$), and the third over all $n \ge 2k+2$. Taking the sum and evaluating at $m = -3/2$, the above expression equals to

$$q + (1+q^{-1}+q^{-2})(q^{2k+1}-q) + (q^{2k+1}-q) + (1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{2k+3} \\ -(1+3q^{-1}+3q^{-2}+q^{-3})q^{2k+3}.$$

Once simplified, this is equal to

$$-q^{-1}(1+q+q^2)(1+2q^{2k+1}).$$

When $d \ge 1$, by Lemma D.6.2, we have

$$\sum_{n=0}^{d+1} q^{3n} \operatorname{vol}(V^0_{n,d}) = 1 + q^3(q^{-2}-q^{-3}) + (1-q^{-2})q^{-1}\sum_{n=2}^{d+1} q^{2n} = q^{2d+1}.$$

Thus, when $d \ge 1$, we have

$$\sum_{n=0}^{d+1} q^{3n} \operatorname{vol}(V^0_{n,d}) = q^{2d}(\operatorname{vol}(V^0_{0,0}) + q^3 \operatorname{vol}(V^0_{1,0})).$$

Moreover, since $\operatorname{vol}(V^0_{n,d}) = q^{-d}\operatorname{vol}(V^0_{n-d,0})$ for $n \ge d+2$, we apply same computations as in section 5.2 to obtain that at $s = 0$, we have

$$I_{0,d} = q^{2d} I_{0,0}.$$

Thus we see that, when $d \ge 0$, $I_0(\boldsymbol{\pi} a, \boldsymbol{\pi} c) = I_{0,d}$ is equal to

$$-q^{-1}(1+q+q^2)(1-\kappa(1-u)2|1-u|^{-1})|a_2|^{-1}|c_2|^{-1},$$

where $\kappa(1-u)$ is 1 when $|1-u| = q^{-2k}$ and -1 when $|1-u| = q^{-2k-1}$.

In our case $|\boldsymbol{\pi}^2 ac|_E^{1/2} = q^{-2}$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2|$ and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2| = |a_2||1-u|$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1+q+q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a,1,c)$. Hence $\xi_0(\boldsymbol{\pi} a, 1, \boldsymbol{\pi} c) I_0(\boldsymbol{\pi} a, \boldsymbol{\pi} c)$ is equal to

$$|a_2c_1 - a_1c_2| - 2\kappa(1-u)|a_2|.$$

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1 - a_1c_2| - \kappa(a_2(a_2c_1-a_1c_2))|c_2| - \kappa(c_2(a_2c_1-a_1c_2))|a_2|.$$

Note that $\kappa(a_2(a_2c_1 - a_1c_2)) = \kappa(1-u)$, $|a_2| = |c_2|$ and so $\kappa(a_2c_2) = 1$. Thus the expression above is equal to $|a_2c_1 - a_1c_2| - 2\kappa(1-u)|a_2|$. This establishes the theorem for $n_1 = 1$, $n_2 = 1$, a_1, c_1, $a_2/c_2 \in R^\times$ and $|1-u| < 1$. □

6.3 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 1$, a_1, $c_2 \in R^\times$, $|c_1| \le 1$, $|a_2| < 1$.

Proof of theorem 2. The integral $I_s(\boldsymbol{\pi} a, \boldsymbol{\pi} c)$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm} \operatorname{vol}(V_n^0).$$

Applying Lemma D.6.3, the sum is $\operatorname{vol}(V_0^0) + \frac{1}{q^{2m}} \operatorname{vol}(V_1^0)$. When $s = 0$, thus $m = -3/2$, this equals

$$1 + (q^{-2} + q^{-4}) \cdot q^3 = q^{-1}(1 + q + q^2).$$

Moreover $|\boldsymbol{\pi}^2 ac|_E^{1/2} = q^{-2}$, $|a/\overline{a} - 1| = |a_2|$, $|c/\overline{c} - 1| = |c_2| = 1$, and $|a/\overline{a} - c/\overline{c}| = |a_2c_1 - a_1c_2| = 1$. Here q is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is cancelled by the factor $-q_E^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(\boldsymbol{\pi} a, 1, \boldsymbol{\pi} c)$. Hence $\xi_0(\boldsymbol{\pi} a, 1, \boldsymbol{\pi} c) I_0(\boldsymbol{\pi} a, \boldsymbol{\pi} c) = -|a_2|$.

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1 - a_1c_2| - \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$

Note that $\kappa(a_2(a_2c_1 - a_1c_2)) = \kappa(a_2c_2) = \kappa(a_2)$, and $\kappa(c_2(a_2c_1 - a_1c_2)) = 1$, thus the H-side is equal to $-|a_2|$.

This establishes the theorem for $n_1 = 1$, $n_2 = 1$, a_1, $c_2 \in R^\times$, $|a_2| < 1$. □

6.4 We proceed to prove the theorem for $n_1 = 1$, $n_2 = 1$, a_1, $c_1 \in R^\times$, $|a_2| < 1$, $|c_2| < 1$ and $|a_2| \neq |c_2|$. Set $|c_2| = q^{-k}$.

Proof of theorem 2. The integral $I_s(\boldsymbol{\pi} a, \boldsymbol{\pi} c)$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm} \operatorname{vol}(V_n^0).$$

Applying Lemma D.6.4, when $s = 0$, thus $m = -3/2$, this equals

$$\sum_{n=0}^{k+1} q^{3n} \operatorname{vol}(V_n^0).$$

When $|c_2| = q^{-k}$, we have the sum

$$1 + q^3(q^{-2} - q^{-3}) + (1 - q^{-2})q^{-1} \sum_{n=2}^{k} q^{2n} + (1 + q^{-1})q^{-1}q^{2(k+1)},$$

which is equal to

$$q+(q^2-1)q^{-3}q^4(q^{2k-2}-1)(q^2-1)^{-1}+(1+q^{-1})q^{-1}q^{2k+2}$$

$$=q^{2k}q^{-1}(1+q+q^2)=q^{-1}(1+q+q^2)|c_2|^{-2}.$$

Moreover $|\boldsymbol{\pi}^2 ac|_E^{1/2}=q^{-2}$, $|a/\overline{a}-1|=|a_2|$, $|c/\overline{c}-1|=|c_2|$, and $|a/\overline{a}-c/\overline{c}|=|a_2c_1-a_1c_2|=|c_2|$ (since we assumed $|a_2|<|c_2|$). Here q is $q_F=q_E^{1/2}$, so $-q^{-3}(1+q+q^2)$ is canceled by the factor $-q_E^{3/2}(q_E+q_E^{1/2}+1)^{-1}$ in $\xi_0(\boldsymbol{\pi} a,1,\boldsymbol{\pi} c)$. Hence $\xi_0(\boldsymbol{\pi} a,1,\boldsymbol{\pi} c)I_0(\boldsymbol{\pi} a,\boldsymbol{\pi} c)=-|a_2|$.

The H-side is equal to

$$\kappa(a_2c_2)|a_2c_1-a_1c_2|-\kappa(a_2(a_2c_1-a_1c_2))|c_2|-\kappa(c_2(a_2c_1-a_1c_2))|a_2|.$$

Note that $|a_2c_1-a_1c_2|=|c_2|$, and $\kappa(c_2(a_2c_1-a_1c_2))=1$, thus the H-side is equal to $-|a_2|$. This establishes the theorem for $n_1=1$, $n_2=1$, a_1, $c_1\in R^\times$, $|a_2|<1$, $|c_2|<1$, and $|a_2|\neq|c_2|$. □

6.5 We proceed to prove the theorem for $n_1=1$, $n_2=1$, a_2, $c_2\in R^\times$, $|c_1|<1$.

Proof of theorem 2. According to Lemma D.6.5, the proof is identical to that of section 6.3. This establishes the theorem for $n_1=1$, $n_2=1$, a_2, $c_2\in R^\times$, $|c_1|<1$. □

A. Appendix

In this section, we state and prove various auxiliary lemmas which are used in the proof.

Lemma A.1. *When E/F is unramified, $c\in R^\times$, $n\geq 1$ and $0\leq 2k<n$, we have*

$$\mathrm{vol}_{dz}(\{z\in E;|N(z)-c\boldsymbol{\pi}^{2k}|\leq q^{-n}\})=q^{-n}(1+q^{-1})$$

and $\mathrm{vol}_{dz}(\{z\in E;|N(z)-c\boldsymbol{\pi}^{2k}|=q^{-n}\})=q^{-n}(1-q^{-2})$.

Proof. Using the exact sequence $1\to E^1\to R_E^\times\to R^\times\to 1$ we see that $\mathrm{vol}_{dz}(E^1)$ is $1+q^{-1}$. Hence $\mathrm{vol}_{dz}(\{z\in E;|N(z)-1|\leq q^{-n}\})$ is $(1+q^{-1})q^{-n}$. This deals with the case of $k=0$. But

$$\int_{\{z\in E;|N(z)-c\boldsymbol{\pi}^{2k}|\leq|\boldsymbol{\pi}^n|\}}dz=q^{-2k}\int_{\{z\in E;|N(z)-c|\leq|\boldsymbol{\pi}^{n-2k}|\}}dz=q^{-n}(1+q^{-1}),$$

as required. □

Lemma A.2. *When E/F is unramified, $c \in R^\times$, $k \geq 1$. Let $u \in R^\times$ such that $|u-1| = q^{-k}$. Then we have*

$$\mathrm{vol}_{dz}(\{z \in E; |N(z)+cu| = q^{-k}, |N(z)+c| = q^{-k-t}\}) = q^{-k-t}(1-q^{-2}),$$

when $t \geq 1$, and

$$\mathrm{vol}_{dz}(\{z \in E; |N(z)+cu| = q^{-k}, |N(z)+c| = q^{-k}\}) = q^{-k}(1-(q+2)q^{-2}).$$

Proof. Let $u = 1 + \boldsymbol{\pi}^k \varepsilon$, where $\varepsilon \in R^\times$. According to Lemma A.1, we have

$$\mathrm{vol}_{dz}(\{z \in E; |N(z)+cu| = q^{-k}\}) = q^{-k}(1-q^{-2}).$$

Note that $|N(z)+c| \leq q^{-k}$, so our set equals to the disjoint union

$$\bigcup_{t=0}^{\infty} \{z \in E; |N(z)+cu| = q^{-k}, |N(z)+c| = q^{-k-t}\}.$$

Suppose that $N(z) + cu = \boldsymbol{\pi}^k N(w)$ where $w \in R_E^\times$. Then

$$N(z) + c = \boldsymbol{\pi}^k (N(w) - \varepsilon c).$$

Thus the equality $|N(z)+cu| = q^{-k}$ is equivalent to $N(w) = 1$ and $|N(z)+c| = q^{-k-t}$ is equivalent to $|N(w) - \varepsilon c| = q^{-t}$, where $|\varepsilon c| = 1$. By Lemma A.1, when $t \geq 1$ we have

$$\mathrm{vol}_{dw}(\{w \in E; |N(w) - \varepsilon c| = q^{-t}\}) = q^{-t}(1-q^{-2})$$

and moreover

$$\mathrm{vol}_{dw}(\{w \in E; |N(w)| = 1, |N(w) - \varepsilon c| = 1\}) = 1 - (q+2)q^{-2}.$$

The lemma follows. □

Definition. For any integer n, we set $\sigma(n)$ to be 0 if n is even and 1 if n is odd.

Our method of computation of volumes is that of "divide and rule". A first example is:

Lemma A.3. *When E/F is unramified and $n \geq j \geq 1$, the volume of the set*

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(y)+N(z)| \leq q^{j-n}, |N(x)+N(y)+N(z)| = q^{-n}\},$$

is equal to

$$(1-q^{-2})^2 q^{j-\sigma(n-j)-2n}.$$

Proof. The volume of this set is given by the integral

$$\int_{|N(y)|=1}\int_{|N(y)+N(z)|\le q^{j-n}}\int_{|N(x)+c|=q^{-n}} dxdzdy,$$

where $c = c(y,z) = N(y) + N(z)$. It is equal to

$$\sum_{t=n-j}^{\infty}\int_{|N(y)|=1}\int_{|N(y)+N(z)|=q^{-t}}\int_{|N(x)+c_t|=q^{-n}} dxdzdy,$$

where $|c_t| = q^{-t}$. Note that the integral over x depends only on $|c_t|$. Thus we can assume that $c_t = \boldsymbol{\pi}^t c_0$, where c_0 is an arbitrary element of $R^\times$ and take the integral over y and z, so that it is now equal to

$$(1-q^{-2})^2\sum_{t=n-j}^{\infty} q^{-t}\int_{|N(x)+c_t|=q^{-n}} dx.$$

When $t \ge n+1$, thus $|c_t| < q^{-n}$ and the volume of the set $|N(x)| = q^{-n}$ is equal to $q^{-n}(1-q^{-2})$ when n is even and to zero when n is odd. The contribution from this term is zero when n is odd and

$$(1-q^{-2})^2(1+q^{-1})q^{-1-2n}.$$

when n is even. When $t = n$, the volume of the set $\{|N(x) + c_n| = q^{-n}\}$ is equal to $q^{-1-n}(1-(q+1)q^{-2})$ when n is even and to q^{-n-2} when n is odd. Thus, the contribution of all terms with $t \ge n-1$ is the product of $(1-q^{-2})^2$ and

$$(1-(1+q)q^{-2})q^{-2n} + (q^{-1}+q^{-2})q^{-2n} = q^{-2n}$$

when n is even and q^{-2n-1} when n is odd.

When $n-j \le t \le n-1$, the volume of the set $\{|N(x)+c_t| = q^{-n}\}$ is equal to zero when t is odd and to $q^{-n}(1-q^{-2})$ when t is even. Thus, when n is even, the even values of t $(n-j \le t \le n-2)$ are of the form $n-j+\sigma(n-j), \dots, n-4, n-2$, and the sum is

$$q^{-n}(q^2+q^4+\ldots+q^{j-\sigma(n-j)}) = q^{-n}(1-q^{-2})^{-1}(q^{j-\sigma(n-j)}-1).$$

When n is odd, the even values of t $(n-j \le t \le n-1)$ are of the form $n-j+\sigma(n-j), \dots, n-3, n-1$, and the sum is

$$q^{-n}(q^1+q^3+\ldots+q^{j-\sigma(n-j)}) = q^{-n-1}(1-q^{-2})^{-1}(q^{j-\sigma(n-j)+1}-1).$$

Thus, the contribution of all terms with $n-j \le t \le n-2$ is the product of $(1-q^{-2})^2$ and

$$q^{-2n}(q^{j-\sigma(n-j)}-1),$$

when n is even and

$$q^{-2n-1}(q^{j-\sigma(n-j)-1}+1)$$

when n is odd. The lemma follows. □

Lemma A.4. *When E/F is unramified and $n \geq j \geq 1$, the volume of the set*

$$\{(x,y,z) \in R_E^3;\, |N(y)| = 1, |N(y)+N(z)| \leq q^{j-n}, |N(x)+\boldsymbol{\pi}(N(y)+N(z))| = q^{-n}\},$$

is equal to

$$(1-q^{-2})^2 q^{j+\sigma(n-j)-2n-1}.$$

Proof. The volume of this set is given by the integral

$$\int_{|N(y)|=1}\int_{|N(y)+N(z)|\leq q^{j-n}}\int_{|N(x)+c|=q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$. It is equal to

$$\sum_{t=n-j}^{\infty}\int_{|N(y)|=1}\int_{|N(y)+N(z)|=q^{-t}}\int_{|N(x)+c_t|=q^{-n}} dxdzdy,$$

where $|c_t| = q^{-t-1}$. Note that the integral over x depends only on $|c_t|$. Thus we can assume that c_t is an arbitrary element so that $|c_t| = q^{-t-1}$ and take the integral over y and z, so that it is now equal to

$$(1-q^{-2})^2\sum_{t=n-j}^{\infty} q^{-t}\int_{|N(x)+c_t|=q^{-n}} dx.$$

When $t \geq n$, thus $|c_t| < q^{-n}$ and the volume of the set $|N(x)| = q^{-n}$ is equal to $q^{-n}(1-q^{-2})$ when n is even and to zero when n is odd. The contribution from this term is zero when n is odd and

$$(1-q^{-2})^2(1+q^{-1})q^{-2n}.$$

when n is even. When $t = n-1$, the volume of the set $\{|N(x)+c_{n-1}| = q^{-n}\}$ is equal to $q^{-n}(1-(q+1)q^{-2})$ when n is even and to q^{-n-1} when n is odd. Thus, the contribution of all terms with $t \geq n-1$ is the product of $(1-q^{-2})^2$ and

$$(1-(1+q)q^{-2})q^{1-2n} + (q^{-1}+q^{-2})q^{1-2n} = q^{1-2n}$$

when n is even and q^{-2n} when n is odd.

