Let $E/F$ be a quadratic extension of $p$-adic fields, $p \neq 2$. Let $x \mapsto \overline{x}$ be the involution of $E$ over $F$. The representation $\pi$ of $\text{GL}(3, E)$ normalizedly induced from the trivial representation of the maximal parabolic subgroup is invariant under the involution $\sigma(g) = Jg^{-1}J$. We compute — by purely local means — the $\sigma$-twisted character $\chi_{\sigma \pi}$ of $\pi$. We show that it is $\sigma$-unstable, namely its value at one $\sigma$-regular-elliptic conjugacy class within a stable such class is equal to negative its value at the other such conjugacy class within the stable class, or zero when the $\sigma$-regular-elliptic stable conjugacy class consists of a single such conjugacy class. Further, we relate this twisted character to the twisted endoscopic lifting from the trivial representation of the “unstable” twisted endoscopic group $U(2, E/F)$ of $\text{GL}(3, E)$. In particular $\pi$ is $\sigma$-elliptic, that is, $\chi_{\sigma \pi}$ is not identically zero on the $\sigma$-elliptic set.

**Keywords:** Admissible representations of a $p$-adic group; twisted characters; endoscopy; stable conjugacy.

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### 1. Introduction

Let $F$ be a finite field extension of $\mathbb{Q}_p$, $G'$ be a reductive connected linear algebraic group over $F$ and $\sigma$ be an involution of $G'$ over $F$. Put $G' = G'(F)$ for the group of $F$-rational points on $G'$. Denote by $\sigma$ also the induced involution on $G'$. Let $\pi$ be an admissible (see [1, 2]) irreducible representation of $G'$ in a complex vector space. It is called $\sigma$-invariant if it is equivalent to the representation $\sigma \pi$, defined
by $\pi(g) = \pi(\sigma g)$, $g \in G'$. In this case there is an intertwining operator $A$ on the space of $\pi$ with $\pi(g)A = A\pi(\sigma g)$ for all $g \in G'$. Since $\sigma^2 = 1$ we have $\pi(g)A^2 = A^2\pi(g)$ for all $g \in G'$. As $\pi$ is irreducible $A^2$ is a scalar by Schur’s lemma [2]. Multiplying by a scalar we may choose $A$ with $A^2 = 1$. This determines $A$ up to a sign. Extend $\pi$ to $G' \times \langle \sigma \rangle$ by $\pi(\sigma) = A$.

Let $K'$ be a maximal compact subgroup of $G'$ with $\sigma K' = K'$. A representation $\pi$ is called unramified if the space of $\pi$ contains a nonzero $K'$-fixed vector. The dimension of the space of $K'$-fixed vectors is bounded by one if $\pi$ is irreducible. If $\pi$ is $\sigma$-invariant, irreducible and unramified, and $v_0 \neq 0$ is a $K'$-fixed vector in the space of $\pi$, then $A v_0$ is also $K'$-fixed, hence is a multiple of $v_0$. Thus $A v_0 = c v_0$, with $c = \pm 1$. Replace $A$ by $c A$ to have $A v_0 = v_0$.

Since $\pi$ is irreducible, by Schur’s lemma there is a character $\omega'$ of the center $Z'$ of $G'$ with $\pi(zg) = \omega'(z)\pi(g)$ for all $z \in Z'$, $g \in G'$. Fix a Haar measure $dg'$ on $G'/Z'$ [2]. Denote by $H'$ the convolution algebra of complex-valued locally constant measures $\phi dg'$ on $G'$ which are compactly supported on $G'$ modulo $Z'$ and which transform under $Z'$ by $\omega'^{-1}$.

For any admissible representation $\pi$ and any $\phi dg' \in H'$ the convolution operator $\pi(\phi dg') = \int_{G'/Z'} \phi(g)\pi(g)dg'$ has finite rank. If $\pi$ is $\sigma$-invariant, denote by $tr \pi(\phi dg' \times \sigma)$ the trace of

$$tr \pi(\phi dg' \times \sigma) = \pi(\phi dg') \times \pi(\sigma) = \int_{G'/Z'} \phi(g') \pi(g') \pi(\sigma) dg'.$$

The distribution $\phi \mapsto tr \pi(\phi dg' \times \sigma)$ is represented by a function $\chi_{Z'}^\pi$, named the $\sigma$-twisted character of $\pi$. Thus $\chi_{Z'}^\pi$ is a locally constant complex-valued function on the $\sigma$-regular set of $G'$ (where $\sigma(g)$ is regular) which transforms under the center $Z'$ via $\omega'$ and is $\sigma$-conjugacy invariant (thus $\chi_{Z'}^\pi(h^{-1}g\sigma(h)) = \chi_{Z'}^\pi(g)$ for all $h \in G'$), which satisfies $tr(\phi dg' \times \sigma) = \int_{G'/Z'} \chi_{Z'}^\pi(g') \phi(g') dg'$ for all $\phi \in H'$ supported on the $\sigma$-regular set (in fact for all $\phi \in H'$ as $\chi_{Z'}^\pi$ is locally integrable on $G'$ (see [3, 9, 10]), hence it is uniquely determined by its restriction to the $\sigma$-regular set).

It is a basic question in representation theory to compute $\chi_{Z'}^\pi$. Methods to carry out such computations are scarce. The question is related to the theory of liftings, and can be studied by global techniques of comparison of trace formulae. We study here one particular interesting example by a purely local technique, based on the usage of a convenient model for the representation in question. This extends a technique first introduced in [5] and then extended by us in [6–8] in more involved cases, the case considered in this paper being the most advanced of them all. It will be interesting to develop a local proof of our theorems to cover the archimedean case as well. Naturally our purely local approach is useful to compare with the global approach [4].

Our example concerns $G' = GL(3, E)$, and the involution $\sigma(g) = J^t \overline{g}^{-1} J$, where $E/F$ is a quadratic extension of $p$-adic fields, $x \mapsto \overline{x}$ denotes the nontrivial automorphism of $E$ over $F$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\overline{g}(h) = (\overline{g}(h))$ if $g = (g_{ij})$. Then $G' = G'(F)$ where $G'$ is the $F$-group $R_{E/F} \mathbf{G}$ obtained from the unitary group $\mathbf{G} = U(3, E/F)$
Put stable conjugacy classes in \( \tilde{\pi} \) of \( E \) in \( \pi \) conjugic to \( GL(2, E) \times E^\times \). A representation \( \pi_1 \) of \( GL(2, E) \) extends to \( P' \) to have a trivial central character and be 1 on the unipotent radical \( N' \) of \( P' \). Let \( \delta \) denote the character of \( P' \) which is trivial on \( N' \) and whose value at \( p = mn \) is \( |\det(a^{-1}h)| \) if \( m \) corresponds to \((h, a)\) in \( GL(2, E) \times E^\times \). Explicitly, if \( P' \) is the upper triangular parabolic subgroup of type \((2,1)\), and \( m \) in \( M' \) is represented in \( GL(3, E) \) by \((m' \ 0 \ \ 0 \ \ 0)\), then \( \delta(m) = |(\det m')/m''|^2 \) \( (m' \in GL(2, E), m'' \in GL(1, E)) \). Denote by \( I(\pi_1) = \text{Ind}(\delta^{1/2}; \pi_1; P', G') \) the representation of \( G' \) normalizedly induced from \( \pi_1 \) on \( P' \) to \( G' \). It is clear from [2] that when \( I(\pi_1) \) is irreducible it is \( \sigma \)-invariant if and only if \( \pi_1 \simeq \tilde{\pi}_1 \), and \( I(\pi_1) \) is unramified if and only if \( \pi_1 \) is unramified. Here \( K'_{GR} = GL(2, R_E) \) and \( K' = PGL(3, R_E) \), where \( R_E \) denotes the ring of integers of \( E \).

We regard the twisted character \( \chi'_{\pi} \) as a function on the set of regular \( \sigma \)-conjugacy classes in \( G' \). These are naturally packed in stable \( \sigma \)-conjugacy classes. We explain the classification of these later, but note here that the norm map, defined using \( g \mapsto g\sigma(g) \), defines a bijection from the set of regular stable \( \sigma \)-conjugacy classes in \( G' \) to the set of regular stable conjugacy classes in \( G = U(3, E/F) = \{g \in G' ; \sigma(g) = g\} \). The centralizers of such regular classes in \( G \) are tori of four types. Put \( E^1 = \{x \in E^\times ; x\tilde{\pi} = 1\} \).

- (0) The diagonal torus \( T^* = \{\text{diag}(a,b,\tilde{\pi}^{-1}) ; a \in E^\times, b \in E^1\} \).
- (1) Elliptic torus \( T = (E^1)^3 \).
- (2) Elliptic torus \( T = (E\times)^2 \times E^1 \). Let \( h \) be an element of the subgroup \( \tilde{H} = \{(a_{ij}) ; a_{ij} = 0 \text{ if } i + j \text{ is odd}\} \simeq H \times E^1 \) of \( G = U(3, E/F) \). Its eigenvalues are \( \alpha, \beta = \alpha_{22}, \gamma \). We view \( H = U(2, E/F) \), the unitary group of the form \( w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), as the subgroup \( H = E^1 \) of \( G \). Then \( \tilde{H} = HZ \), where \( Z \cong E^1 \) is the center of \( G \). An element, conjugacy class, or stable conjugacy class in \( \tilde{H} \) defines one in \( G \). But note that the three distinct stable conjugacy classes in \( \tilde{H} \) with eigenvalues \( \alpha, \beta, \gamma \) in \( E^1 \) define the same stable conjugacy class in \( G \).

Fix a character \( \kappa \) of \( E^\times \) which is trivial on \( NE^\times \), but nontrivial on \( E^\times \). Put \( \kappa_1(h) = \kappa(-1 - \alpha/\beta(1 - \gamma/\beta)) \) if \( h = \text{diag}(\alpha, \beta, \gamma) \) then \( \gamma = \tilde{\pi}^{-1} \), and \( \kappa_1(h) = \kappa(\alpha/\beta) \). A \( \sigma \)-conjugacy class of type (0), (1) or (2) has a representative \( g = (g_1, b) \) in \( ZG' \)-\((\text{diag}(1, -1, 1)) \simeq \text{GL}(2, E) \times E^\times \), which is unique up to \( \sigma \)-conjugacy in \( ZG' \)-\((\text{diag}(1, -1, 1)) \). Put \( \det g \) for \( \det(g_1/b) \). Then \( \kappa(\det_1 g) = \kappa(\det_1(h'\sigma(h')^{-1})) \) \( (h' \in ZG'-(\text{diag}(1, -1, 1))) \) is well-defined. Note that \(|x|_E = |N(x)|_F = |x|^2_F \) defines...
κ that the transfer factor is σ. This extension is not induced from any proper

If the theorem is (properly) parabolically induced (see [1, 2]), we study in the theorem the character of its extension to GL(3, E) × \langle σ \rangle. This extension is not induced from any proper σ-invariant parabolic subgroup of GL(3, E). Hence its twisted character is nonzero on the regular σ-elliptic set. Elliptic representations (ones whose character is nonzero on the regular elliptic set) of a connected reductive p-adic group include the square integrable representations, in particular the cuspidal representations. Cuspidal representations of connected

\textbf{Theorem 1.} If π₁ is the trivial representation of GL(2, E), π = I(π₁), and g is an element of GL(3, E) with elliptic regular norm Ng of type (1) or (2), then 

\[ \Delta(Ng)\chi^\sigma_\pi(g) = \sum \Deltag_1(Nq^\alpha)\kappa_1(Nq^\alpha)/\kappa(\text{det}_1 g^\alpha). \]

The sum is trivial except in case (1), where it ranges over the Weyl group S_3 of permutations of the eigenvalues of g, quotient (S_2\setminus S_3) by the analogous Weyl group S_2 of GL(2, E) where g is viewed in ZG((diag(1,−1,1)). If g is an element of GL(3, E) of type (3) then χ^\sigma_\pi(g) = 0.

In this paper we deal only with case (1), as our main interest is in showing that the transfer factor is κ₁(Ng)/κ(\text{det}_1 g) and the corresponding representation of H = U(2, E/F) is the trivial representation, so that it can be compared with the transfer factors formulated by Langlands, Kottwitz–Shelstad, Waldspurger, in general settings. We checked the theorem for case (2), but a full treatment of case (2) would require another paper.

The theorem was found by global techniques, using the trace formula, as a case of the unstable twisted endoscopic lifting from H = U(2, E/F) to G' = GL(3, E), see [4, pp. 207–208] (note that in [4, p. 208, line 25], e is meant to be b o e) which is the counterpart of the endoscopic lifting from H = U(2, E/F) to G = U(3, E/F) and the basechange lifting from G to G' = GL(3, E); see [4, Proposition II.5.1.1, p. 340]. The proof in [4] uses the Weyl integration formula and its twisted form. Our work provides an alternative, purely local and elementary, verification of this highly complicated relation. The works [5, 6] deal with an analogous computation related to (the unstable counterpart of) the symmetric square lifting from SL(2) to PGL(3), while [7, 8] deal with a computation related to the lifting from the (semisimple) rank two symplectic group GSp(2), to GL(4).

Although the representation π of the theorem is (properly) parabolically induced (see [1, 2]), we study in the theorem the character of its extension to GL(3, E) × \langle σ \rangle. This extension is not induced from any proper σ-invariant parabolic subgroup of GL(3, E). Hence its twisted character is nonzero on the regular σ-elliptic set. Elliptic representations (ones whose character is nonzero on the regular elliptic set) of a connected reductive p-adic group include the square integrable representations, in particular the cuspidal representations. Cuspidal representations of connected
reductive groups can be constructed by compact induction (see the recent survey [12], with references to previous works by Bushnell–Kutzko, Moy–Prasad, Yu and others). This construction can be used to compute their characters. It will be interesting to extend this theory to nonconnected groups, as in our example. Computations of twisted characters are useful to compare with character identities obtained by global techniques, as in [4]. Our example of an unramified nontempered representation is of course another extreme in a direction away from cuspidal representations.

For transfer to local fields of positive characteristic see [11].

To compute the character of \( \pi \) we shall express \( \pi \) as an integral operator in a convenient model, and integrate the kernel over the diagonal. Denote by \( \psi \) the character \( \psi(x) = |x|_{E}^{(s+1)/2} \) of \( E^{\times} \). It defines a character \( \psi_{s, P} \) of \( P' \), trivial on \( N' \), by

\[
\psi_{s, P}(p) = \psi((\det m')/m'^{2}) \text{ if } p = mn \text{ and } m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix} \text{ in } \text{GL}(2, E), \text{ } m'' \in \text{GL}(1, E). \text{ If } s = 0, \text{ then } \psi_{s, P} = \delta^{1/2}. \text{ Let } W_{s} \text{ be the space of complex-valued smooth functions } \psi \text{ on } G' \text{ with } \psi(pg) = \psi_{s, P}(p)\psi(g) \text{ for all } p \in P' \text{ and } g \in G'. \text{ The group } G' \text{ acts on } W_{s} \text{ by right translation: } (\pi_{s}(g)\psi)(h) = \psi(hg). \text{ By definition, } I(\pi_{s}) \text{ is the } G'-\text{module } W_{s} \text{ with } s = 0. \text{ The parameter } s \text{ is introduced for purposes of analytic continuation.}

We prefer to work in another model \( V_{s} \) of the \( G' \)-module \( W_{s} \). Let \( V \) denote the space of column 3-vectors over \( E \). Let \( V_{s} \) be the space of smooth complex-valued functions \( \phi \) on \( V - \{0\} \) with \( \phi(\lambda v) = \mu_{s}(\lambda)^{-3}\phi(v) \). The expression \( \mu_{s}(\det g)\phi(\tau_{g}v) \), which is initially defined for \( g \) in \( \text{GL}(3, E) \), depends only on the image of \( g \) in \( \text{PGL}(3, E) \). The group \( \text{PGL}(3, E) \) acts on \( V_{s} \) by \( (\tau_{s}(g)\phi)(v) = \mu_{s}(\det g)\phi(\tau_{g}v) \). Let \( v_{0} \neq 0 \) be a vector of \( V \) such that the line \( \{ \lambda v_{0}; \lambda \in E \} \) is fixed under the action of \( ^{t}P' \). Explicitly, we take \( v_{0} = (0, 0, 1) \). It is clear that the map \( V_{s} \to W_{s}, \phi \mapsto \psi = \psi_{\phi} \), where \( \psi(g) = (\tau_{s}(g)\phi)(v_{0}) = \mu_{s}(\det g)\phi(\tau_{g}v_{0}) \), is a \( \text{PGL}(3, E) \)-module isomorphism, with inverse \( \psi \mapsto \phi = \psi_{\phi}, \phi(v) = \mu_{s}(\det g)^{-1}\psi(g) \text{ if } v = \tau_{g}v_{0} \text{ (PGL}(3, E) \text{ acts transitively on } V - \{0\}) \).

In this section (only), \( | \cdot | \) denotes \( | \cdot |_{E} \). For \( v = (x, y, z) \in V \) put \( \| v \| = \max(|x|, |y|, |z|) \). Let \( V^{0} \) be the quotient of the set of \( v \) in \( V \) with \( \| v \| = 1 \) by the equivalence relation \( v \sim \alpha v \) if \( \alpha \) is a unit in \( R_{E} \). Denote by \( \mathbb{P}V \) the projective space of lines in \( V - \{0\} \). If \( \Phi \) is a function on \( V - \{0\} \) with \( \Phi(\lambda v) = |\lambda|^{-3}\Phi(v) \) and \( dv = dx dy dz \), then \( \Phi(v)dv \) is homogeneous of degree zero. Define

\[
\int_{\mathbb{P}V} \Phi(v)dv \text{ to be } \int_{V^{0}} \Phi(v)dv.
\]

Clearly we have

\[
\int_{\mathbb{P}V} \Phi(v)dv = \int_{\mathbb{P}V} \Phi(gv)g(v)dv = |\det g| \int_{\mathbb{P}V} \Phi(gv)dv.
\]

Put \( v(x) = |x| \) and \( m = 3(s - 1)/2 \). Note that \( v/\mu_{s} = \mu_{-s} \). Put \( \langle v, w \rangle = \tau_{v}w \). Then \( (gv, \sigma(g)w) = \langle v, w \rangle \).
Lemma 1.1. The operator $T_s : V_s \to V_{-s}$,
\[
(T_s\phi)(v) = \int_{V'} \phi(w) |\langle w, v \rangle|^m dw,
\]
converges when $\Re s > 1/3$ and satisfies $T_s\tau_s(g) = \tau_{-s}(g)T_s$ for all $g$ in $\operatorname{PGL}(3, E)$ where it converges.

Proof. We have
\[
(T_s(\tau_s(g)\phi))(v) = \int \tau_s(g)\phi(w) |wJ\varphi|^m dw = \mu_s(\det g) \int \phi(tg^\dagger w) |wJ\varphi|^m dw
\]
\[
= |\det g|^{-1} \mu_s(\det g) \int \phi(w) |t^\dagger g^{-1} wJ\varphi|^m dw
\]
\[
= (\mu_s/\nu)(\det g) \int \phi(w) |wJ^{-1} Jg |^m dw
\]
\[
= (\mu_s/\nu)(\det g) \int \phi(w) |\langle w, \sigma(t^\dagger g)v \rangle|^m dw
\]
\[
= (\nu/\mu_s)(\det \sigma g) \cdot (T_s\phi)(\sigma(t^\dagger g)v)
\]
\[
= [\tau_{-s}(\sigma g)](T_s\phi)(v).
\]

Further, $\int_{|w|\leq 1} |w|^m dw = (1-q^{-1}) \int_{|w|\leq 1} |w|^{m+1} dw = (1-q^{-1}) \sum_{n\geq 0} q^{-(m+1)n}$ converges if and only if $\Re(m) + 1 > 0$.

The spaces $V_s$ are isomorphic to the space $W$ of locally constant complex-valued functions on $V^0$, and $T_s$ is equivalent to an operator $T_s^0$ on $W$. The proof of Lemma 1.1 implies also the following corollary.

Corollary 1.2. The operator $T_s^0 \circ \tau_s(g^{-1})$ is an integral operator with kernel
\[
(\mu_s/\nu)(\det \sigma g) |\langle w, \sigma(t^\dagger g)v \rangle|^m
\]
and trace
\[
\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})] = (\nu/\mu_s)(\det g) \int_{V^0} |\langle v, g \varphi \rangle|^m dv.
\]

Remark. (1) In the domain where the integral converges, it is clear that $\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})]$ depends only on the $\sigma$-conjugacy class of $g$ if and only if $s = 0$. (2) We evaluate below this integral at $s = 0$ in a case where it converges for all $s$, and no analytic difficulties occur. However, we claim that to compute the trace of the analytic continuation of $T_s^0 \circ \tau_s(g^{-1})$ it suffices to compute this trace for $s$ in the domain of convergence, and then evaluate the resulting expression at the desired $s$. Indeed, for each compact open $\sigma$-invariant subgroup $K$ of $\operatorname{PGL}(3, E)$ the space $W_K$ of $K$-biinvariant functions on $W$ is finite dimensional. Denote by $p_K : W \to W_K$ the natural projection. Then $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ acts on $W_K$, and the trace of the
analytic continuation of \( p_K \circ T_s^0 \circ \tau_s(g^{-1}) \) is the analytic continuation of the trace of \( p_K \circ T_s^0 \circ \tau_s(g^{-1}) \). Since \( K \) can be taken to be arbitrarily small the claim follows.

Next we normalize the operator \( T_s \) so that it acts trivially on the one-dimensional space of \( K' \)-fixed vectors in \( V_s \). This space is spanned by the function \( \phi_0 \) in \( V_s \) with \( \phi_0(v) = 1 \) for all \( v \) in \( V^0 \). Fix a local uniformizer \( \pi_E \) in \( R_E \). Let \( q = q_E \) be the cardinality of the quotient field of \( R_E \). Normalize the valuation \( |\cdot| \) by \( |\pi_E| = q^{-1} \).

Normalize the measure \( dx \) by \( \int_{|x| \leq 1} dx = 1 \), so that \( \int_{|x| = 1} dx = 1 - q^{-1} \). In particular, the volume of \( V^0 \) is
\[
(1 - q^{-3})/(1 - q^{-1}) = 1 + q^{-1} + q^{-2}.
\]

**Lemma 1.3.** We have \( (T_s \phi_0)(v_0) = (1 - q^{-3(s+1)/2})(1 - q^{(1 - 3s/2)})^{-1} \phi_0(v_0) \). When \( s = 0 \) the constant is \(-q^{-1/2}(1 + q^{-1/2} + q^{-1}) = -q^{-3/2}(1 + q^{1/2} + q)\).

**Proof.** We have
\[
\int \phi_0(v)|^m v^J E_0|^m dv = \int_{V_0} |x|^m dxdydz
\]
\[
= (1 - q^{-3(s+1)/2}) \int_{|x| \leq 1} |x|^m dx \int_{|x| = 1} dx,
\]
as required.

To prove the theorem we have to compute \( \text{tr}[T \circ \tau_s(g^{-1})] \) where \( T = \frac{q^{3/2}}{1 - q^{1/2} + q} T_s^0 \).

For this purpose, in Sec. 2, we describe the \( \sigma \)-stable conjugacy classes and find explicit representatives for the \( \sigma \)-conjugacy classes within the stable classes. We begin with a description of conjugacy and stable conjugacy, after which we will come the study of the twisted analogue. In Sec. 3, we state the main theorem of this paper. In particular we introduce the parameters \( n_1 \) and \( n_2 \), and observe that it suffices to deal with three cases, where \((n_1, n_2) = (0, 0), (1, 0) \) and \((1, 1)\). The proof requires numerous and lengthy computations that took us many years to complete. To improve the readability and the clarity of this exposition, we put the computations later in the paper. They are put into the corresponding Appendices. Thus, in Sec. 4, we deal with the case \( n_1 = 0, n_2 = 0 \), postponing the needed computations to Appendix B. In Sec. 5, we deal with the case \( n_1 = 1, n_2 = 0 \), postponing the needed computations to Appendix C. Finally, in Sec. 6, we deal with the case \( n_1 = 1, n_2 = 1 \), postponing the needed computations to Appendix D.

### 2. Conjugacy Classes

Let \( G \) be a connected reductive group defined over a local or global field \( F \). Fix an algebraic closure \( \overline{F} \). Denote by \( \overline{G} = G(\overline{F}) \) the group of \( \overline{F} \)-points on the variety \( G \). Now \( \text{Gal}(\overline{F}/F) \) acts on \( \overline{G} \). The group \( G(F) \) of fixed points is denoted by \( G \). An \( F \)-torus \( T \) in \( G \) is a maximal \( F \)-subgroup \( \overline{F} \)-isomorphic to a power of \( \mathbb{G}_m \). Its group \( T \) of \( F \)-points is also called a torus. An element \( t \) of \( G \) is regular if the centralizer
the Galois group on the torus. Let us make this explicit. All tori are conjugate in \( G \). Tori \( T \) and \( T' \) are stably conjugate if there is such a \( g \) in \( \overline{G} \). Since \( gTg^{-1} = T' \), so that the map \( \text{Int}(g) : T \rightarrow T' \), \( \text{Int}(g)(t) = gtg^{-1} \), is defined over \( F \). Then \( g \cdot \tau = g^{-1}\tau(g) \) centralizes \( T \) for all \( \tau \) in \( \text{Gal}(\overline{F}/F) \), hence lies in \( \overline{T} \), since \( G \) is connected and reductive.

Of course the notion of stable conjugacy can be defined by \( t' = g^{-1}tg \), which will lead to the definition of the cocycle as \( g_{t'} = gt\tau(g^{-1}) \). The change from \( g \) to \( g^{-1} \) should lead to no confusion, and we use both conventions.

We shall now list all stable conjugacy classes of tori in \( G \). Let \( T^* \) be a fixed \( F \)-torus, \( N \) be its normalizer in \( G \) and \( W = T^* \setminus N = N/T^* \) be the absolute Weyl group. For each \( T \) there is \( g \) in \( G(\overline{F}) \) with \( T = gT^*g^{-1} \). Since \( T \) is defined over \( F \), \( g \cdot T \) normalizes \( T^* \), and the cocycle \( \tau 
Rightarrow g_{\tau} \) defines a class in the first cohomology group \( H^1(F, N) \) of \( \text{Gal}(\overline{F}/F) \) with coefficients in \( N(\overline{F}) \). Denote by \( \{g_{\tau}\} \) the image of \( \{g_{\tau}\} \) under the natural map \( H^1(F, N) \rightarrow H^1(F, W) \), obtained from \( N \rightarrow W \).

The stable conjugacy classes are determined by means of the following.

**Proposition 2.1.** The map \( T \mapsto \{g_{\tau}\} \) injects the set of stable conjugacy classes of tori in \( G \) into the image in \( H^1(F, W) \) of \( \ker[H^1(F, N) \rightarrow H^1(F, G)] \). This map is also surjective when \( G \) is quasisplit.

**Proof.** If \( T = gT^*g^{-1} \) and \( T' \) are stably conjugate, then there is \( x \) in \( \overline{G} \) with \( T' = xT'x^{-1} = xyT^*(xy)^{-1} \), and \( (xy)_\tau = g^{-1}x_\tau g \cdot g_{\tau} \), has the image \( g'_{\tau} \) in \( H^1(F, W) \), since \( g^{-1}x_\tau g \) lies in \( \overline{T} \) (\( x_\tau \) in \( \overline{T} \)). Hence the map of the proposition is well-defined.

Conversely, if \( T = gT^*g^{-1} \), \( T' = gT'g^{-1} \) and \( g_{\tau} = a(\tau)g'_{\tau} \) with \( a(\tau) \) in \( \overline{T} \), then \( a(\tau) = g^{-1}x_\tau g'_{\tau} \) with \( x(\tau) \) in \( \overline{T} \), and the map \( t \mapsto gg^{-1}tg'_{\tau}^{-1} \) (\( t \) in \( \overline{T} \)) is defined over \( F \). Hence the map of the proposition is injective.

For the second claim, if \( \{g_{\tau}\} \) lies in \( \ker[H^1(F, N) \rightarrow H^1(F, G)] \), then it defines a new \( \text{Gal}(\overline{F}/F) \)-action by \( \tilde{\tau}(h) = g^{-1}\tau(h)g_{\tau} \) (\( h = t^\tau \) in \( \overline{T} \)). If \( h \) is a fixed \( \tilde{T} \)-invariant regular element, then \( \tilde{\tau}(h) = g_{\tau}h_\tau g_{\tau}^{-1} \), and the conjugacy class of \( h \) in \( \overline{G} \) is defined over \( F \). When \( G \) is quasisplit, a theorem of Steinberg and Kottwitz [13] implies the existence of \( h' \) in \( G \) which is conjugate to \( h \) in \( \overline{G} \), since the field \( F \) is perfect. The centralizer of \( h' \) in \( G \) is a torus whose stable conjugacy class corresponds to \( \{g_{\tau}\} \).

Hence the map is surjective.

**Remark.** Implicit in the proof is a description — used below — of the action of the Galois group on the torus. Let us make this explicit. All tori are conjugate in \( \overline{G} \), thus \( T = g^{-1}Tg \) for some \( g \) in \( \overline{G} \). For any \( t \) in \( \overline{T} \) there is \( t^* \) in \( \overline{T} \) with \( t = g^{-1}t^*g \).

For \( t \) in \( T \), we have

\[
\sigma g^{-1} \sigma t \sigma g = at = t = g^{-1}t^*g.
\]
hence $\sigma t^* = g^{-1}_\sigma t^* g_\sigma \in T'$. Taking regular $t$ (and $t^*$), $g_\sigma \in \mathcal{W}$ is uniquely determined modulo $T'$, namely in $\mathcal{W}$. For any $t^*$ in $T'$ we then have

$$\sigma(g^{-1} t^* g) = g^{-1}(g_\sigma(g^{-1}))\sigma(t^*)(\sigma(g)g^{-1})g,$$

hence the induced action on $T'$ is given by

$$\sigma^* (t^*) = g_\sigma \sigma(t^*) g_\sigma^{-1}.$$

The cocycle $\rho = \rho(T) : \text{Gal}(\overline{F}/F) \to \mathcal{W}$, given by $\rho(\sigma) = g_\sigma \pmod{T'}$, determines $T$ up to stable conjugacy.