When $n-j \leq t \leq n-2$, the volume of the set $\{|N(x)+c_t| = q^{-n}\}$ is equal to zero when t is even and to $q^{-n}(1-q^{-2})$ when t is odd. Thus, when n is even, the odd values of t ($n-j \leq t \leq n-2$) are of the form $n-j+1-\sigma(n-j), \ldots, n-5, n-3$, and the sum is

$$q^{-n}(q^3+q^5+\ldots+q^{j-1+\sigma(n-j)}) = q^{1-n}(1-q^{-2})^{-1}(q^{j+\sigma(n-j)-2}-1).$$

When n is odd, the odd values of t ($n-j \leq t \leq n-2$) are of the form $n-j+1-\sigma(n-j), \ldots, n-4, n-2$, and the sum is

$$q^{-n}(q^2+q^4+\ldots+q^{j-1+\sigma(n-j)}) = q^{-n}(1-q^{-2})^{-1}(q^{j+\sigma(n-j)-1}-1).$$

Thus, the contribution of all terms with $n-j \leq t \leq n-2$ is the product of $(1-q^{-2})^2$ and

$$q^{1-2n}(q^{j+\sigma(n-j)-2}-1),$$

when n is even and

$$q^{-2n}(q^{j+\sigma(n-j)-1}-1)$$

when n is odd. The lemma follows. □

Lemma A.5. *When E/F is unramified, the volume of the set*

$$\{(x,y,z)\in R_E^3; ||(x,y,z)||_E=1,\ |N(x)+N(y)+N(z)|=q^{-n}\},$$

is equal to

$$(1-q^{-2})^2q^{-n}(1-q^{-1}+q^{-2}),$$

when $n\geq 1$ and to

$$(1-q^{-2})(1-q^{-1}+q^{-2}),$$

when $n=0$.

Proof. Consider the case of $n\geq 1$. Then this set is the union of the following subsets according to $|N(x)|=1$, and three subsets corresponding to $|N(x)+N(y)|>q^{-n}$, $|N(x)+N(y)|=q^{-n}$, and $|N(x)+N(y)|<q^{-n}$, and the subset which corresponds to $|N(x)|<1$.
(A) *Case of* $|N(x)|=1$ *and* $|N(x)+N(y)|>q^{-n}$. Note that when $|N(x)+N(y)|=q^{-j}>q^{-n}$, then j should be even and the volume of this subset is given by the following integral

$$\sum_{k=0}^{(n+\sigma(n))/2-1}\int_{|N(x)|=1}\int_{|N(x)+N(y)|=q^{-2k}}\int_{|N(z)+c|=q^{-n}}dzdxdy,$$

where $c=N(x)+N(y)$. Note that when $k=0$, the integral over y over the set $|N(x)+N(y)|=1$ is equal to $(1-(q+1)q^{-2})$. When $k\geq 5$, applying Lemma A.1, the sum over k is equal to

$$(1-q^{-2})^3q^{-n}\left(\sum_{k=1}^{(n+\sigma(n))/2-1}q^{-2k}\right)=(1-q^{-2})^2q^{-n}(q^{-2}-q^{-n-\sigma(n)})$$

and to $(1-q^{-2})^3q^{-n}\times q^{-2}$ when $n\in\{3,4\}$. Note that this sum is empty when $n\in\{1,2\}$. When $n\geq 5$, the value of the integral is equal to

$$(1-q^{-2})^2q^{-n}(1-q^{-1}-q^{-2})+(1-q^{-2})^2q^{-n}(q^{-2}-q^{-n-\sigma(n)}).$$

When $n=3$ or 4, this integral is

$$(1-q^{-2})^2q^{-n}(1-q^{-1}-q^{-2})+(1-q^{-2})^3q^{-n}\times q^{-2}.$$

When $n=1$ or 2, this integral is

$$(1-q^{-2})^2q^{-n}(1-q^{-1}-q^{-2}).$$

(B) *Case of* $|N(x)|=1$ *and* $|N(x)+N(y)|=q^{-n}$, and $n\geq 1$. The volume of this subset is given by the following integral

$$\int_{|N(x)|=1}\int_{|N(x)+N(y)|=q^{-n}}\int_{|N(z)+c|=q^{-n}}dzdxdy,$$

where $c = N(x) + N(y)$. Applying Lemma A.1, when n is even this integral is equal to

$$(1 - q^{-2})^2 q^{-2n}(1 - q^{-1} - q^{-2}).$$

When n is odd the integral over z runs over the set $|N(z)| \leq q^{-n-1}$, and this integral is equal to

$$(1 - q^{-2})^2 q^{-2n-1}.$$

(C) *Case of* $|N(x)| = 1$ *and* $|N(x) + N(y)| < q^{-n}$ and $n \geq 2$. The volume of this subset is given by the following integral

$$\int_{|N(x)|=1} \int_{|N(x)+N(y)| \leq q^{-n-1}} \int_{|N(z)|=q^{-n}} dz dx dy.$$

Applying Lemma A.1, this integral is equal to

$$(1 - q^{-2})^2 (1 + q^{-1}) q^{-2n-1}.$$

When n is odd the contribution from this subcase is zero.
(D) *Case of* $|N(x)| < 1$ and $n \geq 1$. The volume of this subset is given by the following integral

$$\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(z)+c|=q^{-n}} dz dx dy,$$

where $|c| = |N(x) + N(y)| = 1$. Applying Lemma A.1, this integral is equal to

$$(1 - q^{-2})^2 q^{-2} q^{-n}.$$

Thus the contribution from cases (B),(C) and (D) is equal to

$$(1 - q^{-2})^2 q^{-2n} + (1 - q^{-2})^2 q^{-2-n}.$$

Adding the contribution from (A) when $n = 2$, $n = 4$, and $n \geq 6$ the lemma follows.

When $n \geq 5$ is odd, we have the contribution from (A), (B) and (D) is the product of $(1 - q^{-2})^2 q^{-n}$ and

$$(1 - q^{-1} - q^{-2}) + (q^{-2} - q^{-n-1}) + q^{-n-1} + q^{-2} = 1 - q^{-1} + q^{-2}.$$

When $n = 3$, the contribution from (A),(B) and (D) is the product of $(1 - q^{-2})^2 q^{-n}$ and

$$(1 - q^{-1} - q^{-2}) + (1 - q^{-2}) q^{-2} + q^{-4} + q^{-2}.$$

When $n = 1$, the contribution from (A),(B) and (D) is

$$(1 - q^{-2})^2 q^{-n} (1 - q^{-1} - q^{-2} + q^{-2} + q^{-2}).$$

Consider the case of $n = 0$. Then this set is the union of the following subsets according to $|N(x) + N(y)| < 1$ and $|N(x) + N(y)| = 1$. The corresponding integral is equal to

$$\int_{|N(x)|<1}\int_{|N(y)|<1}\int_{|N(z)|=1} dzdydx + \int_{|N(x)|=1}\int_{|N(x)+N(y)|<1}\int_{|N(z)|=1} dzdydx +$$

$$\int_{|N(x)|<1}\int_{|N(y)|=1}\int_{|N(z)+c|=1} dzdydx + \int_{|N(x)|=1}\int_{|N(y)+N(y)|=1}\int_{|N(z)+c|=1} dzdydx,$$

where $c = N(x) + N(y)$ so that $|c| = 1$. This is equal to the product of $1 - q^{-2}$ and

$$q^{-4} + (1 - q^{-2})q^{-1}(1 + q^{-1}) + (1 - q^{-1} - q^{-2})(q^{-2} + 1 - q^{-1} - q^{-2}) = 1 - q^{-1} + q^{-2}.$$

The lemma follows. □

B. Appendix to Section 4

This section contains the computations used in section 4.

Lemma B.4.1. *Suppose that c_1, u, $u - 1$ are units in $R^\times$ and $|a_2| = |c_2| = q^{-d}$, where $d \geq 0$. Then*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1 - 2q^{-3}, & \text{if } n = 0,\ d = 0, \\ 1 - q^{-1} + q^{-2}, & \text{if } n = 0,\ d \geq 1, \\ (1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}), & \text{if } 1 \leq n \leq d - 1, \\ (1 - q^{-2} - q^{-3} - q^{-4})q^{-n}, & \text{if } n = d,\ d \geq 1, \\ (1 - q^{-2})(1 + q^{-1})^2 q^{d-2n}, & \text{if } n \geq d + 1. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. Thus we have that $Q_1 = Q_1(x, y, z) = N(x) + N(y) + N(z)$ and $Q_2 = Q_2(x, y, z) = a_2 a_1^{-1} N(y) + c_2 c_1^{-1} N(z)$. As $|a_2/a_1| = |c_2/c_1| = q^{-d}$, and we are interested only in $|Q_2|$, we can assume that $Q_2(x, y, z) = \boldsymbol{\pi}^d(N(y) + uN(z))$, where $u \in R^\times$ has $|u - 1| = 1$.

We first compute $\mathrm{vol}(V^0_{0,d})$. Here

$$V_{0,d} = \{(x, y, z) \in R_E^3; \max\{|N(x) + N(y) + N(z)|, |\boldsymbol{\pi}^d(N(y) + uN(z))|\} = 1\}.$$

When $d = 0$, we compute the volume of the complement of $V_{0,0}$ in $\{v \in R_E^3; ||v||_E = 1\}$:

$$\{(x, y, z) \in R_E^3; ||(x, y, z)||_E = 1, |N(x) + N(y) + N(z)| < 1, |N(y) + uN(z)| < 1\}.$$

Note that if $|N(x)| < 1$ then $\max\{|N(y)|, |N(z)|\} = 1$ implies

$$\max\{|N(x)+N(y)+N(z)|, |N(y)+uN(z)|\} = 1.$$

Thus $|N(x)| = 1$, and consequently $|N(y)| = |N(z)| = 1$. Thus the volume of the complement is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|<1} \int_{|N(x)+c|<1} dxdydz,$$

where $c = c(y,z) = N(y)+N(z)$ and $|c| = 1$. From Lemma A.1 it follows that the volume of the subset $\{|N(x)+c| < 1\}$ where $|c| = 1$ is equal to $(q+1)/q^2 = q^{-1}(1+q^{-1})$. The above integral is equal to

$$(1-q^{-2})q^{-2}(1+q^{-1})^2.$$

As the volume of $\{v \in R_E^3; ||v||_E = 1\}$ is $(1-q^{-2})(1+q^{-2}+q^{-4})$, the volume of $V_{0,0}$ equals to the product of $1-q^{-2}$ and the difference

$$1+q^{-2}+q^{-4}-q^{-2}(1+q^{-1})^2 \;=\; 1-2q^{-3}.$$

When $d > 0$, we have

$$V_{0,d} = \{(x,y,z) \in R_E^3; |N(x)+N(y)+N(z)| = 1\}.$$

To compute its volume we apply Lemma A.5.

To compute $\mathrm{vol}(V_{n,d}^0)$ $(1 \le n \le d-1)$ recall that since $|\boldsymbol{\pi}^d(N(y)+uN(z))| < q^{-n}$, we have that

$$V_n = \{(x,y,z) \in R_E^3;\ |N(x)+N(y)+N(z)| = q^{-n},\ ||(x,y,z)||_E = 1\}.$$

To compute its volume, we apply Lemma A.5.

To compute $\mathrm{vol}(V_{n,d}^0)$ $(n = d)$ recall that $V_{n,d}$ is

$$\{(x,y,z) \in R_E^3; \max\{|N(x)+N(y)+N(z)|, |\boldsymbol{\pi}^d(N(y)+uN(z))|\} = q^{-n},\ ||(x,y,z)||_E = 1\}.$$

The set $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3;\ |N(x)+N(y)+N(z)| \le q^{-n-1}, |N(y)+uN(z)| = 1\}.$$

and

$$\{(x,y,z) \in R_E^3;\ |N(x)+N(y)+N(z)| = q^{-n}, |N(y)+uN(z)| \le 1\}.$$

We will compute the volume of each subset.

(A) *Case of* $|N(x)+N(y)+N(z)| \le q^{-n-1}$ and $|N(y)+uN(z)| = 1$. We consider the subcases of $|N(z)| = 1$ and $|N(z)| < 1$. When $|N(z)| = 1$ then $|N(y)+N(z)|$ can be equal or less than 1. Moreover the condition $|N(y)+N(z)| < 1$ implies that $|N(y)+uN(z)| = 1$.

The volume of the set $\{|N(y)+N(z)|=1, |N(y)+uN(z)|=1\}$, when $|1-u|=1$ is equal to $1-2(q+1)q^{-2}$. Thus, the contribution from $|N(z)|=1$ is given by two integrals.

The first integral is

$$\int_{|N(z)|=1}\int_{|N(y)+N(z)|\le q^{-1}}\int_{|N(x)+N(y)+N(z)|\le q^{-n-1}} dxdydz.$$

Applying Lemma A.1 to the integral over y and Lemma A.3, with $j=n-1$, to the integral over x, this is equal to the infinite sum

$$(1-q^{-2})^2q^{-2}\sum_{m=n+1}^{\infty}q^{-m}=(1-q^{-2})q^{-n-3}(1+q^{-1}).$$

The second integral is

$$\int_{|N(z)|=1}\int_{|N(y)+N(z)|=1}\int_{|N(x)+N(y)+N(z)|\le q^{-n-1}} dxdydz.$$

Similarly to the first integral, applying Lemma A.3, with $j=n-1$ to the integral over x, and taking the infinite sum, its value is

$$(1-q^{-2})q^{-n-1}(1+q^{-1})(1-2q^{-1}-2q^{-2}).$$

The contribution from $|N(z)|<1$ is given by the integral

$$\int_{|N(z)|<1}\int_{|N(y)|=1}\int_{|N(x)+N(y)+N(z)|\le q^{-n-1}} dxdydz,$$

which is equal to

$$(1-q^{-2})q^{-n-3}(1+q^{-1}).$$

(B) *Case of* $|N(x)+N(y)+N(z)|=q^{-n}$ and $|N(y)+uN(z)|\le 1$. Note that $|N(y)+uN(z)|\le 1$ is always satisfied, and according to Lemma A.5, the volume of this set is equal to

$$(1-q^{-2})^2q^{-n}(1-q^{-1}+q^{-2}).$$

Adding the contributions from cases (A) and (B), then dividing by $1-q^{-2}$, we obtain the product of q^{-n} and

$$2(q^{-3}+q^{-4})+(q^{-1}+q^{-2})(1-2q^{-1}-2q^{-2})+(1-q^{-2})(1-q^{-1}+q^{-2}),$$

which is equal to

$$1-q^{-2}-q^{-3}-q^{-4}.$$

To compute $\mathrm{vol}(V^0_{n,d})$ $(n \geq d+1)$ recall that $V_{n,d}$ is the set of $(x,y,z) \in R^3_E$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x)+N(y)+N(z)|, |\boldsymbol{\pi}^d(N(y)+uN(z))|\} = q^{-n}.$$

Assume that $|N(z)| < 1$. Then $|N(y)+uN(z)| < 1$ implies that $|N(y)| < 1$ and from $|N(x)+N(y)+N(z)| < 1$ it follows that $|N(x)| < 1$ which is a contradiction. Thus $|N(z)| = 1$ and $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z) \in R^3_E;\, |N(z)| = 1, |N(x)+N(y)+N(z)| \leq q^{-n}, |N(y)+uN(z)| = q^{d-n}\}$$

and

$$\{(x,y,z) \in R^3_E;\, |N(z)| = 1, |N(x)+N(y)+N(z)| = q^{-n}, |N(y)+uN(z)| \leq q^{d-n-1}\}.$$

We will compute the volume of each of them.

Note that since $|u-1| = 1$ and $|N(y)+uN(z)| < 1$, we have

$$|N(y)+N(z)| = |N(y)+uN(z)+(1-u)N(z)| = |(1-u)N(z)| = 1.$$

(A) *Case of* $|N(x)+N(y)+N(z)| \leq q^{-n}$ and $|N(y)+uN(z)| = q^{d-n}$. The volume of this subset is equal to the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=q^{d-n}} \int_{|N(x)+c| \leq q^{-n}} dxdydz,$$

where $c = c(y,z) = N(y)+N(z)$ so that $|c| = 1$. By Lemma A.1 the integral is equal to:

$$(1-q^{-2}) \times (1-q^{-2})q^{d-n} \times (1+q^{-1})q^{-n}.$$

(B) *Case of* $|N(x)+N(y)+N(z)| = q^{-n}$ and $|N(y)+uN(z)| \leq q^{d-n-1}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)| \leq q^{d-n-1}} \int_{|N(x)+c| = q^{-n}} dxdydz,$$

where again $c = c(y,z) = N(z)+N(y)$ and $|c| = 1$. Applying Lemma A.1 and its Corollary the above integral is equal to:

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n-1} \times (1-q^{-2})q^{-n}.$$

Adding the expressions of (A) and (B), then dividing by $(1-q^{-2})$, the $\mathrm{vol}(V^0_{n,d})$ is equal to

$$(1-q^{-2})(1+q^{-1})^2 q^{d-2n}.$$

The lemma follows. □

Lemma B.4.2. *Suppose that c_1, u, are units in $R^\times$, and $|a_2| = |c_2| = q^{-d}$, where $d \geq 0$. When $|u-1| = q^{-2k}$ $(k \geq 1)$, we have*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1+q^{-2}-q^{-3}, & \text{if } n=0,\ d=0, \\ 1-q^{-1}+q^{-2}, & \text{if } n=0,\ d>0, \\ (1-q^{-2})q^{-n}(1-q^{-1}+q^{-2}), & \text{if } 1 \leq n < d, \\ (1-q^{-4})q^{-n}, & \text{if } n=d,\ d>0, \\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d-2n}, & \text{if } d<n<2k+d, \quad 2|n-d, \\ (1-q^{-2})q^{d-2n-1}, & \text{if } d<n<2k+d, \quad 2\nmid n-d, \\ (1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{d-2n}, & \text{if } n=2k+d, \\ (1-q^{-2})(1+q^{-1})^2 q^{d-2n}, & \text{if } n>2k+d. \end{cases}$$

When $|u-1| = q^{-2k-1}$ $(k \geq 1)$, we have

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1+q^{-2}-q^{-3}, & \text{if } n=0,\ d=0, \\ 1-q^{-1}+q^{-2}, & \text{if } n=0,\ d>0, \\ (1-q^{-2})q^{-n}(1-q^{-1}+q^{-2}), & \text{if } 1 \leq n < d, \\ (1-q^{-4})q^{-n}, & \text{if } n=d,\ d>0, \\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d-2n}, & \text{if } d<n<2k+d+1, \quad 2|n-d, \\ (1-q^{-2})q^{d-2n-1}, & \text{if } d<n<2k+d+1, \quad 2\nmid n-d, \\ (1+q^{-1})q^{d-2n-1}, & \text{if } n=2k+1+d, \\ 0, & \text{if } n>2k+1+d. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. Recall that $u = (a_1c_2)/(a_2c_1)$. Thus we have that $Q_1(x,y,z) = N(x)+N(y)+N(z)$ and $Q_2(x,y,z) = \frac{a_2}{a_1}(N(y)+uN(z))$. As a_2/a_1 is assumed to be a unit, we can assume that $Q_2(x,y,z) = N(y)+uN(z)$, where $u \in R^\times$ has $|u-1| = 1/q^{2k+\varepsilon}$, where $2k+\varepsilon \geq 1$ and $\varepsilon = 0,1$.

To compute $\mathrm{vol}(V^0_{0,d})$, when $d=0$, recall that

$$V_{0,0} = \{(x,y,z) \in R_E^3; \max\{|N(x)+N(y)+N(z)|, |N(y)+uN(z)|\} = 1\}.$$

(Note that the condition $||(x,y,z)||_E = 1$ is satisfied.) This set is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3; |N(x)+N(y)+N(z)| \leq 1, |N(y)+uN(z)| = 1\}$$

and

$$\{(x,y,z) \in R_E^3; |N(x)+N(y)+N(z)| = 1, |N(y)+uN(z)| < 1\}.$$

We will compute the volume of each of these subsets.

(A) *Case of* $|N(x)+N(y)+N(z)| \leq 1$ and $|N(y)+uN(z)| = 1$. Note that since $|1-u| < 1$ the equality $|N(y)+uN(z)| = 1$ is equivalent to $|N(y)+N(z)| = 1$ whereas the inequality becomes $|N(x)| \leq 1$. Then the volume of subset (A) is given by the integral

$$\left(\int_{|N(y)|<1}\int_{|N(z)|=1} + \int_{|N(y)|=1}\int_{|N(y)+N(z)|=1}\right)\int_{|N(x)|\leq 1} dxdzdy.$$

This is equal to

$$q^{-2}(1-q^{-2}) + (1-q^{-2})(1-(q+1)q^{-2}) = (1-q^{-2})(1-q^{-1}).$$

(B) *Case of* $|N(x)+N(y)+N(z)| = 1$ and $|N(y)+uN(z)| < 1$. Since $|1-u| < 1$ we have

$$|N(y)+N(z)| = |N(y)+uN(z)+(1-u)N(z)| \leq \max\{|N(y)+uN(z)|, |1-u||N(z)|\} < 1,$$

and thus $|N(x)| = 1$. Then the volume of subset (B) is given by the integral

$$\left(\int_{|N(y)|<1}\int_{|N(z)|<1} + \int_{|N(y)|=1}\int_{|N(y)+uN(z)|<1}\right)\int_{|N(x)|=1} dxdzdy,$$

which (by Lemma A.1 with $k=0$, $n=1$) is equal to the product of $1-q^{-2}$ and

$$q^{-2}\cdot q^{-2} + (1-q^{-2})(1+q^{-1})q^{-1} = q^{-1}+q^{-2}-q^{-3}.$$

Adding the expressions of (A) and (B) we obtain that $\mathrm{vol}(V^0_{0,d})$ is equal to

$$(1-q^{-1}) + (q^{-1}+q^{-2}-q^{-3}) \;=\; 1+q^{-2}-q^{-3}.$$

When $d>1$, the claim of the lemma follows from Lemma A.5.

To compute $\mathrm{vol}(V^0_{1,d})$, when $d=0$, recall that

$$V_{1,0} = \{(x,y,z)\in R_E^3; \max\{|N(x)+N(y)+N(z)|, |N(y)+uN(z)|\} = q^{-1}\}.$$

Note that the condition $||(x,y,z)||_E = 1$ is satisfied. If not, $|N(x)| < q^{-2}$, $|N(y)| < q^{-2}$, and $|N(z)| < q^{-2}$ implies that $|N(x)+N(y)+N(z)| < q^{-2}$ and $|N(y)+uN(z)| < q^{-2}$. This set is the disjoint union of two subsets

$$\{(x,y,z)\in R_E^3; |N(x)+N(y)+N(z)| \leq q^{-1}, |N(y)+uN(z)| = q^{-1}\}$$

and

$$\{(x,y,z)\in R_E^3; |N(x)+N(y)+N(z)| = q^{-1}, |N(y)+uN(z)| < q^{-1}\}.$$

Note that from

$$|N(y)+N(z)| \leq \max\{|N(y)+uN(z)|, |(1-u)N(z)|\},$$

it follows that $|N(x)| < 1$. In particular $|N(x)| < 1/q$. This implies that $|u-1| = q^{-1}$, so the second subset is empty when $|u-1| < 1/q$. When $|u-1| = 1/q$ its volume is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=1} \int_{|N(x)+c|=q^{-1}} dxdydz,$$

where $|c| = |N(y)+N(z)| = q^{-1}$ (thus $|N(x)| \le q^{-2}$). The value of the integral is

$$(1-q^{-2}) \times q^{-2}(1+q^{-1}) \times q^{-2}.$$

For the first subset we have $|N(y)| = 1$ and its volume is given by the integral

$$\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(y)+uN(z)|=q^{-1}} dzdydx.$$

The value of the integral is

$$q^{-2}(1-q^{-2}) \times q^{-1}(1-q^{-2}).$$

We obtain that $\mathrm{vol}(V^0_{1,0})$ is equal to $q^{-3}(1-q^{-2})$ when $|u-1| < 1/q$ and to $q^{-3}(1+q^{-1})$ when $|u-1| = 1/q$.

When $d > 1$, the claim of the lemma follows from Lemma A.5.

To compute $\mathrm{vol}(V^0_{n,d})$ $(0 \le n \le d-1)$ recall that since $|\boldsymbol{\pi}^d(N(y)+uN(z))| < q^{-n}$, we have that

$$V_{n,d} = \{(x,y,z) \in R_E^3; |N(x)+N(y)+N(z)| = q^{-n},\ ||(x,y,z)||_E = 1\}.$$

To compute this volume, we apply Lemma A.5.

To compute $\mathrm{vol}(V^0_{n,d})$ $(n = d)$ recall that $V_{n,d}$ is

$$\{(x,y,z) \in R_E^3; \max\{|N(x)+N(y)+N(z)|, |\boldsymbol{\pi}^d(N(y)+uN(z))|\} = q^{-n}\}.$$

where $||(x,y,z)||_E = 1$. The set $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3;\ |N(x)+N(y)+N(z)| \le q^{-n}, |N(y)+uN(z)| = 1\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(x)+N(y)+N(z)| = q^{-n}, |N(y)+uN(z)| < 1\}.$$

We will compute the volume of each subset.

(A) *Case of* $|N(x)+N(y)+N(z)| \le q^{-n}$ and $|N(y)+uN(z)| = 1$. Note that since $|1-u| < 1$ we have $|N(y)+N(z)| = |N(y)+uN(z)| = 1$. The volume is given by the integral

$$\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(z)+c|\le q^{-n}} dzdydx + \int_{|N(x)|=1} \int_{|N(x)+N(y)|=1} \int_{|N(z)+c|\le q^{-n}} dzdydx,$$

where $c = N(x) + N(y)$ so that $|c| = 1$. This is equal to

$$(1 - q^{-2})q^{-n}(1 + q^{-1})(q^{-2} + 1 - q^{-1} - q^{-2}) = (1 - q^{-2})^2 q^{-n}.$$

(B) *Case of* $|N(x) + N(y) + N(z)| = q^{-n}$ and $|N(y) + uN(z)| < 1$. Since $|1 - u| < 1$ the second inequality is equivalent to $|N(y) + N(z)| < 1$. Moreover, if $|N(y)| < 1$ then $|N(z)| < 1$ and consequently $|N(x)| < 1$ which is a contradiction. Thus $|N(y)| = 1$, and applying Lemma A.3 with $j = n-1$ so that $\sigma(n-j) = 1$, the contribution from this subcase is equal to

$$(1 - q^{-2})^2 q^{-n-2}.$$

Adding the contributions from cases (A) and (B), then dividing by $1 - q^{-2}$, we obtain $(1 - q^{-2})(1 + q^{-2})q^{-n}$.

To compute $\mathrm{vol}(V^0_{n,d})$ ($n > d$ and $q^{-n} > |\boldsymbol{\pi}^d(1 - u)|$) recall that

$$V_{n,d} = \{(x, y, z) \in R^3_E; \max\{|N(x) + N(y) + N(z)|, |\boldsymbol{\pi}^d(N(y) + uN(z))|\} = q^{-n}\},$$

where $||(x, y, z)||_E = 1$. Assume that $|N(z)| < 1$. Then $|N(y) + uN(z)| < 1$ implies that $|N(y)| < 1$ and from $|N(x) + N(y) + N(z)| < 1$ it follows that $|N(x)| < 1$ which is a contradiction. Thus $|N(z) = 1|$ and $V_{n,d}$ is the disjoint union of two subsets

$$\{(x, y, z) \in R^3_E;\ ||N(x) + N(y) + N(z)| \leq q^{-n}, |N(y) + uN(z)| = q^{d-n}\}$$

and

$$\{(x, y, z) \in R^3_E;\ |N(x) + N(y) + N(z)| = q^{-n}|, |N(y) + uN(z)| \leq q^{d-n-1}\}.$$

We will compute the volume of each of them.

(A) *Case of* $|N(x) + N(y) + N(z)| \leq q^{-n}$ and $|N(y) + uN(z)| = q^{d-n}$. The volume of this subset is equal to the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=q^{d-n}} \int_{|N(x)+c|\leq q^{-n}} dxdydz,$$

where $c = c(y, z) = N(y) + N(z)$ so that $|c| = q^{d-n}$.

When $d = 0$, note that since $|1-u| < q^{-n}$ we have $|N(y)+N(z)| = |N(y)+uN(z)| = q^{-n}$. It follows that the first inequality is equivalent to $|N(x)| \leq q^{-n}$. Thus the volume of this subset is equal to the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=q^{-n}} \int_{|N(x)|\leq q^{-n}} dxdydz.$$

The volume of set $\{|N(x)| \leq q^{-n}\}$ is equal to q^{-n} if n is even and to q^{-n-1} if n is odd. Thus the integral is equal to the product of

$$(1 - q^{-2}) \times (1 - q^{-2})q^{-2n}$$

and 1 if n is even and q^{-1} if n is odd.

When $d>0$, the volume of the set $\{|N(x)+c|\le q^{-n}\}$ is zero when $n-d$ is odd. Thus, the integral is equal to $(1-q^{-2})\times(1-q^{-2})q^{d-n}\times q^{-n}(1+q^{-1})$, when $n-d$ is even, and zero when $n-d$ is odd.

(B) *Case of* $|N(x)+N(y)+N(z)|=q^{-n}$ and $|N(y)+uN(z)|<q^{d-n}$. Note that inequality $|N(y)+uN(z)|<q^{d-n}$ is equivalent to $|N(y)+N(z)|<q^{d-n}$. The volume of this subset equals the integral

$$\int_{|N(z)|=1}\int_{|N(y)+N(z)|\le q^{d-n-1}}\int_{|N(x)+c|=q^{-n}} dxdydz,$$

where $c=c(y,z)=N(y)+N(z)$.

When $d=0$, thus $|c|<q^{-n}$, and the volume of this subset equals the integral

$$\int_{|N(z)|=1}\int_{|N(y)+N(z)|\le q^{-n-1}}\int_{|N(x)|=q^{-n}} dxdydz.$$

This integral is equal to zero when n is odd and to

$$(1-q^{-2})\times(1+q^{-1})q^{-n-1}\times(1-q^{-2})q^{-n}$$

when n is even. Thus, $\mathrm{vol}(V^0_{n,0})$ is equal to

$$(1-q^{-2})q^{-2n}\times(1+q^{-1}+q^{-2}),$$

when n is even and to

$$(1-q^{-2})q^{-2n-1},$$

when n is odd.

When $d>0$ the volume is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+N(z)|\le q^{d-n-1}}\int_{|N(x)+c|=q^{-n}} dxdydz,$$

where $c=c(y,z)=N(y)+N(z)$. Applying Lemma A.3, it equal to

$$(1-q^{-2})^2q^{d+\sigma(n-d+1)-2n-1}.$$

Thus, adding contributions from cases (A) and (B) and dividing by $1-q^{-2}$, the $\mathrm{vol}(V^0_{n,d})$ is equal to

$$(1-q^{-2})q^{d-2n}\times(1+q^{-1}+q^{-2}),$$

when $n-d$ is even and to

$$(1-q^{-2})q^{d-2n-1},$$

when $n-d$ is odd.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n}=|\boldsymbol{\pi}^d(1-u)|)$ recall that V_n is

$$\{(x,y,z)\in R^3_E; ||(x,y,z)||_E=1, \max\{|N(x)+N(y)+N(z)|, |\boldsymbol{\pi}^d(N(y)+uN(z)|)\}=q^{-n}\}.$$

As in the previous cases we can show that $|N(y)|=|N(z)|=1$ and $V_{n,d}$ is the disjoint union of two subsets.
(A) *Case of* $|N(x)+N(y)+N(z)|\le q^{-n}$, $|N(y)+uN(z)|=q^{d-n}$, and $|N(z)|=1$. When $d=0$, we have

$$|N(y)+N(z)|\le \max\{|N(y)+uN(z)|, |(1-u)N(z)|\}\le q^{d-n}.$$

Thus $|N(x)|\le q^{-n}$ and the volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+uN(z)|=q^{-n}}\int_{|N(x)|\le q^{-n}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1-q^{-2})\times(1-q^{-2})q^{-n}\times q^{-n}.$$

When $d>0$, then for any $t\ge 0$, we define the following subset

$$W_t(z)=\{y\in E; |N(z)|=1, |N(y)+uN(z)|=q^{d-n}, |N(y)+N(z)|=q^{d-t-n}\}.$$

The volume of the set is given by the integral

$$\sum_{t=0}^{\infty}\int_{|N(z)|=1}\int_{W_t(z)}\int_{|N(x)+c_t|\le q^{-n}} dxdydz,$$

where $c_t=N(y)+N(z)$ and $|c_t|=q^{d-t-n}$. Note that the integral over x depends only on $|c_t|$. Thus without loss of generality, we can assume that $c_t=\boldsymbol{\pi}^{d-t-n}c_0$, where $|c_0|=1$ is an arbitrary fixed element of $R^\times$, and take the integral over y and z. The integral is equal to

$$(1-q^{-2})\sum_{t=0}^{\infty}\mathrm{vol}(W_t)\int_{|N(x)+c_t|\le q^{-n}} dx.$$

Applying Lemma A.2, this is equal to the product of $(1-q^{-2})^2$ and

$$(1-(q+2)q^{-2})q^{d-n}\int_{|N(x)+c_0|\le q^{-n}} dx+\sum_{t=1}^{\infty}q^{d-t-n}\int_{|N(x)+c_t|\le q^{-n}} dx.$$