Let $A(T/F)$ be the pointed set of $g$ in $G(\overline{F})$ so that $T' = gTg^{-1}$ is defined over $F$. Then the set

$$B(T/F) = G\backslash A(T/F)/T(\overline{F})$$

parametrizes the morphisms of $T$ into $G$ over $F$, up to inner automorphisms by elements of $G$. If $T$ is the centralizer of $x$ in $G$ then $B(T/F)$ parametrizes the set of conjugacy classes within the stable conjugacy class of $x$ in $G$. The map

$$g \mapsto \{\tau \mapsto g_\tau = g^{-1}\tau(g); \tau \in \text{Gal}(\overline{F}/F)\}$$

defines a bijection

$$B(T/F) \simeq \text{ker}[H^1(F,T) \to H^1(F,G)].$$

Let $p : G^{sc} \to G^{\text{der}}$ denote the simply connected covering group of the derived group $G^{\text{der}}$ of $G$. If $T$ is an $F$-torus in $G$, let $T^{sc} = p^{-1}(T^{\text{der}})$ of $T^{\text{der}} = T \cap G^{\text{der}}$. Then $G = TG^{\text{der}}$ and $G/p(G^{sc}) = T/p(T^{sc})$. Then the pointed set $B(T/F)$ is a subset of the group $C(T/F)$, defined to be the image of $H^1(F,T^{sc})$ in $H^1(F,T)$. If $H^1(F,G^{sc}) = \{0\}$, for example when $F$ is a nonarchimedean local field, then $B(T/F) = C(T/F)$. If $F$ is a global field with a ring $\mathcal{A}$ of adèles, then we put $C(T/\mathcal{A}) = \bigoplus_v C(T/F_v)$, $B(T/\mathcal{A}) = \bigoplus_v B(T/F_v)$. The sums are pointed. They range over all places $v$ of $F$.

Let $K$ be a finite Galois extension of $F$ over which $T$ splits. Denote $H^{-1}(\text{Gal}(K/F),X)$ by $H^{-1}(X)$ and $\text{Hom}(\mathcal{G}_m,T)$ by $X_*(T)$. In the local case the Tate–Nakayama duality (see [14]) identifies $C(T/F)$ with the image of $H^{-1}(X_*(T^{sc}))$ in $H^{-1}(X_*(T))$. In the global case it yields an exact sequence

$$C(T/F) \to C(T/\mathcal{A}) \to \text{Im}[H^{-1}(X_*(T^{sc}))] \to H^{-1}(X_*(T))].$$

The last term here is the quotient of the $\mathbb{Z}$-module of $\mu$ in $X_*(T^{sc})$ with $\sum \tau \mu = 0$ (sum over $\tau$ in $\text{Gal}(K/F)$), by the submodule spanned by $\mu - \tau \mu$, where $\mu$ ranges over $X_*(T)$ and $\tau$ over $\text{Gal}(K/F)$.

We denote by $W(T)$ the Weyl group of $T$ in $G$, by $W = S_3$ the Weyl group of (the algebraic group) $T^*$ in $G$, and by $W'(T)$ the Weyl group of $T$ in $A(T/F)$. We write $\sigma$ for the nontrivial element in $\text{Gal}(E/F)$. 


We shall now discuss the above definitions in our case where \( G = U(3, E/F) \). The centralizer \( E' \) of \( T \) in the algebra \( M(3, E) \) of \( 3 \times 3 \) matrices over \( E \) is a maximal commutative semisimple subalgebra. Hence it is isomorphic to a direct sum of field extensions of \( E \).

There are three possibilities.

1. \( E' = E \oplus E \oplus E \).
2. \( E' = E'' \oplus E \), \( [E'': E] = 2 \).
3. \( E' \) is a cubic extension of \( E \).

The absolute Weyl group \( W \) is the symmetric group on three letters, generated by the reflections \((12), (23), (13)\). Note that we also note that if the eigenvalues of \( \sigma \) whose restriction to \( E \) is nontrivial, maps \( \alpha, \beta, \gamma \) to \( \tau \alpha^{-1}, \tau \beta^{-1}, \tau \gamma^{-1} \). The lattice \( X_\tau(T) \) is the group of \( \mu = (x, y, z) \in \mathbb{Z}^3 \) and \( X_\tau(T^\infty) \) is the subgroup of \( \mu \) with \( x + y + z = 0 \). Indeed, \( G^\infty = SU(3) \). If \( \tau | E \neq 1 \) it maps the set \( \{x, y, z\} \) to the set \( \{-x, -y, -z\} \).

**Proposition 2.2.** (1) There are two stable conjugacy classes of \( F \)-tori in \( G = U(3, E/F) \) which split over \( E \). One, named of type \((0)\), consists of a single conjugacy class, represented by the torus \( T^\star \) with

\[
T^\star = \{ \text{diag}(a, b, \sigma a^{-1}); a \in E^\times, b \in E^1 = \{x \in E^\times; x \sigma x = 1\} \}.
\]

We have \( W'(T^\star) = W(T^\star) = \mathbb{Z}/2 \).

The other stable conjugacy class, named of type \((1)\), consists of tori \( T \) with \( T = (E^1)^3 \), and \( C(T/F) = \{(a, b, c) \in (E^1)^3; abc = 1\} \). We have \( W'(T) = S_3 \). This group acts transitively on the nontrivial elements in \((T, C(T/F))\).

(2) The stable conjugacy classes of \( F \)-tori in \( G \) whose splitting fields are quadratic extensions of \( E \), named of type \((2)\), split over biquadratic extensions \( EL \) of \( F \). Then \( Gal(EL/F) = \mathbb{Z}/2 \times \mathbb{Z}/2 \) is generated by \( \sigma \) which fixes \( L \) and \( \tau \) which fixes \( E \); put \( K = (EL)^{\sigma \tau} \). Each such torus is \( T \simeq \{(a, b, \sigma a^{-1}); a \in (E^1)^3, b \in E^1\} \). Here \( (EL/K)^1 = \{a \in EL; a \sigma a = 1\} \). Further, \( C(T/F) = K^\times/N_{EL/K}(EL)^{\times} = \mathbb{Z}/2 \) and \( W'(T) = \mathbb{Z}/2 \).

(3) The stable conjugacy classes of \( F \)-tori in \( G \) whose splitting fields are cubic extensions of \( E \), named of type \((3)\), are split over cubic extensions \( EM \) of \( E \), where \( M \) is a cubic extension of \( F \). Each stable class consists of a single conjugacy class. If \( EM/F \) is not Galois then \( W'(T) \) is trivial. If \( Gal(EM/F) = S_3 \) or \( \mathbb{Z}/3 \times \mathbb{Z}/2 \) then \( W'(T) \) is \( \mathbb{Z}/3 \).

**Proof.** A cocycle in \( H^1(Gal(E/F), W) \) is determined by \( w_\sigma \) in \( W = S_3 \) with \( 1 = w_{\sigma^2} = w_{\sigma^2} \sigma(w_{\sigma}) \). Thus \( w_\sigma \) is 1 or \((13), (12)(23) \) or \((23)(12)\). As

\[
\sigma((23))[(12)(23)](23) \neq 1 = \sigma((12))[(23)(12)](12),
\]

we obtain that \( W'(T) \) is \( \mathbb{Z}/3 \).
the last two are cohomologous to 1. The cocycle \( w_\sigma = 1 \) defines the action \( \sigma^*(t^*) = \sigma(t^*) \) on \( T^* \). To determine \( C(T^*/F) \), note that \( H^1(F, T^*) = H^1(\text{Gal}(E/F), T^*(E)) \) is the quotient of the cocycles \( t_\sigma = \text{diag}(a, b, c) \in T^*(E) = E^x \), \( t_\sigma(t_{\sigma'}) = t_{\sigma \sigma'} = 1 \), thus \( t_\sigma = \text{diag}(a, b, \sigma a) \), \( a \in E^x \), \( b \in F^x \), by the coboundaries \( t_\sigma \sigma(t_{\sigma}) = \text{diag}(a \sigma c, b \sigma b, c \sigma a) \). Since \( G^{sc} \) is the subgroup of \( G \) of elements of determinant 1, the cocycles which come from \( H^1(F, T^{sc}) \) have the form \( t_\sigma = \text{diag}(a, 1/a \sigma a, \sigma a) \). These are coboundaries: \( u_\sigma \sigma(u_{\sigma}^{-1}) \), with \( u_\sigma = (a, 1/a, 1) \), hence \( C(T^*/F) \) is trivial.

The cocycle \( w_\sigma = (13) \) defines the action \( \sigma^*(\text{diag}(a, b, c)) = (\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1}) \) on \( T^* \). Then \( T = g^{-1}Tg \) for some \( g \in G \) with \( g \sigma(g^{-1}) = J \mod T \), and \( T = T(F) = g^{-1}(E^x)^3g \). A cocycle \( t_\sigma = \text{diag}(a, b, c) \in (E^x)^3 \) of \( \text{Gal}(E/F) \) in \( T^*(E) \) satisfies \( 1 = t_{\sigma^2} = t_\sigma \sigma^*(t_\sigma) = \text{diag}(a/\sigma a, b/\sigma b, c/\sigma c) \), thus \( a, b, c \in F^x \) and it comes from \( T^{sc}(E) \) if \( abc = 1 \). The coboundaries take the form \( t_\sigma \sigma^*(t_{\sigma})^{-1} = \text{diag}(\sigma \sigma a, \sigma b b, \sigma c c) \), hence \( C(T^*/F) = \{(a, b, c) \in (F^x/NE^x)^3; abc = 1\} \).

Consider next an \( F \)-torus \( T \) in \( G \) which splits over a quadratic extension \( L_1 \) of \( E \), but not over \( E \). We claim that \( L_1/F \) is Galois. Indeed, the involution \( \iota(x) = J^xTJ \) stabilizes \( T = T(F) \), and its centralizer \( L_1^x \times E^x \) in \( \text{GL}(3, E) \). It induces on \( L_1 \) an automorphism whose restriction to \( E \) generates \( \text{Gal}(E/F) \). Hence \( L_1/F \) is Galois.

We claim that the Galois group of \( L_1/F \) is not \( Z/4 \). Indeed, if \( \text{Gal}(L_1/F) = Z/4 \) were generated by \( \tau \), then \( \tau^2 \) be trivial on \( E \), \( (w_{\tau^2})^2 = w_{\tau^2} = 1 \) implies \( w_{\tau^2} = 1 \) or (13) up to coboundaries. But \( (13) = w_{\tau^2} = w_{\tau} \tau(w_{\tau}) = w_{\tau}(13)w_{\tau}(13) \) implies \( w_{\tau}^2 = (13) \), which has no solutions, and \( w_{\tau^2} = 1 \) implies that \( T \) splits over \( E \). Then \( \text{Gal}(L_1/F) = Z/2 \times Z/2 \), and \( L_1 \) is the compositum of \( E \) and a quadratic extension \( L \) of \( F \), not isomorphic to \( E \). There are two such \( L \) (up to isomorphism), both ramified if \( E/F \) is unramified.

The Galois group \( \text{Gal}(LE/F) \) is generated by \( \sigma \) whose restriction to \( L \) is trivial, and \( \tau \) whose restriction to \( E \) is trivial. Up to coboundaries, \( w_\sigma = 1 \) or (13). If \( w_\sigma = (13) \), then \( w_\sigma \neq 1 \) is of order 2. Up to coboundary which does not change \( w_\sigma \), we have \( w_\tau = (13) \), and replacing \( \sigma \) by \( \sigma \tau \) (thus changing \( L \)) we may assume \( w_\sigma = 1 \). If \( w_\sigma = 1 \), \( w_\sigma w_\tau = w_{\sigma \tau} = w_{\sigma}(w_\tau) = w_\sigma(w_\tau)(13)w_{\tau}(13) \) implies \( w_{\tau} = (13) \), which has no solutions, and \( w_{\tau^2} = 1 \) implies that \( T \) splits over \( E \). Then \( \text{Gal}(L_1/F) = Z/2 \times Z/2 \), and \( L_1 \) is the compositum of \( E \) and a quadratic extension \( L \) of \( F \), not isomorphic to \( E \). There are two such \( L \) (up to isomorphism), both ramified if \( E/F \) is unramified.

It is simplest to compute \( C(T^*/F) \) using Tate–Nakayama duality. Locally, the image of

\[
\hat{H}^{-1}(F, X_*(T^{sc})) = \{X = (x, y, z) \in \mathbb{Z}^3; x + y + z = 0\}/\langle X - \sigma X, X - \tau X \rangle
\]

in

\[
\hat{H}^{-1}(F, X_*(T)) = \mathbb{Z}^2/\langle X - \tau \sigma X = (2x, 2y, 2z), X - \tau X = (x - z, 0, z - x) \rangle
\]

is \( \mathbb{Z}/2 \).
Here is an explicit computation of $H^1(\text{Gal}(LE/F), T(LE))$. We replace $T$ by $T^*$ if $\rho \in \text{Gal}(LE/F)$ acts by $\rho^*$. To compute note that a cocycle in $H^1(\text{Gal}(LE/F), T^*(LE))$ is defined by $t_\sigma, t_\tau, t_{\sigma\tau}$ satisfying the cocycle relations. Thus $t_\tau = (a, b, c) \in (EL)^{\times 3}$ satisfies $1 = t_\tau \tau^*(t_\tau) = (a, b, c)(\tau c, \tau b, \tau a)$. So $b = b'/\tau b'$ and if $g = (a, b', 1)$, replacing our cocycle $(t_\rho)$ by its product $t_\rho g^{-1} \rho^*(g)$ with a coboundary, we may assume that $t_\tau = 1$. If $t_{\sigma\tau} = (u, v, w)$ then

$$1 = t_{(\sigma\tau)g} = t_{(\sigma\tau)(\sigma\tau)^*}(t_{\sigma\tau}) = (u, v, w)(\tau\sigma u^{-1}, \tau\sigma v^{-1}, \tau\sigma w^{-1}).$$

Hence $(u, v, w) \in K^{\times 3}$. Here $K$ is the fixed field of $\sigma\tau$ in $LE$. Further, $t_{\sigma\tau}(\sigma\tau)^*(t_\tau) = t_\sigma = t_{\tau^*}(t_\sigma)$. Hence $t_{\sigma\tau} = (u, v, w) = (\sigma w, \tau v, \tau u) = (u, v, \tau u)$, $u \in K^\times$, $v \in F^\times$. We can still multiply our cocycle $t_\rho$ by a coboundary $g^{-1} \rho^*(g)$ with $g = (\tau g)$ (to preserve $t_\tau = 1$). Thus $g = (x, y, \tau x)$, $y = \tau y \in E^\times$. Then $g^{-1}(\sigma\tau)^*(g) = (1/u, 1/y\sigma(y)/1/\tau(u)), u = x\sigma(x)$. Now $H^1(\text{Gal}(LE/F), T^*(LE))$ is spanned by the $t_{\sigma\tau} = (u, v, \tau u), u \in K^\times/\text{N}_{EL/K}(EL)^\times, \nu\tau u = 1$. Then $\text{Im}[H^1(F, T^*) \rightarrow H^1(F, T)]$ is represented by

$$(u, 1/\nu\tau u, \tau u), \ u \in K^\times/\text{N}_{EL/K}(EL)^\times \simeq \mathbb{Z}/2.$$  }

Consider next an $F$-torus $T$ in $G$ which splits over a cubic extension $M_1$ of $E$, but not over $E$. The involution $i(x) = J'\mathbf{r}_J J$ stabilizes $T = T(F)$, and its centralizer $M_1^T$ in $\text{GL}(3, E)$. It induces on the field $M_1$ an automorphism, denoted $\sigma$, whose restriction to $E$ generates $\text{Gal}(E/F)$. Define $M$ to be the subfield of $M_1$ whose elements are fixed by $\sigma$. It is a cubic extension of $F$, $M_1 = ME$, and $M_1/F$ is Galois precisely when $M/F$ is. If $M'$ is a Galois closure of $M_1/F$, then there is $\tau$ in $\text{Gal}(M'/F)$ with $\tau(x, y, z) = (z, x, y)$ (up to order). But $X = \tau X = (x, y, -x - y)$ if $X = (x, x + y, 0)$. Hence $C(T/F)$ is $\{0\}$.

There are two possible actions of the Galois group of the Galois closure of $M_1$ over $F$. In both cases we may assume that $\sigma^*(x, y, z) = (\tau z, \tau x, \tau y)$. If $\sigma^*(x, y, z) = (\sigma x^{-1}, \sigma y^{-1}, \sigma x^{-1})$ then $\sigma = \sigma^2$, the Galois group is $S_3$, and $T^*$ consists of $(x, \tau x, \tau^2 x), x \in M_1$ with $\nu x\sigma x = 1$.

If $\sigma^*(x, y, z) = (\sigma x^{-1}, \sigma y^{-1}, \sigma z^{-1})$ then $\sigma = \sigma^2$, the Galois group is $\mathbb{Z}/3 \times \mathbb{Z}/2$, and $T^*$ consists of $(x, \tau x, \tau^2 x), x \in M_1$ with $x\sigma x = 1$.

In the case of $H = U(2)$, each torus $T$ splits over a biquadratic extension of $F$, and $C(T/F)$ is trivial, unless $T$ splits over $E$ and $\sigma$ acts by $\sigma(x, y) = (-x, -y)$, where $C(T/F)$ is $\mathbb{Z}/2$ in the local case.

We also need a twisted analogue of the above discussion. Let $G' = R_{E/F} G$ be the group obtained from $G = U(3, E/F)$ upon restricting scalars from $E$ to $F$. It is defined over $F$. In fact, $G'(\mathcal{T}) = G(\mathcal{T}) \times G(\mathcal{T})$, and $\text{Gal}(F/E)$ acts on $G'(\mathcal{T})$ by $\tau(x, y) = (\tau x, \tau y)$ if $\tau|E = 1$, or by $\tau(x, y) = i(\tau x, \tau y)$ if $\tau|E \neq 1$. Here $i(x, y) = (y, x)$. Further, we have $G'(E) = G(E) \times G(E)$, and $G' = G'(F)$ consists of all $(x, \sigma x)$, $x \in G(E) = \text{GL}(3, E)$. The group $G$ embeds in $G'$ as the diagonal.
Denote by $Z_{G'}(x)$ the $\iota$-centralizer of $x = (x', x'')$ in $G'$. It consists of the elements $y = (y', y'')$ in $G'$ with $(y', y'')(x', x'') = (x', x'')(y', y'')$. These $y$ satisfy $y'x'x'' = x'x'y'$, $y'' = x'^{-1}y'x'$. If $x = (x', \sigma(x'))$ lies in $G'$, $T = Z_{G'}(x)$ is defined over $F$, since $\iota$ is. The group $T$ of $F$-rational points consists of such $y$ with $y'' = \sigma y'$. The $\iota$-centralizer $T$ is isomorphic to the $\sigma$-centralizer of $x'$ in $G$.

The elements $x$ and $x'^{1}$ in $G'$ are called (stably) $\sigma$-conjugate if there is $y$ in $G'$ (respectively, $G'((F))$) so that $yx = x'^{1}i(y)$. In this case $\tau x = x$ for all $\tau$ in Gal($F/F$), and $\tau(x'y) = x'^{1}\tau(x'y)$. Hence the $\sigma$-conjugacy classes within the stable $\sigma$-conjugacy class of $x$ are parametrized by the elements $\{\tau \mapsto y_{\tau} = y^{-1}\tau(y)\}$ of the kernel $B''(T/F)$ of the natural map from $H^{1}(F, T)$ to $H^{1}(F, G')$. Here $T$ denotes the $\iota$-centralizer of $x = (x', x'')$ in $G'$.

The conjugacy class in $G((F))$ of $x'x'' = x'\sigma(x')$ is defined over $F$. Hence it contains a member $Nx$ of $G$ by [13]. The element $Nx$ is determined only up to stable conjugacy. The group $T$ is isomorphic to the centralizer of $Nx$ in $G$, over $F$, by the map $y = (y', y'') \mapsto y'$. Recall that $y$ satisfies $y'x'x'' = x'x'y'$, $y'' = x'^{-1}y'x'$. The pointed set $H^{1}(F, G')$ is trivial. Hence $B''(T/F) = H^{1}(F, T)$.

We introduce the notion of (stable) $\sigma$-conjugacy since we shall use below orbital integrals $\int \phi(\gamma x\sigma(y)^{-1})d\gamma/dt$ over $G'/Z_{G}(x)$ of functions $\phi$ which transform under the center $Z' = E^{\times}$ of $G' = \text{GL}(3, E)$ via a character $\omega'(z) = \omega(z^{1}/z^{2})$ of $z \in E^{\times}$. In particular $\phi$ transforms trivially on $F^{\times}$. Hence the actual notion of stable $\sigma$-conjugacy that we need is $yx(y)^{-1} = zx$, for $z$ in $F^{\times}$, viewed as $(z, \sigma(z) = z^{-1})$ in $G'$.

The map $z \mapsto \{z_{\tau} = (z, 1)\tau(z, 1)^{-1}\}$ embeds $F^{\times}$ in $B''(T/F)$. Here $z_{\tau}$ acts on $x$ in $G'$ by $z_{\tau}(x)z_{1}^{-1}(z, 1) = zxz = (x\sigma(x'), \sigma(zx'))$ if $x = (x', \sigma(x'))$. Thus $z$ maps the member $\{y_{\tau} = y^{-1}\tau(y)\}$ of $B''(T/F)$ to $\{(zy_{\tau})\}$, which sends $x$ to $[(z', 1)\tau(z', 1)^{-1}]y_{\tau}(y_{\tau})^{-1} = (z, z^{-1})yx(y^{-1})$. The quotient of $B''(T/F)$ under this action of $F^{\times}$ is denoted by $B''(T/F)$. Put $B''(T/F/k) = \bigoplus_{v} B''(T/F_{v})$ (pointed sum) if $F$ is global.

The Tate–Nakayama theory implies that $B''(T/F)$ (in the local case) or $B''(T/F/k)/\text{Image } B''(T/F)$ (in the global case) is the quotient of the $Z$-module of the $X$ in $X_{s}(T)$ modulo $Z$ with $\sum_{\tau} \tau X = 0$ ($\tau$ in Gal($K/F$)), by the span of $X - \tau X$ for all $x$ in $X_{s}(T)$ and $\tau$ in Gal($K/F$), where $K$ is a Galois extension of $F$ over which $T$ splits.

The map $x \mapsto Nx$ gives a bijection from the set of stable $\sigma$-conjugacy classes in $G'$ (parametrized by $B''(T/F)$) to the set of stable conjugacy classes in $G$. In fact, for our present work it suffices to consider regular $x$ in $G$ ($x$ with distinct eigenvalues), and $\sigma$-regular $x$ in $G'$ ($Nx$ is regular). Hence there are three types of stable $\sigma$-conjugacy classes of $\sigma$-regular elements in $G'$, denoted by (1), (2), (3) as in the nontwisted case. Using the Tate–Nakayama theory we see (in the local case) that $B''(T/F)$ is trivial if $T$ is $T^{\dagger}$ (the diagonal torus), and in case (3); it is $\mathbb{Z}/2$ in case (2); it is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ if $T$ splits over $E$ but $T$ is not (stably) conjugate to $T^{\dagger}$.

To compute orbital integrals, we need explicit representatives.
Lemma 2.3. If $T$ splits over $E$ but is not $T^*$, $H^1(F, T)/F^*$ is $F^x/NE^x$, where $F^x/NE^x$ embeds diagonally into $F^x/NE^x$. If $T$ splits over a biquadratic extension $LE$ of $F$, $\text{Gal}(LE/F) = \langle \tau, \sigma \rangle$, $L = (LE)\tau$, $E = (LE)\sigma$, $K = (LE)\sigma\tau$ are the quadratic extensions of $F$ in $EL$, then $H^1(F, T)/F^*$ is $K^x/N_{LE/K}(LE)^x$, which is $\mathbb{Z}/2\mathbb{Z}$.

Proof. If $T$ splits over $E$ but is not $T^*$, a cocycle $t_\sigma = (a, b, c)$ in $H^1(E, T(E))$ satisfies

$$1 = t_\sigma^2 = t_\sigma^\sigma(t_\sigma)^{-1} = (a, b, c)(\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1}).$$

Thus $(a, b, c)$ lies in $F^x$. A coboundary has the form $t_\sigma^\sigma(t_\sigma)^{-1} = (a, b, c)(\sigma a, \sigma b, \sigma c)$. Hence we get $NE^x$, and $H^1(F, T)/F^*$ is $F^x/NE^x$.

If $T$ splits over a biquadratic extension $LE$ of $F$, the group $H^1(\text{Gal}(LE/F), T(LE)/F)$ is computed in the proof of Proposition 2. Then $H^1(\text{Gal}(LE/F), T(LE)/F^*)$ is represented by $t_{\tau\sigma} = (u, 1, \tau u)$, $u \in K^x/N_{LE/K}(LE)^x$. \[\square\]

We also need an explicit realizations of the twisted stable conjugacy classes in the cases that they contain several twisted conjugacy classes, namely the cases corresponding to the tori $T = (E^1)^3$ and $T = (EL/K)^1 \times E^1$. This is useful in computations of twisted orbital integrals and twisted characters. We consider only classes whose norm is regular (has distinct eigenvalues) in $G = U(3, E/F)$. We name such classes twisted-regular or $\sigma$-regular.

Proposition 2.4. (i) A set of representatives for the $\sigma$-conjugacy classes within the stable $\sigma$-conjugacy class of a twisted regular $x$ in $\text{GL}(3, E)$ with norm in an anisotropic torus which splits over $E$, thus $Nx = h^{-1}\text{diag}(a/\sqrt{A}, b/\sqrt{D}, c/\sqrt{B})h$ in a torus

$$T_1 = h^{-1}T^*(E^1)h,$$

distinct $a/\sqrt{A}, b/\sqrt{D}, c/\sqrt{B}$, is given by $x_1 = h^{-1}\text{diag}(a, b, c)h$, $x_2 = h^{-1}\text{diag}(a, bp, c)h$, $x_3 = h^{-1}\text{diag}(ap, b, c)h$, $x_4 = h^{-1}\text{diag}(a, b, cp)h$, where $a, b, c$ lie in $E^x$, $\rho \in F - NE$.

If $y = h^{-1}\text{diag}(\alpha, \beta, \gamma)h$ then $yx_1\sigma y^{-1} = h^{-1}\text{diag}(a\alpha, b\beta, c\gamma)h$.

(ii) Let $L = F(\sqrt{A}) = (EL)^\sigma$, $E = (EL)^\tau = F(\sqrt{D})$, and $K = (EL)^{\sigma\tau} = F(\sqrt{DA})$ be the distinct quadratic extensions of $F$. Here $\{A, D, AD\} = \{\pi, u, u\pi\}$ and $u \in R_E - R_E^2$.

A set of representatives for the $\sigma$-conjugacy classes within the stable $\sigma$-conjugacy class of a $\sigma$-regular $x$ in $\text{GL}(3, E)$ with norm in a torus which splits
over a biquadratic extension $EL$ of $F$ can be realized by

$$t = t_\alpha = h^{-1} \begin{pmatrix} (a + b\sqrt{A})\alpha & c \\ (a - b\sqrt{A})\tau(\alpha) & \end{pmatrix} h,$$

$$h = \begin{pmatrix} 1 & \sqrt{A/D} \\ \sqrt{D/A} & \frac{1}{2} \end{pmatrix},$$

where $a, b, c \in E^\times$ and $\alpha \in K^\times/N_{EL/K}(EL)^\times$. Then $(Nt \text{ is regular})$ and

$$Nt = t\sigma(t) = h^{-1} \text{diag}(\frac{a + b\sqrt{A}}{(\tau - b\sqrt{A})}, c/\tau, (a - b\sqrt{A})/(\tau + b\sqrt{A})) h.$$

The norm map is surjective.

**Proof.** First note that $x_1 = h^{-1} \text{diag}(a, b, c) h$ satisfies

$$N x_1 = x_1 \sigma(x_1) = h^{-1} \text{diag}(a, b, c) h \cdot \sigma(h^{-1}) \text{diag}(1/\tau, 1/\overline{\tau}, 1/\overline{\tau}) \sigma(h).$$

Since $\sigma(h^{-1}) = h$ and $h^2 = \text{diag}(2, -1, -2)$, this is

$$= h^{-1} \text{diag}(a/\tau, b/\overline{\tau}, c/\overline{\tau}) \text{diag}(2, -1, -2)^t h^{-1} J.$$

But $\text{diag}(2, -1, -2)^t h^{-1} J = h$. In particular the norm $N$ is onto the torus $T \simeq (E^1)^3$, which we realize as $T_1 = h^{-1} T^* h$.

The stable $\iota$-conjugates of $x_1$ are given by $y' x_1 y'^{-1}$ where $y_\iota = y'^{-1} \sigma(y'')$ lies in $H^1(F, T_1)/F^\times$, $T_1 = h^{-1} T^* h$, where $T^*$ denotes the diagonal torus, as discussed in the paragraph where (stable) $\sigma$-conjugacy is defined, prior to Proposition 2.3.

A set of representatives for the stable $\iota$-conjugates of $x_1$ up to $\iota$-conjugacy is given as $y_\sigma$ ranges over $h^{-1} th$, where $t$ ranges over $T^*(F)/Z(F) N T^*(E)$; $Z$ is the diagonal. Choose $\rho \in F - NE$. Thus we may take $t$ to be $1$, $\text{diag}(1, \rho, 1)$, $\text{diag}(\rho, 1, 1)$, $\text{diag}(1, 1, \rho)$. Taking $y''$ to be $1$, we choose $y' = h^{-1} th$, to get $x_i$ (1 $\leq$ $i$ $\leq$ 4) of the proposition.

In the case of the torus splitting over $EL$ and isomorphic to $\ker N_{EL/K} \times E^1$, note that $\sigma(h) = h$, and that $\sigma^* (a, b, c) = (\sigma a^{-1}, \sigma b^{-1}, \sigma a^{-1})$. We write $\sigma a = \overline{\tau}$, and $\sigma$ fixes $\sqrt{A}$. The $\sigma$-conjugacy classes within the stable $\sigma$-conjugacy class are parametrized in Lemma 3.

**Remark.** (1) Define $a', b'$ in $E$ by $a' + b'\sqrt{A} = (a + b\sqrt{A})\alpha$ in $EL = E(\sqrt{A})$. Then

$$t_\alpha = \begin{pmatrix} a' & b'A/\sqrt{D} \\ b'\sqrt{D} & c \end{pmatrix}.$$
(2) If $E/F$ is unramified then $K/F$ and $L/F$ are ramified. We may take $D$ to be a nonsquare unit in $F$. Hence we may choose the $\alpha \neq 1$ to be $\sqrt{AD}$, which is in $K$ but not in $N_{E/L/K}(EL)$.

3. Elements in $G$ with Norm in the Torus $(E^1)^3$

In the case of an element whose norm in $G$ lies in the torus $(E^1)^3$, the expression $\Delta(N\gamma)\chi_{E}(\gamma)$ of Theorem 1 is the value at $s = 0$ of the expression $(a, b, c$ in $E^\times$, distinct $a/\pi$, $b/\pi$, $c/\pi$ in $E^1$; take $x_1$ in Proposition 2.4, use Corollary 1.2 and Lemma 1.3; $|\cdot| = |\cdot|_E$ and $|\pi|_E = q^{-1}_E$ there; $|x'|_F = |N_{E/F}x'|_F^{1/2} = |x|_F^{1/2}$

$$\xi_s(a,b,c) = -\frac{q^{3/2}_E}{q_E + q^{3/2}_E} + 1 \left| \left( \frac{a}{\pi} - \frac{b}{\pi} \right)^c \frac{c}{\pi} \right|_{E}^{m} |abc|^{1/2}_E$$

times

$$\int_{V_0} \left| -by\pi + \frac{a}{2}(x+z)(\pi + \pi) - \frac{c}{2}(x-z)(\pi - \pi) \right|_{E}^{m} dx dy dz, \quad m = \frac{3}{2}(s - 1).$$

This and the following sections, $q$ is $q_F$, $|\cdot| = |\cdot|_F$, $R$ is $R_F$.

Since the $\sigma$-character depends only on the $\sigma$-conjugacy class, we may replace any of $a, b, c$ by its multiple with an element of $NE^\times$. Suppose that $E/F$ is unramified and that the residual characteristic $p$ is odd ($\neq 2$). Then we can take $\rho \in F^\times - NE^\times$ to be a generator $\pi$ of the maximal ideal in the ring $R = R_F$ of integers in $F$, and $E = F(\sqrt{D})$, $D \in R^\times - R^\times 2$. We may assume that $|b| \geq |a|$, $|c|$, and that $a/b$ and $c/b$ lie in $R_F$ or in $\pi R_F$. Since the central character of the representation $\pi$ is trivial, thus $\chi_{E}(\pi) = \chi_{E}(b)$ for $b \in E^\times$, we may replace $a, c$ by $a/b, c/b$, hence assume that $b = 1$, and that $a, c$ are in $R_F^\times$ or in $\pi R_F^\times$. Fix $\epsilon \in E$ with $N\epsilon = 2$. After the change of variables: $x' = (x+z)/\epsilon$, $z' = (x-z)/\epsilon$, renaming ($x', z'$) back to ($x, z$), and noting that $-1 \in NE^\times$, the integral becomes

$$\int_{V_0} |aN(x) + N(y) + cN(z)|^{m}_F dx dy dz.$$
where max \( |\cdot| \) is \( |\cdot|_F \), \( q \) is \( q_F \), \( R \) is \( R_F \). In Sec. 3 we defined the quadratic forms

\[ Q_1(x, y, z) = N(x) + a_1 N(y) + c_1 N(z), \quad Q_2(x, y, z) = a_2 N(y) + c_2 N(z), \]

where \( \max\{|a_1|, |a_2|\} = \max\{|c_1|, |c_2|\} = 1 \) and the set

\[ V_n = \{(x, y, z) \in R_F^3; \| (x, y, z)\|_E = 1, \max\{|Q_1(x, y, z)|, |Q_2(x, y, z)|\} = q^{-n}\}. \]

Note that if \( |a_1| < 1 \) then \( |a_2| = 1 \) and we can replace \( Q_1 \) with \( Q_1 + Q_2 \). Since in this case \( |a_1 + a_2| = 1 \), without loss of generality we can assume that \( |a_1| = 1 \), i.e. \( a_1 \in R^\times \) is a unit in \( R \). Recall that \( u = a_1 c_2/(a_2 c_1) \). Then, we have the following possible cases:

(4.1) \( c_1, a_2/c_2 \in R^\times \) and \( |1 - u| = 1 \);

(4.2) \( c_1, a_2/c_2 \in R^\times \) and \( |1 - u| < 1 \);
(4.3) $|a_2| < 1$, $|a_2| \neq |c_2|$, $\max\{|c_1|, |c_2|\} = 1$;
(4.4) $c_2, a_2 \in R^\times$, $|c_1| < 1$.

The case $a_1, a_2, c_1 \in R^\times$, $|c_2| < 1$ can be reduced to that of case 4.3 on interchanging $a, c$.

4.1.

We proceed to prove the theorem when $n_1 = 0 = n_2, a_1, c_1, u, u - 1$ are units in $R^\times$. When $|a_2| = |c_2| = q^{−d}$, to emphasize we denote $V_n$ by $V_{n,d}$. Similarly we denote $V_n^0$ by $V_{n,d}^0$, thus $\text{vol}(V_n^0) = \text{vol}(V_n)/(1 − q^{−2})$. These volumes are computed in Lemma B.4.1.

**Proof of Theorem 2.** Denote $I_s(a, c)$ by $I_{s,d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{−2mn} \text{vol}(V_{n,d}^0).$$

When $d = 0$, by Lemma B.4.1, this is equal to

$$1 - 2q^{−3} + (1 - q^{−2})(1 + q^{−1})^2 \sum_{n=1}^{\infty} a^n,$$

where $\alpha = q^{−2n(m+1)}$. This sum is equal to

$$1 - 2q^{−3} + (1 - q^{−2})(1 + q^{−1})^2 \alpha(1 - \alpha)^{−1}.$$  

In our case, when $s = 0$, thus $m = −3/2$ and $\alpha = q$, this equals

$$1 - 2q^{−3} - (1 + q^{−1})^3 = −3q^{−1}(1 + q^{−1} + q^{−2}).$$

When $d \geq 1$, the sum

$$\sum_{n=0}^{d} q^{jn} \text{vol}(V_{n,d}^0),$$

by Lemma B.4.1, is equal to

$$1 - q^{−1} + q^{−2} + (1 - q^{−2})(1 - q^{−1} + q^{−2}) \sum_{n=1}^{d−1} q^{2n} + (1 - q^{−2} - q^{−3} - q^{−4})q^{2d},$$

which is

$$(1 - 2q^{−3})q^{2d} = q^{2d} \text{vol}(V_{0,0}^0).$$

Note that, by Lemma B.4.1, $\text{vol}(V_{n,d}^0) = q^{−d} \text{vol}(V_{n-d,0}^0)$ ($n > d$). Thus

$$I_{s,d} = \sum_{n=0}^{\infty} q^{−2mn} \text{vol}(V_{n,d}^0) = \sum_{n=0}^{d} q^{−2mn} \text{vol}(V_{n,d}^0) + \sum_{n=d+1}^{\infty} q^{−2mn} \text{vol}(V_{n,d}^0)$$

$$= \sum_{n=0}^{d} q^{−2mn} \text{vol}(V_{n,d}^0) + q^{−d(1+2m)} \sum_{j=0}^{\infty} q^{−2jm} \text{vol}(V_{j,0}^0) - q^{−d(1+2m)} \text{vol}(V_{0,0}^0).$$
The sum in the middle of last line is equal to $q^{-d(1+2m)}I_{s,0}$. At $s = 0$, thus $m = -3/2$, the other two terms cancel each other by the equality found in the last paragraph, and we obtain

$$I_{0,d} = q^{2d}I_{0,0} = -3q^{2d} \times q^{-3}(1 + q + q^2).$$

In our case $|ac|_E = 1$, $|a/\pi - 1| = |a_2|$, $|c/\pi - 1| = |c_2|$ and $|a/\pi - c/\pi| = |a_2c_1 - a_1c_2| = |a_2c_1||1 - u| = |a_2|$. We put $|a| = |N_{E/F}a|^{1/2} = |a|^{1/2}$. Here $q$ is $q_E = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is canceled by the factor $q_F^{3/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, b, c)$. Hence $\xi_0(a, 1, c)I_{0}(a, c) = 3|a_2|$.

The $H$-side, namely the displayed formula in Theorem 2, is equal to

$$\kappa(a_2c_2)(a_2c_1 - a_1c_2) + \kappa(a_2c_1 - a_1c_2)|c_2| + \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.$$ 

Note that $\kappa(a_2) = \kappa(c_2) = \kappa(a_2c_1 - a_1c_2) = 1$. Since $|a_2c_1 - a_1c_2| = |c_2| = |a_2|$, each of the three summands is equal to $|a_2|$. Theorem 2 follows for $n_1 = n_2 = 0$, $a_1, c_1, a_2/c_2 \in R^\times, |u - 1| = 1$.

4.2.

Consider the case of $n_1 = 0, n_2 = 0, a_1, c_1, u \in R^\times$, and $|u - 1| < 1$. When $|a_2| = |c_2| = q^{-d}$, as in case 4.1, we denote $V_n$ by $V_{n,d}$. Similarly we denote $V_n^0$ by $V_{n,d}^0$, thus $\text{vol}(V_n^0)$ is $\text{vol}(V_n)/(1 - q^{-2})$. The $\text{vol}(V_{n,d}^0)$ is computed in Lemma B.4.2.

**Proof of Theorem 2.** Denote $I_s(a, c)$ by $I_{s,d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm} \text{vol}(V_{n,d}^0).$$

We split this sum into two: a finite sum over $n$, which we evaluate at once at $s = 0$ ($m = -3/2$), and an infinite sum, in which we first take big $m$, to have convergence, then add up the sum and only then evaluate the result at $m = -3/2$.

Thus when $|1 - u| = q^{-2k}$ and $d = 0$, by Lemma B.4.2, we have the sum of

$$\begin{align*}
(1 + q^{-2} - q^{-3}) + q^3(q^{-3} - q^{-5}) + (1 - q^{-2}) \left( \sum_{n=3}^{2k-1} q^{n-1} + (1 + q^{-1} + q^{-2}) \sum_{n=2}^{2k-2} q^n \right) \\
(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2k} + (1 - q^{-2})(1 + q^{-1})^2 \sum_{n=2k+1}^{\infty} \alpha^n,
\end{align*}$$

where the first sum is taken over the odd $n$ ($3 \leq n \leq 2k - 1$), the second over the even $n$ ($2 \leq n \leq 2k - 2$) and the third over all $n \geq 2k + 1$, and $\alpha = q^{-2n(m+1)}$. 

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We have
\[ \sum_{n=3}^{2k-1} q^{n-1} = \sum_{n=2}^{2k-2} q^n = (1 - q^{-2})^{-1}(q^{2k-2} - 1), \]
and
\[ \sum_{n=2k+1}^{\infty} q^n = -(1 - \alpha^{-1})^{-1} \alpha^{2k}. \]

At \( m = -3/2 \), thus \( \alpha = q \), the above expression equals
\[ 2 - q^{-3} + (2 + q^{-1} + q^{-2})(q^{2k-2} - 1) \]
\[ + (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4} - (1 + q^{-1})^3)q^{2k}. \]

The coefficient of \( q^{2k} \),
\[ 2q^{-2} + q^{-3} + q^{-4} + 1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4} - 1 - 3q^{-1} - 3q^{-2} - q^{-3}, \]
is equal to \( -2q^{-1}(1 + q^{-1} + q^{-2}) \). Hence we obtain
\[ -q^{-3}(1 + q + q^2)(1 + 2q^{2k}). \]

When \( |1 - u| = q^{-2k-1} \) and \( d = 0 \), by Lemma B.4.2, we have the sum
\[ 2 - q^{-3} + (1 - q^{-2}) \left( \sum_{n=3}^{2k-1} q^{n-1} + (1 + q^{-1} + q^{-2}) \sum_{n=2}^{2k} q^n \right) + (1 + q^{-1})q^{2k}, \]
where the first sum is taken over the odd \( n \) and second over the even \( n \). To make both sums of the same length, we split the term \( n = 2k \) from the second sum. Once the sums are computed, our expression is equal to
\[ 2 - q^{-3} + (1 + q^{-1} - q^{-3} - q^{-4})q^{2k} + (2 + q^{-1} + q^{-2})(q^{2k-2} - 1) + (1 + q^{-1})q^{2k} \]
\[ = -q^{-1}(1 + q^{-1} + q^{-2})(1 - 2q^{2k+1}). \]

When \( |1 - u| = q^{-1} \), by Lemma B.4.2, we have the sum
\[ 1 + q^{-2} - q^{-3} + q^{-4}(q^{-3} + q^{-4}) = -q^{-1}(1 + q^{-1} + q^{-2}) + 2(1 + q^{-1} + q^{-2}), \]
which coincides to the odd case when \( k = 0 \). Thus we see that, if \( d = 0 \), then \( I_{0,0} \) is equal to
\[ -(1 + 2\kappa(1 - u)|1 - u|^{-1})q^{-3}(1 + q + q^2), \]
where \( \kappa(1 - u) \) is 1 when \( |1 - u| = q^{-2k} \) and is \( -1 \) when \( |1 - u| = q^{-2k-1} \).

When \( d \geq 1 \), the sum
\[ \sum_{n=0}^{d} q^{3n} \text{vol}(V_{n,d}), \]
by Lemma B.4.2, is equal to
\[ 1 - q^{-1} + q^{-2} + (1 - q^{-2})(1 - q^{-1} + q^{-2}) \sum_{n=1}^{d-1} q^{2n} + (1 - q^{-4})q^{2d}, \]
Consider the case of 4.3.

Moreover, since, by Lemma B.4.2, \( \text{vol}(V_{n,d}^0) = q^{-d} \text{vol}(V_{n-d,0}^0) \) for \( n > d \), we have the same equality as in Lemma B.4.1:

\[
I_{0,d} = q^{2d} I_{0,0}.
\]

In our case \( |bc|_E = 1, |a/\pi - 1| = |a_2|, |c/\pi - 1| = |c_2| \) and \( |a/\pi - c/\bar{\pi}| = |a_2c_1 - a_1c_2| = |a_2c_2|/|1 - u| \). Here \( q = q_E^{1/2} \), so \(-q^{-3}(1 + q + q^2)\) is canceled by the factor \(-q^{3/2}_E(q + q^{1/2}_E + 1)^{-1}\) in \( \xi_0(a, b, c) \). Hence \( \xi_0(a, 1, c)I_0(a, c) \) is equal to

\[
|a_2c_1 - a_1c_2| + 2\kappa(1 - u)|a_2|.
\]

The \( H \)-side is equal to

\[
\kappa(a_2c_2)|a_2c_1 - a_1c_2| + \kappa(a_2(a_2c_1 - a_1c_2))|c_2| + \kappa(c_2(a_2c_1 - a_1c_2))|a_2|.
\]

Note that \( \kappa(a_2c_2) = 1 \), and \( \kappa(a_2(a_2c_1 - a_1c_2)) = \kappa(1 - u) \). Thus the above expression is equal to \( |a_2c_1 - a_1c_2| + 2\kappa(1 - u)|a_2| \). Theorem 2 follows when \( n_1 = 0, n_2 = 0 \), \( a_1, c_1, a_2/c_2 \in \mathbb{R}^\times \), \( |1 - u| < 1 \).

4.3.

Consider the case of \( n_1 = 0, n_2 = 0, a_1 \in \mathbb{R}^\times, \max\{|c_1|, |c_2|\} = 1, |a_2| < 1, |a_2| \neq |c_2| \). Recall that we are dealing with the quadratic forms

\[
Q_1(x, y, z) = N(x) + a_1N(y) + c_1N(z), \quad Q_2(x, y, z) = a_2N(y) + c_2N(z).
\]

Thus we can interchange \((a, y)\) with \((c, z)\) to reduce the case of \( |a_2| = 1, |c_1| < 1 \) to the case \( |c_2| < 1, |c_1| = 1 \) of Lemma B.4.3. When \(|c_2| = q^{-d}\), to emphasize we denote \( V_{n} \) by \( V_{n,d} \). Similarly, as in case 4.1, we denote \( V_{n}^0 \) by \( V_{n,d}^0 \), thus \( \text{vol}(V_{n}^0) \) is \( \text{vol}(V_{n}/(1 - q^{-2})) \). The \( \text{vol}(V_{n,d}^0) \) are computed in Lemma B.4.3.

**Proof of Theorem 2.** Denote \( I_s(a, c) \) by \( I_{s,d} \). Recall that the integral \( I_{s,d} \) is equal to

\[
\sum_{n=0}^{\infty} q^{-2nm} \text{vol}(V_{n,d}^0).
\]

Note that \( \text{vol}(V_{n,d}^0) \) \((d > 0 \text{ and } 0 \leq n \leq d)\) of Lemma B.4.3 are equal to \( \text{vol}(V_{n,d}^0) \) of Lemma B.4.1, thus when \( m = -3/2 \), we have

\[
\sum_{n=0}^{d} q^{3n} \text{vol}(V_{n,d}^0) = q^{2d} \text{vol}(V_{0,0}^0).
\]

Moreover \( \text{vol}(V_{n,d}^0) \) \((n \geq 0)\) of Lemma B.4.3 are related to those of Lemma B.4.2. Namely, when \(|a_2| \) of Lemma B.4.3 is equal to \(|1 - u| \) of Lemma B.4.2, then \( \text{vol}(V_{n,d}^0) \) of Lemma B.4.3 is equal to \( \text{vol}(V_{n,d}^0) \) of Lemma B.4.2. Since
vol(V^n_0) = q^{-d}vol(V^n_{n-d,0}) for n > d, we have the same equality as in Lemma B.4.2:

I_{0,d} = q^{2d}I_{0,0} = -|c_2|^2 (1 + 2K(a_2)|a_2|^{-1})q^{-3}(1 + q + q^2),

where K(a_2) is equal to 1 when |a_2| = q^{-2k} and to -1 when |a_2| = q^{-2k-1}.

In our case |a_2| = 1, |a/\sqrt{q} - 1| = |a_2|, |c/\sqrt{q} - 1| = |c_2|, and |a/\sqrt{q} - c/\sqrt{q}| = |a_2c_1 - a_1c_2| = |c_2| (since |a_2c_1| < 1). Here \( q = q_F, q_E = q_F^2, \) and \(-q^{-3}(1 + q + q^2)\) is canceled by the factor \(-q_{E}^{-3}(q_E + q_{E}^{-1} + 1)^{-1} \) in \( \xi_0(a,b,c). \) Hence \( \xi_0(a,1,c)I_0(a,c) \) is equal to \( |a_2| + 2K(a_2). \) Theorem 2 follows when \( n_1 = 0, n_2 = 0, a_1 \in R^\infty, |a_2| < 1, |a_2| \neq |c_2|, \) and \( \max\{|c_1|, |c_2|\} = 1. \) □

4.4.

Consider the case of \( n_1 = 0, n_2 = 0, a_1, a_2, c_2 \in R^\infty, |c_1| < 1. \) As in the previous sections, the computation of vol(V^n_{n-d,0}) is done in Lemma B.4.4.

**Proof of Theorem 2.** Note that the vol(V^n_0) of Lemma B.4.4 is the same as the vol(V^n_{n,0}) of Lemma B.4.1. Thus, \( I_0(a,c) \) is equal to \(-3q^{-3}(1 + q + q^2)\).

In our case |a_2| = 1, |a/\sqrt{q} - 1| = |a_2| = 1, |c/\sqrt{q} - 1| = |c_2| = 1 and |a/\sqrt{q} - c/\sqrt{q}| = |a_2c_1 - a_1c_2| = 1. Here \( q = q_F = q_F^{1/2}, \) and \(-q^{-3}(1 + q + q^2)\) is canceled by the factor \(-q_{E}^{3/2}(q_E + q_{E}^{1/2} + 1)^{-1} \) in \( \xi_0(a,b,c). \) Hence \( \xi_0(a,1,c)I_0(a,c) = 3. \)

The \( H \)-side is equal to

\[
K(a_2c_2)|a_2c_1 - a_1c_2| + K(a_2c_1 - a_1c_2)|c_2| + K(c_2c_1 - a_1c_2)|a_2|.
\]

As each of the three terms on the right, namely the \( H \)-side, of the identity of the theorem, is equal to 1 (as |a_2| = |c_2| = |a_2c_1 - a_1c_2| = 1), Theorem 2 follows when \( n_1 = n_2 = 0, a_1, a_2, c_2 \in R^\infty \) and |c_1| < 1. □

5. Case of \( n_1 = 1, n_2 = 0 \)

In this section \( |\cdot| \) is \( |\cdot|_F, q \) is \( q_F, R \) is \( R_F, \) and the quadratic forms are

\[
Q_1(x,y,z) = N(x) + a_1\Pi N(y) + c_1N(z), \quad Q_2(x,y,z) = a_2\Pi N(y) + c_2N(z),
\]

where \( \max\{|a_1|, |a_2|\} = \max\{|c_1|, |c_2|\} = 1. \) As in the case of \( n_1 = n_2 = 0, \) without loss of generality we assume that |a_1| = 1. Then, we have the following possible cases:

\[
\begin{align*}
(5.1) & \quad c_2 \in R^\infty; \\
(5.2) & \quad |c_2| < 1, c_1 \in R^\infty, |a_2\Pi| > |c_2|; \\
(5.3) & \quad |c_2| < 1, c_1 \in R^\infty, |a_2\Pi| \leq |c_2|.
\end{align*}
\]
5.1.

We proceed to prove the theorem for \( n_1 = 1, \, n_2 = 0, \, c_2 \in \mathbb{R}^\times \). The computation of \( \text{vol}(V_0^n) \) is done in Lemma C.5.1.

**Proof of Theorem 2.** The integral \( I_s(\pi a, c) \) is equal to

\[
\sum_{n=0}^{\infty} q^{-2m} \text{vol}(V_0^n).
\]

By Lemma C.5.1, the sum is \( \text{vol}(V_0^0) + \frac{1}{1-q} \text{vol}(V_0^0) \). When \( s = 0 \), thus \( m = -3/2 \), this equals

\[
1 + q^{-2} + q^{-4} = 1 + q^{-1} + q^{-2} = q \times q^{-3}(1 + q + q^2).
\]

Here \( q = q_E = q_E^{1/2} \), so \( -q^{3/2}(1 + q + q^2) \) is canceled by the factor \( -q^{3/2}(q_E + q_E^{1/2} + 1) \) in \( \xi_0(\pi a, 1, c) \). Moreover, \( |\pi ac|_E^{-1/2} = q^{1/2} = q^{-1}, |c/\pi - 1| = |a_2|, |c/\pi - 1| = |c_2| = 1, \) and \( |a/\pi - c/\pi| = |a_2 c_1 - a_1 c_2| \). Hence \( \xi_0(\pi a, 1, c)I_0(\pi a, c) \) is equal to \(-|a_2 c_1 - a_1 c_2|^2\) when \( |a_2 c_1| < 1 \) and to \(-|a_2 c_1 - a_1 c_2| \) when \( |a_2 c_1| = 1 \).

The \( H \)-side is equal to

\[
-\kappa(a_2 c_2)|a_2 c_1 - a_1 c_2| + \kappa(a_2(a_2 c_1 - a_1 c_2))|c_2| - \kappa(c_2(a_2 c_1 - a_1 c_2))|a_2|.
\]

When \( |a_2 c_1| < 1 \) thus \( |a_2 c_1 - a_1 c_2| = 1, \kappa(a_2(a_2 c_1 - a_1 c_2)) = \kappa(a_2), \) and \( \kappa(c_2(a_2 c_1 - a_1 c_2)) = 1 \), so the \( H \)-side is equal to

\[
-\kappa(a_2 c_2) + \kappa(a_2) - |a_2| = -|a_2|.
\]

When \( |a_2 c_1| = 1 \) we have \( \kappa(a_2 c_2) = 1, \) and \(-\kappa(a_2(a_2 c_1 - a_1 c_2)) + \kappa(c_2(a_2 c_1 - a_1 c_2)) = 0, \) so the \( H \)-side is equal to

\[
-|a_2 c_1 - a_1 c_2|.
\]

This establishes the theorem for \( n_1 = 1, \, n_2 = 0, \, c_2 \in \mathbb{R}^\times \).

\[ \square \]

5.2.

We proceed to prove the theorem for \( n_1 = 1, \, n_2 = 0, \, a_1, \, c_1 \in \mathbb{R}^\times \) and \(|c_2| < |a_2 \pi|\). When \( |a_2| = q^{-d} \), to emphasize we denote \( V_n \) by \( V_{n,d} \). Similarly we denote \( V_0^n \) by \( V_0^{n,d} \). The computation of \( \text{vol}(V_{n,d}^0) \) is done in Lemma C.5.2.

**Proof of Theorem 2.** Denote \( I_s(\pi a, c) \) by \( I_{s,d} \), where \( |a_2| = q^{-d} \). Recall that the integral \( I_{s,d} \) is equal to

\[
\sum_{n=0}^{\infty} q^{-2mn} \text{vol}(V_{n,d}^0).
\]

We split (as in Lemma B.4.2) this sum into two: a finite sum over \( n \), which we evaluate at once at \( s = 0 \) (\( m = -3/2 \)), and an infinite sum, in which we first take
big \( m \), to have convergence, then add up the sum and only then evaluate the result at \( m = -3/2 \).

Thus, when \(|c_2/(a_2 \pi)| = q^{-2k} \) and \( d = 0 \), by Lemma C.5.2, we have the sum of
\[
(1 - q^{-1}) + q^3(q^{-1} + q^{-2} - q^{-5})
\]
\[
+ (1 - q^{-2}) \left( (1 + q^{-1} + q^{-2}) \sum_{n=3}^{2k-1} q^{n+1} + \sum_{n=2}^{2k} q^n \right)
\]
and
\[
(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4}) q^{2k+2} + (1 - q^{-2})(1 + q^{-1})^2 \sum_{n=2k+2}^{\infty} \alpha^{n+1}.
\]
The first sum is taken over the odd \( n (3 \leq n \leq 2k - 1) \), the second over the even \( n (2 \leq n \leq 2k) \) and the third over all \( n \geq 2k + 2 \), and \( \alpha = q^{-2n(m+1)} \). We have
\[
\sum_{n=3}^{2k-1} q^{n+1} = (1 - q^{-2})^{-1}q^2(q^{2k-2} - 1), \quad \sum_{n=2}^{2k} q^n = (1 - q^{-2})^{-1}(q^{2k} - 1),
\]
and
\[
\sum_{n=2k+2}^{\infty} \alpha^{n+1} = -(1 - \alpha^{-1})^{-1} \alpha^{2k+2}.
\]
At \( m = -3/2 \), thus \( \alpha = q \), the above expression equals to the sum of
\[
1 - q^{-1} + q^2 + q - q^{-2} + (q^{2k} - 1) + (1 + q^{-1} + q^{-2})(q^{2k} - q^2)
\]
and
\[
((1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4}) - (1 + q^{-1})^3) q^{2k+2}
\]
Once simplified, this is equal to
\[-q^{-2}(1 + q + q^2)(1 + 2q^{2k+1}).\]
When \(|c_2/(a_2 \pi)| = q^{-2k-1} \) and \( d = 0 \), by Lemma C.5.2, we have the sum of
\[
(1 - q^{-1}) + q^3(q^{-1} + q^{-2} - q^{-5}) \text{ and}
\]
\[
(1 - q^{-2})(1 + q^{-1} + q^{-2}) \sum_{n=3}^{2k+1} q^{n+1} + (1 - q^{-2}) \sum_{n=2}^{2k} q^n + (1 + q^{-1}) q^{2k+2},
\]
where the first sum is taken over the odd \( n (3 \leq n \leq 2k + 1) \), the second over the even \( n (2 \leq n \leq 2k) \) and the third over all \( n \geq 2k + 2 \). Once simplified, this is equal to
\[-q^{-2}(1 + q + q^2)(1 - 2q^{2k}).
\]
Thus we see that, when \( d = 0 \), \( I_s(a, c) \) is equal to
\[-q^{-2}(1 + q + q^2)(1 + 2\kappa(c_2/(a_2 \pi)) q^{2k+1}),
\]
where $\kappa(c_2/(a_2\pi))$ is 1 when $|c_2/(a_2\pi)| = q^{-2k}$ ($k \geq 1$) and $-1$ when $|c_2/(a_2\pi)| = q^{-2k-1}$ ($k \geq 0$).

When $|a_2| = q^{-d}$, $d \geq 1$, the sum

$$\sum_{n=0}^{d+1} q^{3n} \text{vol}(V_{n,d}^0),$$

by Lemma C.5.2, is equal to

$$1 - q^{-1} + q^3(q^{-1} - q^{-3} + q^{-4}) + (1 - q^{-2}) \sum_{n=2}^{d} q^{2n}$$

$$+ (1 - q^{-2})(1 + q^{-1} + q^{-2})q^{2d+2},$$

which is $q^{2d} + (1 - q^{-2})(1 + q + q^2)q^{2d}$

$$= (q^2 + q + 1 - q^{-1} - q^{-2})q^{2d} = q^{2d}(\text{vol}(V_{0,0}^0) + q^3 \text{vol}(V_{1,0}^0)).$$

Moreover, since $\text{vol}(V_{n,d}^0) = q^{-d} \text{vol}(V_{n-1,d}^0)$ for $n \geq d + 2$, we have

$$I_{s,d} = \sum_{n=0}^{\infty} q^{3n} \text{vol}(V_{n,d}^0) = \sum_{n=0}^{d+1} q^{3n} \text{vol}(V_{n,d}^0) + \sum_{n=d+2}^{\infty} q^{-2mn} \text{vol}(V_{n,d}^0)$$

$$= \sum_{n=0}^{d+1} q^{3n} \text{vol}(V_{n,d}^0) + q^{-(1+2m)} \sum_{j=0}^{\infty} q^{-2jm} \text{vol}(V_{j,0}^0)$$

$$- q^{-(1+2m)}(\text{vol}(V_{0,0}^0) + q^3 \text{vol}(V_{1,0}^0)).$$

The sum in the middle is $q^{-d(1+2m)}I_{s,0}$. At $s = 0$, thus $m = -3/2$, the other two terms cancel each other by the equality above, and we obtain that $I_{0,d} = q^{2d}I_{0,0}$, is equal to

$$-q^{-2}[a_2]^{-2}(1 + q + q^2)(1 + 2\kappa(c_2/(a_2\pi))q^{2k+1})$$

$$= -q^{-2}(1 + q + q^2)[a_2]^{-1}(|a_2|^{-1} + 2\kappa(c_2/(a_2\pi))|c_2|^{-1}).$$

In our case $|\pi a_1c_1E|^{1/2} = q^{-1}$, $|a/\pi - 1| = |a_2|$, $|c/\pi - 1| = |c_2|$ and $|a/\pi - c/\pi| = |a_2c_1 - a_1c_2| = |a_2|$. Here $q$ is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is canceled by the factor $-q_{E}^{1/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, b, c)$. Hence $\xi_0(\pi a, 1, c)I_0(\pi a, c)$ is equal to

$$|c_2| - 2\kappa(c_2/a_2)|a_2|,$$

where $\kappa(c_2/(a_2\pi)) = -\kappa(c_2/a_2)$.