If $t\ge d$ so that $|c_t|\le q^{-n}$, and thus $|N(x)|\le q^{-n}$. The integral over x is equal to $q^{-n-\sigma(n)}$. Recall that we put $\sigma(n)=n-2[n/2]$ ($=0$ if n is even, 1 if n is odd) before Lemma A.3. The sum of the terms which correspond to $t\ge d$ is equal to

$$(1-q^{-2})^2q^{-2n-\sigma(n)}\sum_{t=d}^{\infty}q^{d-t}=(1-q^{-2})(1+q^{-1})q^{-2n-\sigma(n)}.$$

If $1 \le t \le d-1$ then the set $|N(x)+c_t| \le q^{-n}$ is empty when $n-d+t$ is odd and is $(1+q^{-1})q^{-n}$ when $n-d+t$ is even. Set $j = n-d+t$, then $n-d+1 \le j \le n-1$ and the even j are of the form

$$n-d+1+\sigma(n-d+1), n-d+3+\sigma(n-d+1), \dots, n-1-\sigma(n-1).$$

Since $\sigma(n-d+1) = 1-\sigma(n-d)$ and $\sigma(n-1) = 1-\sigma(n)$, this sequence can be written as

$$n-d-\sigma(n-d)+2, n-d-\sigma(n-d)+4, \dots, n+\sigma(n)-2.$$

So the sum is the product of $(1+q^{-1})q^{-n}$ and

$$q^{-(n-d-\sigma(n-d)+2)} + q^{-(n-d-\sigma(n-d)+4)} + \dots + q^{-(n+\sigma(n)-2)}$$

$$= (1-q^{-2})^{-1}(q^{-(n-d-\sigma(n-d)+2)} - q^{-(n+\sigma(n)-2)-2}).$$

Thus, the contribution from $1 \le t \le d-1$ is equal to

$$(1+q^{-1})(1-q^{-2})(q^{-2-2n+d+\sigma(n-d)} - q^{-2n-\sigma(n)})$$

and the the contribution from $t \ge 1$ is equal to

$$(1+q^{-1})(1-q^{-2})q^{-2-2n+d+\sigma(n-d)}.$$

If $t=0$, applying Lemma A.2, the contribution from this term is

$$(1-q^{-2})(1-(q+2)q^{-2})(1+q^{-1})q^{d-n},$$

when $n-d$ is even and zero when $n-d$ is odd.
(B) *Case of* $|N(x)+N(y)+N(z)| = q^{-n}$, $|N(y)+uN(z)| < q^{d-n}$, and $|N(y)| = |N(z)| = 1$. Since $|N(y)+uN(z)| < q^{d-n}$, we obtain

$$|N(y)+N(z)| = |N(y)+uN(z)+(1-u)N(z)| = |1-u| = q^{d-n}.$$

Thus, when $d=0$ the volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|<q^{-n}} \int_{|N(x)+c|=q^{-n}} dxdydz,$$

where $c = c(y,z) = N(y)+N(z)$ and $|c| = q^{-n}$ when n is even and by

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|<q^{-n}} \int_{|N(x)|\le q^{-n-1}} dxdydz,$$

when n is odd. Applying Lemma A.1 the value of the integral is

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n-1} \times (1-(q+1)q^{-2})q^{-n}$$

when n is even and to

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n-1} \times q^{-n-1}$$

when n is odd. When $d > 0$ the volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|<q^{d-n}} \int_{|N(x)+c|=q^{-n}} dxdydz,$$

where $c = c(y,z) = N(y) + N(z)$ and $|c| = q^{d-n}$. Applying Lemma A.1 we see that the value of the integral is

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n-1} \times (1-q^{-2})q^{-n}$$

when $n-d$ is even. It is zero when $n-d$ is odd.

Adding the contributions from cases (A) and (B) when $d = 0$, then dividing by $1 - 1/q^2$, we obtain

$$(1+q^{-1})(1-q^{-1}+q^{-1}(1-(q+1)q^{-2}))q^{-2n}$$

when n is even and

$$(1+q^{-1})q^{-2n-1}$$

when n is odd.

Adding the contributions from cases (A) and (B) when $d \geq 1$, then dividing by $(1-1/q^2)$, we obtain

$$\begin{aligned}(1+q^{-1})q^{d-2n}&(q^{-2}+1-(q+2)q^{-2}+(1-q^{-2})q^{-1}) \\ &= (1+q^{-1})(1-q^{-2}-q^{-3})q^{d-2n}.\end{aligned}$$

when $n-d$ is even and

$$(1+q^{-1})q^{d-1-2n}$$

when $n-d$ is odd.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n} < |\boldsymbol{\pi}^d(1-u)|)$ recall that

$$V_{n,d} = \{(x,y,z) \in R^3_E; \max\{|N(x)+N(y)+N(z)|, |\boldsymbol{\pi}^d(N(y)+uN(z))|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. Then $|N(z)| = 1$ and $|N(y)+uN(z)| \leq q^{d-n}$, we have

$$|N(y)+N(z)| = |(1-u)N(z)| = |1-u|.$$

Thus $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z) \in R^3_E;\ |N(z)| = 1, |N(x)+N(y)+N(z)| \leq q^{-n}, |N(y)+uN(z)| = q^{-n}\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(z)| = 1, |N(x)+N(y)+N(z)| = q^{-n}, |N(y)+uN(z)| \le q^{-n-1}\}.$$

We will compute the volume of each of them.
(A) *Case of* $|N(x)+N(y)+N(z)| \le q^{-n}$, $|N(y)+uN(z)| = q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=q^{d-n}} \int_{|N(x)+c| \le q^{-n}} dxdydz,$$

where $c = c(y,z) = N(y) + N(z)$ and $|c| = |1-u| = q^{-2k-\varepsilon}$, where ε is 0 or 1. Applying Lemma A.1 the value of the integral is

$$(1-q^{-2}) \times (1-q^{-2})q^{d-n} \times (1+q^{-1})q^{-n}$$

when $\varepsilon = 0$ and to zero when $\varepsilon = 1$.
(B) *Case of* $|N(x)+N(y)+N(z)| = q^{-n}$, $|N(y)+uN(z)| < q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|<q^{d-n}} \int_{|N(x)+c| = q^{-n}} dxdzdy,$$

where $c = c(y,z) = N(y) + N(z)$ and $|c| = |1-u|$. Applying Lemma A.1 we conclude that the integral is equal to

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n-1} \times (1-q^{-2})q^{-n}$$

We obtain that $\mathrm{vol}(V^0_{n,d})$ (when $n > 2k$) is equal to

$$q^{d-2n}(1-q^{-2})(1+q^{-1})^2$$

when $\varepsilon = 0$. It is zero when $\varepsilon = 1$. The lemma follows. □

Lemma B.4.3. *Suppose that* $n_1 = 0$, $n_2 = 0$, $a_1 \in R^\times$, $|c_1| \le 1$, $|c_2| = q^{-d}$, $d \ge 0$. *When* $|a_2| = q^{-2k}|c_2|$ $(k \ge 1)$, *we have*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1+q^{-2}-q^{-3}, & \text{if } n=0,\ d=0, \\ 1-q^{-1}+q^{-2}, & \text{if } n=0,\ d\ge 1, \\ (1-q^{-2})q^{-n}(1-q^{-1}+q^{-2}), & \text{if } 1 \le n \le d-1, \\ (1-q^{-2}-q^{-3}-q^{-4})q^{-n}, & \text{if } n=d,\ d \ge 1, \\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d-2n}, & \text{if } 0<n-d<2k,\quad 2|n-d, \\ (1-q^{-2})q^{d-2n-1}, & \text{if } 0<n-d<2k,\quad 2\nmid n-d, \\ (1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{d-2n}, & \text{if } n=2k+d, \\ (1-q^{-2})(1+q^{-1})^2 q^{d-2n}, & \text{if } n>2k+d. \end{cases}$$

When $|a_2| = q^{-2k-1}|c_2|$ $(k \geq 0)$, *we have*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1+q^{-2}-q^{-3}, & \textit{if } n=0,\ d=0,\\ 1-q^{-1}+q^{-2}, & \textit{if } n=0,\ d\geq 1,\\ (1-q^{-2})q^{-n}(1-q^{-1}+q^{-2}), & \textit{if } 1\leq n\leq d,\\ (1-q^{-2}-q^{-3}-q^{-4})q^{-n}, & \textit{if } n=d,\ d\geq 1,\\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d-2n}, & \textit{if } 0<n-d<2k+1,\quad 2|n-d,\\ (1-q^{-2})q^{d-2n-1}, & \textit{if } 0<n-d<2k+1,\quad 2\nmid n-d,\\ (1+q^{-1})q^{d-2n-1}, & \textit{if } n=2k+d+1,\\ 0, & \textit{if } n>2k+d+1. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Since we are interested in $|Q_2|$, we can assume that $c_2 = \boldsymbol{\pi}^d$. Thus we have that $Q_1(x,y,z) = N(x)+N(y)+c_1N(z)$ and $Q_2(x,y,z) = a_2N(y)+\boldsymbol{\pi}^dN(z)$. Note that $d>0$ (thus $|c_2|<1$) implies $|c_1| = 1$.

To compute $\mathrm{vol}(V^0_{0,d})$ recall that

$$V_{0,d} = \{(x,y,z)\in R_E^3; \max\{|N(x)+N(y)+c_1N(z)|, |a_2N(y)+\boldsymbol{\pi}^dN(z)|\} = 1\},$$

where $||(x,y,z)||_E = 1$. When $d=0$, this set is the disjoint union of two subsets

$$\{(x,y,z)\in R_E^3; |N(x)|\leq 1, |N(y)|\leq 1, |N(z)|=1\},$$

and

$$\{(x,y,z)\in R_E^3; |N(x)+N(y)+c_1N(z)|=1, |N(z)|<1\},$$

where the second one splits further according to $|N(x)|<1$ or $|N(x)|=1$. Then the volume of the first subset is $(1-q^{-2})$ and that of the second

$$\int_{|N(z)|<1}\int_{|N(x)|<1}\int_{|N(y)|=1} dydxdz + \int_{|N(z)|<1}\int_{|N(x)|=1}\int_{|N(x)+N(y)+c_1N(z)|=1} dydxdz.$$

Then $\mathrm{vol}(V_{0,0})$ is equal to

$$(1-q^{-2})+q^{-2}(1-q^{-2})(q^{-2}+(1-(q+1)q^{-2})) = (1-q^{-2})(1+q^{-2}-q^{-3}).$$

When $d\geq 1$, we have that

$$V_{0,d} = \{(x,y,z)\in R_E^3; |N(x)+N(y)+N(z)|=1\},$$

and we apply Lemma A.5.

To compute $\mathrm{vol}(V^0_{n,d})$ $(1 \le n \le d-1)$ recall that since $|a_2 N(y) + \boldsymbol{\pi}^d N(z)| < q^{-n}$, and $|c_1| = 1$ (thus can assume it to be 1) we have

$$V_{n,d} = \{(x,y,z) \in R_E^3; |N(x) + N(y) + N(z)| = q^{-n}\}.$$

where $||(x,y,z)||_E = 1$. To compute its volume, we apply Lemma A.5.

To compute $\mathrm{vol}(V^0_{n,d})$ $(n = d)$ recall that since $|a_2| < q^{-n}$, we have

$$V_{n,d} = \{(x,y,z) \in R_E^3; \max\{|N(x) + N(y) + c_1 N(z)|, |\boldsymbol{\pi}^d N(z)|\} = q^{-n},\ ||(x,y,z)||_E = 1\}.$$

Since $|c_2| < 1$, we have $|c_1| = 1$ and without loss of generality can assume it to be 1. The set $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3;\ |N(x) + N(y) + N(z)| \le q^{-n-1}, |N(z)| = 1\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(x) + N(y) + N(z)| = q^{-n}\}.$$

We will compute the volume of each subset.

(A) *Case of* $|N(x) + N(y) + N(z)| \le q^{-n-1}$ and $|N(z)| = 1$. We consider the subcases of $|N(z)| = 1$ and $|N(z)| < 1$. The volume of this set is equal to the sum of two integrals.

The first integral is

$$\int_{|N(z)|=1} \int_{|N(y)+N(z)| \le q^{-1}} \int_{|N(x)+N(y)+N(z)| \le q^{-n-1}} dx dy dz.$$

Note that this is the same integral as in 5.1 $(n = d)$ and is equal to

$$(1 - q^{-2}) q^{-n-3} (1 + q^{-1}).$$

The second integral is

$$\int_{|N(z)|=1} \int_{|N(y)+N(z)| = 1} \int_{|N(x)+N(y)+N(z)| \le q^{-n-1}} dx dy dz.$$

Note that this is the same integral as in 4.1 $(n = d)$ and is equal to

$$(1 - q^{-2}) q^{-n-1} (1 + q^{-1})(1 - 2q^{-1} - 2q^{-2}).$$

(B) Applying Lemma A.5, the volume of this subset is $(1 - q^{-2})^2 q^{-n} (1 - q^{-1} + q^{-2})$.

Adding the contributions from cases (A) and (B) and dividing by $1 - q^{-2}$, we obtain

$$(1 - q^{-2} - q^{-3} - q^{-4}) q^{-n}.$$

To compute $\mathrm{vol}(V_{n,d}^0)$ $(q^{-n} > |a_2|)$ recall that

$$V_{n,d} = \{(x,y,z) \in R_E^3; \max\{|N(x)+N(y)+c_1N(z)|, |a_2N(y)+\boldsymbol{\pi}^dN(z)|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. Note that $|N(y)| = 1$. Indeed, the condition $|N(y)| < 1$ implies $|N(z)| \le |N(y)| < 1$ and $|N(x)| < 1$ which is a contradiction. Since $|a_2| < q^{-n}$ it is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x)+N(y)+c_1N(z)| \le q^{-n}, |N(z)| = q^{d-n}\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x)+N(y)+c_1N(z)| = q^{-n}, |N(z)| \le q^{d-n-1}\}.$$

Note that the term $c_1N(z)$ can be dropped and the volumes are given by the same integrals as in Lemma B.4.2 (case of $q^{-n} > |1-u|$). Thus $\mathrm{vol}(V_{n,d}^0)$ equal to

$$(1-q^{-2})q^{d-2n} \times (1+q^{-1}+q^{-2}),$$

when n is even and

$$(1-q^{-2})q^{d-2n} \times q^{-1},$$

when n is odd.

To compute $\mathrm{vol}(V_{n,d}^0)$ $(q^{-n} = |a_2|)$ recall that $V_{n,d}$ is

$$\{(x,y,z) \in R_E^3; \max\{|N(x)+N(y)+c_1N(z)|, |a_2N(y)+\boldsymbol{\pi}^dN(z))|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. As in previous cases we can show that $N(y) = 1$ and $V_{n,d}$ is the disjoint union of two subsets

(A) *Case of* $|N(x)+N(y)+c_1N(z)| = q^{-n}$, $|a_2N(y)+\boldsymbol{\pi}^dN(z)| \le q^{-n}$, and $|N(y)| = 1$. Since $|a_2N(y)| = q^{-n}$, the volume of this subset is given by the integral

$$\int_{|N(y)|=1} \int_{|N(z)| \le q^{d-n}} \int_{|N(x)+N(y)+c_1N(z)|=q^{-n}} dxdzdy,$$

where $|N(y)+c_1N(z)| = 1$. Applying Lemma A.1 the integral is equal to

$$(1-q^{-2}) \times q^{d-n} \times (1-q^{-2})q^{-n}$$

when $n-d$ is even and

$$(1-q^{-2}) \times q^{d-n-1} \times (1-q^{-2})q^{-n}$$

when $n-d$ is odd.

(B) *Case of* $|N(x)+N(y)+c_1N(z)| \leq q^{-n-1}$, $|a_2N(y)+\boldsymbol{\pi}^dN(z)| = q^{-n}$, and $|N(y)| = 1$. The volume of this subset is given by the integral

$$\int_{|N(y)|=1}\int_{|N(z)+a_2\boldsymbol{\pi}^{-d}N(y)|=q^{d-n}}\int_{|N(x)+N(y)+c_1N(z)|\leq q^{-n-1}} dxdzdy,$$

where $|N(y)+c_1N(z)| = 1$. Applying Lemma A.1 the value of the integral is

$$(1-q^{-2}) \times (1-q^{-1}-q^{-2})q^{d-n} \times (1+q^{-1})q^{-n-1}$$

when $n-d$ is even and

$$(1-q^{-2}) \times q^{d-n-1} \times (1+q^{-1})q^{-n-1}$$

when $n-d$ is odd.