The $H$-side is equal to

$$-\kappa(a_2c_2)|a_2c_2 - a_1c_2| + \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|$$

$$= -2\kappa(a_2c_2)|a_2| + \kappa(a_2^2)|c_2| = |c_2| - 2\kappa(a_2c_2)|a_2|.$$

Note that $\kappa(c_2/a_2) = \kappa(a_2c_2)$. Theorem 2 follows for $n_1 = 1$, $n_2 = 0$, $a_1, c_1 \in R^X$, $|c_2| < 1.$
5.3.

We proceed to prove the theorem for \( n_1 = 1, n_2 = 0, a_1, c_1 \in \mathbb{R}^\times, |c_2| < 1 \) and \( |\pi a_2| \leq |c_2| \).

**Proof of Theorem 2.** Denote \( I_s(\pi a, c) \) by \( I_{s,d} \), where \( |c_2| = q^{-d} \). Recall that the integral \( I_{s,d} \) is equal to

\[
\sum_{n=0}^{\infty} q^{-2nm} \text{vol}(\mathcal{V}_{n,d}^{0}).
\]

Applying Lemma C.5.3, when \( |a_2\pi| < |c_2| \) and \( d = 1 \), we have

\[
(1 - q^{-1}) + q^3(q^{-1} + q^{-2} + q^{-4}) = q^2(1 + q^{-1} + q^{-2}).
\]

Applying Lemma C.5.3, when \( d \geq 2 \), we have

\[
1 - q^{-1} + q^3(q^{-1} - q^{-3} + q^{-4}) + (1 - q^{-2}) \sum_{n=2}^{d-1} q^{2n} + (1 + q^{-1})q^{2d}.
\]

This is equal to

\[
1 - q^{-1} + q^2 - 1 + q^{-1} + (q^{2d-2} - q^2) + (1 + q^{-1})q^{2d} = (1 + q^{-1} + q^{-2})q^{2d}.
\]

When \( |a_2\pi| = |c_2| \), we split this sum into two: a finite sum over \( n \), which we evaluate at once at \( s = 0 \) \( (m = -3/2) \), and an infinite sum, in which we first take big \( m \), to have convergence, then add up the sum and only then evaluate the result at \( m = -3/2 \). Set \( \alpha = q^{-2n(m+1)} \), thus when \( m = -3/2 \) we have \( \alpha = q \). When \( d = 1 \), we have

\[
(1 - q^{-1}) + q^3(q^{-1} + q^{-2} - q^{-3} - q^{-4} - q^{-5}) + (1 - q^{-2})(1 + q^{-1})^2 \sum_{n=2}^{\infty} \alpha^{1+n}.
\]

\[
= 1 - q^{-1} + q^2 + q^3 - 1 - q^{-1} - q^{-2} - q^2(1 + q^{-1})^3
\]

\[
= -(1 + q^{-1} + q^{-2})(1 + 2q).
\]

When \( d \geq 2 \), we have

\[
1 - q^{-1} + q^3(q^{-1} - q^{-3} + q^{-4}) + (1 - q^{-2}) \sum_{n=2}^{d-1} q^{2n}
\]

\[
+ (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2d} + (1 - q^{-2})(1 + q^{-1})^2 q^d \sum_{n=d+1}^{\infty} \alpha^n.
\]

This is equal to

\[
q^2 + (q^{d-2} - q^2) + (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2d} - (1 + q^{-1})^3 q^{2d}
\]

\[
= -(1 + q^{-1} + q^{-2})(1 + 2q)q^{2d-2}.
\]
Thus we see that, when \(|c_2| = q^{-d}, d \geq 0\), the integral \(I_{0,d}\) is equal to
\[
q^{-2}(1 + q + q^2)q^{2d} = q^{-2}(1 + q + q^2)|c_2|^{-2}
\]
when \(\|a_2\| < |c_2|\) and to
\[
-q^{-2}(1 + q + q^2)(1 + 2q)|2d|^{-2} = -q^{-2}(1 + q + q^2)(1 + 2|a_2||c_2|^{-1})|a_2|^{-2}
\]
when \(\|a_2\| = |c_2|\).

In our case \(|\pi a c|^{1/2} = q^{-1}, |a/\pi - 1| = |a_2|, |c/\pi - 1| = |c_2|\) and \(|a/\pi - c/\pi| = |a_2c_1 - a_1c_2|\). Here \(q\) is \(q_F = q^{1/2}\) so \(-q^{-3}(1 + q + q^2)\) is canceled by the factor \(-q^{1/2}(qE + qF^{1/2} + 1)^{-1}\) in \(\xi_0(a, b, c)\). Hence \(\xi_0(\pi a, 1, c)I_0(\pi a, c)\) is equal to
\[
-|a_2||c_2|^{-1}a_2c_1 - a_1c_2| = -|a_2|,
\]
when \(|a_2| < |c_2|\) (thus \(|a_2c_1 - a_1c_2| = |c_2|\)), to
\[
-|a_2||c_2|^{-1}a_2c_1 - a_1c_2 = -|a_2c_1 - a_1c_2|,
\]
when \(|a_2| = |c_2|\) and to
\[
|c_2| + 2|a_2|,
\]
when \(\|a_2\| = |c_2|\) (thus \(|a_2c_1 - a_1c_2| = |a_2|\)).

The \(H\)-side is equal to
\[
-k(a_2c_2)|a_2c_1 - a_1c_2| + k(a_2(a_2c_1 - a_1c_2))|c_2| - k(c_2(a_2c_1 - a_1c_2))|a_2|.
\]

When \(|a_2| < |c_2|\), thus \(k(a_2(a_2c_1 - a_1c_2)) = k(a_2c_2)\), and \(k(c_2(a_2c_1 - a_1c_2)) = k(c_2) = 1\), the \(H\)-side is equal to
\[
-k(a_2c_2)|c_2| + k(a_2c_2)|c_2| - k(c_2)|a_2| = -|a_2|.
\]

When \(|a_2| = |c_2|\), thus \(k(a_2(a_2c_1 - a_1c_2))|c_2| - k(c_2(a_2c_1 - a_1c_2))|a_2| = 0\), and \(k(a_2c_2) = 1\), the \(H\)-side is equal to
\[
-|a_2c_1 - a_1c_2|.
\]

When \(\|\pi a_2| = |c_2|\) we have that \(k(a_2c_2) = -1, k(a_2(a_2c_1 - a_1c_2)) = k(c_2) = 1\) and \(k(c_2(a_2c_1 - a_1c_2)) = k(c_2a_2) = 1\). Then the \(H\)-side is equal to
\[
|a_2c_1 - a_1c_2| + |c_2| + |a_2| = |c_2| + 2|a_2|.
\]

Theorem 2 follows for \(n_1 = 1, n_2 = 0, a_1, c_1 \in R^x, |c_2| < 1\) and \(\|\pi a_2| \leq |c_2|\).

6. Case of \(n_1 = 1, n_2 = 1\)

In this section \(|x| = |x| \cdot q_F, R| = R_F|. The quadratic forms are
\[
Q_1(x, y, z) = N(x) + a_1\pi N(y) + c_1\pi N(z),\quad Q_2(x, y, z) = a_2\pi N(y) + c_2\pi N(z),
\]
where without loss of generality \(|a_1| = 1\), max\(|c_1|, |c_2|\} = 1\) and, as in Sec. 3,
\[
V_n = \{(x, y, z) \in R_F; \|\langle x, y, z\rangle\|_F = 1, \max\{|Q_1(x, y, z)|, |Q_2(x, y, z)|\} = q^{-n}\}.
\]

Recall that \(u = a_1c_2/(a_2c_1)\). Then, we have the following possible cases:

(6.1) \(c_1, a_2/c_2, u \in R^x\) and \(|1 - u| = 1;\)
(6.2) \(c_1, a_2/c_2, u \in R^x\) and \(|1 - u| < 1;\)
We proceed to prove the theorem for $6.2$. 

The case $a_1, a_2, c_1 \in R^{\times}, |c_2| < 1$ can be reduced to that of case 6.3 on interchanging $a, c$.

6.1.

We proceed to prove the theorem for $n_1 = 1, n_2 = 1, a_1, c_1 \in R^{\times}, |u - 1| = 1$, where $u = a_1c_2/a_2c_1$.

Proof of Theorem 2. Denote $I_s(\pi a, \pi c)$ by $I_{s,d}$, where $|a_2| = |c_2| = q^{-d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^{\infty} q^{-2nm} \text{vol}(V_{n,d}^0).$$

By Lemma D.6.1, this sum is finite. Hence we may evaluate at $s = 0$ already at this stage. Thus we take $m = -3/2$. When $d = 0$, we have

$$\text{vol}(V_0^0) + q^3 \text{vol}(V_1^0) = 1 + q^3(q^{-2} + q^{-4}) = q^{-1}(1 + q + q^2).$$

By Lemma D.6.1, when $d \geq 1$, we have

$$1 + q^3(q^{-2} - q^{-3}) + (1 - q^{-2}) \sum_{n=2}^{d} q^{2n} + (1 + q^{-1})q^{2d+1}.$$ 

This is equal to

$$q + q^{-1}(q^{2d} - q^2) + (q + 1)q^{2d} = q^{-1}(1 + q + q^2)|a_2|^{-1}|c_2|^{-1}.$$ 

In our case $|\pi^2 a c_1|_E = q^{-2}$, $|a/c - 1| = |a_2|$, $|c/c - 1| = |c_2|$ and $|a/c - c/c| = |a_2 c_1 - a_1 c_2| = |a_2|$. Here $q$ is $q_E = q_1^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is canceled by the factor $-q_3^{3/2}(q_3 + q_3^{1/2} + 1)^{-1}$ in $\xi_0(a, 1, c)$. Hence $\xi_0(\pi a, 1, \pi c)I_0(\pi a, \pi c)$ is equal to $-|a_2|$. The $H$-side is equal to

$$\kappa(a_2 c_2 |a_2 c_1 - a_1 c_2| - \kappa(a_2 a_2 c_1 - a_1 c_2)|c_2| - \kappa(c_2 a_2 c_1 - a_1 c_2)|a_2|.$$ 

Since $|a_2| = |c_2|$, we have $\kappa(a_2 c_2) = \kappa(a_2 a_2 c_1 - a_1 c_2)) = 1$, so the above expression is equal to $-|a_2|$. This establishes the theorem for $n_1 = 1, n_2 = 1, a_1, c_1, a_2/c_2, u - 1 \in R^{\times}$. 

6.2.

We proceed to prove the theorem for $n_1 = 1, n_2 = 1, a_1, c_1 \in R^{\times}$ and $|u - 1| = q^{-2k-\varepsilon}$ $(k \geq 1, \varepsilon = 0, 1)$ and $u = a_1 c_2/a_2 c_1$. When $|a_2| = |c_2| = q^{-d} \leq 1$ $(d \geq 0)$
to emphasize as usual we denote $V_n$ by $V_{n,d}$. Similarly we denote $V_n^0$ by $V_{n,d}^0$, thus $\text{vol}(V_n^0)$ is $\text{vol}(V_n)/(1-q^2)$. It is computed in Lemma D.6.2.

**Proof of Theorem 2.** Denote $I_s(\pi_0, \pi c)$ by $I_{s,d}$, where $|a_2| = |c_2| = q^{-d}$. Recall that the integral $I_{s,d}$ is equal to

$$\sum_{n=0}^\infty q^{-2nm} \text{vol}(V_{n,d}^0).$$

We split this sum into two: a finite sum over $n$, which we evaluate at once at $s = 0$ ($m = -3/2$), and an infinite sum, in which we first take big $m$, to have convergence, then add up the sum and only then evaluate the result at $m = -3/2$.

Applying Lemma D.6.2, when $|1-u| = q^{-2k}$ and $d = 0$, we have

$$1 + q^3(q^{-2} - q^{-3}) + (1-q^{-2}) \left(1 + q^{-1} + q^{-2} \sum_{n=3}^{2k-1} q^n + \sum_{n=2}^{2k} q^{n+1}\right)$$

$$+(1 + q^{-1})q^{2k+1},$$

where the first sum is taken over the odd $n$ ($3 \leq n \leq 2k-1$), the second over the even $n$ ($2 \leq n \leq 2k$) and the third over all $n \geq 2k + 2$. The above expression equals to

$$q + (1 + q^{-1} + q^{-2})(q^{2k+1} - q) + (q^{2k-1} - q) + (1 + q^{-1})q^{2k+1}.$$ 

Once simplified, this is equal to

$$-q^{-3}(1 + q + q^2)(1 - 2q^{2k}).$$

Applying Lemma D.6.2, when $|1-u| = q^{-2k-1}$ and $d = 0$, we have the sum of

$$1 + q^3(q^{-2} - q^{-3}) + (1-q^{-2}) \left(1 + q^{-1} + q^{-2} \sum_{n=3}^{2k+1} q^n + \sum_{n=2}^{2k} q^{n+1}\right)$$

$$+(1 + q^{-1})q^{2k+1}$$

and

$$(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2k+3} + (1 - q^{-2})(1 + q^{-1})^2q \sum_{n=2k+3}^\infty q^{-2(m+1)},$$

where the first sum is taken over the odd $n$ ($3 \leq n \leq 2k+1$), the second over the even $n$ ($2 \leq n \leq 2k$) and the third over all $n \geq 2k + 2$. Taking the sum and evaluating at $m = -3/2$, the above expression equals to

$$q + (1 + q^{-1} + q^{-2})(q^{2k+1} - q) + (q^{2k+1} - q) + (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{2k+3}$$

$$-(1 + 3q^{-1} + 3q^{-2} + q^{-3})q^{2k+3}.$$ 

Once simplified, this is equal to

$$-q^{-1}(1 + q + q^2)(1 + 2q^{2k+1}).$$
When $d \geq 1$, by Lemma D.6.2, we have
\[
\sum_{n=0}^{d+1} q^{3n} \operatorname{vol}(V_{n,d}^0) = 1 + q^3(1 - q^{-2}) + (1 - q^{-2}) q^{-1} \sum_{n=2}^{d+1} q^{2n} = q^{2d+1}.
\]

Thus, when $d \geq 1$, we have
\[
\sum_{n=0}^{d+1} q^{3n} \operatorname{vol}(V_{n,d}^0) = q^{2d}(\operatorname{vol}(V_{0,0}^0) + q^3 \operatorname{vol}(V_{1,0}^0)).
\]

Moreover, since $\operatorname{vol}(V_{n,d}^0) = q^{-d} \operatorname{vol}(V_{n-d,0}^0)$ for $n \geq d + 2$, we apply same computations as in Sec. 5.2 to obtain that at $s = 0$, we have
\[
I_{0,d} = q^{2d} I_{0,0}.
\]

Thus we see that, when $d \geq 0$, $I_0(\pi a, \pi c) = I_{0,d}$ is equal to
\[
-q^{-1} (1 + q + q^2) (1 - \kappa(1 - u)[1 - u]^{-1}) |a_2|^{-1} |c_2|^{-1},
\]
where $\kappa(1 - u)$ is 1 when $|1 - u| = q^{-2k}$ and $-1$ when $|1 - u| = q^{-2k-1}$.

In our case $|\pi^2 ac|_E^{1/2} = q^{-2}$, $|a/\pi - 1| = |a_2|$, $|c/\pi - 1| = |c_2|$ and $|a/\pi - c/\pi| = |a_2 c_1 - a_1 c_2| = |a_2| |1 - u|$. Here $q$ is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is canceled by the factor $-q_E^{-3}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a, 1, c)$. Hence $\xi_0(\pi a, 1, \pi c) I_0(\pi a, \pi c)$ is equal to $|a_2 c_1 - a_1 c_2| - 2\kappa(1 - u) |a_2|$.

The $H$-side is equal to
\[
\kappa(a_2 c_2)(a_2 c_1 - a_1 c_2) - \kappa(a_2(a_2 c_1 - a_1 c_2)) |c_2| - \kappa(c_2(a_2 c_1 - a_1 c_2)) |a_2|.
\]
Note that $\kappa(a_2(a_2 c_1 - a_1 c_2)) = \kappa(1 - u) |a_2| |c_2| = |c_2|^{2}$ and so $\kappa(a_2 c_2) = 1$. Thus the expression above is equal to $|a_2 c_1 - a_1 c_2| - 2\kappa(1 - u) |a_2|$. This establishes the theorem for $n_1 = 1$, $n_2 = 1$, $a_1, c_1, a_2, c_2 \in R^\infty$ and $|1 - u| < 1$.

### 6.3.

We proceed to prove the theorem for $n_1 = 1$, $n_2 = 1$, $a_1, c_2 \in R^\infty$, $|c_1| \leq 1$, $|a_2| < 1$.

**Proof of Theorem 2.** The integral $I_1(\pi a, \pi c)$ is equal to
\[
\sum_{n=0}^{\infty} q^{-2n} \operatorname{vol}(V_n^0) = q^{-2} + \frac{1}{q^4} \operatorname{vol}(V_1^0).
\]

Applying Lemma D.6.3, the sum is $\operatorname{vol}(V_0^0) + \frac{1}{q^4} \operatorname{vol}(V_1^0)$. When $s = 0$, thus $m = -3/2$, this equals
\[
1 + (q^{-2} + q^{-4}) (q^{-2}) = q^{-1}(1 + q + q^2).
\]

Moreover $|\pi^2 ac|_E^{1/2} = q^{-2}$, $|a/\pi - 1| = |a_2|$, $|c/\pi - 1| = |c_2| = 1$ and $|a/\pi - c/\pi| = |a_2 c_1 - a_1 c_2| = 1$. Here $q$ is $q_F = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is canceled by the factor $-q_E^{-3}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(\pi a, 1, \pi c)$. Hence $\xi_0(\pi a, 1, \pi c) I_0(\pi a, \pi c) = -|a_2|$.
The $H$-side is equal to
\[ \kappa(a_2c_2)|a_2c_1 - a_1c_2| - \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|. \]

Note that $\kappa(a_2(a_2c_1 - a_1c_2)) = \kappa(a_2c_2) = \kappa(a_2)$, and $\kappa(c_2(a_2c_1 - a_1c_2)) = 1$, thus the $H$-side is equal to $-|a_2|$. This establishes the theorem for $n_1 = 1$, $a_1$, $c_2 \in R^*$, $|a_2| < 1$. □

6.4.

We proceed to prove the theorem for $n_1 = 1$, $n_2 = 1$, $a_1$, $c_2 \in R^*$, $|a_2| < 1$, $|c_2| < 1$ and $|a_2| \neq |c_2|$. Set $|c_2| = q^{-k}$.

**Proof of Theorem 2.** The integral $I_4(\pi a, \pi c)$ is equal to
\[ \sum_{n=0}^{\infty} q^{-2kn} \text{vol}(V_n^0). \]

Applying Lemma D.6.4, when $s = 0$, thus $m = -3/2$, this equals
\[ \sum_{n=0}^{k+1} q^{3n} \text{vol}(V_n^0). \]

When $|c_2| = q^{-k}$, we have the sum
\[ 1 + q^3(q^{-2} - q^{-3}) + (1 - q^{-2})q^{-1} \sum_{n=2}^{k} q^{2n} + (1 + q^{-1})q^{-1} q^{2(k+1)}, \]

which is equal to
\[ q + (q^2 - 1)q^{-3}q^4(q^{2k-2} - 1)(q^2 - 1)^{-1} + (1 + q^{-1})q^{-1} q^{2k+2} \]
\[ = q^{2k}q^{-1}(1 + q + q^2) = q^{-1}(1 + q + q^2)|c_2|^{-2}. \]

Moreover $|\pi^2 a c|^1/2 = q^{-2}$, $|a/\pi - 1| = |a_2|$, $|c/\pi - 1| = |c_2|$ and $|a/\pi - c/\pi| = |a_2c_1 - a_1c_2| = |c_2|$ (since we assumed $|a_2| < |c_2|$). Here $q = q_E = q_E^{1/2}$, so $-q^{-3}(1 + q + q^2)$ is canceled by the factor $-q_E^{1/2}(q_E + q_E^{1/2} + 1)^{-1}$ in $\xi_0(a_1, \pi c)$, hence $\xi_0(\pi a, 1, \pi c)I_4(\pi a, \pi c) = -|a_2|$. The $H$-side is equal to
\[ \kappa(a_2c_2)|a_2c_1 - a_1c_2| - \kappa(a_2(a_2c_1 - a_1c_2))|c_2| - \kappa(c_2(a_2c_1 - a_1c_2))|a_2|. \]

Note that $|a_2c_1 - a_1c_2| = |c_2|$, and $\kappa(c_2(a_2c_1 - a_1c_2)) = 1$, thus the $H$-side is equal to $-|a_2|$. This establishes the theorem for $n_1 = 1$, $n_2 = 1$, $a_1$, $c_2 \in R^*$, $|a_2| < 1$, $|c_2| < 1$ and $|a_2| \neq |c_2|$. □
6.5.
We proceed to prove the theorem for \( n_1 = 1, n_2 = 1, a_2, c_2 \in R^\times, |c_1| < 1 \).

**Proof of Theorem 2.** According to Lemma D.6.5, the proof is identical to that of Sec. 4.3. This establishes the theorem for \( n_1 = 1, n_2 = 1, a_2, c_2 \in R^\times, |c_1| < 1 \). \( \square \)

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**Appendix A. Auxiliary Lemmas**

In this section, we state and prove various auxiliary lemmas which are used in the proof.

**Lemma A.1.** When \( E/F \) is unramified, \( c \in R^\times, n \geq 1 \) and \( 0 \leq 2k < n \), we have
\[
\text{vol}_{d_2}(\{z \in E; |N(z) - c\pi^{2k}| \leq q^{-n}\}) = q^{-n}(1 + q^{-1})
\]
and \( \text{vol}_{d_2}(\{z \in E; |N(z) - c\pi^{2k}| = q^{-n}\}) = q^{-n}(1 - q^{-2}) \).

**Proof.** Using the exact sequence \( 1 \to E^1 \to R_E^\times \to R^\times \to 1 \) we see that \( \text{vol}_{d_2}(E^1) \) is \( 1 + q^{-1} \). Hence \( \text{vol}_{d_2}(\{z \in E; |N(z) - 1| \leq q^{-n}\}) \) is \( (1 + q^{-1})q^{-n} \). This deals with the case of \( k = 0 \).
\[
\int_{\{z \in E; |N(z) - c\pi^{2k}| \leq q^{-n}\}} dz = q^{-2k} \int_{\{z \in E; |N(z) - c| \leq \pi^{n-2k}\}} dz = q^{-n}(1 + q^{-1}),
\]
as required. \( \square \)

**Lemma A.2.** When \( E/F \) is unramified, \( c \in R^\times, k \geq 1 \). Let \( u \in R^\times \) such that \( |u - 1| = q^{-k} \). Then we have
\[
\text{vol}_{d_2}(\{z \in E; |N(z) + cu| = q^{-k}, |N(z) + c| = q^{-k-t}\}) = q^{-k-t}(1 - q^{-2}),
\]
when \( t \geq 1 \), and
\[
\text{vol}_{d_2}(\{z \in E; |N(z) + cu| = q^{-k}, |N(z) + c| = q^{-k}\}) = q^{-k}(1 - (q + 2)q^{-2}).
\]

**Proof.** Let \( u = 1 + \pi^k\varepsilon \), where \( \varepsilon \in R^\times \). According to Lemma A.1, we have
\[
\text{vol}_{d_2}(\{z \in E; |N(z) + cu| = q^{-k}\}) = q^{-k}(1 - q^{-2}).
\]
Thus the equality
\[ q \geq t \]
\[ \text{Definition.} \]
\[ \text{For any integer } n, \text{ we set } \sigma(n) \text{ to be 0 if } n \text{ is even and 1 if } n \text{ is odd.} \]

Our method of computation of volumes is that of “divide and rule”. A first example is the following.

Lemma A.3. When \( E/F \) is unramified and \( n \geq j \geq 1 \), the volume of the set
\[ \{ (x, y, z) \in R_E^3; |N(y)| = 1, |N(y) + N(z)| \leq q^{j-n}, |N(x) + N(y) + N(z)| = q^{-n} \}, \]
is equal to
\[ (1 - q^{-2})^2 q^{j-\sigma(n-j)-2n}. \]

Proof. The volume of this set is given by the integral
\[ \int_{|N(y)|=1} \int_{|N(y)+N(z)| \leq q^{-n}} \int_{|N(x)+c|=q^{-n}} dx \, dz \, dy, \]
where \( c = c(y, z) = N(y) + N(z) \). It is equal to
\[ \sum_{t=n-j}^{\infty} \int_{|N(y)|=1} \int_{|N(y)+N(z)| = q^{-t}} \int_{|N(x)+c_t|=q^{-n}} dx \, dz \, dy, \]
where \( |c_t| = q^{-t} \). Note that the integral over \( x \) depends only on \( |c_t| \). Thus we can assume that \( c_t = \pi^tc_0 \), where \( c_0 \) is an arbitrary element of \( R^\times \) and take the integral over \( y \) and \( z \), so that it is now equal to
\[ (1 - q^{-2})^2 \sum_{t=n-j}^{\infty} q^{-t} \int_{|N(x)+c_t|=q^{-n}} dx. \]
When $t \geq n+1$, thus $|c_t| < q^{-n}$ and the volume of the set $|N(x)| = q^{-n}$ is equal to $q^{-n}(1 - q^{-2})$ when $n$ is even and to zero when $n$ is odd. The contribution from this term is zero when $n$ is odd and

$$(1 - q^{-2})^2(1 + q^{-1})q^{-1-2n}.$$ 

When $n$ is even. When $t = n$, the volume of the set $|N(x) + c_n| = q^{-n}$ is equal to $q^{-n}(1 - (q + 1)q^{-2})$ when $n$ is even and to $q^{-n-2}$ when $n$ is odd. Thus, the contribution of all terms with $t \geq n-1$ is the product of $(1 - q^{-2})^2$ and

$$(1 - (1 + q)q^{-2})q^{-2n} + (q^{-1} + q^{-2})q^{-2n} = q^{-2n}$$

when $n$ is even and $q^{-2n-1}$ when $n$ is odd.

When $n - j \leq t \leq n - 1$, the volume of the set $|N(x) + c_t| = q^{-n}$ is equal to zero when $t$ is odd and to $q^{-n}(1 - q^{-2})$ when $t$ is even. Thus, when $n$ is even, the even values of $t (n - j \leq t \leq n - 2)$ are of the form $n - j + \sigma(n-j), \ldots, n-4, n-2$, and the sum is

$$q^{-n}(q^2 + q^4 + \cdots + q^{j-\sigma(n-j)}) = q^{-n}(1 - q^{-2})^{-1}(q^{j-\sigma(n-j)} - 1).$$

When $n$ is odd, the even values of $t (n - j \leq t \leq n - 1)$ are of the form $n - j + \sigma(n-j), \ldots, n-3, n-1$, and the sum is

$$q^{-n}(q^1 + q^3 + \cdots + q^{j-\sigma(n-j)}) = q^{-n-1}(1 - q^{-2})^{-1}(q^{j-\sigma(n-j)+1} - 1).$$

Thus, the contribution of all terms with $n - j \leq t \leq n - 2$ is the product of $(1 - q^{-2})^2$

and

$$q^{-2n}(q^{j-\sigma(n-j)} - 1)$$

when $n$ is even and

$$q^{-2n-1}(q^{j-\sigma(n-j)-1} + 1)$$

when $n$ is odd. The lemma follows.

**Lemma A.4.** When $E/F$ is unramified and $n \geq j \geq 1$, the volume of the set

$$\{(x, y, z) \in R^3_E; |N(y)| = 1, |N(y) + N(z)| \leq q^{2-n}, |N(x) + \pi(N(y) + N(z))| = q^{-n}\}$$

is equal to

$$(1 - q^{-2})^2q^{j+\sigma(n-j)-2n-1}.$$ 

**Proof.** The volume of this set is given by the integral

$$\int_{|N(y)|=1} \int_{|N(y)+N(z)| \leq q^{-n}} \int_{|N(x) + \pi c| = q^{-n}} dx dz dy,$$

where $c = c(y, z) = \pi(N(y) + N(z))$. It is equal to

$$\sum_{t=n-j}^{\infty} \int_{|N(y)|=1} \int_{|N(y)+N(z)| = q^{-t}} \int_{|N(x) + \pi c| = q^{-n}} dx dz dy,$$
where $|c_t| = q^{-t-1}$. Note that the integral over $x$ depends only on $|c_t|$. Thus we can assume that $c_t$ is an arbitrary element so that $|c_t| = q^{-t-1}$ and take the integral over $y$ and $z$, so that it is now equal to

$$(1 - q^{-2})^2 \sum_{t=n-j}^{\infty} q^{-t} \int_{|N(x) + c_t| = q^{-n}} dx.$$  

When $t \geq n$, thus $|c_t| < q^{-n}$ and the volume of the set $|N(x) + c_t| = q^{-n}$ is equal to $q^{-n}(1 - q^{-2})$ when $n$ is even and to zero when $n$ is odd. The contribution from this term is zero when $n$ is odd and

$$(1 - q^{-2})^2(1 + q^{-1})q^{-2n}$$

when $n$ is even. When $t = n - 1$, the volume of the set $\{|N(x) + c_{n-1}| = q^{-n}\}$ is equal to $q^{-n}(1 - (q + 1)q^{-2})$ when $n$ is even and to $q^{-n-1}$ when $n$ is odd. Thus, the contribution of all terms with $t \geq n - 1$ is the product of $(1 - q^{-2})^2$ and

$$(1 - (1 + q)q^{-2})q^{1-2n} + (q^{-1} + q^{-2})q^{1-2n} = q^{1-2n}$$

when $n$ is even and $q^{-2n}$ when $n$ is odd.

When $n - j \leq t \leq n - 2$, the volume of the set $\{|N(x) + c_t| = q^{-n}\}$ is equal to zero when $t$ is even and to $q^{-n}(1 - q^{-2})$ when $t$ is odd. Thus, when $n$ is even, the odd values of $t$ ($n - j \leq t \leq n - 2$) are of the form $n - j + 1 - \sigma(n - j), \ldots, n - 5, n - 3$, and the sum is

$q^{-n}(q^3 + q^5 + \cdots + q^{j+\sigma(n-j)}) = q^{1-n}(1 - q^{-2})^{-1}(q^{i+\sigma(n-j)} - 1).$

When $n$ is odd, the odd values of $t$ ($n - j \leq t \leq n - 2$) are of the form $n - j + 1 - \sigma(n - j), \ldots, n - 4, n - 2$, and the sum is

$q^{-n}(q^3 + q^4 + \cdots + q^{j+\sigma(n-j)}) = q^{-n}(1 - q^{-2})^{-1}(q^{i+\sigma(n-j)} - 1).$

Thus, the contribution of all terms with $n - j \leq t \leq n - 2$ is the product of $(1 - q^{-2})^2$ and

$q^{1-2n}(q^{i+\sigma(n-j)} - 1),$

when $n$ is even and

$q^{-2n}(q^{i+\sigma(n-j)} - 1)$

when $n$ is odd. The lemma follows. 

\[ \square \]

**Lemma A.5.** When $E/F$ is unramified, the volume of the set 

$$\{(x, y, z) \in R_E^3 : \| (x, y, z) \|_E = 1, |N(x) + N(y) + N(z)| = q^{-n}\}$$

is equal to

$$(1 - q^{-2})^2 q^{-n}(1 - q^{-1} + q^{-2})$$

when $n \geq 1$ and to

$$(1 - q^{-2})(1 - q^{-1} + q^{-2})$$

when $n = 0$. 

Proof. Consider the case of $n \geq 1$. Then this set is the union of the following subsets according to $|N(x)| = 1$, and three subsets corresponding to $|N(x) + N(y)| > q^{-n}$, $|N(x) + N(y)| = q^{-n}$ and $|N(x) + N(y)| < q^{-n}$, and the subset which corresponds to $|N(x)| < 1$.

(A) Case of $|N(x)| = 1$ and $|N(x) + N(y)| > q^{-n}$. Note that when $|N(x) + N(y)| = q^{-j} > q^{-n}$, then $j$ should be even and the volume of this subset is given by the following integral

$$
\sum_{k=0}^{(n+\sigma(n))/2-1} \int_{|N(x)|=1} \int_{|N(x) + N(y)|=q^{-2k}} \int_{|N(x) + c| = q^{-n}} dzdxdy,
$$

where $c = N(x) + N(y)$. Note that when $k = 0$, the integral over $y$ runs over the set $|N(x) + N(y)| = 1$ is equal to $(1 - (q + 1)q^{-2})$. When $k \geq 5$, applying Lemma A.1, the sum over $k$ is equal to

$$
(1 - q^{-2})^3 q^{-n} \left( \sum_{k=1}^{(n+\sigma(n))/2-1} q^{-2k} \right) = (1 - q^{-2})^2 q^{-n} (q^{-2} - q^{-n+\sigma(n)})
$$

and to $(1 - q^{-2})^3 q^{-n} \times q^{-2}$ when $n \in \{3, 4\}$. Note that this sum is empty when $n \in \{1, 2\}$. When $n \geq 5$, the value of the integral is equal to

$$
(1 - q^{-2})^2 q^{-n} (1 - q^{-1} - q^{-2}) + (1 - q^{-2})^2 q^{-n} (q^{-2} - q^{-n+\sigma(n)}).
$$

When $n = 3$ or $4$, this integral is

$$
(1 - q^{-2})^2 q^{-n} (1 - q^{-1} - q^{-2}) + (1 - q^{-2})^3 q^{-n} \times q^{-2}.
$$

When $n = 1$ or $2$, this integral is

$$
(1 - q^{-2})^2 q^{-n} (1 - q^{-1} - q^{-2}).
$$

(B) Case of $|N(x)| = 1$ and $|N(x) + N(y)| = q^{-n}$ and $n \geq 1$. The volume of this subset is given by the following integral

$$
\int_{|N(x)|=1} \int_{|N(x) + N(y)|=q^{-n}} \int_{|N(x) + c| = q^{-n}} dzdxdy,
$$

where $c = N(x) + N(y)$. Applying Lemma A.1, when $n$ is even this integral is equal to

$$
(1 - q^{-2})^2 q^{-2n} (1 - q^{-1} - q^{-2}).
$$

When $n$ is odd the integral over $z$ runs over the set $|N(z)| \leq q^{-n-1}$, and this integral is equal to

$$
(1 - q^{-2})^2 q^{-2n-1}.
$$

(C) Case of $|N(x)| = 1$ and $|N(x) + N(y)| < q^{-n}$ and $n \geq 2$. The volume of this subset is given by the following integral

$$
\int_{|N(x)|=1} \int_{|N(x) + N(y)| \leq q^{-n-1}} \int_{|N(x)| = q^{-n}} dzdxdy.
$$
Applying Lemma A.1, this integral is equal to

$$(1 - q^{-2})^2 (1 + q^{-1}) q^{-2n-1}.$$

When $n$ is odd the contribution from this subcase is zero.

(D) Case of $|N(x)| < 1$ and $n \geq 1$. The volume of this subset is given by the following integral

$$\int_{|N(x)| < 1} \int_{|N(y)| = 1} \int_{|N(z) + c| = q^{-n}} dz \, dy \, dx,$$

where $|c| = |N(x) + N(y)| = 1$. Applying Lemma A.1, this integral is equal to

$$(1 - q^{-2})^2 q^{-n}.$$

Thus the contribution from cases (B), (C) and (D) is equal to

$$(1 - q^{-2})^2 q^{-2n} + (1 - q^{-2})^2 q^{-2n-1}.$$

Adding the contribution from (A) when $n = 2, n = 4$ and $n \geq 6$, the lemma follows.

When $n \geq 5$ is odd, we have the contribution from (A), (B) and (D) is the product of $(1 - q^{-2})^2 q^{-n}$ and

$$(1 - q^{-1} - q^{-2}) + (q^{-2} - q^{-n-1}) + q^{-n-1} + q^{-2} = 1 - q^{-1} + q^{-2}.$$

When $n = 3$, the contribution from (A), (B) and (D) is the product of $(1 - q^{-2})^2 q^{-n}$ and

$$(1 - q^{-1} - q^{-2}) + (1 - q^{-2}) q^{-2} + q^{-4} + q^{-2}.$$

When $n = 1$, the contribution from (A), (B) and (D) is

$$(1 - q^{-2})^2 q^{-n} (1 - q^{-1} - q^{-2} + q^{-2} + q^{-2}).$$

Consider the case of $n = 0$. Then this set is the union of the following subsets according to $|N(x) + N(y)| < 1$ and $|N(x) + N(y)| = 1$. The corresponding integral is equal to

$$\int_{|N(x)| < 1} \int_{|N(y)| < 1} \int_{|N(z)| = 1} dz \, dy \, dx + \int_{|N(x)| = 1} \int_{|N(y) + N(y)| < 1} \int_{|N(z)| = 1} dz \, dy \, dx$$

$$+ \int_{|N(x)| < 1} \int_{|N(y)| = 1} \int_{|N(z) + c| = 1} dz \, dy \, dx$$

$$+ \int_{|N(x)| = 1} \int_{|N(y) + N(y)| = 1} \int_{|N(z) + c| = 1} dz \, dy \, dx,$$

where $c = N(x) + N(y)$ so that $|c| = 1$. This is equal to the product of $1 - q^{-2}$ and

$$(1 - q^{-2}) q^{-1} (1 + q^{-1}) + (1 - q^{-1} - q^{-2}) (q^{-2} + 1 - q^{-1} - q^{-2})$$

$$= 1 - q^{-1} + q^{-2}.$$

The lemma follows.
Appendix B. Section 4 Computations

This section contains the computations used in Sec. 4.

Lemma B.4.1. Suppose that \( c_1, u, u - 1 \) are units in \( R^\times \) and \( |a_2| = |v_2| = q^{-d} \), where \( d \geq 0 \). Then

\[
\text{vol}(V^0_{n,d}) = \begin{cases} 
1 - 2q^{-3} & \text{if } n = 0, \quad d = 0, \\
1 - q^{-1} + q^{-2} & \text{if } n = 0, \quad d \geq 1, \\
(1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}) & \text{if } 1 \leq n \leq d - 1, \\
(1 - q^{-2} - q^{-3} - q^{-4})q^{-n} & \text{if } n = d, \quad d \geq 1, \\
(1 - q^{-2})(1 + q^{-1})^2q^{d-2n} & \text{if } n \geq d + 1.
\end{cases}
\]

Proof. Making a change of variables \( y' = \alpha y \) where \( \alpha \in R^\times_E \) such that \( N(\alpha) = a_1 \) and renaming \( y' \) back to \( y \), without loss of generality we can assume that \( a_1 = 1 \).

Similarly, we can assume that \( c_1 = 1 \). Thus we have that \( Q_1 = Q_1(x, y, z) = N(x) + N(y) + N(z) \) and \( Q_2 = Q_2(x, y, z) = a_2a_1^{-1}N(y) + c_2c_1^{-1}N(z) \). As \( |a_2/a_1| = |c_2/c_1| = q^{-d} \), and we are interested only in \( |Q_2| \), we can assume that \( Q_2(x, y, z) = \pi^d(N(y) + uN(z)) \), where \( u \in R^\times \) has \( |u - 1| = 1 \).

We first compute \( \text{vol}(V^0_{n,d}) \). Here

\[
V_{0,d} = \{(x, y, z) \in R^3_E; \max\{|N(x) + N(y) + N(z)|, |\pi^d(N(y) + uN(z))|\} = 1\}.
\]

When \( d = 0 \), we compute the volume of the complement of \( V_{0,0} \) in \( \{v \in R^3_E; ||v||_E = 1\} \):

\[
\{(x, y, z) \in R^3_E; ||(x, y, z)||_E = 1, |N(x) + N(y) + N(z)| < 1, |N(y) + uN(z)| < 1\}.
\]

Note that if \( |N(x)| < 1 \) then \( \max\{|N(y)|, |N(z)|\} = 1 \) implies

\[
\max\{|N(x) + N(y) + N(z)|, |N(y) + uN(z)|\} = 1.
\]

Thus \( |N(x)| = 1 \), and consequently \( |N(y)| = |N(z)| = 1 \). Thus the volume of the complement is given by the integral

\[
\int_{|N(x)|=1} \int_{|N(y)+uN(z)|<1} \int_{|N(x)+c|<1} dx dy dz,
\]

where \( c = c(y, z) = N(y) + N(z) \) and \( |c| = 1 \). From Lemma A.1 it follows that the volume of the subset \( \{|N(x) + c| < 1\} \) where \( |c| = 1 \) is equal to \( (q + 1)/q^2 = q^{-1}(1 + q^{-1}) \). The above integral is equal to

\[
1 - 2q^{-3}q^{-2}(1 + q^{-1})^2.
\]

As the volume of \( \{v \in R^3_E; ||v||_E = 1\} \) is \( (1 - q^{-2})(1 + q^{-2} + q^{-4}) \), the volume of \( V_{0,0} \) equals to the product of \( 1 - q^{-2} \) and the difference

\[
1 + q^{-2} + q^{-4} - q^{-2}(1 + q^{-1})^2 = 1 - 2q^{-3}.
\]
When $d > 0$, we have

$$V_{0,d} = \{(x,y,z) \in R^3_E; |N(x) + N(y) + N(z)| = 1\}.$$  

To compute its volume, we apply Lemma A.5.

To compute $\text{vol}(V_{n,d}^0) \ (1 \leq n \leq d-1)$ recall that since $|\pi^d(N(y)+uN(z))| < q^{-n}$, we have that

$$V_n = \{(x,y,z) \in R^3_E; |N(x) + N(y) + N(z)| = q^{-n}, \|(x,y,z)\|_E = 1\}.$$  

To compute its volume we apply Lemma A.5.

We will compute the volume of each subset.

(A) Case of $|N(x) + N(y) + N(z)| \leq q^{-n-1}$ and $|N(y) + uN(z)| = 1$. We consider the subcases of $|N(z)| = 1$ and $|N(z)| < 1$. When $|N(z)| = 1$ then $|N(y) + N(z)|$ can be equal or less than 1. Moreover the condition $|N(y) + N(z)| < 1$ implies that $|N(y) + uN(z)| = 1$. The volume of the set $\{|N(y) + N(z)| = 1, |N(y) + uN(z)| = 1\}$, when $|1 - u| = 1$ is equal to $1 - 2(q+1)q^{-2}$. Thus, the contribution from $|N(z)| = 1$ is given by two integrals.

The first integral is

$$\int_{|N(z)|=1} \int_{|N(y) + N(z)| \leq q^{-n-1}} \int_{|N(x) + N(y) + N(z)| \leq q^{-n-1}} dxdydz.$$  

Applying Lemma A.1 to the integral over $y$ and Lemma A.3, with $j = n - 1$, to the integral over $x$, this is equal to the infinite sum

$$(1 - q^{-2})^2q^{-2} \sum_{m=n+1}^{\infty} q^{-m} = (1 - q^{-2})q^{-n-3}(1 + q^{-1}).$$  

The second integral is

$$\int_{|N(z)|=1} \int_{|N(y) + N(z)|=1} \int_{|N(x) + N(y) + N(z)| \leq q^{-n-1}} dxdydz.$$  

Similarly to the first integral, applying Lemma A.3, with $j = n - 1$, to the integral over $x$, and taking the infinite sum, its value is

$$(1 - q^{-2})q^{-n-1}(1 + q^{-1})(1 - 2q^{-1} - 2q^{-2}).$$
The contribution from $|N(z)| < 1$ is given by the integral
\[ \int_{|N(z)|<1} \int_{|N(y)|=1} \int_{|N(x)+N(y)+N(z)| \leq q^{-r-1}} dxdydz, \]
which is equal to
\[ (1 - q^{-2})q^{-n-3}(1 + q^{-1}). \]

(B) Case of $|N(x) + N(y) + N(z)| = q^{-n}$ and $|N(y) + uN(z)| \leq 1$. Note that $|N(y) + uN(z)| \leq 1$ is always satisfied, and according to Lemma A.5, the volume of this set is equal to
\[ (1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}). \]

Adding the contributions from cases (A) and (B), then dividing by $1 - q^{-2}$, we obtain the product of $q^{-n}$ and
\[ 2(q^{-3} + q^{-4}) + (q^{-1} + q^{-2})(1 - 2q^{-1} - 2q^{-2}) + (1 - q^{-2})(1 - q^{-1} + q^{-2}), \]
which is equal to
\[ 1 - q^{-2} - q^{-3} - q^{-4}. \]

To compute $\text{vol}(V_{n,d}^n)$ ($n \geq d + 1$) recall that $V_{n,d}$ is the set of $(x, y, z) \in R^3_E$ with $\|(x, y, z)\|_E = 1$ and
\[ \max\{|N(x) + N(y) + N(z)|, |\pi^d(N(y) + uN(z))|\} = q^{-n}. \]

Assume that $|N(z)| < 1$. Then $|N(y) + uN(z)| < 1$ implies that $|N(y)| < 1$ and from $|N(x) + N(y) + N(z)| < 1$ it follows that $|N(x)| < 1$ which is a contradiction. Thus $|N(z)| = 1$ and $V_{n,d}$ is the disjoint union of two subsets
\[ \{(x, y, z) \in R^3_E; |N(z)| = 1, |N(x) + N(y) + N(z)| \leq q^{-n}, \]
\[ |N(y) + uN(z)| = q^{d-n}\} \]
and
\[ \{(x, y, z) \in R^3_E; |N(z)| = 1, |N(x) + N(y) + N(z)| = q^{-n}, \]
\[ |N(y) + uN(z)| \leq q^{d-n-1}\}. \]

We will compute the volume of each of them.
Note that since $|u - 1| = 1$ and $|N(y) + uN(z)| < 1$, we have
\[ |N(y) + N(z)| = |N(y) + uN(z) + (1 - u)N(z)| = |(1 - u)N(z)| = 1. \]
(A) Case of $|N(x) + N(y) + N(z)| \leq q^{-n}$ and $|N(y) + uN(z)| = q^{d-n}$. The volume of this subset is equal to the integral
\[
\int_{|N(z)|=1} |N(y)+uN(z)|=q^{d-n} \int_{|N(x)+c|\leq q^{-n}} dx dy dz,
\]
where $c = c(y,z) = N(y) + N(z)$ so that $|c| = 1$. By Lemma A.1 the integral is equal to
\[(1 - q^{-2}) \times (1 - q^{-2})q^{d-n} \times (1 + q^{-1})q^{-n}.
\]

(B) Case of $|N(x) + N(y) + N(z)| = q^{-n}$ and $|N(y) + uN(z)| \leq q^{d-n-1}$. The volume of this subset is given by the integral
\[
\int_{|N(z)|=1} |N(y)+uN(z)|\leq q^{d-n-1} \int_{|N(x)+c|=q^{-n}} dx dy dz,
\]
where again $c = c(y,z) = N(z) + N(y)$ and $|c| = 1$. Applying Lemma A.1 and its corollary the above integral is equal to
\[(1 - q^{-2}) \times (1 + q^{-1})q^{d-n-1} \times (1 - q^{-2})q^{-n}.
\]

Adding the expressions of (A) and (B), then dividing by $(1 - q^{-2})$, the vol($V_{n,d}^0$) is equal to
\[(1 - q^{-2})(1 + q^{-1})^2q^{d-2n}.
\]

The lemma follows.

\textbf{Lemma B.4.2.} Suppose that $c_1, u, a_2$ are units in $R^\times$, and $|a_2| = |c_2| = q^{-d}$, where $d \geq 0$. When $|u - 1| = q^{-2k}$ ($k \geq 1$), we have

\[
\text{vol}(V_{n,d}^0) = \begin{cases} 
1 + q^{-2} - q^{-3} & \text{if } n = 0, \ d = 0, \\
1 - q^{-1} + q^{-2} & \text{if } n = 0, \ d > 0, \\
(1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}) & \text{if } 1 \leq n < d, \\
(1 - q^{-4})q^{-n} & \text{if } n = d, \ d > 0, \\
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d-2n} & \text{if } d < n < 2k + d, \ 2 \mid n - d, \\
(1 - q^{-2})q^{d-2n-1} & \text{if } d < n < 2k + d, \ 2 \nmid n - d, \\
(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{d-2n} & \text{if } n = 2k + d, \\
(1 - q^{-2})(1 + q^{-1})^2q^{d-2n} & \text{if } n > 2k + d.
\end{cases}
\]
When $|u - 1| = q^{-2k-1}(k \geq 1)$, we have
\[
\text{vol}(V_{n,a}^0) = \begin{cases} 
1 + q^{-2} - q^{-3} & \text{if } n = 0, \quad d = 0, \\
1 - q^{-1} + q^{-2} & \text{if } n = 0, \quad d > 0, \\
(1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}) & \text{if } 1 \leq n < d, \\
(1 - q^{-4})q^{-n} & \text{if } n = d, \quad d > 0, \\
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d-2n} & \text{if } d < n < 2k + d + 1, \quad 2 \nmid n-d, \\
(1 - q^{-2})q^{d-2n-1} & \text{if } d < n < 2k + d + 1, \quad 2 \mid n-d, \\
(1 + q^{-1})q^{d-2n-1} & \text{if } n = 2k + 1 + d, \\
0 & \text{if } n > 2k + 1 + d.
\end{cases}
\]

**Proof.** Making a change of variables $y' = \alpha y$ where $\alpha \in R^*_E$ such that $N(\alpha) = a_1$ and renaming $y'$ back to $y$, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. Recall that $u = (a_1c_2)/(a_2c_1)$. Thus we have that $Q_1(x, y, z) = N(x) + N(y) + N(z)$ and $Q_2(x, y, z) = \frac{N(y) + uN(z)}{2}$ as $a_2/a_1$ is assumed to be a unit, we can assume that $Q_2(x, y, z) = N(y) + uN(z)$, where $u \in R^*_E$ has $|u-1| = 1/q^{2k+\varepsilon}$, where $2k + \varepsilon \geq 1$ and $\varepsilon = 0, 1$.

To compute $\text{vol}(V_{0,a}^0)$, when $d = 0$, recall that
\[
V_{0,0} = \{(x, y, z) \in R^*_E; \max\{|N(x) + N(y) + N(z)|, |N(y) + uN(z)|\} = 1\}.
\]
(Note that the condition $\|\{x, y, z\}\|_E = 1$ is satisfied.) This set is the disjoint union of two subsets
\[
\{(x, y, z) \in R^*_E; |N(x) + N(y) + N(z)| \leq 1, |N(y) + uN(z)| = 1\}
\]
and
\[
\{(x, y, z) \in R^*_E; |N(x) + N(y) + N(z)| = 1, |N(y) + uN(z)| < 1\}.
\]

We will compute the volume of each of these subsets.

(A) Case of $|N(x) + N(y) + N(z)| \leq 1$ and $|N(y) + uN(z)| = 1$. Note that since $|1 - u| < 1$ the equality $|N(y) + uN(z)| = 1$ is equivalent to $|N(y) + N(z)| = 1$ whereas the inequality becomes $|N(x)| \leq 1$. Then the volume of subset (A) is given by the integral
\[
\left(\int_{|N(y)| < 1} 1_{|N(z)| = 1} + \int_{|N(y)| = 1} 1_{|N(y) + N(z)| = 1}\right) \int_{|N(x)| \leq 1} 1 \, dx \, dy \, dz.
\]
This is equal to
\[
q^{-2}(1 - q^{-2}) + (1 - q^{-2})(1 - (q + 1)q^{-2}) = (1 - q^{-2})(1 - q^{-1}).
\]
(B) Case of $|N(x) + N(y) + N(z)| = 1$ and $|N(y) + uN(z)| < 1$. Since $|1 - u| < 1$ we have
\[
|N(y) + N(z)| = |N(y) + uN(z) + (1 - u)N(z)| \\
\leq \max\{|N(y) + uN(z)|, |1 - u||N(z)|\} < 1,
\]
and thus $|N(x)| = 1$. Then the volume of subset (B) is given by the integral
\[
\left(\int_{|N(y)|<1} \int_{|N(z)|<1} + \int_{|N(y)|=1} \int_{|N(y)+uN(z)|<1} \right) \int_{|N(z)|=1} dx dy dz,
\]
which (by Lemma A.1 with $k = 0, n = 1$) is equal to the product of $1 - q^{-2}$ and
\[
q^{-2} \cdot q^{-2} + (1 - q^{-2})(1 + q^{-1})q^{-1} = q^{-1} + q^{-2} - q^{-3}.
\]
Adding the expressions of (A) and (B) we obtain that $\text{vol}(V_{0,q})$ is equal to
\[
(1 - q^{-1}) + (q^{-1} + q^{-2} - q^{-3}) = 1 + q^{-2} - q^{-3}.
\]
When $d > 1$, the claim of the lemma follows from Lemma A.5.

To compute $\text{vol}(V_{d,q})$, when $d = 0$, recall that
\[
V_{1,0} = \{(x, y, z) \in R_E^3; \max\{|N(x) + N(y) + N(z)|, |N(y) + uN(z)|\} = q^{-1}\}.
\]
Note that the condition $\| (x, y, z) \|_E = 1$ is satisfied. If not, $|N(x)| < q^{-2}$, $|N(y)| < q^{-2}$ and $|N(z)| < q^{-2}$ implies that $|N(x) + N(y) + N(z)| < q^{-2}$ and $|N(y) + uN(z)| < q^{-2}$. This set is the disjoint union of two subsets
\[
\{(x, y, z) \in R_E^3; |N(x) + N(y) + N(z)| \leq q^{-1}, |N(y) + uN(z)| = q^{-1}\}
\]
and
\[
\{(x, y, z) \in R_E^3; |N(x) + N(y) + N(z)| = q^{-1}, |N(y) + uN(z)| < q^{-1}\}.
\]
Note that from
\[
|N(y) + N(z)| \leq \max\{|N(y) + uN(z)|, |(1 - u)N(z)|\},
\]
it follows that $|N(x)| < 1$. In particular $|N(x)| < 1/q$. This implies that $|u - 1| = q^{-1}$, so the second subset is empty when $|u - 1| < 1/q$. When $|u - 1| = 1/q$ its volume is given by the integral
\[
\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=1} \int_{|N(x)+c|=q^{-1}} dx dy dz,
\]
where $|c| = |N(y) + N(z)| = q^{-1}$ (thus $|N(x)| \leq q^{-2}$). The value of the integral is
\[
(1 - q^{-2}) \times q^{-2}(1 + q^{-1}) \times q^{-2}.
\]
For the first subset we have $|N(y)| = 1$ and its volume is given by the integral
\[
\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(y)+uN(z)|=q^{-1}} dz 
\]
The value of the integral is
\[
q^{-2}(1-q^{-2}) \times q^{-1}(1-q^{-2}).
\]
We obtain that $\text{vol}(V_{1,0})$ is equal to $q^{-3}(1-q^{-2})$ when $|u-1| < 1/q$ and to
$q^{-3}(1+1/q)$ when $|u-1| = 1/q$.

When $d > 1$, the claim of the lemma follows from Lemma A.5.

To compute $\text{vol}(V_{n,d})$ ($0 \leq n \leq d-1$) recall that since $|\pi^d(N(y)+uN(z))| < q^{-n}$, we have that
\[
V_{n,d} = \{(x,y,z) \in R^n_E; |N(x) + N(y) + N(z)| = q^{-n}, \|(x,y,z)\|_E = 1\}.
\]

To compute this volume, we apply Lemma A.5.

To compute $\text{vol}(V^n_{n,d})$ ($n = d$) recall that $V_{n,d}$ is
\[
\{(x,y,z) \in R^n_E; \max\{|N(x) + N(y) + N(z)|, |\pi^d(N(y)+uN(z))|\} = q^{-n}\},
\]
where $\|(x,y,z)\|_E = 1$. The set $V_{n,d}$ is the disjoint union of two subsets
\[
\{(x,y,z) \in R^n_E; |N(x) + N(y) + N(z)| \leq q^{-n}, |N(y) + uN(z)| = 1\}
\]
and
\[
\{(x,y,z) \in R^n_E; |N(x) + N(y) + N(z)| = q^{-n}, |N(y) + uN(z)| < 1\}.
\]

We will compute the volume of each subset.

(A) **Case of** $|N(x) + N(y) + N(z)| \leq q^{-n}$ **and** $|N(y) + uN(z)| = 1$. Note that since $|1-u| < 1$ we have $|N(y) + N(z)| = |N(y) + uN(z)| = 1$. The volume is given by the integral
\[
\int_{|N(x)|<1} \int_{|N(y)|=1} \int_{|N(y)+c| \leq q^{-n}} dz dy dx
\]
\[
+ \int_{|N(x)|=1} \int_{|N(y)+N(z)|=1} \int_{|N(z)+c| \leq q^{-n}} dz dy dx,
\]
where $c = N(x) + N(y)$ so that $|c| = 1$. This is equal to
\[
(1-q^{-2})q^{-n}(1+q^{-1})(q^{-2}+1-q^{-1}-q^{-2}) = (1-q^{-2})^2 q^{-n}.
\]

(B) **Case of** $|N(x) + N(y) + N(z)| = q^{-n}$ **and** $|N(y) + uN(z)| < 1$. Since $|1-u| < 1$ the second inequality is equivalent to $|N(y) + N(z)| < 1$. Moreover, if $|N(y)| < 1$ then $|N(z)| < 1$ and consequently $|N(x)| < 1$ which is a contradiction. Thus $|N(y)| = 1$, and applying Lemma A.3 with $j = n-1$ so that $\sigma(n-j) = 1$, the
contribution from this subcase is equal to

\[(1 - q^{-2})^2 q^{-n-2}.\]

Adding the contributions from cases (A) and (B), then dividing by \(1 - q^{-2}\), we obtain \((1 - q^{-2}) (1 + q^{-2}) q^{-n}\).

To compute \(\text{vol}(V_{n,d}^d)\) \((n > d)\) and \(q^{-n} > |\pi^d(1 - u)|\) recall that \(V_{n,d} = \{(x, y, z) \in R^3; \max(|N(x) + N(y) + N(z)|, |\pi^d(N(y) + uN(z))|) = q^{-n}\}\), where \(||(x, y, z)||_{E} = 1\). Assume that \(|N(z)| < 1\). Then \(|N(y) + uN(z)| < 1\) implies that \(|N(y)| < 1\) and from \(|N(x) + N(y) + N(z)| < 1\) it follows that \(|N(x)| < 1\) which is a contradiction. Thus \(|N(z) = 1\) and \(V_{n,d}\) is the disjoint union of two subsets

\[\{(x, y, z) \in R^3; ||N(x) + N(y) + N(z)| = q^{-n}, |N(y) + uN(z)| = q^{d-n}\}\]

and

\[\{(x, y, z) \in R^3; |N(x) + N(y) + N(z)| = q^{-n}, |N(y) + uN(z)| \leq q^{d-n-1}\}\].

We will compute the volume of each of them.

(A) Case of \(|N(x) + N(y) + N(z)| \leq q^{-n}\) and \(|N(y) + uN(z)| = q^{d-n}\). The volume of this subset is equal to the integral

\[\int_{|N(z)| = 1} \int_{|N(y) + uN(z)| = q^{d-n}} \int_{|N(x) + c| \leq q^{-n}} dx dy dz,\]

where \(c = c(y, z) = N(y) + N(z)\) so that \(|c| = q^{d-n}\).

When \(d = 0\), note that since \(|1 - u| < q^{-n}\) we have \(|N(y) + N(z)| = |N(y) + uN(z)| = q^{-n}\). It follows that the first inequality is equivalent to \(|N(x)| \leq q^{-n}\). Thus the volume of this subset is equal to the integral

\[\int_{|N(z)| = 1} \int_{|N(y) + uN(z)| = q^{-n}} \int_{|N(x)| \leq q^{-n}} dx dy dz.\]

The volume of set \(|N(x)| \leq q^{-n}\) is equal to \(q^{-n}\) if \(n\) is even and to \(q^{-n-1}\) if \(n\) is odd. Thus the integral is equal to the product of \((1 - q^{-2}) x (1 - q^{-2}) q^{-2n}\) and 1 if \(n\) is even and \(q^{-1}\) if \(n\) is odd.

When \(d > 0\), the volume of the set \(|N(x) + c| \leq q^{-n}\) is zero when \(n - d\) is odd. Thus, the integral is equal to \((1 - q^{-2}) x (1 - q^{-2}) q^{d-n} \times q^{-n}(1 + q^{-1})\), when \(n - d\) is even, and zero when \(n - d\) is odd.

(B) Case of \(|N(x) + N(y) + N(z)| = q^{-n}\) and \(|N(y) + uN(z)| < q^{d-n}\). Note that inequality \(|N(y) + uN(z)| < q^{d-n}\) is equivalent to \(|N(y) + N(z)| < q^{d-n}\). The volume of this subset equals the integral

\[\int_{|N(z)| = 1} \int_{|N(y) + N(z)| \leq q^{d-n-1}} \int_{|N(x) + c| = q^{-n}} dx dy dz,\]

where \(c = c(y, z) = N(y) + N(z)\).
When $d = 0$, thus $|c| < q^{-n}$, and the volume of this subset equals the integral

$$\int_{|N(z)|=1} \int_{|N(y)+N(z)| \leq q^{-n-1}} \int_{|N(x)|=q^n} dx dy dz.$$ 

This integral is equal to zero when $n$ is odd and to

$$(1 - q^{-2}) \times (1 + q^{-1}) q^{-n-1} \times (1 - q^{-2}) q^{-n}$$

when $n$ is even. Thus, $\text{vol}(V_{n,0}^0)$ is equal to

$$(1 - q^{-2}) q^{-2n} \times (1 + q^{-1} + q^{-2})$$

when $n$ is even and to

$$(1 - q^{-2}) q^{-2n-1},$$

when $n$ is odd.