Adding the expressions of (A) and (B), and dividing by $1-1/q^2$, we obtain

$$q^{d-2n}(1+q^{-1})(1-q^{-2}-q^{-3})$$

when $n-d$ is even and

$$q^{d-2n-1}(1+q^{-1})$$

when $n-d$ is odd.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n} < |a_2|)$ recall that $V_{n,d}$ is

$$\{(x,y,z) \in R^3_E; \max\{|N(x)+N(y)+c_1N(z)|, |a_2N(y)+\boldsymbol{\pi}^dN(z)|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. By the same argument as in previous cases, we can assume $|N(y)| = 1$. Set $a_2' = a_2\boldsymbol{\pi}^{-d}$. Thus $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z) \in R^3_E;\ |N(y)| = 1, |N(x)+N(y)+c_1N(z)| = q^{-n}, |a_2'N(y)+N(z)| \leq q^{d-n}\}$$

and

$$\{(x,y,z) \in R^3_E;\ |N(y)| = 1, |N(x)+N(y)+c_1N(z)| \leq q^{-n-1}, |a_2'N(y)+N(z)| = q^{d-n}\}.$$

We will compute the volume of each of them.

(A) *Case of* $|N(x)+N(y)+c_1N(z)| = q^{-n}$, $|a_2'N(y)+N(z)| \leq q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(y)|=1}\int_{|N(z)+a_2N(y)|\leq q^{d-n}}\int_{|N(x)+c|\leq q^{-n}} dxdzdy,$$

where $c = c(y,z) = N(y)+c_1N(z)$ and $|c| = 1$. Applying Lemma A.1 the integral equals

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n} \times (1-q^{-2})q^{-n}$$

when $|a_2/c_2| = q^{-2k}$ and zero when $|a_2/c_2| = q^{-2k-1}$.
(B) *Case of* $|N(x)+N(y)+c_1N(z)| \le q^{-n-1}$, $|a_2'N(y)+N(z)| = q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(y)|=1}\int_{|N(z)+a_2N(y)|=q^{d-n}}\int_{|N(x)+c|\le q^{-n-1}} dxdzdy,$$

where $c = c(y,z) = N(y) + c_1N(z)$ and $|c| = 1$. Applying Lemma A.1 the integral equals

$$(1-q^{-2}) \times (1-q^{-2})q^{d-n} \times (1+q^{-1})q^{-n-1}$$

when $|a_2/c_2| = q^{-2k}$ and zero when $|a_2/c_2| = q^{-2k-1}$.

We obtain that $\mathrm{vol}(V^0_{n,d})$ (when $n > 2k$) is equal to

$$q^{d-2n}(1-q^{-2})(1+q^{-1})^2$$

when $|a_2/c_2| = q^{-2k}$ and to zero when $|a_2/c_2| = q^{-2k-1}$. The lemma follows. □

Lemma B.4.4. *Suppose that* $n_1 = 0$, $n_2 = 0$, a_1, a_2, $c_2 \in R^\times$. *When* $|c_1| < 1$, *we have*

$$\mathrm{vol}(V^0_n) = \begin{cases} 1-2q^{-3}, & \textit{if } n = 0, \\ (1-q^{-2})(1+q^{-1})^2q^{-2n}, & \textit{if } n > 0. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_2 = 1$. Thus we have that $Q_1(x,y,z) = N(x) + N(y) + c_1N(z)$ and $Q_2(x,y,z) = a_2N(y) + N(z)$.

We first compute $\mathrm{vol}(V_0)$. Here we can drop the term $c_1N(z)$ so that

$$V_0 = \{(x,y,z) \in R_E^3; \max\{|N(x)+N(y)|, |a_2N(y)+N(z)|\} = 1\}.$$

We first compute the volume of the complement of V_0 in $\{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1\}$:

$$\{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, |N(x)+N(y)| < 1, |a_2N(y)+N(z)| < 1\}.$$

Note that if $|N(x)| < 1$ then $|N(y)| < 1$ and $|N(z)| < 1$ which is a contradiction. Thus $|N(x)| = 1$, and consequently $|N(y)| = |N(z)| = 1$. Consequently the volume of the complement is given by the integral

$$\int_{|N(y)|=1}\int_{|a_2N(y)+N(z)|<1}\int_{|N(x)+N(y)|<1} dxdzdy.$$

This integral is the same as in Lemma B.4.1, case of $n = d = 0$. Thus, the volume of V_0 equals to $(1-q^{-2})(1-2q^{-3})$.

To compute $\mathrm{vol}(V_n^0)$ $(n \geq 1)$ recall that

$$V_n = \{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, \max\{|Q_1|, |Q_2|\} = q^{-n}\}.$$

Assume that $|N(y)| < 1$. Then $|a_2N(y) + N(z)| < 1$ implies that $|N(z)| < 1$ and from $|N(x) + N(y) + c_1N(z)| < 1$ it follows that $|N(x)| < 1$ which is a contradiction. Thus $|N(y)| = 1$ and V_n is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |Q_1| \leq q^{-n}, |Q_2| = q^{-n}\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |Q_1| = q^{-n}, |Q_2| \leq q^{-(n+1)}\}.$$

We will compute the volume of each of them.

(A) *Case of* $|N(x) + N(y) + c_1N(z)| \leq q^{-n}$ and $|a_2N(y) + N(z)| = q^{-n}$. Thus the volume of this subset is equal to the integral

$$\int_{|N(y)|=1} \int_{|a_2N(y)+N(z)|=q^{-n}} \int_{|N(x)+c| \leq q^{-n}} dxdzdy,$$

where $c = c(y,z) = N(y) + c_1N(z)$ and $|c| = 1$. By Lemma A.1 the integral is equal to:

$$(1-q^{-2}) \times (1-q^{-2})q^{-n} \times (1+q^{-1})q^{-n}.$$

(B) *Case of* $|N(x) + N(y) + c_1N(z)| = q^{-n}$ and $|a_2N(y) + N(z)| \leq q^{-n-1}$. The volume of this subset is given by the integral

$$\int_{|N(y)|=1} \int_{|a_2N(y)+N(z)| \leq q^{-n-1}} \int_{|N(x)+c| = q^{-n}} dxdzdy,$$

where again $c = c(y,z) = N(y) + c_1N(z)$ and $|c| = 1$. Applying Lemma A.1, the above integral is equal to:

$$(1-q^{-2}) \times (1+q^{-1})q^{-n-1} \times (1-q^{-2})q^{-n}.$$

Adding the expressions from (A) and (B), simplifying and dividing by $(1-q^{-2})$ gives

$$\mathrm{vol}(V_n^0) = (1-q^{-2})(1+q^{-1})^2q^{-2n}.$$

The lemma follows. $\square$

C. Appendix to Section 5

This section contains the computations used in section 5.

Lemma C.5.1. *Suppose that* $c_2 \in R^\times$. *Then*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1+q^{-2}, & \text{if } n=0,\\ q^{-4}, & \text{if } n=1,\\ 0, & \text{if } n\ge 2.\end{cases}$$

Proof. We first show that V_n $(n \geq 2)$ *is empty.*

Suppose $(x, y, z) \in V_n$. Then $|N(z)| < 1$, since $|c_2N(z)| = |N(z)| = 1$ implies $|Q_2| = 1 > q^{-n}$. Also $|N(x)| < 1$ since $|N(x)| = 1$ implies $|Q_1| = 1$. Hence $|N(y)| = 1$. As E/F is unramified, $|N(x)| < q^{-1}$, $|N(z)| < q^{-1}$, hence

$$\max\{|Q_1|, |Q_2|\} = \max\{|a_1|, |a_2|\} \times |\boldsymbol{\pi}| = |\boldsymbol{\pi}| > q^{-n}.$$

Hence V_n $(n \geq 2)$ is empty.

To compute $\mathrm{vol}(V_0^0)$ note that

$$V_0 = \{(x, y, z) \in R_E^3; \max\{|N(x) + c_1N(z)|, |N(z)|\} = 1\}.$$

This is the disjoint union of the subset $|N(z)| = 1$, $|N(x)| \leq 1$, of volume $1 - q^{-2}$, and the subset $|N(z)| < 1$, $|N(x)| = 1$, of volume $q^{-2}(1 - q^{-2})$. Hence the volume of V_0 is the product of $(1 - q^{-2})$ and $(1 + q^{-2})$.

To compute $\mathrm{vol}(V_1^0)$ recall that V_1 is the set

$$\{(x, y, z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi} a_1N(y) + c_1N(z)|, |\boldsymbol{\pi} a_2N(y) + c_2N(z)|\} = 1/q\},$$

where $||(x, y, z)||_E = 1$. If $(x, y, z) \in V_1$ then $|N(z)| < 1$, $|N(x)| < 1$, $|N(y)| = 1$. Then the volume of V_1 is $(1 - q^{-2})q^{-4}$, as required. ☐

Lemma C.5.2. *Suppose that* a_1, $c_1 \in R^\times$. *Let* $|a_2| = q^{-d}$, *where* $d \geq 0$, *When* $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k-1}$ $(k \geq 0)$, *we have*

$$\mathrm{vol}(V_{n,d}^0) = \begin{cases} 1-q^{-1}, & \text{if } n=0,\\ q^{-1}+q^{-2}-q^{-5}, & \text{if } n=1, d=0,\\ q^{-1}-q^{-3}+q^{-4}, & \text{if } n=1, d\ge 1,\\ (1-q^{-2})q^{-n}, & \text{if } 2\le n\le d,\\ ((1-q^{-2})(1+q^{-1}+q^{-2})q^{-n}, & \text{if } n=d+1,\\ (1-q^{-2})q^{d-2n}, & \text{if } 2\le n-d\le 2k+1,\quad 2|n-d,\\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d+1-2n}, & \text{if } 2\le n-d\le 2k+1,\quad 2\nmid n-d,\\ (1+q^{-1})q^{d-2n}, & \text{if } n=2k+d+2,\\ 0, & \text{if } n>2k+d+2.\end{cases}$$

When $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k}$ $(k \geq 1)$, *we have*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1-q^{-1}, & \text{if } n=0,\\ q^{-1}+q^{-2}-q^{-5}, & \text{if } n=1, d=0,\\ q^{-1}-q^{-3}+q^{-4}, & \text{if } n=1, d\geq 1,\\ (1-q^{-2})q^{-n}, & \text{if } 2\leq n< d+1,\\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{-n}, & \text{if } n=d+1,\\ (1-q^{-2})q^{d-2n}, & \text{if } 2\leq n-d\leq 2k, \quad 2|n-d,\\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d+1-2n}, & \text{if } 2\leq n-d\leq 2k, \quad 2\nmid n-d,\\ (1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{d+1-2n}, & \text{if } n=2k+d+1,\\ (1-q^{-2})(1+q^{-1})^2 q^{d+1-2n}, & \text{if } n>2k+d+1. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. We have that $Q_1(x,y,z) = N(x)+\boldsymbol{\pi}N(y)+N(z)$ and $Q_2(x,y,z) = \boldsymbol{\pi}a_2N(y) + c_2N(z)$. Since we are interested in $|Q_2|$ we can assume that a_2 is $\boldsymbol{\pi}^d$. Thus without loss of generality we can assume $Q_2(x,y,z) = \boldsymbol{\pi}^{d+1}N(y) + c_2N(z)$.

To compute $\mathrm{vol}(V^0_{0,d})$ note that

$$V_{0,d} = \{(x,y,z) \in R_E^3; |N(x) + N(z)| = 1\}.$$

This is the disjoint union of the subset $|N(z)| = 1$, $|N(x) + N(z)| = 1$, of volume $(1 - q^{-2})(1-q^{-1}-q^{-2})$, and the subset $|N(z)| < 1$, $|N(x)| = 1$, of volume $q^{-2}(1-q^{-2})$. Hence the volume of V_0 is the product of $(1-q^{-2})$ and $(1-q^{-2})$.

To compute $\mathrm{vol}(V^0_{1,d})$ recall that $V_{1,d}$ is the set

$$\{(x,y,z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}N(y) + N(z)|, |\boldsymbol{\pi}^{d+1}N(y) + c_2N(z)|\} = q^{-1}\},$$

where $||(x,y,z)||_E = 1$. When $d > 0$, this set is the disjoint union of two subsets

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x)| < 1, |N(z)| < 1\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(y)| \leq 1, |N(x) + N(z)| = q^{-1}\}.$$

The volume of the first is equal to $(1-q^{-2})q^{-4}$ and that of the second to $(1-q^{-2}) \times q^{-1} \times (1-q^{-2})$.

When $d = 0$, this set is the disjoint union of three subsets

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x)| = 1, |N(x) + N(z)| \leq q^{-1}\},$$

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x)| < 1, |N(z)| < 1\}$$

and

$$\{(x,y,z)\in R_E^3;\ |N(y)|<1, |N(x)+N(z)|=q^{-1}\}.$$

The volume of the first is equal to $(1-q^{-2})^2q^{-1}(1+q^{-1})$, of the second to $(1-q^{-2})q^{-4}$ and of the third $(1-q^{-2})^2q^{-3}$.

To compute $\mathrm{vol}(V_{n,d}^0)$ $(2\le n<d+1)$ recall that since $|\boldsymbol{\pi}^{d+1}N(y)+c_2N(z))|<q^{-n}$, we have

$$V_{n,d}=\{(x,y,z)\in R_E^3; |N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n}\},$$

where $||(x,y,z)||_E=1$. As in the previous cases $|N(x)|=|N(z)|=1$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(x)+N(z)|=q^{-n}} dxdz.$$

Applying Lemma A.1 this is equal to

$$(1-q^{-2})\times(1-q^{-2})q^{-n}.$$

To compute $\mathrm{vol}(V_{n,d}^0)$ $(n=d+1)$ recall that $V_{n,d}$ is the set of $(x,y,z)\in R_E^3$ with $||(x,y,z)||_E=1$ and

$$\max\{|N(x)+\boldsymbol{\pi}N(y)+N(z)|, |\boldsymbol{\pi}^{d+1}N(y)+c_2N(z)|\}=q^{-n}.$$

As in the previous cases we have that $|N(x)|=|N(z)|=1$. Moreover $|\boldsymbol{\pi}^{d+1}N(y)+c_2N(z)|\le q^{-n}$. So the set $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z)\in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n}\}$$

and

$$\{(x,y,z)\in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)|\le q^{-n-1}, |N(y)|=1\}.$$

We will compute the volume of each of them.

(A) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(x)+N(z)|=q^{-n}} dxdz.$$

Applying Lemma A.1 this is equal to

$$(1-q^{-2})\times(1-q^{-2})q^{-n}.$$

(B) *Case of* $|N(x)+N(y)+N(z)|\le q^{-n-1}$, $|N(y)|=1$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)|=1}\int_{|N(x)+N(z)|\le q^{-n-1}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1-q^{-2})\times(1-q^{-2})\times(1+q^{-1})q^{-n-1}.$$

Adding the contributions from (A) and (B) and dividing by $1-q^{-2}$ we obtain

$$(1-q^{-2})(1+q^{-1}+q^{-2})q^{-n}.$$

To compute $\mathrm{vol}(V^0_{n,d})$ $(d+1<n<2k+d+1)$ recall that

$$V_n=\{(x,y,z)\in R_E^3;\max\{|N(x)+\boldsymbol{\pi}N(y)+N(z)|,|\boldsymbol{\pi}^{d+1}N(y)+c_2N(z))|\}=q^{-n}\},$$

where $||(x,y,z)||_E=1$. As in the previous cases $|N(x)|=|N(z)|=1$. Moreover $|c_2N(z))|\leq q^{-n}$. So $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z)\in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n},|N(y)|\leq q^{d+1-n}\}$$

and

$$\{(x,y,z)\in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)|\leq q^{-n-1},|N(y)|=q^{d+1-n}\}.$$

We will compute the volume of each of them.

(A) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n}$, $|N(y)|\leq q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)|\leq q^{d+1-n}}\int_{|N(x)+N(z)|=q^{-n}}dxdydz.$$

Applying Lemma A.1 and since the volume of the set $|N(y)|\leq q^{-n}$ is $q^{-n-\sigma(n)}$, the value of the integral is

$$(1-q^{-2})\times q^{d+1-n-\sigma(n-d-1)}\times(1-q^{-2})q^{-n}.$$

(B) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)|\leq q^{-n-1}$, $|N(y)|=q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)|=q^{d+1-n}}\int_{|N(x)+N(z)|\leq q^{-n-1}}dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1-q^{-2})\times q^{d+1-n}(1-q^{-2})\times(1+q^{-1})q^{-n-1}$$

when $n-d-1$ is even and zero when $n-d-1$ is odd.