When $d > 0$ the volume is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+N(z)| \leq q^{d-n-1}} \int_{|N(x)+c|=q^n} dx dy dz,$$

where $c = c(y, z) = N(y) + N(z)$. Applying Lemma A.3, it is equal to

$$(1 - q^{-2})^2 q^{d+\sigma(n-d)+1-2n-1}.$$ 

Thus, adding contributions from cases (A) and (B) and dividing by $1 - q^{-2}$, the $\text{vol}(V_{n,d}^0)$ is equal to

$$(1 - q^{-2}) q^{d-2n} \times (1 + q^{-1} + q^{-2}),$$

when $n - d$ is even and to

$$(1 - q^{-2}) q^{d-2n-1},$$

when $n - d$ is odd.

To compute $\text{vol}(V_{n,d}^0)$ ($q^{-n} = |\pi^d(1-u)|$) recall that $V_n$ is

$$\{(x, y, z) \in R^3_\mathbb{E}; \|\langle x, y, z \rangle\|_E = 1,$$

$$\max\{|N(x) + N(y) + N(z)|, |\pi^d(N(y) + uN(z))|\} = q^{-n}\}.$$

As in the previous cases we can show that $|N(y)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of two subsets.

(A) Case of $|N(x) + N(y) + N(z)| \leq q^{-n}$, $|N(y) + uN(z)| = q^{d-n}$ and $|N(z)| = 1$.

When $d = 0$, we have

$$|N(y) + N(z)| \leq \max\{|N(y) + uN(z)|, |(1-u)N(z)|\} \leq q^{d-n}.$$

Thus $|N(x)| \leq q^{-n}$ and the volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=q^{-n}} \int_{|N(x)|\leq q^{-n}} dx dy dz.$$
Applying Lemma A.1 the value of the integral is

\[(1 - q^{-2}) \times (1 - q^{-2})q^{-n} \times q^{-n} \cdot \]

When \(d > 0\), then for any \(t \geq 0\), we define the following subset

\[W_t(z) = \{ y \in E; |N(z)| = 1, |N(y) + uN(z)| = q^{d-n}, |N(y) + N(z)| = q^{d-t-n} \}. \]

The volume of the set is given by the integral

\[ \sum_{t=0}^{\infty} \int_{|N(z)|=1} \int_{|N(y) + c_t| \leq q^{-n}} dxdydz, \]

where \(c_t = N(y) + N(z)\) and \(|c_t| = q^{d-t-n}\). Note that the integral over \(x\) depends
only on \(|c_t|\). Thus without loss of generality, we can assume that 
\(c_t = \pi^{d-t-n}c_0\), where \(|c_0| = 1\) is an arbitrary fixed element of \(R^x\), and take the integral over \(y\)
and \(z\). The integral is equal to

\[ (1 - q^{-2}) \sum_{t=0}^{\infty} \text{vol}(W_t) \int_{|N(x) + c_t| \leq q^{-n}} dx. \]

Applying Lemma A.2, this is equal to the product of \((1 - q^{-2})^2\) and

\[ (1 - (q + 2)q^{-2})q^{d-n} \int_{|N(x) + c_0| \leq q^{-n}} dx + \sum_{t=1}^{\infty} q^{d-t-n} \int_{|N(x) + c_t| \leq q^{-n}} dx. \]

If \(t \geq d\) so that \(|c_t| \leq q^{-n}\), and thus \(|N(x)| \leq q^{-n}\). The integral over \(x\) is equal
to \(q^{-n-\sigma(n)}\). Recall that we put \(\sigma(n) = n - 2[n/2] (= 0 \text{ if } n \text{ is even, } 1 \text{ if } n \text{ is odd})\)
before Lemma A.3. The sum of the terms which correspond to \(t \geq d\) is equal to

\[ (1 - q^{-2})^2 q^{-2n-\sigma(n)} \sum_{t=d}^{\infty} q^{d-t} = (1 - q^{-2})(1 + q^{-1})q^{-2n-\sigma(n)}. \]

If \(1 \leq t \leq d - 1\) then the set \(|N(x) + c_t| \leq q^{-n}\) is empty when \(n - d + t\) is odd and
is \((1 + q^{-1})q^{-n}\) when \(n - d + t\) is even. Set \(j = n - d + t\), then \(n - d + 1 \leq j \leq n - 1\)
and the even \(j\) are of the form

\[ n - d + 1 + \sigma(n - d + 1), n - d + 3 + \sigma(n - d + 1), \ldots, n - 1 - \sigma(n - 1). \]

Since \(\sigma(n - d + 1) = 1 - \sigma(n - d)\) and \(\sigma(n - 1) = 1 - \sigma(n)\), this sequence can be
written as

\[ n - d - \sigma(n - d) + 2, n - d - \sigma(n - d) + 4, \ldots, n + \sigma(n) - 2. \]

So the sum is the product of \((1 + q^{-1})q^{-n}\) and

\[ q^{-(n-d-\sigma(n-d)+2)} + q^{-(n-d-\sigma(n-d)+4)} + \ldots + q^{-(n+\sigma(n)-2)} \]

\[ = (1 - q^{-2})^{-1} (q^{- (n-d-\sigma(n-d)+2)} - q^{- (n+\sigma(n)-2)}). \]
Thus, the contribution from $1 \leq t \leq d - 1$ is equal to
$$(1 + q^{-1})(1 - q^{-2})(q^{-2n + d + \sigma(n-d)} - q^{-2n - \sigma(n)})$$
and the contribution from $t \geq 1$ is equal to
$$(1 + q^{-1})(1 - q^{-2})q^{-2n + d + \sigma(n-d)}.$$
Thus when $\varepsilon = 0$ and to zero when $\varepsilon = 1$. 

when $n$ is even and 

$$(1 + q^{-1})q^{-2n-1}$$

when $n$ is odd.

Adding the contributions from cases (A) and (B) when $d \geq 1$, then dividing by $(1 - 1/q^2)$, we obtain 

$$(1 + q^{-1})q^{d-2n}(q^{-2} + 1 - (q + 2)q^{-2} + (1 - q^{-2})q^{-1})$$

$$= (1 + q^{-1})(1 - q^{-2} - q^{-3})q^{d-2n}.$$ 

when $n - d$ is even and 

$$(1 + q^{-1})q^{d-1-2n}$$

when $n - d$ is odd.

To compute $\text{vol}(V_{n,d}^0) \,(q^{-n} < |\pi^d(1 - u)|)$ recall that 

$V_{n,d} = \{(x, y, z) \in R^3_E; \max\{|N(x) + N(y) + N(z)|, |\pi^d(N(y) + uN(z))|\} = q^{-n}\}$, 

where $\|(x, y, z)\|_E = 1$. Then $|N(z)| = 1$ and $|N(y) + uN(z)| \leq q^{d-n}$, we have 

$$|N(y) + N(z)| = |(1 - u)N(z)| = |1 - u|.$$ 

Thus $V_{n,d}$ is the disjoint union of two subsets 

$$\{(x, y, z) \in R^3_E; |N(z)| = 1, |N(x) + N(y) + N(z)| \leq q^{-n},$$

$$|N(y) + uN(z)| = q^{-n}\}$$

and 

$$\{(x, y, z) \in R^3_E; |N(z)| = 1, |N(x) + N(y) + N(z)| = q^{-n},$$

$$|N(y) + uN(z)| \leq q^{-n-1}\}.$$ 

We will compute the volume of each of them.

(A) Case of $|N(x) + N(y) + N(z)| \leq q^{-n}$, $|N(y) + uN(z)| = q^{d-n}$. The volume of this subset is given by the integral 

$$\int_{|N(z)|=1} \int_{|N(y)+uN(z)|=q^{d-n}} \int_{|N(x)+c|\leq q^{-n}} dx dy dz,$$

where $c = c(y, z) = N(y) + N(z)$ and $|c| = |1 - u| = q^{-2k-\varepsilon}$, where $\varepsilon$ is 0 or 1. 

Applying Lemma A.1 the value of the integral is equal to 

$$(1 - q^{-2}) \times (1 - q^{-2})q^{d-n} \times (1 + q^{-1})q^{-n}$$

when $\varepsilon = 0$ and to zero when $\varepsilon = 1$. 

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(B) Case of $|N(x) + N(y) + N(z)| = q^{-n}$, $|N(y) + uN(z)| < q^{d-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y) + uN(z)| < q^{d-n}} \int_{|N(z) + c| = q^{-n}} dx dz dy,$$

where $c = c(y, z) = N(y) + N(z)$ and $|c| = |1-u|$. Applying Lemma A.1 we conclude that the integral is equal to

$$(1 - q^{-2}) \times (1 + q^{-1}) q^{d-n-1} \times (1 - q^{-2}) q^{-n}$$

We obtain that $\text{vol}(V_{n,d}^0)$ (when $n > 2k$) is equal to

$$q^{d-2n}(1 - q^{-2})(1 + q^{-1})^2$$

when $\varepsilon = 0$ and zero when $\varepsilon = 1$. The lemma follows.

\[\Box\]

**Lemma B.4.3.** Suppose that $n_1 = 0, n_2 = 0, a_1 \in R^\times, |c_1| \leq 1, |c_2| = q^{-d}, d \geq 0$. When $|a_2| = q^{-2k}|c_2|$ ($k \geq 1$), we have

$$\text{vol}(V_{n,d}^0) = \begin{cases} 
1 + q^{-2} - q^{-3} & \text{if } n = 0, \ d = 0, \\
1 - q^{-1} + q^{-2} & \text{if } n = 0, \ d \geq 1, \\
(1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}) & \text{if } 1 \leq n \leq d - 1, \\
(1 - q^{-2} - q^{-3} - q^{-4})q^{-n} & \text{if } n = d, \ d \geq 1, \\
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d-2n} & \text{if } 0 < n - d < 2k, \ 2 \mid n - d, \\
(1 - q^{-2})q^{d-2n-1} & \text{if } 0 < n - d < 2k, \ 2 \nmid n - d, \\
(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{d-2n} & \text{if } n = 2k + d, \\
(1 - q^{-2})(1 + q^{-1})^2q^{d-2n} & \text{if } n > 2k + d. 
\end{cases}$$

When $|a_2| = q^{-2k-1}|c_2|$ ($k \geq 0$), we have

$$\text{vol}(V_{n,d}^0) = \begin{cases} 
1 + q^{-2} - q^{-3} & \text{if } n = 0, \ d = 0, \\
1 - q^{-1} + q^{-2} & \text{if } n = 0, \ d \geq 1, \\
(1 - q^{-2})q^{-n}(1 - q^{-1} + q^{-2}) & \text{if } 1 \leq n \leq d, \\
(1 - q^{-2} - q^{-3} - q^{-4})q^{-n} & \text{if } n = d, \ d \geq 1, \\
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d-2n} & \text{if } 0 < n - d < 2k + 1, \ 2 \mid n - d, \\
(1 - q^{-2})q^{d-2n-1} & \text{if } 0 < n - d < 2k + 1, \ 2 \nmid n - d, \\
(1 + q^{-1})q^{d-2n-1} & \text{if } n = 2k + d + 1, \\
0 & \text{if } n > 2k + d + 1. 
\end{cases}$$
Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in \mathbb{R}_E^x$ such that $N(\alpha) = a_1$ and renaming $y'$ back to $y$, without loss of generality we can assume that $a_1 = 1$. Since we are interested in $|Q_2|$, we can assume that $c_2 = \pi^d$. Thus we have that $Q_1(x, y, z) = N(x) + N(y) + c_1 N(z)$ and $Q_2(x, y, z) = a_2 N(y) + \pi^d N(z)$. Note that $d > 0$ (thus $|c_2| < 1$) implies $|c_1| = 1$.

To compute $\text{vol}(V_{0,d}^n)$ recall that
\[ V_{0,d} = \{(x, y, z) \in R_E^3; \max\{|N(x) + N(y) + c_1 N(z)|, |a_2 N(y) + \pi^d N(z)|\} = 1\}, \]
where $\|(x, y, z)\|_E = 1$. When $d = 0$, this set is the disjoint union of two subsets
\[ \{(x, y, z) \in R_E^3; |N(x)| \leq 1, |N(y)| \leq 1, |N(z)| = 1\}, \]
and
\[ \{(x, y, z) \in R_E^3; |N(x) + N(y) + c_1 N(z)| = 1, |N(z)| < 1\}, \]
where the second one splits further according to $|N(x)| < 1$ or $|N(x)| = 1$. Then the volume of the first subset is $(1 - q^{-2})$ and that of the second
\[ \int_{|N(z)|<1} \int_{|N(x)|<1} \int_{|N(y)|=1} dydx dz \]
\[ + \int_{|N(z)|<1} \int_{|N(x)|=1} \int_{|N(x)+N(y)+c_1 N(z)|=1} dydx dz. \]
Then $\text{vol}(V_{0,0})$ is equal to
\[ (1 - q^{-2}) + q^{-2}(1 - q^{-2})(q^{-2} + (1 - (q + 1)q^{-2})) = (1 - q^{-2})(1 + q^{-2} - q^{-3}). \]

When $d \geq 1$, we have that
\[ V_{0,d} = \{(x, y, z) \in R_E^3; |N(x) + N(y) + N(z)| = 1\}, \]
and we apply Lemma A.5.

To compute $\text{vol}(V_{n,d}^0)$ ($1 \leq n \leq d - 1$) recall that since $|a_2 N(y) + \pi^d N(z)| < q^{-n}$, and $|c_1| = 1$ (thus can assume it to be 1) we have
\[ V_{n,d} = \{(x, y, z) \in R_E^3; |N(x) + N(y) + N(z)| = q^{-n}\}, \]
where $\|(x, y, z)\|_E = 1$. To compute its volume, we apply Lemma A.5.

To compute $\text{vol}(V_{n,d}^n)$ ($n = d$) recall that since $|a_2| < q^{-n}$, we have
\[ V_{n,d} = \{(x, y, z) \in R_E^3; \max\{|N(x) + N(y) + c_1 N(z)|, |\pi^d N(z)|\} = q^{-n}, \]
\[ \|(x, y, z)\|_E = 1\}. \]

Since $|c_1| < 1$, we have $|c_1| = 1$, and without loss of generality we can assume it to be 1. The set $V_{n,d}$ is the disjoint union of two subsets
\[ \{(x, y, z) \in R_E^3; |N(x) + N(y) + N(z)| \leq q^{-n-1}, |N(z)| = 1\} \]
and 
\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(x) + N(y) + N(z)| = q^{-n}\}.
\]

We will compute the volume of each subset.

(A) Case of \(|N(x) + N(y) + N(z)| \leq q^{-n-1}\) and \(|N(z)| = 1\). We consider the subcases of \(|N(z)| = 1\) and \(|N(z)| < 1\). The volume of this set is equal to the sum of two integrals.

The first integral is
\[
\int_{|N(z)|=1} \int_{|N(y)+N(z)|=1} \int_{|N(x)+N(y)+N(z)|=q^{-n-1}} dxdydz.
\]
Note that this is the same integral as in case 5.1 \((n = d)\) and is equal to \((1-q^{-2})q^{-n-3}(1+q^{-1})\).

The second integral is
\[
\int_{|N(z)|=1} \int_{|N(y)+N(z)|=1} \int_{|N(x)+N(y)+N(z)|=q^{-n-1}} dxdydz.
\]
Note that this is the same integral as in case 4.1 \((n = d)\) and is equal to 
\[(1-q^{-2})q^{-n-1}(1+q^{-1})(1-2q^{-1} - 2q^{-2}).\]

(B) Applying Lemma A.5, the volume of this subset is \((1-q^{-2})^2 q^{-n} (1-q^{-1} + q^{-2}).\)

Adding the contributions from cases (A) and (B) and dividing by \(1 - q^{-2}\), we obtain 
\[(1-q^{-2} - q^{-3} - q^{-4}) q^{-n}.\]

To compute \(\text{vol}(V_{n,d}^0)\) \((q^{-n} > |a_2|)\) recall that 
\[V_{n,d} = \{(x, y, z) \in \mathbb{R}^3_E; \max\{|N(x) + N(y) + c_1 N(z)|, |a_2 N(y) + \pi^d N(z)|\} = q^{-n}\},\]
where \(\|(x, y, z)\|_E = 1\). Note that \(|N(y)| = 1\). Indeed, the condition \(|N(y)| < 1\) implies \(|N(z)| \leq |N(y)| < 1\) and \(|N(x)| < 1\) which is a contradiction. Since \(|a_2| < q^{-n}\) it is the disjoint union of two subsets
\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| = 1, |N(x) + N(y) + c_1 N(z)| \leq q^{-n}, |N(z)| = q^{d-n}\}
\]
and
\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| = 1, |N(x) + N(y) + c_1 N(z)| = q^{-n}, |N(z)| \leq q^{d-n-1}\}.
\]
Note that the term \(c_1 N(z)\) can be dropped and the volumes are given by the same integrals as in Lemma B.4.2 \((\text{case of } q^{-n} \geq |1-u|)\). Thus \(\text{vol}(V_{n,d}^0)\) is equal to 
\[(1-q^{-2})q^{d-2n} \times (1+q^{-1} + q^{-2})\]
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when \( n \) is even and

\[
(1 - q^{-2})q^{d-2n} \times q^{-1}
\]

when \( n \) is odd.

To compute \( \text{vol}(V^0_{n,d}) \) \((q^{-n} = |a_2|)\) recall that \( V_{n,d} \) is

\[
\{(x, y, z) \in \mathbb{R}^3; \max\{|N(x) + N(y) + c_1 N(z)|, |a_2 N(y) + \pi^d N(z)|\} = q^{-n}\},
\]

where \(|(x, y, z)|_E = 1\). As in previous cases we can show that \( N(y) = 1 \) and \( V_{n,d} \) is the disjoint union of two subsets

(A) Case of \(|N(x) + N(y) + c_1 N(z)| = q^{-n}, |a_2 N(y) + \pi^d N(z)| \leq q^{-n} \) and \(|N(y)| = 1\).

Since \(|a_2 N(y)| = q^{-n}\), the volume of this subset is given by the integral

\[
\int_{|N(y)|=1} \int_{|N(z)| \leq q^{-n}} \int_{|N(x) + N(y) + c_1 N(z)| = q^{-n}} dx \, dz \, dy,
\]

where \(|N(y) + c_1 N(z)| = 1\). Applying Lemma A.1 the integral is equal to

\[
(1 - q^{-2}) \times q^{d-n} \times (1 - q^{-2})q^{-n}
\]

when \( n - d \) is even and

\[
(1 - q^{-2}) \times q^{d-n-1} \times (1 - q^{-2})q^{-n}
\]

when \( n - d \) is odd.

(B) Case of \(|N(x) + N(y) + c_1 N(z)| \leq q^{-n-1}, |a_2 N(y) + \pi^d N(z)| = q^{-n} \) and \(|N(y)| = 1\). The volume of this subset is given by the integral

\[
\int_{|N(y)|=1} \int_{|N(z) + a_2 \pi^d N(y)| = q^{-n}} \int_{|N(x) + N(y) + c_1 N(z)| \leq q^{-n-1}} dx \, dz \, dy,
\]

where \(|N(y) + c_1 N(z)| = 1\). Applying Lemma A.1 the value of the integral is

\[
(1 - q^{-2}) \times (1 - q^{-1} - q^{-2})q^{d-n} \times (1 + q^{-1})q^{-n-1}
\]

when \( n - d \) is even and

\[
(1 - q^{-2}) \times q^{d-n-1} \times (1 + q^{-1})q^{-n-1}
\]

when \( n - d \) is odd.

Adding the expressions of (A) and (B), and dividing by \( 1 - 1/q^2 \), we obtain

\[
q^{d-2n}(1 + q^{-1})(1 - q^{-2} - q^{-3})
\]

when \( n - d \) is even and

\[
q^{d-2n-1}(1 + q^{-1})
\]

when \( n - d \) is odd.

To compute \( \text{vol}(V^0_{n,d}) \) \((q^{-n} < |a_2|)\) recall that \( V_{n,d} \) is

\[
\{(x, y, z) \in \mathbb{R}^3; \max\{|N(x) + N(y) + c_1 N(z)|, |a_2 N(y) + \pi^d N(z)|\} = q^{-n}\},
\]
Lemma B.4.4. Suppose that \( n_1 = 0, n_2 = 0, a_1, a_2, c_2 \in R^x \). When \( |c_1| < 1 \), we have

\[
\text{vol}(V_{n}^0) = \begin{cases} 
 1 - 2q^{-3} & \text{if } n = 0, \\
 (1 - q^{-2})(1 + q^{-1})^2 q^{-2n} & \text{if } n > 0.
\end{cases}
\]
Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R^3_E$ such that $N(\alpha) = a_1$ and renaming $y'$ back to $y$, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_2 = 1$. Thus we have that $Q_1(x, y, z) = N(x) + N(y) + c_1 N(z)$ and $Q_2(x, y, z) = a_2 N(y) + N(z)$.

We first compute $\text{vol}(V_0)$. Here we can drop the term $c_1 N(z)$ so that

$$V_0 = \{(x, y, z) \in R^3_E; \max\{|N(x) + N(y)|, |a_2 N(y) + N(z)|\} = 1\}.$$  

We first compute the volume of the complement of $V_0$ in $\{(x, y, z) \in R^3_E; \|x, y, z\|_E = 1\}$:

$$\{(x, y, z) \in R^3_E; \|x, y, z\|_E = 1, |N(x) + N(y)| < 1, |a_2 N(y) + N(z)| < 1\}.$$  

Note that if $|N(x)| < 1$ then $|N(y)| < 1$ and $|N(z)| < 1$ which is a contradiction. Thus $|N(x)| = 1$, and consequently $|N(y)| = |N(z)| = 1$. Consequently the volume of the complement is given by the integral

$$\int_{|N(y)| = 1} \int_{|a_2 N(y) + N(z)| < 1} \int_{|N(x) + N(y)| < 1} dx dz dy.$$  

This integral is the same as in Lemma B.4.1, case of $n = d = 0$. Thus, the volume of $V_0$ is equal to $(1 - q^{-2})(1 - 2q^{-3})$.

To compute $\text{vol}(V_n^a)$ ($n \geq 1$) recall that

$$V_n = \{(x, y, z) \in R^3_E; \|x, y, z\|_E = 1, \max\{|Q_1|, |Q_2|\} = q^{-n}\}.$$  

Assume that $|N(y)| < 1$. Then $|a_2 N(y) + N(z)| < 1$ implies that $|N(z)| < 1$ and from $|N(x) + N(y)| + c_1 N(z)| < 1$ it follows that $|N(x)| < 1$ which is a contradiction. Thus $|N(y)| = 1$ and $V_n$ is the disjoint union of two subsets

$$\{(x, y, z) \in R^3_E; |N(y)| = 1, |Q_1| \leq q^{-n}, |Q_2| = q^{-n}\}$$  

and

$$\{(x, y, z) \in R^3_E; |N(y)| = 1, |Q_1| = q^{-n}, |Q_2| \leq q^{-(n+1)}\}.$$  

We will compute the volume of each of them.

(A) Case of $|N(x) + N(y) + c_1 N(z)| \leq q^{-n}$ and $|a_2 N(y) + N(z)| = q^{-n}$. Thus the volume of this subset is equal to the integral

$$\int_{|N(y)| = 1} \int_{|a_2 N(y) + N(z)| = q^{-n}} \int_{|N(x) + c| \leq q^{-n}} dx dz dy,$$

where $c = c(y, z) = N(y) + c_1 N(z)$ and $|c| = 1$. By Lemma A.1 the integral is equal to

$$(1 - q^{-2}) \times (1 - q^{-2}) q^{-n} \times (1 + q^{-1}) q^{-n}.$$
(B) Case of \(|N(x) + N(y) + c_1 N(z)| = q^{-n} \) and \(|a_2 N(y) + N(z)| \leq q^{-n-1} \). The volume of this subset is given by the integral

\[
\int_{|N(y)|=1} \int_{|a_2 N(y) + N(z)| \leq q^{-n-1}} \int_{|N(x) + c| = q^{-n}} dxdzdy,
\]

where again \( c = c(y, z) = N(y) + c_1 N(z) \) and \( |c| = 1 \). Applying Lemma A.1, the above integral is equal to

\[
(1 - q^{-2}) \times (1 + q^{-1})q^{-n-1} \times (1 - q^{-2})q^{-n}.
\]

Adding the expressions from (A) and (B), simplifying and dividing by \((1 - q^{-2})\) gives

\[
\text{vol}(V^0_n) = (1 - q^{-2})(1 + q^{-1})^2 q^{-2n}.
\]

The lemma follows. \( \square \)

### Appendix C. Section 5 Computations

This section contains the computations used in Sec. 5.

**Lemma C.5.1.** Suppose that \( c_2 \in R^\infty \). Then

\[
\text{vol}(V^0_n) = \begin{cases} 
1 + q^{-2} & \text{if } n = 0, \\
q^{-4} & \text{if } n = 1, \\
0 & \text{if } n \geq 2.
\end{cases}
\]

**Proof.** We first show that \( V_0 \ (n \geq 2) \) is empty.

Suppose \((x, y, z) \in V_0 \). Then \(|N(z)| < 1 \), since \(|c_2 N(z)| = |N(z)| = 1 \) implies \(|Q_2| = 1 > q^{-n} \). Also \(|N(x)| < 1 \) since \(|N(x)| = 1 \) implies \(|Q_1| = 1 \). Hence \(|N(y)| = 1 \). As \( E/F \) is unramified, \(|N(x)| < q^{-1}, |N(z)| < q^{-1} \), hence

\[
\max\{ |Q_1|, |Q_2| \} = \max\{ |a_1|, |a_2| \} \times |\pi| = |\pi| > q^{-n}.
\]

Hence \( V_0 \ (n \geq 2) \) is empty.

To compute \( \text{vol}(V^0_n) \) note that

\( V_0 = \{(x, y, z) \in R^3_F; \max\{ |N(x) + c_1 N(z)|, |N(z)| \} = 1 \} \).

This is the disjoint union of the subset \(|N(z)| = 1, |N(x)| \leq 1 \), of volume \(1 - q^{-2} \), and the subset \(|N(z)| < 1, |N(x)| = 1 \), of volume \(q^{-2}(1 - q^{-2}) \). Hence the volume of \( V_0 \) is the product of \((1 - q^{-2})\) and \((1 + q^{-2})\).

To compute \( \text{vol}(V^1_n) \) recall that \( V_1 \) is the set

\( \{(x, y, z) \in R^3_F; \max\{ |N(x) + \pi a_1 N(y) + c_1 N(z)|, |\pi a_2 N(y) + c_2 N(z)| \} = 1/q \} \),

where \( \|(x, y, z)\|_E = 1 \). If \((x, y, z) \in V_1 \) then \(|N(z)| < 1, |N(x)| < 1, |N(y)| = 1 \). Then the volume of \( V_1 \) is \((1 - q^{-2})q^{-4} \), as required. \( \square \)
Lemma C.5.2. Suppose that $a_1, c_1 \in R^\times$. Let $|a_2| = q^{-d}$, where $d \geq 0$. When $|c_2/(a_2\pi)| = q^{-2k-1}$ ($k \geq 0$), we have

\[
\begin{cases}
1 - q^{-1} & \text{if } n = 0, \\
q^{-1} + q^{-2} - q^{-5} & \text{if } n = 1, \quad d = 0, \\
q^{-1} - q^{-3} + q^{-4} & \text{if } n = 1, \quad d \geq 1, \\
(1 - q^{-2})q^{-n} & \text{if } 2 \leq n \leq d,
\end{cases}
\]

\[
\text{vol}(V_{n,d}^0) = \begin{cases}
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{-n} & \text{if } n = d + 1, \\
(1 - q^{-2})q^{-2n} & \text{if } 2 \leq n - d \leq 2k + 1, \quad 2 \mid n - d, \\
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d+1-2n} & \text{if } 2 \leq n - d \leq 2k + 1, \quad 2 \nmid n - d, \\
(1 + q^{-1})q^{-d-2n} & \text{if } n = 2k + d + 2, \\
0 & \text{if } n > 2k + d + 2.
\end{cases}
\]

When $|c_2/(a_2\pi)| = q^{-2k}$ ($k \geq 1$), we have

\[
\begin{cases}
1 - q^{-1} & \text{if } n = 0, \\
q^{-1} + q^{-2} - q^{-5} & \text{if } n = 1, \quad d = 0, \\
q^{-1} - q^{-3} + q^{-4} & \text{if } n = 1, \quad d \geq 1, \\
(1 - q^{-2})q^{-n} & \text{if } 2 \leq n < d + 1,
\end{cases}
\]

\[
\text{vol}(V_{n,d}^0) = \begin{cases}
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{-n} & \text{if } n = d + 1, \\
(1 - q^{-2})q^{-2n} & \text{if } 2 \leq n - d \leq 2k, \quad 2 \mid n - d, \\
(1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d+1-2n} & \text{if } 2 \leq n - d \leq 2k, \quad 2 \nmid n - d, \\
(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{d+1-2n} & \text{if } n = 2k + d + 1, \\
(1 - q^{-2})(1 + q^{-1})q^{d+1-2n} & \text{if } n > 2k + d + 1.
\end{cases}
\]

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming $y'$ back to $y$, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. We have that $Q_1(x, y, z) = N(x) + \pi N(y) + N(z)$ and $Q_2(x, y, z) = \pi a_2 N(y) + c_2 N(z)$. Since we are interested in $|Q_2|$ we can assume that $a_2$ is $\pi^d$. Thus without loss of generality we can assume $Q_2(x, y, z) = \pi^{d+1} N(y) + c_2 N(z)$.

To compute $\text{vol}(V_{0,d}^0)$ note that

\[
V_{0,d} = \{(x, y, z) \in R_E^3; |N(x) + N(z)| = 1\}.
\]

This is the disjoint union of the subset $|N(z)| = 1$, $|N(x) + N(z)| = 1$, of volume $(1 - q^{-2})(1 - q^{-1} - q^{-2})$, and the subset $|N(z)| < 1$, $|N(x)| = 1$, of volume $q^{-2} (1 - q^{-2})$. Hence the volume of $V_0$ is the product of $(1 - q^{-2})$ and $(1 - q^{-2})$.