Adding the contributions from (A) and (B) and dividing by $1-q^{-2}$ we obtain

$$(1-q^{-2})(1+q^{-1}+q^{-2})q^{d+1-2n}$$

when $n-d-1$ is even and

$$(1-q^{-2})q^{d-2n}$$

when $n-d-1$ is odd.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n}=|c_2|)$ recall that

$$V_{n,d}=\{(x,y,z)\in R_E^3; \max\{|N(x)+\boldsymbol{\pi}N(y)+N(z)|,|\boldsymbol{\pi}^{d+1}N(y)+c_2N(z))|\}=q^{-n}\},$$

where $||(x,y,z)||_E=1$. If $|N(z)|<1$ then $|N(x)|<1$ which is a contradiction. Thus $|N(x)|=|N(z)|=1$ and since $|c_2|=q^{-n}$, $V_{n,d}$ is the disjoint union of two subsets

$$\{(x,y,z)\in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n},|N(y)|\le q^{d+1-n}\}$$

and

$$\{(x,y,z)\in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)|\le q^{-n-1},|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)|=q^{d+1-n}\}.$$

We will compute the volume of each of them.

(A) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)|=q^{-n}$, $|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)|\le q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)|\le q^{d+1-n}}\int_{|N(x)+N(z)|=q^{-n}} dxdydz.$$

Applying Lemma A.1, the value of the integral is

$$(1-q^{-2})\times q^{d+1-n}\times(1-q^{-2})q^{-n}$$

when $|c_2/(a_2\boldsymbol{\pi})|=q^{-2k}$ (thus $n-d-1=2k$ is even), and

$$(1-q^{-2})\times q^{d-n}\times(1-q^{-2})q^{-n}$$

when $|c_2/(a_2\boldsymbol{\pi})|=q^{-2k-1}$ (thus $n-d-1=2k+1$ is odd).

(B) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)|\le q^{-n-1}$, $|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)|=q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)|=q^{d+1-n}}\int_{|N(x)+N(z)|\le q^{-n-1}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1-q^{-2})\times(1-q^{-1}-q^{-2})q^{d+1-n}\times(1+q^{-1})q^{-n-1}.$$

when $|c_2/(a_2\boldsymbol{\pi})|=q^{-2k}$ and

$$(1-q^{-2})\times q^{d-n}\times(1+q^{-1})q^{-n-1},$$

when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k-1}$.

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

$$(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{d+1-2n}$$

when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k}$ and

$$(1 + q^{-1})q^{d-2n},$$

when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k-1}$.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n} > |c_2|)$ recall that

$$V_{n,d} = \{(x, y, z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}N(y) + N(z)|, |\boldsymbol{\pi}^{d+1}N(y) + c_2N(z))|\} = q^{-n}\},$$

where $||(x, y, z)||_E = 1$. As in the previous case, we have that $|N(x)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of two subsets

$$\{(x, y, z) \in R_E^3;\ |N(x) + \boldsymbol{\pi}N(y) + N(z)| = q^{-n}, |N(y) + c_2\boldsymbol{\pi}^{-d-1}N(z)| \le q^{d+1-n}\}$$

and

$$\{(x, y, z) \in R_E^3;\ |N(x) + \boldsymbol{\pi}N(y) + N(z)| \le q^{-n-1}, |N(y) + c_2\boldsymbol{\pi}^{-d-1}N(z)| = q^{d+1-n}\}.$$

Note that according to Lemma A.1, this set is empty when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k-1}$.

We will compute the volume of each of them when $|c_2/(a_2\boldsymbol{\pi})| = q^{-2k}$.

(A) *Case of* $|N(x) + \boldsymbol{\pi}N(y) + N(z)| = q^{-n}$, $|N(y) + c_2\boldsymbol{\pi}^{-d-1}N(z)| \le q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)|\le q^{d+1-n}} \int_{|N(x)+N(z)|=q^{-n}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times (1 + q^{-1})q^{d+1-n} \times (1 - q^{-2})q^{-n}.$$

(B) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)| \le q^{-n-1}$, $|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)| = q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+c_2\boldsymbol{\pi}^{-d-1}N(z)|=q^{d+1-n}} \int_{|N(x)+N(z)|\le q^{-n-1}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times (1 - q^{-2})q^{d+1-n} \times (1 + q^{-1})q^{-n-1}.$$

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

$$(1 - q^{-2})(1 + q^{-1})^2q^{d+1-2n}.$$

□

Lemma C.5.3. *Suppose that* a_1, $c_1 \in R^\times$. *Let* $|c_2| = q^{-d}$, *where* $d \geq 1$. *When* $|a_2\boldsymbol{\pi}| < |c_2|$, *we have*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1 - q^{-1}, & \text{if } n = 0, \\ q^{-1} + q^{-2} + q^{-4}, & \text{if } n = 1,\ d = 1, \\ q^{-1} - q^{-3} + q^{-4}, & \text{if } n = 1,\ d \geq 2, \\ (1 - q^{-2})q^{-n}, & \text{if } 1 < n < d, \\ (1 + q^{-1})q^{-n}, & \text{if } n = d,\ d \geq 2, \\ 0, & \text{if } n > d. \end{cases}$$

When $|a_2\boldsymbol{\pi}| = |c_2|$, *we have*

$$\mathrm{vol}(V^0_{n,d}) = \begin{cases} 1 - q^{-1}, & \text{if } n = 0, \\ q^{-1} + q^{-2} - q^{-3} - q^{-4} - q^{-5}, & \text{if } n = 1,\ d = 1, \\ q^{-1} - q^{-3} + q^{-4}, & \text{if } n = 1,\ d \geq 2 \\ (1 - q^{-2})q^{-n}, & \text{if } 1 < n < d, \\ (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{-n}, & \text{if } n = d,\ d \geq 2, \\ (1 - q^{-2})(1 + q^{-1})^2 q^{d-2n}, & \text{if } n > d. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. We have that $Q_1(x, y, z) = N(x) + \boldsymbol{\pi} N(y) + N(z)$ and $Q_2(x, y, z) = \boldsymbol{\pi} a_2 N(y) + c_2 N(z)$. Since we are interested in $|Q_2|$ we can assume that a_2 is $\boldsymbol{\pi}^d$. Thus without loss of generality we can assume $Q_2(x, y, z) = \boldsymbol{\pi}^{d+1} N(y) + c_2 N(z)$.

To compute $\mathrm{vol}(V^0_{0,d})$ note that $|a_2\boldsymbol{\pi} N(y) + c_2 N(z)| < 1$ and

$$V_{0,d} = \{(x, y, z) \in R_E^3; |N(x) + N(z)| = 1\}.$$

The volume of this subset is given by the following integral

$$\int_{|N(z)|=1} \int_{|N(x)+N(z)|=1} dxdz + \int_{|N(z)|<1} \int_{|N(x)|=1} dxdz,$$

which is equal to the product of $1 - q^{-2}$ and $1 - (q + 1)/q^2 + 1/q^2 = 1 = 1 - 1/q$.

To compute $\mathrm{vol}(V^0_{1,d})$ recall that, when $d > 1$ we have $|a_2\boldsymbol{\pi}| < |c_2| < q^{-1}$ thus

$$V_{n,d} = \{(x, y, z) \in R_E^3; |N(x) + \boldsymbol{\pi} N(y) + N(z)| = q^{-1},\ ||(x, y, z)||_E = 1\}.$$

Note that the condition $|N(x)| < 1$ implies $|N(z)| < 1$ and $|N(y)| = 1$. Thus the volume of this subset is given by the following sum of two integrals

$$\int_{|N(x)|=1} \int_{|N(y)|\leq 1} \int_{|N(z)+c|=q^{-1}} dzdydx + \int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(z)|<1} dzdydx,$$

where $c = c(y,z) = N(x) + \boldsymbol{\pi} N(y)$, $|c| = 1$. This is equal to $(1-q^{-2})(q^{-1}-q^{-3}+q^{-4})$.

When $d = 1$, the volume of this subset is given by the following sum of two integrals

$$\int_{|N(x)|=1}\int_{|N(y)|\leq 1}\int_{|N(z)+c|\leq q^{-1}} dzdydx + \int_{|N(x)|<1}\int_{|N(y)|=1}\int_{|N(z)|<1} dzdydx,$$

where $c = c(y,z) = N(x) + \boldsymbol{\pi} N(y)$, $|c| = 1$. This is equal to $(1-q^{-2})(q^{-1}+q^{-2}+q^{-4})$.

To compute $\mathrm{vol}(V^0_{n,d})$ $(1 < n < d)$ recall that $|a_2\boldsymbol{\pi}| < |c_2| < q^{-n}$ and

$$V_{n,d} = \{(x,y,z) \in R_E^3;\ |N(x) + \boldsymbol{\pi} N(y) + N(z)| = q^{-n},\ ||(x,y,z)||_E = 1\}.$$

Note that the condition $|N(x)| < 1$ implies $|N(z)| < 1$ and $|N(y)| = 1$, thus $|N(x)+\boldsymbol{\pi} N(y)+N(z)| = q^{-1}$ which is a contradiction. The volume of this subset is given by the following integral

$$\int_{|N(x)|=1}\int_{|N(y)|\leq 1}\int_{|N(z)+c|=q^{-n}} dzdydx,$$

where $c = c(y,z) = N(x) + \boldsymbol{\pi} N(y)$, $|c| = 1$. This is equal to $(1-q^{-2})q^{-n}(1-q^{-2})$.

To compute $\mathrm{vol}(V^0_{n,d})$ $(n = d)$, when $|a_2\boldsymbol{\pi}| < |c_2| = q^{-n}$ recall that $V_{n,d}$ consist of $(x,y,z) \in R_E^3$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x) + \boldsymbol{\pi} N(y) + N(z)|, |c_2 N(z))|\} = q^{-n}.$$

If $|N(z)| < 1$ then $|N(x)| < 1$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and since $|c_2| = q^{-n}$, $V_{n,d}$ is given by

$$\{(x,y,z) \in R_E^3;\ |N(x) + \boldsymbol{\pi} N(y) + N(z)| \leq q^{-n}, |N(z)| = 1\}.$$

The volume of this subset is given by the following integral

$$\int_{|N(z)|=1}\int_{|N(y)|\leq 1}\int_{|N(x)+c|=q^{-n}} dxdydz = (1-q^{-2})q^{-n}(1+q^{-1}).$$

When $|a_2\boldsymbol{\pi}| = |c_2| = q^{-n}$, (set $c_2' = c_2(a_2\boldsymbol{\pi})^{-1}$), the set $V_{n,d}$ consists of the $(x,y,z) \in R_E^3$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x) + \boldsymbol{\pi} N(y) + N(z)|, |N(y) + c_2' N(z))|\} = q^{-n}.$$

If $|N(z)| < 1$ then $|N(x)| < 1$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x,y,z) \in R_E^3;\ |N(x) + \boldsymbol{\pi} N(y) + N(z)| = q^{-n}, |N(y) + c_2' N(z)| \leq 1\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(x) + \boldsymbol{\pi} N(y) + N(z)| \leq q^{-n-1}, |N(y) + c_2' N(z)| = 1\}.$$

We will compute the volume of each of them.
(A) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)| = q^{-n}$, (the $|N(y)+c_2'N(z)| \le 1$ is trivial). The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)|\le 1}\int_{|N(x)+N(z)|=q^{-n}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1-q^{-2}) \times (1-q^{-2})q^{-n}.$$

(B) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)| \le q^{-n-1}$, $|N(y)+c_2'N(z)| = 1$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+c_2'N(z)|=1}\int_{|N(x)+N(z)|\le q^{-n-1}} dxdydz.$$

Applying Lemma A.1, the value of the integral is

$$(1-q^{-2}) \times (1-q^{-1}-q^{-2}) \times (1+q^{-1})q^{-n-1}.$$

Adding the contributions from (A) and (B) and dividing by $1-q^{-2}$ we obtain

$$(1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{-n}.$$

To compute $\mathrm{vol}(V^0_{n,d})$ $(n>d)$ note that the condition $|N(z)|<1$ implies that $|N(x)|<1$ and thus $|N(y)|=1$ and $|N(x)+\boldsymbol{\pi}N(y)+N(z)| = q^{-1}$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and when $|a_2\boldsymbol{\pi}| < |c_2| = q^{-d}$ we have

$$|a_2\boldsymbol{\pi}N(y)+c_2N(z)| = |c_2N(z)| = q^{-d} < q^{-n}.$$

Thus the set is empty.

When $|a_2\boldsymbol{\pi}| = |c_2| = q^{-d}$, (set $c_2' = c_2(a_2\boldsymbol{\pi})^{-1}$) we have that $|N(x)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of its two subsets

$$\{(x,y,z) \in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)| = q^{-n}, |N(y)+c_2'N(z)| \le q^{d-n}\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(x)+\boldsymbol{\pi}N(y)+N(z)| \le q^{-n-1}, |N(y)+c_2'N(z)| = q^{d-n}\}.$$

We will compute the volume of each of them.
(A) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)| = q^{-n}$, $|N(y)+c_2'N(z)| \le q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+c_2'N(z)|\le q^{d-n}}\int_{|N(x)+N(z)|=q^{-n}} dxdydz.$$

Applying Lemma A.1 the value of the integral is

$$(1-q^{-2})\times(1+q^{-1})q^{d-n}\times(1-q^{-2})q^{-n}.$$

(B) *Case of* $|N(x)+\boldsymbol{\pi}N(y)+N(z)|\le q^{-n-1}$, $|N(y)+c_2'N(z)|=q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+c_2'N(z)|=q^{d-n}}\int_{|N(x)+N(z)|\le q^{-n-1}} dxdydz.$$

Applying Lemma A.1, the value of the integral is

$$(1-q^{-2})\times(1-q^{-2})q^{d-n}\times(1+q^{-1})q^{-n-1}.$$

Adding the contributions from (A) and (B) and dividing by $1-q^{-2}$ we obtain

$$(1-q^{-2})(1+q^{-1})^2q^{d-2n}.$$

The lemma follows. □

D. Appendix to Section 6

This section contains the computations used in section 6.

Lemma D.6.1. *Suppose that* $c_1\in R^\times$ *and* $|1-u|=1$. *Suppose* $|a_2|=|c_2|=q^{-d}$, *where* $d\ge 0$. *Then we have*

$$\mathrm{vol}(V_n^0)=\begin{cases}1, & \text{if } n=0,\\ q^{-2}+q^{-4}, & \text{if } n=1,\ d=0,\\ q^{-2}-q^{-3}, & \text{if } n=1,\ d\ge 1,\\ (1-q^{-2})q^{-n-1}, & \text{if } 2\le n\le d,\\ (1+q^{-2})q^{-n-1}, & \text{if } n=d+1,\ d\ge 1,\\ 0, & \text{if } n\ge d+2.\end{cases}$$

Proof. Making a change of variables $y'=\alpha y$ where $\alpha\in R_E^\times$ such that $N(\alpha)=a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1=1$. Similarly, we can assume that $c_1=1$. We have that $Q_1(x,y,z)=N(x)+\boldsymbol{\pi}(N(y)+N(z))$ and $Q_2(x,y,z)=a_2\boldsymbol{\pi}N(y)+c_2\boldsymbol{\pi}N(z)$. Since we are interested in $|Q_2|$ we can assume that a_2 is $\boldsymbol{\pi}^d$. Thus without loss of generality we can assume $Q_2(x,y,z)=\boldsymbol{\pi}^{d+1}(N(y)+uN(z))$.

We first show that V_n $(n\ge d+2)$ *is empty.*

Suppose $(x,y,z) \in V_n$. Then $|N(x)| < 1$, since $|N(x)| = 1$ implies $|Q_1| = 1 > q^{-n}$. Since $|1-u| = 1$ we obtain that

$$|N(y) + uN(z)| = |N(y) + N(z) + (1-u)N(z)| = \max\{|N(y) + N(z)|, |N(z)|\}.$$

This is equal to

$$\max\{|N(y)|, |N(z)|\} = 1.$$

Hence $\max\{|Q_1|, |Q_2|\} = |\boldsymbol{\pi}^{d+1}| > q^{-n}$. We conclude that V_n $(n \geq 2)$ is empty.