To compute $\text{vol}(V_{1,d}^0)$ recall that $V_{1,d}$ is the set

\[
\{(x, y, z) \in R_E^3; \max\{|N(x) + \pi N(y) + N(z)|, |\pi^{d+1} N(y) + c_2 N(z)|\} = q^{-1}\},
\]

\[
\text{vol}(V_{1,d}^0) = \begin{cases}
\end{cases}
\]
where \( \| (x, y, z) \|_E = 1 \). When \( d > 0 \), this set is the disjoint union of two subsets

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| = 1, |N(x)| < 1, |N(z)| < 1\}
\]

and

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| \leq 1, |N(x) + N(z)| = q^{-1}\}.
\]

The volume of the first is equal to \((1 - q^{-2})q^{-4}\) and that of the second to \((1 - q^{-2})q^{-1} \times (1 - q^{-2})\).

When \( d = 0 \), this set is the disjoint union of three subsets

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| = 1, |N(x)| = 1, |N(x) + N(z)| \leq q^{-1}\},
\]

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| = 1, |N(x)| < 1, |N(z)| < 1\}
\]

and

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(y)| < 1, |N(x) + N(z)| = q^{-1}\}.
\]

The volume of the first is equal to \((1 - q^{-2})q^{-2}q^{-1}(1 + q^{-1})\), of the second to \((1 - q^{-2})q^{-4}\) and of the third to \((1 - q^{-2})^2 q^{-3}\).

To compute \( \text{vol}(V_{n,d}^0) \) \( (2 \leq n < d + 1) \) recall that since \( |\pi^{d+1}N(y) + c_2N(z)| < q^{-n} \), we have

\[
V_{n,d} = \{(x, y, z) \in \mathbb{R}^3_E; |N(x) + \pi N(y) + N(z)| = q^{-n}\},
\]

where \( \| (x, y, z) \|_E = 1 \). As in the previous cases \( |N(x)| = |N(z)| = 1 \). The volume of this subset is given by the integral

\[
\int_{|N(z)|=1}^{1} \int_{|N(x)+N(z)|=q^{-n}} dx dz.
\]

Applying Lemma A.1 this is equal to

\[(1 - q^{-2}) \times (1 - q^{-2})q^{-n}.
\]

To compute \( \text{vol}(V_{n,d}^0) \) \( (n = d + 1) \) recall that \( V_{n,d} \) is the set of \((x, y, z) \in \mathbb{R}^3_E\) with \( \| (x, y, z) \|_E = 1 \) and

\[
\max\{|N(x) + \pi N(y) + N(z)|, |\pi^{d+1}N(y) + c_2N(z)|\} = q^{-n}.
\]

As in the previous cases we have that \( |N(x)| = |N(z)| = 1 \). Moreover \( |\pi^{d+1}N(y) + c_2N(z)| \leq q^{-n} \). So the set \( V_{n,d} \) is the disjoint union of two subsets

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(x) + \pi N(y) + N(z)| = q^{-n}\}
\]

and

\[
\{(x, y, z) \in \mathbb{R}^3_E; |N(x) + \pi N(y) + N(z)| \leq q^{-n-1}, |N(y)| = 1\}.
\]

We will compute the volume of each of them.
(A) Case of $|N(x) + \pi N(y) + N(z)| = q^{-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(x)+N(z)|=q^{-n}} dx dz.$$ 

Applying Lemma A.1 this is equal to

$$(1 - q^{-2}) \times (1 - q^{-2}) q^{-n}.$$ 

(B) Case of $|N(x) + N(y) + N(z)| \leq q^{-n-1}$, $|N(y)| = 1$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)|=1} \int_{|N(x)+N(z)|\leq q^{-n-1}} dx dy dz.$$ 

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times (1 - q^{-2}) \times (1 + q^{-1}) q^{-n-1}.$$ 

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

$$(1 - q^{-2})(1 + q^{-1} + q^{-2}) q^{-n}.$$ 

To compute $\text{vol}(V_{n,d}^0)$ ($d + 1 < n < 2k + d + 1$) recall that $V_n = \{(x, y, z) \in \mathbb{R}_E^3; \quad \max\{N(x) + \pi N(y) + N(z), |\pi^{d+1} N(y) + c_2 N(z)|\} = q^{-n}\}$, where $\|(x, y, z)\|_E = 1$. As in the previous cases $|N(x)| = |N(z)| = 1$. Moreover $|c_2 N(z)| \leq q^{-n}$. So $V_{n,d}$ is the disjoint union of two subsets

$$\{(x, y, z) \in \mathbb{R}_E^3; |N(x) + \pi N(y) + N(z)| = q^{-n}, |N(y)| \leq q^{d+1-n}\}$$ 

and

$$\{(x, y, z) \in \mathbb{R}_E^3; |N(x) + \pi N(y) + N(z)| \leq q^{-n-1}, |N(y)| = q^{d+1-n}\}.$$ 

We will compute the volume of each of them.

(A) Case of $|N(x) + \pi N(y) + N(z)| = q^{-n}$, $|N(y)| \leq q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)|\leq q^{d+1-n}} \int_{|N(x)+N(z)|=q^{-n}} dx dy dz.$$ 

Applying Lemma A.1 and since the volume of the set $|N(y)| \leq q^{-n}$ is $q^{-n-s(n)}$, the value of the integral is

$$(1 - q^{-2}) \times q^{d+1-n-s(n-d-1)} \times (1 - q^{-2}) q^{-n}.$$
(B) Case of $|N(x) + \pi N(y) + N(z)| \leq q^{-n-1}$, $|N(y)| = q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)|=q^{d+1-n}} \int_{|N(x)+N(z)| \leq q^{-n-1}} dxdydz.$$  

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times q^{d+1-n} (1 - q^{-2}) \times (1 + q^{-1}) q^{-n-1}$$

when $n - d - 1$ is even and zero when $n - d - 1$ is odd.

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

$$(1 - q^{-2})(1 + q^{-1} + q^{-2}) q^{d+1-2n}$$

when $n - d - 1$ is even and

$$(1 - q^{-2}) q^{d-2n}$$

when $n - d - 1$ is odd.

To compute $\text{vol}(V_{n,d}^0) \ (q^{-n} = |c_2|)$ recall that

$$V_{n,d} = \{(x, y, z) \in R_E^3; \max\{|N(x) + \pi N(y) + N(z)|, \ |\pi^{d+1} N(y) + c_2 N(z)|\} = q^{-n}\},$$

where $\|(x, y, z)\|_E = 1$. If $|N(z)| < 1$ then $|N(x)| < 1$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and since $|c_2| = q^{-n}$, $V_{n,d}$ is the disjoint union of two subsets

$$\{(x, y, z) \in R_E^3; |N(x) + \pi N(y) + N(z)| = q^{-n}, |N(y)| \leq q^{d+1-n}\}$$

and

$$\{(x, y, z) \in R_E^3; |N(x) + \pi N(y) + N(z)| \leq q^{-n-1},$$

$$|N(y) + c_2 \pi^{d-1} N(z)| = q^{d+1-n}\}.$$

We will compute the volume of each of them.

(A) Case of $|N(x) + \pi N(y) + N(z)| = q^{-n}$, $|N(y) + c_2 \pi^{-d-1} N(z)| \leq q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)|=q^{d+1-n}} \int_{|N(x)+N(z)| = q^{-n}} dxdydz.$$  

Applying Lemma A.1, the value of the integral is

$$(1 - q^{-2}) \times q^{d+1-n} \times (1 - q^{-2}) q^{-n}$$

when $|c_2/(a_2 \pi)| = q^{-2k}$ (thus $n - d - 1 = 2k$ is even), and

$$(1 - q^{-2}) \times q^{d-n} \times (1 - q^{-2}) q^{-n}$$

when $|c_2/(a_2 \pi)| = q^{-2k-1}$ (thus $n - d - 1 = 2k + 1$ is odd).
(B) Case of $|N(x) + \pi N(y) + N(z)| \leq q^{-n-1}$, $|N(y) + c_2 \pi^{-d-1} N(z)| = q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+c_2 \pi^{-d-1}N(z)|=q^{d+1-n}} \int_{|N(x)+N(z)|\leq q^{-n-1}} dx dy dz.$$ 

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times (1 - q^{-1} - q^{-2}) q^{d+1-n} \times (1 + q^{-1}) q^{-n-1}$$

when $|c_2/(a_2 \pi)| = q^{-2k}$ and

$$(1 - q^{-2}) \times q^{d-n} \times (1 + q^{-1}) q^{-n-1}$$

when $|c_2/(a_2 \pi)| = q^{-2k-1}$. 

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

$$(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4}) q^{d+1-2n}$$

when $|c_2/(a_2 \pi)| = q^{-2k}$ and

$$(1 + q^{-1}) q^{d-2n}$$

when $|c_2/(a_2 \pi)| = q^{-2k-1}$. 

To compute $\text{vol}(V_n,d) (q^{-n} > |c_2|)$ recall that

$$V_{n,d} = \{(x, y, z) \in R^3_E; \max\{|N(x) + \pi N(y) + N(z)|, |\pi^{-d} N(y) + c_2 N(z)|\} = q^{-n}\},$$

where $||(x, y, z)||_E = 1$. As in the previous case, we have that $|N(x)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of two subsets

$$\{(x, y, z) \in R^3_E; |N(x) + \pi N(y) + N(z)| = q^{-n},$$

$$|N(y) + c_2 \pi^{-d-1} N(z)| \leq q^{d+1-n}\}$$

and

$$\{(x, y, z) \in R^3_E; |N(x) + \pi N(y) + N(z)| \leq q^{-n-1},$$

$$|N(y) + c_2 \pi^{-d-1} N(z)| = q^{d+1-n}\}.$$ 

Note that according to Lemma A.1, this set is empty when $|c_2/(a_2 \pi)| = q^{-2k-1}$. 

We will compute the volume of each of them when $|c_2/(a_2 \pi)| = q^{-2k}$. 

(A) Case of $|N(x) + \pi N(y) + N(z)| = q^{-n}$, $|N(y) + c_2 \pi^{-d-1} N(z)| \leq q^{d+1-n}$. The volume of this subset is given by the integral

$$\int_{|N(z)|=1} \int_{|N(y)+c_2 \pi^{-d-1}N(z)|\leq q^{d+1-n}} \int_{|N(x)+N(z)|=q^{-n}} dx dy dz.$$ 

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times (1 + q^{-1}) q^{d+1-n} \times (1 - q^{-2}) q^{-n}.$$
(B) Case of $|N(x) + \pi N(y) + N(z)| \leq q^{-n-1}$, $|N(y) + c_2 \pi^{-d-1} N(z)| = q^{d+1-n}$. The volume of this subset is given by the integral

\[
\int_{|N(x)|=1} \int_{|N(y) + c_2 \pi^{-d-1} N(z)| = q^{d+1-n}} \int_{|N(x) + N(z)| \leq q^{-n-1}} dx dy dz.
\]

Applying Lemma A.1 the value of the integral is

\[
(1 - q^{-2}) \times (1 - q^{-2}) q^{d+1-n} \times (1 + q^{-1}) q^{-n-1}.
\]

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

\[
(1 - q^{-2})(1 + q^{-1})^2 q^{d+1-2n}.
\]

Lemma C.5.3. Suppose that $a_1, c_1 \in R^\times$. Let $|c_2| = q^{-d}$, where $d \geq 1$. When $|a_2 \pi| < |c_2|$, we have

\[
\text{vol}(V^0_{n,d}) = \begin{cases} 
1 - q^{-1} & \text{if } n = 0, \\
q^{-1} + q^{-2} + q^{-4} & \text{if } n = 1, \ d = 1, \\
q^{-1} - q^{-3} + q^{-4} & \text{if } n = 1, \ d \geq 2, \\
(1 - q^{-2}) q^{-n} & \text{if } 1 < n < d, \\
(1 + q^{-1}) q^{-n} & \text{if } n = d, \ d \geq 2, \\
0 & \text{if } n > d.
\end{cases}
\]

When $|a_2 \pi| = |c_2|$, we have

\[
\text{vol}(V^0_{n,d}) = \begin{cases} 
1 - q^{-1} & \text{if } n = 0, \\
q^{-1} + q^{-2} - q^{-3} - q^{-4} - q^{-5} & \text{if } n = 1, \ d = 1, \\
q^{-1} - q^{-3} + q^{-4} & \text{if } n = 1, \ d \geq 2, \\
(1 - q^{-2}) q^{-n} & \text{if } 1 < n < d, \\
(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4}) q^{-n} & \text{if } n = d, \ d \geq 2, \\
(1 - q^{-2})(1 + q^{-1})^2 q^{d-2n} & \text{if } n > d.
\end{cases}
\]

Proof. Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming $y'$ back to $y$, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. We have that $Q_1(x,y,z) = N(x) + \pi N(y) + N(z)$ and $Q_2(x,y,z) = \pi a_2 N(y) + c_2 N(z)$. Since we are interested in $|Q_2|$ we can assume that $a_2$ is $\pi^d$. Thus without loss of generality we can assume $Q_2(x,y,z) = \pi^{d+1} N(y) + c_2 N(z)$.

To compute $\text{vol}(V^0_{n,d})$ note that $|a_2 \pi N(y) + c_2 N(z)| < 1$ and

\[
V_{0,d} = \{(x,y,z) \in R_E^{3}; |N(x) + N(z)| = 1\}.
\]
The volume of this subset is given by the following integral

$$\int_{|N(z)|=1} \int_{|N(x)+N(z)|=1} dx dz + \int_{|N(z)|<1} \int_{|N(x)|=1} dx dz,$$

which is equal to the product of $1 - q^{-2}$ and $1 - (q + 1)/q^2 + 1/q^2 = 1 - 1/q$.

To compute $\text{vol}(V_{1,q}^0)$ recall that, when $d > 1$ we have $|a_2\mathbf{1}| < |c_2| < q^{-1}$ thus

$$V_{n,d} = \{(x, y, z) \in \mathbb{R}_E^3; |N(x) + \mathbf{1}N(y) + N(z)| = q^{-1}, \|(x, y, z)\|_E = 1\}.$$

Note that the condition $|N(x)| < 1$ implies $|N(z)| < 1$ and $|N(y)| = 1$. Thus the volume of this subset is given by the following sum of two integrals

$$\int_{|N(x)|=1} \int_{|N(y)|\leq 1} \int_{|N(z)+c|=q^{-1}} dxdydz + \int_{|N(z)|<1} \int_{|N(y)|=1} \int_{|N(x)|<1} dxdydz,$$

where $c = c(y, z) = N(x) + \mathbf{1}N(y), |c| = 1$. This is equal to $(1-q^{-2})(q^{-1} - q^{-3} + q^{-4})$.

When $d = 1$, the volume of this subset is given by the following sum of two integrals

$$\int_{|N(x)|=1} \int_{|N(y)|\leq 1} \int_{|N(z)+c|\leq q^{-1}} dxdydz + \int_{|N(z)|<1} \int_{|N(y)|=1} \int_{|N(x)|<1} dxdydz,$$

where $c = c(y, z) = N(x) + \mathbf{1}N(y), |c| = 1$. This is equal to $(1-q^{-2})(q^{-1} - q^{-2} + q^{-4})$.

To compute $\text{vol}(V_{n,d}^0)$ $(1 < n < d)$ recall that $|a_2\mathbf{1}| < |c_2| < q^{-n}$ and

$$V_{n,d} = \{(x, y, z) \in \mathbb{R}_E^3; |N(x) + \mathbf{1}N(y) + N(z)| = q^{-n}, \|(x, y, z)\|_E = 1\}.$$

Note that the condition $|N(x)| < 1$ implies $|N(z)| < 1$ and $|N(y)| = 1$, thus $|N(x) + \mathbf{1}N(y) + N(z)| = q^{-1}$ which is a contradiction. The volume of this subset is given by the following integral

$$\int_{|N(x)|=1} \int_{|N(y)|\leq 1} \int_{|N(z)+c|=q^{-n}} dxdydz,$$

where $c = c(y, z) = N(x) + \mathbf{1}N(y), |c| = 1$. This is equal to $(1-q^{-2})q^{-n}(1-q^{-2})$.

To compute $\text{vol}(V_{n,d}^0)$ $(n = d)$, when $|a_2\mathbf{1}| < |c_2| = q^{-n}$ recall that $V_{n,d}$ consist of $(x, y, z) \in \mathbb{R}_E^3$ with $\|(x, y, z)\|_E = 1$ and

$$\max\{|N(x) + \mathbf{1}N(y) + N(z)|, |c_2N(z)|\} = q^{-n}.$$

If $|N(z)| < 1$ then $|N(x)| < 1$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and since $|c_2| = q^{-n}$, $V_{n,d}$ is given by

$$\{(x, y, z) \in \mathbb{R}_E^3; |N(x) + \mathbf{1}N(y) + N(z)| \leq q^{-n}, |N(z)| = 1\}.$$

The volume of this subset is given by the following integral

$$\int_{|N(z)|=1} \int_{|N(y)|\leq 1} \int_{|N(x)+c|=q^{-n}} dxdydz = (1-q^{-2})q^{-n}(1 + q^{-1}).$$
When $|a_2\pi| = |c_2| = q^{-n}$ (set $c'_2 = c_2(a_2\pi)^{-1}$), the set $V_{n,d}$ consists of the $(x, y, z) \in R^3_E$ with $||(x, y, z)||_E = 1$ and

$$\max\{|N(x) + \pi N(y) + N(z)|, |N(y) + c'_2 N(z)|\} = q^{-n}.$$ 

If $|N(z)| < 1$ then $|N(x)| < 1$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x, y, z) \in R^3_E; |N(x) + \pi N(y) + N(z)| = q^{-n}, |N(y) + c'_2 N(z)| \leq 1\}$$

and

$$\{(x, y, z) \in R^3_E; |N(x) + \pi N(y) + N(z)| \leq q^{-n-1}, |N(y) + c'_2 N(z)| = 1\}.$$ 

We will compute the volume of each of them.

(A) Case of $|N(x) + \pi N(y) + N(z)| = q^{-n}$ (the $|N(y) + c'_2 N(z)| \leq 1$ is trivial). The volume of this subset is given by the integral

$$\int_{|N(x)| = 1} \int_{|N(y)| \leq 1} \int_{|N(x) + N(z)| = q^{-n}} dx dy dz.$$ 

Applying Lemma A.1 the value of the integral is

$$(1 - q^{-2}) \times (1 - q^{-2}) q^{-n}.$$ 

(B) Case of $|N(x) + \pi N(y) + N(z)| \leq q^{-n-1}, |N(y) + c'_2 N(z)| = 1$. The volume of this subset is given by the integral

$$\int_{|N(x)| = 1} \int_{|N(y) + c'_2 N(z)| = 1} \int_{|N(x) + N(z)| \leq q^{-n-1}} dx dy dz.$$ 

Applying Lemma A.1, the value of the integral is

$$(1 - q^{-2}) \times (1 - q^{-1} - q^{-2}) \times (1 + q^{-1}) q^{-n-1}.$$ 

Adding the contributions from (A) and (B) and dividing by $1 - q^{-2}$ we obtain

$$(1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4}) q^{-n}.$$ 

To compute $\text{vol}(V_{n,d}^0)$ ($n > d$) note that the condition $|N(z)| < 1$ implies that $|N(x)| < 1$ and thus $|N(y)| = 1$ and $|N(x) + \pi N(y) + N(z)| = q^{-1}$ which is a contradiction. Thus $|N(x)| = |N(z)| = 1$ and when $|a_2\pi| < |c_2| = q^{-d}$ we have

$$|a_2\pi N(y) + c_2 N(z)| = |c_2 N(z)| = q^{-d} < q^{-n}.$$ 

Thus the set is empty.

When $|a_2\pi| = |c_2| = q^{-d}$ (set $c'_2 = c_2(a_2\pi)^{-1}$) we have that $|N(x)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of its two subsets

$$\{(x, y, z) \in R^3_E; |N(x) + \pi N(y) + N(z)| = q^{-n}, |N(y) + c'_2 N(z)| \leq q^{d-n}\}.$$
and

\[ \{(x, y, z) \in R_E^3: |N(x) + \pi N(y) + N(z)| \leq q^{-n-1}, |N(y) + c_2'N(z)| = q^{d-n}\}. \]

We will compute the volume of each of them.

(A) Case of \(|N(x) + \pi N(y) + N(z)| = q^{-n}, |N(y) + c_2'N(z)| \leq q^{d-n}\). The volume of this subset is given by the integral

\[ \int_{|N(z)|=1} \int_{|N(y)+c_2'N(z)|\leq q^{d-n}} \int_{|N(x)+N(z)|=q^{-n}} dx dy dz. \]

Applying Lemma A.1 the value of the integral is

\[ (1 - q^{-2}) \times (1 + q^{-1})q^{d-n} \times (1 - q^{-2})q^{-n}. \]

(B) Case of \(|N(x) + \pi N(y) + N(z)| \leq q^{-n-1}, |N(y) + c_2'N(z)| = q^{d-n}\). The volume of this subset is given by the integral

\[ \int_{|N(z)|=1} \int_{|N(y)+c_2'N(z)|=q^{d-n}} \int_{|N(x)+N(z)|\leq q^{-n-1}} dx dy dz. \]

Applying Lemma A.1, the value of the integral is

\[ (1 - q^{-2}) \times (1 - q^{-2})q^{d-n} \times (1 + q^{-1})q^{-n-1}. \]

Adding the contributions from (A) and (B) and dividing by \(1 - q^{-2}\) we obtain

\[ (1 - q^{-2})(1 + q^{-1})q^{d-2n}. \]

The lemma follows.

\[ \square \]

**Appendix D. Section 6 Computations**

This section contains the computations used in Sec. 6.

**Lemma D.6.1.** Suppose that \(c_1 \in R^\times\) and \(|1 - u| = 1\). Suppose \(|a_2| = |c_2| = q^{-d}\), where \(d \geq 0\). Then we have

\[
\text{vol}(V_n^0) = \begin{cases} 
1 & \text{if } n = 0, \\
q^{-2} + q^{-4} & \text{if } n = 1, \ d = 0, \\
q^{-2} - q^{-3} & \text{if } n = 1, \ d \geq 1, \\
(1 - q^{-2})q^{n-1} & \text{if } 2 \leq n \leq d, \\
(1 + q^{-2})q^{-n-1} & \text{if } n = d + 1, \ d \geq 1, \\
0 & \text{if } n \geq d + 2.
\end{cases}
\]

**Proof.** Making a change of variables \(y' = \alpha y\) where \(\alpha \in R_E^\times\) such that \(N(\alpha) = a_1\) and renaming \(y'\) back to \(y\), without loss of generality we can assume that \(a_1 = 1\). Similarly, we can assume that \(c_1 = 1\). We have that \(Q_1(x, y, z) = N(x) + \pi\)
\[(N(y) + N(z)) \text{ and } Q_2(x, y, z) = a_2 \pi N(y) + c_2 \pi N(z). \] Since we are interested in \(|Q_2|\) we can assume that \(a_2 = \pi^d\). Thus without loss of generality we can assume \(Q_2(x, y, z) = \pi^{d+1}(N(y) + uN(z)).\)

We first show that \(V_n (n \geq d + 2)\) is empty.

Suppose \((x, y, z) \in V_n\). Then \(|N(x)| < 1\), since \(|N(x)| = 1\) implies \(|Q_1| = 1 > q^{-n}\). Since \(|1 - u| = 1\) we obtain that

\[
|N(y) + uN(z)| = |N(y) + N(z) + (1 - u)N(z)| = \max\{|N(y) + N(z)|, |N(z)|\}. 
\]

This is equal to

\[
\max\{|N(y)|, |N(z)|\} = 1. 
\]

Hence \(\max\{|Q_1|, |Q_2|\} = |\pi^{d+1}| > q^{-n}\). We conclude that \(V_n (n \geq 2)\) is empty.

To show that \(\text{vol}(V_n^0) = 1 - 1/q^2\) note that

\[
V_0 = \{(x, y, z) \in \mathbb{R}_E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}. 
\]

To compute \(\text{vol}(V_1^0)\) recall that, when \(d \geq 1\) we have \(|\pi^{d+1}(N(y) + uN(z))| < q^{-1}\) thus

\[
V_n \{ (x, y, z) \in \mathbb{R}_E^3; |N(x) + \pi(N(y) + N(z))| = q^{-1}\},
\]

where \(||(x, y, z)||_E = 1\). Thus the volume of this subset is given by the following sum of two integrals

\[
\int_{|N(x)| < 1} \int_{|N(y)| = 1} \int_{|N(z) + N(y)| = q^{-1}} dzdydx \\
+ \int_{|N(x)| < 1} \int_{|N(y)| < 1} \int_{|N(z)| = 1} dzdydx. 
\]

This is equal to \((1 - q^{-2})q^{-2}(1 - q^{-1})\). When \(d = 1\), we have

\[
\{(x, y, z) \in \mathbb{R}_E^3; \max\{|N(x) + \pi(N(y) + N(z)), |\pi(N(y) + uN(z))|\} = 1/q\},
\]

where \(||(x, y, z)||_E = 1\). If \((x, y, z) \in V_1\) then \(|N(x)| < 1\), and this set is equal to

\[
\{ (x, y, z) \in \mathbb{R}_E^3; \max\{|N(y) + N(z), |N(y) + uN(z)|\} = 1\},
\]

where \(||(x, y, z)||_E = 1\). Since \(|1 - u| = 1\), the volume of this set is

\[
\left(\int_{|N(y)| < 1} \int_{|N(z)| = 1} + \int_{|N(y)| = 1} \int_{|N(z)| \leq 1}\right) \int_{|N(x)| < 1} dx dz dy,
\]

which is equal to

\[
q^{-2}(1 - q^{-2})(1 + q^{-2}).
\]

To compute \(\text{vol}(V_n^0) (2 \leq n \leq d)\) recall that \(|\pi^{d+1}(N(y) + uN(z))| < q^{-n}\) and

\[
V_n \{ (x, y, z) \in \mathbb{R}_E^3; |N(x) + \pi(N(y) + N(z))| = q^{-n}\},
\]
Thus implies that $\| (x, y, z) \|_E = 1$. The volume of this subset is given by the following integral

$$\int_{|N(y)|=1} \int_{|N(z)|<1} \int_{|N(z)+c|=q^{1-n}} dxdydz,$$

where $c = c(x, y) = N(y) + N(x)/\pi$, $|c| = 1$. This is equal to $(1-q^{-2})q^{-2} \times q^{1-n}(1-q^{-2})$.

To compute $\text{vol}(V^n)$ ($n = d + 1$), recall that we have

$$V_n = \{(x, y, z) \in R_E^3; \max\{|N(x) + \pi(N(y) + N(z))|, |\pi^{d+1}(N(y) + uN(z))| = q^{-n}\},$$

where $\|(x, y, z)\|_E = 1$. If $|N(z)| < 1$ then $|N(y)| < 1$ which is a contradiction. Thus $|N(y)| = |N(z)| = 1$ and $|N(z)| < 1$. Note that the condition $|N(y) + N(z)| < 1$, implies that

$$|N(y) + uN(z)| = |N(y) + N(z) + (u-1)N(z)| = |(1-u)N(z)|.$$

Thus $V_n$ is given by

$$\{(x, y, z) \in R_E^3; |N(x) + \pi(N(y) + N(z))| \leq q^{-n}, |N(z)| = 1\}.$$

The volume of this subset is given by the following integral

$$\int_{|N(z)|=1} \int_{|N(x)|<1} \int_{|N(z)+c|\leq q^{1-n}} dydxdz = (1-q^{-2})q^{-2} \times q^{1-n}(1+q^{-1}),$$

where $c = c(x, z) = N(z) + N(x)/\pi$ so that $|c| = 1$. The lemma follows.

**Lemma D.6.2.** Suppose that $n_1 = 1, n_2 = 1, a_1, c_1 \in R^\times$. Suppose that $|a_2| = |c_2| = q^{-d}, d \geq 0$. When $|1-u| = q^{-2k}$ ($k \geq 1$), we have

$$\text{vol}(V^n_{n,d}) = \begin{cases} 1 & \text{if } n = 0, \\ q^{-2} - q^{-3} & \text{if } n = 1, \\ (1-q^{-2})q^{-n-1} & \text{if } 2 \leq n \leq d, \\ (1-q^{-2})q^{-n-1} & \text{if } n = d+1, \\ (1-q^{-2})(1+q^{-1}+q^{-2})q^{d+1-2n} & \text{if } 2 \leq n-d \leq 2k, 2|n-d, \\ (1-q^{-2})q^{-d-2n} & \text{if } n = 2k+1+d, \\ (1+q^{-1})q^{-d-2n} & \text{if } n > 2k+1+d. \end{cases}$$
When $|1 - u| = q^{-2k-1}(k \geq 0)$, we have

$$\text{vol}(V_{n,d}^0) = \begin{cases} 1 & \text{if } n = 0, \\ q^{-2} - q^{-3} & \text{if } n = 1, \\ (1 - q^{-2})q^{-n-1} & \text{if } 1 \leq n \leq d, \\ (1 - q^{-2})q^{-n-1} & \text{if } n = d + 1, \\ (1 - q^{-2})(1 + q^{-1} + q^{-2})q^{d+1-2n} & \text{if } 2 \leq n - d \leq 2k, 2 | n - d, \\ (1 - q^{-2})q^{d-2n} & \text{if } 2 \leq n - d \leq 2k + 1, \\ (1 + q^{-1} - q^{-2} - 2q^{-3} - q^{-4})q^{1+d-2n} & \text{if } n = 2k + 2 + d, \\ (1 - q^{-2})(1 + q^{-1})^2q^{1+d-2n} & \text{if } n > 2k + 2 + d. \end{cases}$$

**Proof.** Making a change of variables $y' = \alpha y$ where $\alpha \in R_E^\times$ such that $N(\alpha) = a_1$ and renaming $y'$ back to $y$, without loss of generality we can assume that $a_1 = 1$. Similarly, we can assume that $c_1 = 1$. We have that $Q_1(x, y, z) = N(x) + \pi(N(y) + N(z))$ and $Q_2(x, y, z) = a_2 \pi N(y) + c_2 \pi N(z)$. Since we are interested in $|Q_2|$, we can assume that $a_2 = \pi^d$. Thus without loss of generality we can assume $Q_2(x, y, z) = \pi^{d+1}(N(y) + uN(z))$.

To see that $\text{vol}(V_{0, d}^0) = 1 - 1/q^2$ note that

$$V_{0, d} = \{(x, y, z) \in R_E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}.$$

To compute $\text{vol}(V_{1, d}^0)$ recall that $V_{1, d}$ is the set

$$\{(x, y, z) \in R_E^3; \max(|N(x) + \pi(N(y) + N(z))|, |\pi^{1+d}(N(y) + uN(z))|) = 1/q\},$$

where $\|(x, y, z)\|_E = 1$. When $d = 0$ then $|N(x)| < 1$, and since $|1 - u| < 1$ this set is equal to

$$\{(x, y, z) \in R_E^3; \|(x, y, z)\|_E = 1, |N(y) + N(z)| = 1\}.$$

Then its volume is given by the integral

$$\int_{|N(y)| < 1} \int_{|N(z)| = 1} \int_{|N(y) + N(z)| = 1} dxdzdy,$$

whose value is

$$q^{-d}(1 - q^{-2}) + (1 - q^{-2})(1 - (q + 1)q^{-2})q^{-2} = (1 - q^{-2})(1 - q^{-1})q^{-2}.$$

To compute $\text{vol}(V_{n, d}^0)$ $(n \geq 2$ and $q^{-n} > |\pi^{1+d}|)$ recall that since $d > 0$ and $|\pi^{1+d}(N(y) + uN(z))| < q^{-n}$, we have that

$$V_{n, d} = \{(x, y, z) \in R_E^3; |N(x) + \pi(N(y) + N(z))| = q^{-n}, \|(x, y, z)\|_E = 1\}.$$
Assume that $|N(y)| < 1$. Since $|N(x)| \neq |\pi N(z)|$, it implies that $|N(z)| < 1$ and thus $|N(x)| < 1$, which is a contradiction. Thus $|N(x)| < 1$, $|N(y)| = 1$ and

$$V_{n,d} = \{(x, y, z) \in R^3_E; |N(y)| = 1, |N(x)| < 1, |N(x) + \pi(N(y) + N(z))| = q^{-n}\}.$$ 

So, the volume is given by the integral

$$\int_{|N(x)| < 1} \int_{|N(y)| = 1} \int_{|N(z)| = 1} \int_{|N(z) + c| = q^{-n}} dzdydx = q^{-2}(1 - q^{-2})q^{1-n}(1 - q^{-2}),$$

where $c = c(x, y) = N(x)/\pi + N(y)$ so that $|c| = |N(y)| = 1$. Thus, vol($V_{n,d}$) is equal to $(1 - q^{-2})q^{1-n}$.

To compute vol($V_{n,d}$) ($q^{-n} = |\pi^{1+d}|$) recall that $V_{n,d}$ consists of the $\{(x, y, z) \in R^3_E$ with $\|(x, y, z)\|_E = 1$ and

$$\max\{|N(x) + \pi(N(y) + N(z))|, |\pi^{1+d}(N(y) + uN(z))\} = q^{-n}\}.$$ 

Assume that $|N(z)| < 1$. Since $|N(x)| \neq |\pi N(y)|$, it implies that $|N(y)| < 1$ and thus $|N(x)| < 1$, which is a contradiction. Thus $|N(x)| < 1$, $|N(y)| = |N(z)| = 1$ and $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x, y, z) \in R^3_E; |N(z)| = 1, |N(x) + \pi(N(y) + N(z))| \leq q^{-n}, |N(y) + uN(z)| = 1\}
$$

and

$$\{(x, y, z) \in R^3_E; |N(z)| = 1, |N(x) + \pi(N(y) + N(z))| = q^{-n}, |N(y) + uN(z)| < 1\}.$$ 

We will compute the volume of each subset.

(A) Case of $|N(x) + \pi(N(y) + N(z))| \leq q^{-n}$ and $|N(y) + uN(z)| = 1$. Note that since $|1 - u| < 1$ we have $|N(y) + N(z)| = |N(y) + uN(z)| = 1$. And the set $|N(x) + \pi(N(y) + N(z))| < q^{-n}$ is empty.

(B) Case of $|N(x) + \pi(N(y) + N(z))| = q^{-n}$ and $|N(y) + uN(z)| < 1$. Since $|1 - u| < 1$ the second inequality is equivalent to $|N(y) + N(z)| < 1$. Applying Lemma A.4, the volume of this set is equal to

$$(1 - q^{-2})^2q^{-2n}q^{-n-1}.$$ 

To compute vol($V_{n,d}$) ($n \geq d + 2$ and $q^{-n} > |\pi^{1+d}(1 - u)|$) recall that $V_{n,d}$ is the set of $(x, y, z) \in R^3_E$ with $\|(x, y, z)\|_E = 1$ and

$$\max\{|N(x) + \pi(N(y) + N(z))|, |\pi^{1+d}(N(y) + uN(z))\} = q^{-n}.$$ 

Assume that $|N(y)| < 1$. Then $|N(y) + uN(z)| < 1$ implies that $|N(z)| < 1$; from $|N(x)/\pi + N(y) + N(z)| < 1/q^2$ it follows that $|N(x)| < 1$, which is a contradiction.
Moreover, the inequality $|\pi^{1+d}(N(y) + uN(z))| \leq q^{-n}$ is equivalent to $|\pi^{1+d}(N(y) + N(z))| \leq q^{-n}$. Thus $|N(y)| = 1$ and $V_{n,d}$ is the disjoint union of the two subsets

$$\{(x, y, z) \in R^3; |N(y)| = 1, |N(x) + \pi(N(y) + N(z))| \leq q^{-n}, |N(y) + N(z)| = q^{1+d-n}\}$$

and

$$\{(x, y, z) \in R^3; |N(y)| = 1, |N(x) + \pi(N(y) + N(z))| = q^{-n}, |N(y) + N(z)| < q^{1+d-n}\}.$$

We will compute the volume of each subset.

(A) Case of $|N(x) + \pi(N(y) + N(z))| \leq q^{-n}$ and $|N(y) + N(z)| = q^{1+d-n}$. The volume of this subset equals the integral

$$\int_{|N(y)|=1} \int_{|N(x)+N(z)|=q^{1+d-n}} \int_{|N(x)|\leq q^{-n}} dx dz dy,$$

where $c = c(y, z) = \pi(N(y) + N(z))$ so that $|c| = q^{d-n}$.

When $d = 0$, thus $|c| = q^{-n}$ and the volume of this subset equals the integral

$$\int_{|N(y)|=1} \int_{|N(x)+N(z)|=q^{1+d-n}} \int_{|N(z)|\leq q^{-n}} dx dz dy.$$

The volume of the set $\{|N(x)| \leq q^{-n}\}$ is equal to $q^{-n}$ if $n$ is even and to $q^{-n-1}$ if $n$ is odd. Thus the integral is equal to the product of

$$(1 - q^{-2}) \times (1 - q^{-2}) q^{1-2n}$$

and 1 if $n$ is even and $q^{-1}$ if $n$ is odd.

When $d > 0$, the volume of the set $\{|N(x) + c| \leq q^{-n}\}$ is zero when $n - d$ is odd. Thus, the integral is equal to $(1 - q^{-2}) \times (1 - q^{-2}) q^{1+d-n} \times q^{-n} (1 - q^{-1})$, when $n - d$ is even, and zero when $n - d$ is odd.

(B) Case of $|N(x) + \pi(N(y) + N(z))| = q^{-n}$ and $|N(y) + N(z)| < q^{1+d-n}$. The volume of this subset equals the integral

$$\int_{|N(y)|=1} \int_{|N(x)+N(z)|\leq q^{1+d-n}} \int_{|N(x)|=q^{-n}} dx dz dy,$$

where $c = c(y, z) = \pi(N(y) + N(z))$.

When $d = 0$, thus $|c| < q^{-n}$, and the volume of this subset equals the integral

$$\int_{|N(y)|=1} \int_{|N(x)+N(z)|\leq q^{-n}} \int_{|N(x)|=q^{-n}} dx dz dy.$$

This integral is equal to zero when $n$ is odd and to

$$(1 - q^{-2}) \times (1 - q^{-1}) q^{-n} \times (1 - q^{-2}) q^{-n}$$

when $n$ is even. Thus, $\text{vol}(V_{n,0})$ is equal to

$$(1 - q^{-2}) q^{1-2n} \times (1 + q^{-1} + q^{-2})$$
when \( n \) is even and to
\[
(1 - q^{-2})q^{1-2n} \times q^{-1}
\]
when \( n \) is odd.

When \( d > 0 \) the volume is given by the integral
\[
\int_{|N(y)|=1} \int_{|N(y)+N(z)| \leq q^{d-n}} \int_{|N(x)+c|=q^{-n}} dx dy dz,
\]
where \( c = c(y,z) = \pi(N(y) + N(z)) \). It is equal to
\[
\sum_{t=n-d}^{\infty} \int_{|N(y)|=1} \int_{|N(y)+N(z)| = q^{-t}} \int_{|N(x)+c_t|=q^{-n}} dx dy dz,
\]
where \( |c_t| = q^{1-t-1} \). Applying Lemma A.4, it is equal to
\[
(1 - q^{-2})^2q^{d+n-d-2n-1}.
\]
Thus, \( \text{vol}(V_{n,d}) \) is equal to
\[
(1 - q^{-2})q^{d+1-2n} \times (1 + q^{-1} + q^{-2})
\]
when \( n - d \) is even and to
\[
(1 - q^{-2})q^{d-2n}
\]
when \( n - d \) is odd.

To compute \( \text{vol}(V_{n,d}) \) \((q^{-n} = |\pi^{1+d}(1-u)|)\) recall that \( V_{n,d} \) consists of the \((x,y,z) \in R_{K}^{3}\) with \( \| (x,y,z) \|_{E} = 1 \) and
\[
\max \{|N(x)+\pi(N(y)+N(z))|, |\pi^{1+d}(N(y)+uN(z))|\} = q^{-n}.
\]
As in the previous cases we can show that \(|N(y)| = |N(z)| = 1, \) and \( V_{n,d} \) is the disjoint union of the two subsets.

(A) Case of \(|N(x)+\pi(N(y)+N(z))| \leq q^{-n}, \) \(|N(y)+uN(z)| = q^{1+d-n} \) and \(|N(y)| = |N(z)| = 1. \) When \( d = 0, \) we have
\[
|N(y)+N(z)| \leq \max\{|N(y)+uN(z)|,|(1-u)N(z)|\} \leq q^{1-n}.
\]
Thus \(|N(x)| \leq q^{-n} \) and the volume of this subset is given by the integral
\[
\int_{|N(y)|=1} \int_{|N(y)+N(z)| = q^{-n}} \int_{|N(z)| \leq q^{-n}} dx dy dz,
\]
which is equal to
\[
(1 - q^{-2}) \times (1 - q^{-2})q^{1-n} \times q^{-n-\sigma(n)}.
\]
When \( d > 0, \) then for any \( t \geq 0, \) we define the following subset
\[
W_t(z) = \{ y \in E ; |N(y)| = 1, |N(y)+uN(z)| = q^{1+d-n},
\]
\[
|N(y)+N(z)| = q^{1+d-t-n}\}.
\]
The volume of the set is given by the integral

\[ \sum_{t=0}^{\infty} \int_{|N(z)|=1} \int_{W_t(z)} \int_{|N(x)+c_t| \leq q^{-n}} dx \, dz \, dy, \]

where \( c_t = \pi(N(y)+N(z)) \) and \( |c_t| = q^{d-t-n} \). Note that the integral over \( x \) depends only on the \( |c_t| \). Thus we can assume that \( c_t \) is an arbitrary element so that \( |c_t| = q^{d-t-n} \) and take the integral over \( y \) and \( z \). The integral is equal to

\[ (1-q^{-2}) \sum_{t=0}^{\infty} \text{vol}(W_t) \int_{|N(x)+c_t| \leq q^{-n}} dx. \]

Applying Lemma A.2, this is equal to the product of \((1-q^{-2})^2\) and

\[ (1-(q+2)q^{-2})q^{1+d-n} \int_{|N(x)+c_t| \leq q^{-n}} dx + \sum_{t=1}^{\infty} q^{1+d-t-n} \int_{|N(x)+c_t| \leq q^{-n}} dx. \]

If \( t \geq d \) so that \( |c_t| \leq q^{-n} \), and thus \( |N(x)| \leq q^{-n} \). The integral over \( x \) is equal to \( q^{-n-\sigma(n)} \). The sum of the terms which correspond to \( t \geq d \) is equal to

\[ (1-q^{-2})^2 q^{1-2n-\sigma(n)} \sum_{t=d}^{\infty} q^{-d-t} = (1-q^{-2})(1+q^{-1})q^{1-2n-\sigma(n)}. \]

If \( 1 \leq t \leq d-1 \) then the set \( |N(x)+c_t| \leq q^{-n} \) is empty when \( n-d+t \) is odd and is \( (1+q^{-1})q^{-n} \) when \( n-d+t \) is even. Set \( j = n-d+t \), then \( n-d+1 \leq j \leq n-1 \) and the even \( j \) are of the form over

\[ n-d+1 + \sigma(n-d+1), n-d+3 + \sigma(n-d+1), \ldots, n-1 - \sigma(n-1). \]

Since \( \sigma(n-d+1) = 1 - \sigma(n-d) \) and \( \sigma(n-1) = 1 - \sigma(n) \), this sequence can be written as

\[ n-d - \sigma(n-d) + 2, n-d - \sigma(n-d) + 4, \ldots, n + \sigma(n) - 2. \]

So the sum is the product of \((1+q^{-1})q^{1-n}\) and

\[ q^{-(n-d-\sigma(n-d)+2)} + q^{-(n-d-\sigma(n-d)+4)} + \ldots + q^{-(n+\sigma(n)-2)} \]

\[ = (1-q^{-2})^{-1}(1-q^{-2})^{-1} q^{-(n-d-\sigma(n-d)+2)} - q^{-(n+\sigma(n)-2)-2}. \]

Thus, the contribution from \( 1 \leq t \leq d-1 \) is equal to

\[ (1+q^{-1})(1-q^{-2})(q^{-1-2n+d+\sigma(n-d)} - q^{-1-2n-\sigma(n)}). \]

If \( t = 0 \), the contribution from this term is

\[ (1-(q+2)q^{-2})(1+q^{-1})q^{-n}, \]

when \( n-d \) is even and zero when \( n-d \) is odd.

(B) Case of \( |N(x)+\pi(N(y)+N(z))| = q^{-n}, |N(y)+uN(z)| < q^{1+d-n} \) and \( |N(y)| = |N(z)| = 1. \) Since \( |N(x)| = 1 \) and \( |N(y)+uN(z)| < q^{1+d-n} \), we obtain

\[ |N(y)+N(z)| = |N(y)+uN(z)+(1-u)N(z)| = |1-u| = q^{1+d-n}. \]
Thus, the volume of this subset is given by the integral
\[
\int_{|N(z)|=1} \int_{|N(y)+uN(z)| \leq q^{d-n}} \int_{|N(x)+c| = q^{-n}} dx dz dy,
\]
where \( c = c(y, z) = \pi(N(y) + N(z)) \) and \( |c| = q^{d-n} \). When \( d = 0 \), the integral over \( x \) is taken over the set \( |N(x)| \leq q^{-n-1} \) when \( n \) is odd and over the set \( |N(x)+c| \leq q^{-n} \) when \( n \) is even. Thus, the integral is equal to
\[
(1 - q^{-2}) \times (1 + q^{-1})q^{-n} \times q^{-n-1}
\]
when \( n \) is odd and to
\[
(1 - q^{-2}) \times (1 + q^{-1})q^{-n} \times (1 - (q + 1)q^{-2})q^{-n}
\]
when \( n \) is even.

When \( d > 0 \) and \( n - d \) is odd the set \( |N(x)+c| = q^{-n} \) is empty and the integral is zero. In the case of \( n - d \) being even, the volume is equal to
\[
(1 - q^{-2}) \times (1 + q^{-1})q^{d-n} \times (1 - q^{-2})q^{-n}.
\]
When \( d = 0 \), adding the contributions from cases (A) and (B) (and dividing by \( (1 - 1/q^2) \)), we obtain
\[
q^{-2n}(1 + q^{-1})
\]
when \( n \) is odd and
\[
q^{-2n}(q + 1 - q^{-1} - 2q^{-2} - q^{-3})
\]
when \( n \) is even.

When \( d > 0 \), adding the contributions from cases (A) and (B) (and dividing by \( (1 - 1/q^2) \)), we obtain
\[
(1 + q^{-1})q^{1-2n-\sigma(n)} + (1 + q^{-1})(q^{-1-2n} - q^{1-2n-\sigma(n)})
\]
\[
+ (1 + q^{-1})(1 - q^{-2})q^{d-2n} + (1 - q^{-2})(1 - (q + 2)q^{-2})(1 + q^{-1})q^{1+d-2n}
\]
\[
= (1 + q^{-1})q^{-2n}(q^{d-1} + q^d - q^{-d-2} + q^{1+d} - q^{d} - 2q^{d-1})
\]
\[
= (1 + q^{-1})q^{d-2n}(q - q^{-1} - q^{-2}).
\]
when \( n - d \) is even and
\[
(1 + q^{-1})q^{1-2n-\sigma(n)} + (1 + q^{-1})(q^{-2n} - 1)q^{d-2n} = (1 + q^{-1})q^{d-2n}
\]
when \( n - d \) is odd.

To compute \( \text{vol}(V_{n,d}^0) \) \( (q^{-n} < |\pi^{1+d}(1-u)|) \) recall that \( V_{n,d} \) consists of the \( (x, y, z) \in R_q^3 \) with \( \|(x, y, z)\|_E = 1 \) and
\[
\max\{|N(x) + \pi(N(y) + N(z))|, |\pi^{1+d}(N(y) + uN(z))|\} = q^{-n}.
\]
Then we have \( |N(y)| = 1 \). Since \( |N(z)| = 1 \), \( q^{1+d-n} < |1-u| \) and \( |N(y) + uN(z)| \leq q^{1+d-n} \), we have
\[
|N(y) + N(z)| = |(1-u)N(z)| = |1-u|.
\]
Thus $V_{n,d}$ is the disjoint union of the two subsets
\[
\{(x, y, z) \in R_E^3; |N(y)| = 1, |N(x) + \pi(N(y) + N(z))| \leq q^{-n},
\]
\[
|N(y) + uN(z)| = q^{1+d-n}\}
\]
and
\[
\{(x, y, z) \in R_E^3; |N(y)| = 1, |N(x) + \pi(N(y) + N(z))| = q^{-n},
\]
\[
|N(y) + uN(z)| \leq q^{1+d-n}\}.
\]

We will compute the volume of each of subset.

(A) Case of $|N(x) + \pi(N(y) + N(z))| \leq q^{-n}, |N(y) + uN(z)| = q^{1+d-n}$. Then $|N(y)| = 1$. The volume of this subset is given by the integral
\[
\int_{|N(y)|=1} \int_{|N(y)/u+N(z)|=q^{1+d-n}} \int_{|N(z)+c|=q^{-n}} dx dz dy,
\]
where $c = c(y, z) = \pi(N(y) + N(z))$ and $|c| = |\pi(1-u)| = q^{-2k-1-\varepsilon} < q^{-n}$. Applying Lemma A.1 the integral is equal to zero when $\varepsilon = 0$ and when $\varepsilon = 1$, to
\[
(1 - q^{-2}) \times (1 - q^{-2}) q^{1+d-n} \times (1 + q^{-1}) q^{-n}.
\]

(B) Case of $|N(x) + \pi(N(y) + N(z))| = q^{-n}, |N(y) + uN(z)| < q^{1+d-n}$. Then $|N(y)| = 1$. The volume of this subset is given by the integral
\[
\int_{|N(y)|=1} \int_{|N(y)/u+N(z)|<q^{1+d-n}} \int_{|N(z)+c|=q^{-n}} dx dz dy,
\]
where $c = c(y, z) = \pi(N(y) + N(z))$ and $|c| = q^{-2k-1-\varepsilon}$. Applying Lemma A.1 the integral is equal to zero when $\varepsilon = 0$ and when $\varepsilon = 1$, to
\[
(1 - q^{-2}) \times (1 + q^{-1}) q^{d-n} \times (1 - q^{-2}) q^{-n}.
\]
We obtain that $\text{vol}(V_{n,d}^0)$ (when $n > 2k + 1 + \varepsilon$) is equal to zero if $\varepsilon = 0$ and if $\varepsilon = 1$, to
\[
(1 - q^{-2})(1 + q^{-1})^2 q^{1+d-2n}.
\]

The lemma follows.

\[\text{Lemma D.6.3. Suppose that } a_1, c_2 \in R^\times, |c_1| \leq 1, |a_2| < 1. \text{ Then}
\]
\[
\text{vol}(V_n^0) = \begin{cases} 1 & \text{if } n = 0, \\ q^{-2} + q^{-4} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}
\]

\[\text{Proof. Making a change of variables } y' = ay \text{ where } a \in R_E^\times \text{ such that } N(a) = a_1 \text{ and renaming } y' \text{ back to } y, \text{ without loss of generality we can assume that } a_1 = 1.\]
Thus we have that \( Q_1(x, y, z) = N(x) + \pi(N(y) + c_1 N(z)) \) and \( Q_2(x, y, z) = \pi(a_2 N(y) + c_2 N(z)) \). Moreover, if \(|c_1| < 1\) and since \(|a_2| < 1\) we can replace \( Q_1 \) with \( Q_1 + Q_2 \). So without loss of generality we can assume that \(|c_1| = 1\) and thus \( c_1 = 1 \).

Since we are interested only in \(|Q_2|\), we can replace \( Q_2 \) with \( Q_2/c_2 \) and put \( a_2/c_2 \) to be \( a_2 \). So \( Q_1(x, y, z) = N(x) + \pi(N(y) + N(z)) \) and \( Q_2(x, y, z) = \pi(a_2 N(y) + N(z)) \).

We first show that \( V_n \) (\( n \geq 2 \)) is empty. Suppose \( (x, y, z) \in V_n \). Then \(|N(x)| < 1\), since \(|N(x)| = 1\) implies \(|Q_1| = 1 > q^{-n}\). Moreover, from \(|Q_1| \leq q^{-2}\) the condition \(|N(y)| < 1\) implies \(|N(z)| < 1\), which leads to a contradiction. Thus \(|N(y)| = |N(z)| = 1\). Since \(|a_2| < 1\) we obtain \(|\pi(a_2 N(y) + N(z))| = |\pi N(z)| = q^{-1}\). Hence max\(\{|Q_1|, |Q_2|\} < q^{-1} > q^{-n}\). We conclude that \( V_n \) (\( n \geq 2 \)) is empty.

To see that \( \text{vol}(V_0^0) = 1 - 1/q^2 \) note that

\[
V_0 = \{(x, y, z) \in \mathbb{R}^3_E; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}.
\]

To compute \( \text{vol}(V_0^0) \) recall that \( V_1 \) is the set

\[
\{(x, y, z) \in \mathbb{R}^3_E; \max\{|N(x) + \pi(N(y) + c_1 N(z)), |\pi(a_2 N(y) + N(z))|\} = 1/q\}.
\]

where \( \|(x, y, z)\|_E = 1 \). If \((x, y, z) \in V_1\) then \(|N(x)| < 1\), and since \(|a_2| < 1\) this set is equal to

\[
\{(x, y, z) \in \mathbb{R}^3_E; \|(x, y, z)\|_E = 1, \max\{|N(y) + N(z)|, |N(z)|\} = 1\}.
\]

Then its volume is given by the integral

\[
\left( \int_{|N(z)|=1} + \int_{|N(z)|<1} \right) \int_{|N(y)|=1} dx dy dz,
\]

whose value is

\[
q^{-1}(1 - q^{-2}) + (1 - q^{-2})q^{-2} = (1 - q^{-2})(1 + q^{-2})q^{-2}.
\]

**Lemma D.6.4.** Suppose \( a_1, c_1 \in \mathbb{R}^\times, |a_2| < 1, |c_2| < 1 \) and \( |a_2| \neq |c_2| \). Set \(|c_2| = q^{-k}\). Then

\[
\text{vol}(V_0^0) = \begin{cases} 
1 & \text{if } n = 0, \\
q^{-2} - q^{-3} & \text{if } n = 1, \\
q^{-1-n}(1 - q^{-2}) & \text{if } 2 \leq n \leq k, \\
q^{-1-n}(1 + q^{-1}) & \text{if } n = k + 1, \\
0 & \text{if } n \geq k + 2.
\end{cases}
\]

**Proof.** Making a change of variables \( y' = \alpha y \) where \( \alpha \in \mathbb{R}^\times_+ \) such that \( N(\alpha) = a_1 \) and renaming \( y' \) back to \( y \), without loss of generality we can assume that \( a_1 = 1 \) and similarly \( c_1 = 1 \). Thus we have that \( Q_1(x, y, z) = N(x) + \pi(N(y) + N(z)) \) and \( Q_2(x, y, z) = \pi(a_2 N(y) + c_2 N(z)) \). Without loss of generality, we can assume that \(|a_2| < |c_2|\).
Then its volume is given by the integral
\[ \text{vol}(V_0) = 1 - 1/q^2 \]
where
\[ \|x, y, z\|_{E^3} = 1 \] implies
\[ \max(|Q_1|, |Q_2|) = q^{-k-1} > q^{-n} \]. We conclude that \( V_n \) \((n \geq k + 2)\) is empty.

To see that \( \text{vol}(V_0) = 1 - 1/q^2 \) note that
\[ V_0 = \{(x, y, z) \in E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}. \]

To compute \( \text{vol}(V_1) \) recall that \( V_1 \) is the set
\[ \{(x, y, z) \in E^3; |N(x) + \pi(N(y) + N(z))| = 1/q\}, \]
where \( \|(x, y, z)\|_E = 1 \). If \((x, y, z) \in V_1 \) then \( |N(x)| < 1 \), and this set is equal to
\[ \{(x, y, z) \in E^3; \|(x, y, z)\|_E = 1, |N(x)| < 1, |N(y) + N(z)| = 1\}. \]

Then its volume is given by the integral
\[ \left( \int_{|N(y)| = 1} \int_{|N(y) + N(z)| = 1} + \int_{|N(y)| < 1} \int_{|N(z)| = 1} \right) \int_{|N(z)| < 1} dxdzd, \]
whose value is
\[ q^{-2}(1 - q^{-2})(1 - (q + 1)q^{-2}) + (1 - q^{-2})q^{-4} = (1 - q^{-2})(q^{-2} - q^{-3}). \]

To compute \( \text{vol}(V_0) \) \((n > 1\) and \( q^{-n} > |\pi c_2|)\) recall that \( V_n \) is the set
\[ \{(x, y, z) \in E^3; \max(|N(x) + \pi(N(y) + N(z))|, |\pi a_2 N(y) + c_2 N(z))| = q^{-n}\}, \]
where \( \|(x, y, z)\|_E = 1 \). It implies that \( |N(x)| < 1 \) and thus \( |N(y)| = |N(z)| = 1 \). Then \( |Q_2| = |\pi c_2| = q^{-k-1} < q^{-n} \), so \( V_n \) is the set
\[ \{(x, y, z) \in E^3; \|(x, y, z)\|_E = 1, |N(x)| < 1, |N(y)| = 1, |N(y) + N(z)| = q^{1-n}\}. \]

Its volume is given by the integral
\[ \int_{|N(x)| < 1} \int_{|N(y)| = 1} \int_{|N(y) + N(z)| = 1/a^{-n}} dxdydz, \]
whose value is
\[ q^{-2}(1 - q^{-2})q^{1-n}(1 - q^{-2}). \]

To compute \( \text{vol}(V_0) \) \((q^{-n} = |\pi c_2|)\) recall that \( V_n \) is the set
\[ \{(x, y, z) \in E^3; \max(|N(x) + \pi(N(y) + N(z))|, |\pi a_2 N(y) + c_2 N(z))| = q^{-n}\}, \]
where \((x, y, z)\|_E = 1\). It implies that \(|N(x)| < 1\) and thus \(|N(y)| = |N(z)| = 1\). Then \(Q_2 = |\mathbf{a}_2| = q^{-k-1} = q^{-n}\), so \(V_n\) is the set
\[
\{(x, y, z) \in R_E^3; \| (x, y, z) \|_E = 1, |N(x)| < 1, |N(y)| = |N(z)| = 1, |N(y) + N(z)| \leq q^{-1-n}\}.
\]
Its volume is given by the integral
\[
\int_{|N(x)| < 1} \int_{|N(y)| = 1} \int_{|N(y) + N(z)| \leq q^{-1-n}} dx dy dz,
\]
whose value is
\[
q^{-2}(1 - q^{-2})q^{-1-n}(1 + q^{-1}).
\]

\textbf{Lemma D.6.5.} Suppose that \(a_2, c_2 \in R^x, |c_1| < 1\). Then
\[
\text{vol}(V_n^0) = \begin{cases} 1 & \text{if } n = 0, \\ q^{-2} + q^{-4} & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}
\]

\textbf{Proof.} Making a change of variables \(y' = \alpha y\) where \(\alpha \in R_E^x\) such that \(N(\alpha) = a_1\) and renaming \(y'\) back to \(y\), without loss of generality we can assume that \(a_1 = 1\). Thus we have that \(Q_1(x, y, z) = N(x) + \mathbf{p}(N(y) + c_1 N(z))\) and \(Q_2(x, y, z) = \mathbf{p}(a_2 N(y) + c_2 N(z))\). Since we are interested only in \(|Q_2|\), we can replace \(Q_2\) with \(Q_2/c_2\) and put \(a_2/c_2\) to be \(a_2\). So \(Q_1(x, y, z) = N(x) + \mathbf{p}(N(y) + c_1 N(z))\) and \(Q_2(x, y, z) = \mathbf{p}(a_2 N(y) + N(z))\).

We first show that \(V_n (n \geq 2)\) is empty. Suppose \((x, y, z) \in V_n\). Then \(|N(x)| < 1\), since \(|N(x)| = 1\) implies \(|Q_1| = 1 > q^{-n}\). Moreover, from \(|Q_2| \leq q^{-2}\) the condition \(|N(y)| < 1\) implies \(|N(z)| < 1\), which leads to a contradiction. Thus \(|N(y)| = |N(z)| = 1\) and \(|N(x)| < 1\). Since \(|c_1| < 1\) we obtain \(|N(x) + \mathbf{p}(N(y) + c_1 N(z))| = |\mathbf{p} N(y)| = q^{-1}\). Hence \(\max\{|Q_1|, |Q_2|\} = q^{-1} > q^{-n}\). We conclude that \(V_n (n \geq 2)\) is empty.

To see that \(\text{vol}(V_0^0) = 1 - 1/q^2\) note that
\[
V_0 = \{(x, y, z) \in R_E^3; |N(x)| = 1, |N(y)| \leq 1, |N(z)| \leq 1\}.
\]
To compute \(\text{vol}(V_0^0)\) recall that \(V_1\) is the set
\[
\{(x, y, z) \in R_E^3; \max\{|N(x) + \mathbf{p}(N(y) + c_1 N(z))|, |\mathbf{p}(a_2 N(y) + N(z))|\} = 1/q\},
\]
where \(|(x, y, z)|_E = 1\). If \((x, y, z) \in V_1\) then \(|N(x)| < 1\), and since \(|c_1| < 1\) this set is equal to
\[
\{(x, y, z) \in R_E^3; |(x, y, z)|_E = 1, \max\{|N(y) + N(z)|, |N(y)|\} = 1\},
\]
which is the same set as in case 6.3, \(n = 1\).
References