To show that $\mathrm{vol}(V_0^0) = 1 - 1/q^2$ note that

$$V_0 = \{(x,y,z) \in R_E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}.$$

To compute $\mathrm{vol}(V_1^0)$ recall that, when $d \geq 1$ we have $|\boldsymbol{\pi}^{d+1}(N(y) + uN(z))| < q^{-1}$ thus

$$V_n\{(x,y,z) \in R_E^3; |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-1}\},$$

where $||(x,y,z)||_E = 1$. Thus the volume of this subset is given by the following sum of two integrals

$$\int_{|N(x)|<1}\int_{|N(y)|=1}\int_{|N(z)+N(y)|=q^{-1}} dzdydx + \int_{|N(x)|<1}\int_{|N(y)|<1}\int_{|N(z)|=1} dzdydx.$$

This is equal to $(1-q^{-2})q^{-2}(1-q^{-1})$. When $d = 1$, the

$$\{(x,y,z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}(N(y) + uN(z))|\} = 1/q\},$$

where $||(x,y,z)||_E = 1$. If $(x,y,z) \in V_1$ then $|N(x)| < 1$, and this set is equal to

$$\{(x,y,z) \in R_E^3; \max\{|N(y) + N(z)|, |N(y) + uN(z)|\} = 1\},$$

where $||(x,y,z)||_E = 1$. Since $|1-u| = 1$, the volume of this set is

$$\left(\int_{|N(y)|<1}\int_{|N(z)|=1} + \int_{|N(y)|=1}\int_{|N(z)|\leq 1}\right)\int_{|N(x)|<1} dxdzdy,$$

which is equal to

$$q^{-2}(1-q^{-2})(1+q^{-2}).$$

To compute $\mathrm{vol}(V_n^0)$ $(2 \leq n \leq d)$ recall that $|\boldsymbol{\pi}^{d+1}(N(y) + uN(z))| < q^{-n}$ and

$$V_n\{(x,y,z) \in R_E^3; |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. The volume of this subset is given by the following integral

$$\int_{|N(y)|=1}\int_{|N(x)|<1}\int_{|N(z)+c|=q^{1-n}} dzdxdy,$$

where $c = c(x,y) = N(y) + N(x)/\boldsymbol{\pi}$, $|c| = 1$. This is equal to $(1-q^{-2})q^{-2} \times q^{1-n}(1-q^{-2})$.

To compute $\mathrm{vol}(V_n^0)$ $(n = d+1)$, recall that we have

$$V_n = \{(x,y,z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}^{d+1}(N(y) + uN(z)))|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. If $|N(z)| < 1$ then $|N(y)| < 1$ which is a contradiction. Thus $|N(y)| = |N(z)| = 1$ and $|N(x)| < 1$. Note that the condition $|N(y) + N(z)| < 1$, implies that

$$|N(y) + uN(z)| = |N(y) + N(z) + (u-1)N(z)| = |(1-u)N(z)|.$$

Thus V_n is given by

$$\{(x,y,z) \in R_E^3;\ |N(x) + \boldsymbol{\pi}(N(y) + N(z))| \le q^{-n}, |N(z)| = 1\}.$$

The volume of this subset is given by the following integral

$$\int_{|N(z)|=1} \int_{|N(x)|<1} \int_{|N(y)+c| \le q^{1-n}} dydxdz = (1-q^{-2})q^{-2} \times q^{1-n}(1+q^{-1}),$$

where $c = c(x,z) = N(z) + N(x)/\boldsymbol{\pi}$ so that $|c| = 1$. The lemma follows. □

Lemma D.6.2. *Suppose that* $n_1 = 1$, $n_2 = 1$. $a_1, c_1 \in R^\times$. *Suppose that* $|a_2| = |c_2| = q^{-d}$, $d \ge 0$. *When* $|1-u| = q^{-2k}$ $(k \ge 1)$*, we have*

$$\mathrm{vol}(V_{n,d}^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-2} - q^{-3}, & \text{if } n = 1, \\ (1-q^{-2})q^{-n-1}, & \text{if } 2 \le n \le d, \\ (1-q^{-2})q^{-n-1}, & \text{if } n = d+1, \\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d+1-2n}, & \text{if } 2 \le n-d \le 2k, \quad 2|n-d, \\ (1-q^{-2})q^{d-2n}, & \text{if } 2 \le n-d \le 2k-1, \quad 2 \nmid n-d, \\ (1+q^{-1})q^{d-2n}, & \text{if } n = 2k+1+d, \\ 0, & \text{if } n > 2k+1+d. \end{cases}$$

When $|1-u| = q^{-2k-1}$ $(k \ge 0)$*, we have*

$$\mathrm{vol}(V_{n,d}^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-2} - q^{-3}, & \text{if } n = 1, \\ (1-q^{-2})q^{-n-1}, & \text{if } 1 \le n \le d, \\ (1-q^{-2})q^{-n-1}, & \text{if } n = d+1, \\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d+1-2n}, & \text{if } 2 \le n-d \le 2k, \quad 2|n-d, \\ (1-q^{-2})q^{d-2n}, & \text{if } 2 \le n-d \le 2k+1, \quad 2 \nmid n-d, \\ (1+q^{-1}-q^{-2}-2q^{-3}-q^{-4})q^{1+d-2n}, & \text{if } n = 2k+2+d, \\ (1-q^{-2})(1+q^{-1})^2 q^{1+d-2n}, & \text{if } n > 2k+2+d. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. We have that $Q_1(x,y,z) = N(x) + \boldsymbol{\pi}(N(y) + N(z))$ and $Q_2(x,y,z) = a_2\boldsymbol{\pi}N(y) + c_2\boldsymbol{\pi}N(z)$. Since we are interested in $|Q_2|$ we can assume that a_2 is $\boldsymbol{\pi}^d$. Thus without loss of generality we can assume $Q_2(x,y,z) = \boldsymbol{\pi}^{d+1}(N(y) + uN(z))$.

To see that $\mathrm{vol}(V_{0,d}^0) = 1 - 1/q^2$ note that

$$V_{0,d} = \{(x,y,z) \in R_E^3; |N(x)| = 1, |N(y)| \le 1, |N(z)| \le 1\}.$$

To compute $\mathrm{vol}(V_{1,d}^0)$ recall that $V_{1,d}$ is the set

$$\{(x,y,z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}^{1+d}(N(y) + uN(z))|\} = 1/q\},$$

where $||(x,y,z)||_E = 1$. When $d = 0$ then $|N(x)| < 1$, and since $|1-u| < 1$ this set is equal to

$$\{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, |N(y) + N(z)| = 1\}.$$

Then its volume is given by the integral

$$\left(\int_{|N(y)|<1}\int_{|N(z)|=1} + \int_{|N(y)|=1}\int_{|N(y)+N(z)|=1}\right)\int_{|N(x)|<1} dxdzdy,$$

whose value is

$$q^{-4}(1-q^{-2}) + (1-q^{-2})(1-(q+1)q^{-2})q^{-2} = (1-q^{-2})(1-q^{-1})q^{-2}.$$

To compute $\mathrm{vol}(V_{n,d}^0)$ ($n \ge 2$ and $q^{-n} > |\boldsymbol{\pi}^{1+d}|$) recall that since $d > 0$ and $|\boldsymbol{\pi}^{1+d}(N(y) + uN(z))| < q^{-n}$, we have that

$$V_{n,d} = \{(x,y,z) \in R_E^3; |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-n} \;\; ||(x,y,z)||_E = 1\}.$$

Assume that $|N(y)| < 1$. Since $|N(x)| \ne |\boldsymbol{\pi}N(z)|$, it implies that $|N(z)| < 1$ and thus $|N(x)| < 1$, which is a contradiction. Thus $|N(x)| < 1$, $|N(y)| = 1$ and

$$V_{n,d} = \{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x)| < 1, |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-n}\}.$$

So, the volume is given by the integral

$$\int_{|N(x)|<1}\int_{|N(y)|=1}\int_{|N(z)+c|=q^{1-n}} dzdydx = q^{-2}(1-q^{-2})q^{1-n}(1-q^{-2}),$$

where $c = c(x,y) = N(x)/\boldsymbol{\pi} + N(y)$ so that $|c| = |N(y)| = 1$. Thus, $\mathrm{vol}(V_{n,d}^0)$ is equal to $(1-q^{-2})q^{-1-n}$.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n} = |\boldsymbol{\pi}^{1+d}|)$ recall that $V_{n,d}$ consists of the $\{(x,y,z) \in R_E^3$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}^{1+d}(N(y) + uN(z))|\} = q^{-n}\}.$$

Assume that $|N(z)| < 1$. Since $|N(x)| \neq |\boldsymbol{\pi} N(y)|$, it implies that $|N(y)| < 1$ and thus $|N(x)| < 1$, which is a contradiction. Thus $|N(x)| < 1$, $|N(y)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x,y,z) \in R_E^3;\ |N(z)| = 1, |N(x) + \boldsymbol{\pi}(N(y) + N(z))| \leq q^{-n}, |N(y) + uN(z)| = 1\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(z)| = 1, |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-n}, |N(y) + uN(z)| < 1\}.$$

We will compute the volume of each subset.
(A) *Case of* $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| \leq q^{-n}$ and $|N(y)+uN(z)| = 1$. Note that since $|1-u| < 1$ we have $|N(y)+N(z)| = |N(y)+uN(z)| = 1$. And the set $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| < q^{-n}$ is empty.
(B) *Case of* $|N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-n}$ and $|N(y) + uN(z)| < 1$. Since $|1 - u| < 1$ the second inequality is equivalent to $|N(y) + N(z)| < 1$. Applying Lemma A.4, the volume of this set is equal to

$$(1 - q^{-2})^2 q^{-2n} q^{n-1}.$$

To compute $\mathrm{vol}(V^0_{n,d})$ $(n \geq d + 2$ and $q^{-n} > |\boldsymbol{\pi}^{1+d}(1 - u)|)$ recall that $V_{n,d}$ is the set of $(x,y,z) \in R_E^3$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}^{1+d}(N(y) + uN(z))|\} = q^{-n}.$$

Assume that $|N(y)| < 1$. Then $|N(y)+uN(z)| < 1$ implies that $|N(z)| < 1$; from $|N(x)/\boldsymbol{\pi} + N(y) + N(z)| < 1/q^2$ it follows that $|N(x)| < 1$, which is a contradiction. Moreover, the inequality $|\boldsymbol{\pi}^{1+d}(N(y) + uN(z))| \leq q^{-n}$ is equivalent to $|\boldsymbol{\pi}^{1+d}(N(y) + N(z))| \leq q^{-n}$. Thus $|N(y)| = 1$ and $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x) + \boldsymbol{\pi}(N(y) + N(z))| \leq q^{-n}, |N(y) + N(z)| = q^{1+d-n}\}$$

and

$$\{(x,y,z) \in R_E^3;\ |N(y)| = 1, |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = q^{-n}, |N(y) + N(z)| < q^{1+d-n}\}.$$

We will compute the volume of each subset.
(A) *Case of* $|N(x) + \boldsymbol{\pi}(N(y) + N(z))| \leq q^{-n}$ and $|N(y) + N(z)| = q^{1+d-n}$. The volume of this subset equals the integral

$$\int_{|N(y)|=1} \int_{|N(y)+N(z)|=q^{1+d-n}} \int_{|N(x)+c| \leq q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$ so that $|c| = q^{d-n}$.

When $d=0$, thus $|c| = q^{-n}$ and the volume of this subset equals the integral

$$\int_{|N(y)|=1}\int_{|N(y)+N(z)|=q^{1-n}}\int_{|N(x)|\le q^{-n}} dxdzdy.$$

The volume of the set $\{|N(x)| \le q^{-n}\}$ is equal to q^{-n} if n is even and to q^{-n-1} if n is odd. Thus the integral is equal to the product of

$$(1-q^{-2}) \times (1-q^{-2})q^{1-2n}$$

and 1 if n is even and q^{-1} if n is odd.

When $d>0$, the volume of the set $\{|N(x)+c| \le q^{-n}\}$ is zero when $n-d$ is odd. Thus, the integral is equal to $(1-q^{-2}) \times (1-q^{-2})q^{1+d-n} \times q^{-n}(1+q^{-1})$, when $n-d$ is even, and zero when $n-d$ is odd.

(B) *Case of* $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| = q^{-n}$ and $|N(y)+N(z)| < q^{1+d-n}$. The volume of this subset equals the integral

$$\int_{|N(y)|=1}\int_{|N(y)+N(z)|\le q^{d-n}}\int_{|N(x)+c|=q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$.

When $d=0$, thus $|c| < q^{-n}$, and the volume of this subset equals the integral

$$\int_{|N(y)|=1}\int_{|N(y)+N(z)|\le q^{-n}}\int_{|N(x)|=q^{-n}} dxdzdy.$$

This integral is equal to zero when n is odd and to

$$(1-q^{-2}) \times (1+q^{-1})q^{-n} \times (1-q^{-2})q^{-n}$$

when n is even. Thus, $\mathrm{vol}(V^0_{n,0})$ is equal to

$$(1-q^{-2})q^{1-2n} \times (1+q^{-1}+q^{-2}),$$

when n is even and to

$$(1-q^{-2})q^{1-2n} \times q^{-1},$$

when n is odd.

When $d>0$ the volume is given by the integral

$$\int_{|N(y)|=1}\int_{|N(y)+N(z)|\le q^{d-n}}\int_{|N(x)+c|=q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$. It is equal to

$$\sum_{t=n-d}^{\infty} \int_{|N(y)|=1} \int_{|N(y)+N(z)|=q^{-t}} \int_{|N(x)+c_t|=q^{-n}} dxdzdy,$$

where $|c_t| = q^{-t-1}$. Applying Lemma A.4, it equal to

$$(1-q^{-2})^2 q^{d+\sigma(n-d)-2n-1}.$$

Thus, $\mathrm{vol}(V_{n,d}^0)$ is equal to

$$(1-q^{-2})q^{d+1-2n} \times (1+q^{-1}+q^{-2}),$$

when $n-d$ is even and to

$$(1-q^{-2})q^{d-2n},$$

when $n-d$ is odd.

To compute $\mathrm{vol}(V_{n,d}^0)$ ($q^{-n} = |\boldsymbol{\pi}^{1+d}(1-u)|$) recall that $V_{n,d}$ consists of the $(x,y,z) \in R_E^3$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x)+\boldsymbol{\pi}(N(y)+N(z))|, |\boldsymbol{\pi}^{1+d}(N(y)+uN(z))|\} = q^{-n}.$$

As in the previous cases we can show that $|N(y)| = |N(z)| = 1$, and $V_{n,d}$ is the disjoint union of the two subsets:
(A) *Case of* $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| \le q^{-n}$, $|N(y)+uN(z)| = q^{1+d-n}$, and $|N(y)| = |N(z)| = 1$. When $d=0$, we have

$$|N(y)+N(z)| \le \max\{|N(y)+uN(z)|, |(1-u)N(z)|\} \le q^{1-n}.$$

Thus $|N(x)| \le q^{-n}$ and the volume of this subset is given by the integral

$$\int_{|N(y)|=1} \int_{|N(y)/u+N(z)|=q^{1-n}} \int_{|N(x)|\le q^{-n}} dxdzdy,$$

which is equal to

$$(1-q^{-2}) \times (1-q^{-2})q^{1-n} \times q^{-n-\sigma(n)}.$$

When $d>0$, then for any $t \ge 0$, we define the following subset

$$W_t(z) = \{y \in E; |N(z)| = 1, |N(y)+uN(z)| = q^{1+d-n}, |N(y)+N(z)| = q^{1+d-t-n}\}.$$

The volume of the set is given by the integral

$$\sum_{t=0}^{\infty} \int_{|N(z)|=1} \int_{W_t(z)} \int_{|N(x)+c_t|\le q^{-n}} dxdzdy,$$

where $c_t = \boldsymbol{\pi}(N(y) + N(z))$ and $|c_t| = q^{d-t-n}$. Note that the integral over x depends only on the $|c_t|$. Thus we can assume that c_t is an arbitrary element so that $|c_t| = q^{d-t-n}$ and take the integral over y and z. The integral is equal to

$$(1 - q^{-2}) \sum_{t=0}^{\infty} \mathrm{vol}(W_t) \int_{|N(x)+c_t| \le q^{-n}} dx.$$

Applying Lemma A.2, this equal to the product of $(1 - q^{-2})^2$ and

$$(1 - (q + 2)q^{-2})q^{1+d-n} \int_{|N(x)+c_0| \le q^{-n}} dx + \sum_{t=1}^{\infty} q^{1+d-t-n} \int_{|N(x)+c_t| \le q^{-n}} dx.$$

If $t \ge d$ so that $|c_t| \le q^{-n}$, and thus $|N(x)| \le q^{-n}$. The integral over x is equal to $q^{-n-\sigma(n)}$. The sum of the terms which correspond to $t \ge d$ is equal to

$$(1 - q^{-2})^2 q^{1-2n-\sigma(n)} \sum_{t=d}^{\infty} q^{d-t} = (1 - q^{-2})(1 + q^{-1})q^{1-2n-\sigma(n)}.$$

If $1 \le t \le d - 1$ then the set $|N(x) + c_t| \le q^{-n}$ is empty when $n - d + t$ is odd and is $(1 + q^{-1})q^{-n}$ when $n - d + t$ is even. Set $j = n - d + t$, then $n - d + 1 \le j \le n - 1$ and the even j are of the form over

$$n - d + 1 + \sigma(n - d + 1), n - d + 3 + \sigma(n - d + 1), \dots , n - 1 - \sigma(n - 1).$$

Since $\sigma(n - d + 1) = 1 - \sigma(n - d)$ and $\sigma(n - 1) = 1 - \sigma(n)$, this sequence can be written as

$$n - d - \sigma(n - d) + 2, n - d - \sigma(n - d) + 4, \dots , n + \sigma(n) - 2.$$

So the sum is the product of $(1 + q^{-1})q^{1-n}$ and

$$q^{-(n-d-\sigma(n-d)+2)} + q^{-(n-d-\sigma(n-d)+4)} + \dots + q^{-(n+\sigma(n)-2)}$$
$$= (1 - q^{-2})^{-1}(q^{-(n-d-\sigma(n-d)+2)} - q^{-(n+\sigma(n)-2)-2}).$$

Thus, the contribution from $1 \le t \le d - 1$ is equal to

$$(1 + q^{-1})(1 - q^{-2})(q^{-1-2n+d+\sigma(n-d)} - q^{1-2n-\sigma(n)}).$$

If $t = 0$, the contribution from this term is

$$(1 - (q + 2)q^{-2})(1 + q^{-1})q^{-n},$$

when $n - d$ is even and zero when $n - d$ is odd.

(B) *Case of* $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| = q^{-n}$, $|N(y)+uN(z)| < q^{1+d-n}$, and $|N(y)| = |N(z)| = 1$. Since $|N(z)| = 1$ and $|N(y)+uN(z)| < q^{1+d-n}$, we obtain

$$|N(y)+N(z)| = |N(y)+uN(z)+(1-u)N(z)| = |1-u| = q^{1+d-n}.$$

Thus, the volume of this subset is given by the integral

$$\int_{|N(z)|=1}\int_{|N(y)+uN(z)|\le q^{d-n}}\int_{|N(x)+c|=q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$ and $|c| = q^{d-n}$. When $d = 0$, the integral over x is taken over the set $|N(x)| \le q^{-n-1}$ when n is odd and over the set $|N(x)+c| \le q^{-n}$ when n is even. Thus, the integral is equal to

$$(1-q^{-2}) \times (1+q^{-1})q^{-n} \times q^{-n-1}.$$

when n is odd, and when n is even, to

$$(1-q^{-2}) \times (1+q^{-1})q^{-n} \times (1-(q+1)q^{-2})q^{-n}.$$

When $d > 0$ and $n-d$ is odd the set $|N(x)+c| = q^{-n}$ is empty and the integral is zero. In the case of $n-d$ being even, the volume is equal to

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n} \times (1-q^{-2})q^{-n}.$$

When $d = 0$, adding the contributions from cases (A) and (B) (and dividing by $(1-1/q^2)$), we obtain

$$q^{-2n}(1+q^{-1})$$

when n is odd and

$$q^{-2n}(q+1-q^{-1}-2q^{-2}-q^{-3})$$

when n is even.

When $d > 0$, adding the contributions from cases (A) and (B) (and dividing by $(1-1/q^2)$), we obtain

$$\begin{aligned}
&(1+q^{-1})q^{1-2n-\sigma(n)} + (1+q^{-1})(q^{-1-2n+d} - q^{1-2n-\sigma(n)}) \\
&+(1+q^{-1})(1-q^{-2})q^{d-2n} + (1-q^{-2})(1-(q+2)q^{-2})(1+q^{-1})q^{1+d-2n} \\
&= (1+q^{-1})q^{-2n}(q^{d-1}+q^d-q^{d-2}+q^{1+d}-q^d-2q^{d-1}) \\
&= (1+q^{-1})q^{d-2n}(q-q^{-1}-q^{-2}).
\end{aligned}$$

when $n-d$ is even and

$$(1+q^{-1})q^{1-2n-\sigma(n)} + (1+q^{-1})(q^{-2n+d} - q^{1-2n-\sigma(n)}) = (1+q^{-1})q^{d-2n}$$

when $n-d$ is odd.

To compute $\mathrm{vol}(V^0_{n,d})$ $(q^{-n} < |\boldsymbol{\pi}^{1+d}(1-u)|)$ recall that $V_{n,d}$ consists of the $(x,y,z) \in R^3_E$ with $||(x,y,z)||_E = 1$ and

$$\max\{|N(x)+\boldsymbol{\pi}(N(y)+N(z))|, |\boldsymbol{\pi}^{1+d}(N(y)+uN(z))|\} = q^{-n}\}.$$

Then we have $|N(y)| = 1$. Since $|N(z)| = 1$, $q^{1+d-n} < |1-u|$ and $|N(y)+uN(z)| \le q^{1+d-n}$, we have

$$|N(y)+N(z)| = |(1-u)N(z)| = |1-u|$$

Thus $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x,y,z) \in R^3_E;\ |N(y)| = 1, |N(x)+\boldsymbol{\pi}(N(y)+N(z))| \le q^{-n}, |N(y)+uN(z)| = q^{1+d-n}\}$$

and

$$\{(x,y,z) \in R^3_E;\ |N(y)| = 1, |N(x)+\boldsymbol{\pi}(N(y)+N(z))| = q^{-n}, |N(y)+uN(z)| \le q^{1+d-n}\}.$$

We will compute the volume of each of subset.

(A) *Case of* $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| \le q^{-n}$, $|N(y)+uN(z)| = q^{1+d-n}$. Then $|N(y)| = 1$. The volume of this subset is given by the integral

$$\int_{|N(y)|=1} \int_{|N(y)/u+N(z)|=q^{1+d-n}} \int_{|N(x)+c|\le q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$ and $|c| = |\boldsymbol{\pi}(1-u)| = q^{-2k-1-\varepsilon} < q^{-n}$. Applying Lemma A.1 the integral is equal to zero when ε is 0 and when ε is 1, to

$$(1-q^{-2}) \times (1-q^{-2})q^{1+d-n} \times (1+q^{-1})q^{-n}.$$

(B) *Case of* $|N(x)+\boldsymbol{\pi}(N(y)+N(z))| = q^{-n}$, $|N(y)+uN(z)| < q^{1+d-n}$. Then $|N(y)| = 1$. The volume of this subset is given by the integral

$$\int_{|N(y)|=1} \int_{|N(y)/u+N(z)|<q^{1+d-n}} \int_{|N(x)+c|=q^{-n}} dxdzdy,$$

where $c = c(y,z) = \boldsymbol{\pi}(N(y)+N(z))$ and $|c| = q^{-2k-1-\varepsilon}$. Applying Lemma A.1 the integral is equal to zero when ε is 0 and when ε is 1, to

$$(1-q^{-2}) \times (1+q^{-1})q^{d-n} \times (1-q^{-2})q^{-n}.$$

We obtain that $\mathrm{vol}(V^0_{n,d})$ (when $n > 2k+1+\varepsilon$) is equal to zero if $\varepsilon = 0$ and if $\varepsilon = 1$, to

$$(1-q^{-2})(1+q^{-1})^2 q^{1+d-2n}.$$

The lemma follows. □

Lemma D.6.3. *Suppose that a_1, $c_2 \in R^\times$, $|c_1| \le 1$, $|a_2| < 1$. Then*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-2} + q^{-4}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Thus we have that $Q_1(x, y, z) = N(x) + \boldsymbol{\pi}(N(y) + c_1 N(z))$ and $Q_2(x, y, z) = \boldsymbol{\pi}(a_2 N(y) + c_2 N(z))$. Moreover, if $|c_1| < 1$ and since $|a_2| < 1$ we can replace Q_1 with $Q_1 + Q_2$ So without loss of generality we can assume that $|c_1| = 1$ and thus $c_1 = 1$. Since we are interested only in $|Q_2|$, we can replace Q_2 with Q_2/c_2 and put a_2/c_2 to be a_2. So $Q_1(x, y, z) = N(x) + \boldsymbol{\pi}(N(y) + N(z))$ and $Q_2(x, y, z) = \boldsymbol{\pi}(a_2 N(y) + N(z))$.

We first show that V_n $(n \ge 2)$ *is empty.* Suppose $(x, y, z) \in V_n$. Then $|N(x)| < 1$, since $|N(x)| = 1$ implies $|Q_1| = 1 > q^{-n}$. Moreover, from $|Q_1| \le q^{-2}$ the condition $|N(y)| < 1$ implies $|N(z)| < 1$, which leads to a contradiction. Thus $|N(y)| = |N(z)| = 1$. Since $|a_2| < 1$ we obtain then $|\boldsymbol{\pi}(a_2 N(y) + N(z))| = |\boldsymbol{\pi} N(z)| = q^{-1}$. Hence $\max\{|Q_1|, |Q_2|\} = q^{-1} > q^{-n}$. We conclude that V_n $(n \ge 2)$ is empty.

To see that $\mathrm{vol}(V_0^0) = 1 - 1/q^2$ note that

$$V_0 = \{(x, y, z) \in R_E^3; |N(x)| = 1, |N(y)| \le 1, |N(z)| \le 1\}.$$

To compute $\mathrm{vol}(V_1^0)$ recall that V_1 is the set

$$\{(x, y, z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + c_1 N(z))|, |\boldsymbol{\pi}(a_2 N(y) + N(z))|\} = 1/q\},$$

where $||(x, y, z)||_E = 1$. If $(x, y, z) \in V_1$ then $|N(x)| < 1$, and since $|a_2| < 1$ this set is equal to

$$\{(x, y, z) \in R_E^3; ||(x, y, z)||_E = 1, \max\{|N(y) + N(z)|, |N(z)|\} = 1\}.$$

Then its volume is given by the integral

$$\left(\int_{|N(z)|=1} \int_{|N(y)| \le 1} + \int_{|N(z)|<1} \int_{|N(y)|=1} \right) \int_{|N(x)|<1} dx dy dz,$$

whose value is

$$q^{-4}(1 - q^{-2}) + (1 - q^{-2}) q^{-2} = (1 - q^{-2})(1 + q^{-2}) q^{-2}.$$

□

Lemma D.6.4. *Suppose* a_1, $c_1 \in R^\times$, $|a_2| < 1$, $|c_2| < 1$ *and* $|a_2| \neq |c_2|$. *Set* $|c_2| = q^{-k}$. *Then*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-2} - q^{-3}, & \text{if } n = 1, \\ q^{-1-n}(1 - q^{-2}), & \text{if } 2 \leq n \leq k, \\ q^{-1-n}(1 + q^{-1}), & \text{if } n = k+1, \\ 0, & \text{if } n \geq k+2. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$ and similarly $c_1 = 1$. Thus we have that $Q_1(x, y, z) = N(x) + \boldsymbol{\pi}(N(y) + N(z))$ and $Q_2(x, y, z) = \boldsymbol{\pi}(a_2 N(y) + c_2 N(z))$. Without loss of generality, we can assume that $|a_2| < |c_2|$.

We first show that V_n $(n \geq k+2)$ *is empty.* Suppose $(x, y, z) \in V_n$. Then $|N(x)| < 1$, since $|N(x)| = 1$ implies $|Q_1| = 1 > q^{-n}$. Moreover, from $|Q_1| \leq q^{-2}$ the condition $|N(y)| < 1$ implies $|N(z)| < 1$, which leads to a contradiction. Thus $|N(y)| = |N(z)| = 1$. Since $|a_2| < 1$ we have $|\boldsymbol{\pi}(a_2 N(y) + c_2 N(z))| = |\boldsymbol{\pi} c_2 N(y)| = q^{-k-1}$. Hence $\max\{|Q_1|, |Q_2|\} = q^{-k-1} > q^{-n}$. We conclude that V_n $(n \geq k+2)$ is empty.

To see that $\mathrm{vol}(V_0^0) = 1 - 1/q^2$ note that

$$V_0 = \{(x, y, z) \in R_E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}.$$

To compute $\mathrm{vol}(V_1^0)$ recall that V_1 is the set

$$\{(x, y, z) \in R_E^3; |N(x) + \boldsymbol{\pi}(N(y) + N(z))| = 1/q\},$$

where $||(x, y, z)||_E = 1$. If $(x, y, z) \in V_1$ then $|N(x)| < 1$, and this set is equal to

$$\{(x, y, z) \in R_E^3; ||(x, y, z)||_E = 1, |N(x)| < 1, |N(y) + N(z)| = 1\}.$$

Then its volume is given by the integral

$$\left(\int_{|N(y)|=1} \int_{|N(y)+N(z)|=1} + \int_{|N(y)|<1} \int_{|N(z)|=1} \right) \int_{|N(x)|<1} dx dz dy,$$

whose value is

$$q^{-2}(1 - q^{-2})(1 - (q+1)q^{-2}) + (1 - q^{-2})q^{-4} = (1 - q^{-2})(q^{-2} - q^{-3}).$$

To compute $\mathrm{vol}(V_n^0)$ $(n > 1$ and $q^{-n} > |\boldsymbol{\pi} c_2|)$ recall that V_n is the set

$$\{(x, y, z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}(a_2 N(y) + c_2 N(z))|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. It implies that $|N(x)| < 1$ and thus $|N(y)| = |N(z)| = 1$. Then $|Q_2| = |\boldsymbol{\pi} c_2| = q^{-k-1} < q^{-n}$, so V_n is the set

$$\{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, |N(x)| < 1, |N(y)| = 1, |N(y) + N(z)| = q^{1-n}\}.$$

Its volume is given by the integral

$$\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(y)+N(z)|=q^{1-n}} dxdydz,$$

whose value is

$$q^{-2}(1-q^{-2})q^{1-n}(1-q^{-2}).$$

To compute $\mathrm{vol}(V_n^0)$ $(q^{-n} = |\boldsymbol{\pi} c_2|)$ recall that V_n is the set

$$\{(x,y,z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + N(z))|, |\boldsymbol{\pi}(a_2 N(y) + c_2 N(z))|\} = q^{-n}\},$$

where $||(x,y,z)||_E = 1$. It implies that $|N(x)| < 1$ and thus $|N(y)| = |N(z)| = 1$. Then $|Q_2| = |\boldsymbol{\pi} a_2| = q^{-k-1} = q^{-n}$, so V_n is the set

$$\{(x,y,z) \in R_E^3; ||(x,y,z)||_E = 1, |N(x)| < 1, |N(y)| = 1, |N(y) + N(z)| \le q^{1-n}\}.$$

Its volume is given by the integral

$$\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(y)+N(z)|\le q^{1-n}} dxdydz,$$

whose value is

$$q^{-2}(1-q^{-2})q^{1-n}(1+q^{-1}).$$

□

Lemma D.6.5. *Suppose that a_2, $c_2 \in R^\times$, $|c_1| < 1$. Then*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ q^{-2} + q^{-4}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming y' back to y, without loss of generality we can assume that $a_1 = 1$. Thus we have that $Q_1(x,y,z) = N(x) + \boldsymbol{\pi}(N(y) + c_1 N(z))$ and $Q_2(x,y,z) = \boldsymbol{\pi}(a_2 N(y) + c_2 N(z))$. Since we are interested only in $|Q_2|$, we can replace Q_2 with Q_2/c_2 and put a_2/c_2 to be a_2. So $Q_1(x,y,z) = N(x) + \boldsymbol{\pi}(N(y) + c_1 N(z))$ and $Q_2(x,y,z) = \boldsymbol{\pi}(a_2 N(y) + N(z))$.

We first show that V_n $(n \ge 2)$ *is empty.* Suppose $(x,y,z) \in V_n$. Then $|N(x)| < 1$, since $|N(x)| = 1$ implies $|Q_1| = 1 > q^{-n}$. Moreover, from $|Q_2| \le q^{-2}$ the condition

$|N(y)| < 1$ implies $|N(z)| < 1$, which leads to a contradiction. Thus $|N(y)| = |N(z)| = 1$ and $|N(x)| < 1$. Since $|c_1| < 1$ we obtain then $|N(x) + \boldsymbol{\pi}(N(y) + c_1 N(z))| = |\boldsymbol{\pi} N(y)| = q^{-1}$. Hence $\max\{|Q_1|, |Q_2|\} = q^{-1} > q^{-n}$. We conclude that V_n $(n \geq 2)$ is empty.

To see that $\mathrm{vol}(V_0^0) = 1 - 1/q^2$ note that

$$V_0 = \{(x, y, z) \in R_E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}.$$

To compute $\mathrm{vol}(V_1^0)$ recall that V_1 is the set

$$\{(x, y, z) \in R_E^3; \max\{|N(x) + \boldsymbol{\pi}(N(y) + c_1 N(z))|, |\boldsymbol{\pi}(a_2 N(y) + N(z))|\} = 1/q\},$$

where $||(x, y, z)||_E = 1$. If $(x, y, z) \in V_1$ then $|N(x)| < 1$, and since $|c_1| < 1$ this set is equal to

$$\{(x, y, z) \in R_E^3; ||(x, y, z)||_E = 1, \max\{|N(y) + N(z)|, |N(y)|\} = 1\},$$

which is the same set as in case of 6.3, $n = 1$. □

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