# TWISTED CHARACTER OF A SMALL REPRESENTATION OF PGL(4)

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Abstract. We compute by a purely local method the elliptic $\theta$-twisted character $\chi_\pi$ of the representation $\pi = I_{(3,1)}(1_3)$ of $\mathrm{PGL}(4, F)$. Here $F$ is a $p$-adic field; $\theta$ is the "transpose-inverse" automorphism of $G = \mathrm{PGL}(4, F)$; $\pi$ is the representation of $\mathrm{PGL}(4, F)$ normalizedly induced from the trivial representation of the maximal parabolic subgroup of type $(3, 1)$. Put $\mathbf{C} = \{(g_1, g_2) \in \mathrm{GL}(2) \times \mathrm{GL}(2); \det(g_1) = \det(g_2)\}/\mathbb{G}_m$ ($\mathbb{G}_m$ embeds diagonally). It is a $\theta$-twisted elliptic endoscopic group of PGL(4). We deduce from the computation that $\chi_\pi$ is an unstable function: its value at one twisted regular elliptic conjugacy class with norm in $C = \mathbf{C}(F)$ is minus its value at the other class within the twisted stable conjugacy class, and 0 at the classes without norm in $C$. Moreover $\pi$ is the unstable endoscopic lift of the trivial representation of $C$.

Naturally, this computation plays a role in the theory of lifting from $\mathbf{C}$(="SO(4)") and PGp(2) to $\mathbf{G} = \mathrm{PGL}(4)$ using the trace formula, to be discussed elsewhere.

Our work develops a 4-dimensional analogue of the model of the small representation of $\mathrm{PGL}(3, F)$ introduced with Kazhdan in [FK] in a 3-dimensional case. It uses the classification of twisted stable and unstable regular conjugacy classes in $\mathrm{PGL}(4, F)$ of [F], motivated by Weissauer [W]. It extends the local method of computation introduced by us in [FZ].

## Introduction

Let $\pi$ be an admissible representation (see Bernstein-Zelevinsky [BZ], 2.1), of a $p$-adic reductive group $G$. Its character $\chi_\pi$ is a complex valued function defined by $\mathrm{tr}\,\pi(fdg) = \int_G \chi_\pi(g) f(g) dg$ for all complex valued smooth compactly supported measures $fdg$ ([BZ], 2.17). It is smooth on the regular set of the group $G$. The character is important since it characterizes the representation up to equivalence. A fundamental result of Harish-Chandra [H] establishes that the character is a locally integrable function in characteristic zero.

Let $\theta$ be an automorphism of finite order of the group $G$. Define ${}^\theta\pi$ by ${}^\theta\pi(g) = \pi(\theta(g))$. When $\pi$ is invariant under the action of $\theta$ (thus ${}^\theta\pi$ is equivalent to $\pi$), Shintani and others introduced an extension of $\pi$ to the semidirect product $G \rtimes \langle\theta\rangle$. The twisted character $\chi_\pi(g \times \theta)$ is defined by $\mathrm{tr}\,\pi(fdg \times \theta) = \int_G \chi_\pi(g \times \theta) f(g) dg$ for all $fdg$. It depends only on the $\theta$-conjugacy class $\{hg\theta(h)^{-1}; h \in G\}$ of $g$. It is again smooth on the $\theta$-regular set,

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and characterizes the $\theta$-invariant irreducible $\pi$ up to isomorphism. Moreover, it is locally integrable (see Clozel [C]) in characteristic zero.

Characters provide a very precise tool to express a relation of representations of different groups, called lifting. It was studied extensively by Shintani and others in the case of base change. It was studied also in non base change situations such as twisting by characters (Kazhdan [K], Waldspurger [Wa]), and the symmetric square lifting from SL(2) to PGL(3) ([Fsym], [FK]). In this last case twisted characters of $\theta$-invariant representations of PGL(3) are related to packets of representations of SL(2), and $\theta$ is the involution sending $g$ to its transpose-inverse.

*The aim of the present work is to compute the twisted* (*by* $\theta$) *character of a specific representation* $\pi = I_{(3,1)}(1_3)$, *of the group* $G = \mathrm{PGL}(4, F)$, $F$ *a* $p$*-adic field,* $p$ *odd.* This $\pi$ is normalizedly induced from the trivial representation $1_3$ of the standard (upper triangular) maximal parabolic subgroup $P$ of type $(3,1)$. It is invariant under the involution $\theta(g) = J^{-1}{}^t g^{-1} J$, where $J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

A natural setting for the statement of our result is the theory of liftings to the group $\mathbf{G} = \mathrm{PGL}(4)$ from its $\theta$-twisted endoscopic group (see Kottwitz-Shelstad [KS])

$$\mathbf{C} = \{(g, g') \in \mathrm{GL}(2) \times \mathrm{GL}(2);\ \det g = \det g'\}/\mathbb{G}_m.$$

Here the multiplicative group $\mathbb{G}_m = \mathrm{GL}(1)$ embeds as $z \mapsto (zI_2, zI_2)$, $I_2$ is the identity $2 \times 2$ matrix. The corresponding map $\lambda_1$ of dual groups is simply the natural embedding in $\hat{G} = \mathrm{SL}(4, \mathbb{C})$ of $\hat{C} = Z_{\hat{G}}(\hat{s}\hat{\theta}) =$ "$\mathrm{SO}(4, \mathbb{C})$"

$$= \left\{ g \in \hat{G} = \mathrm{SL}(4, \mathbb{C});\ g\hat{s}J{}^t g = \hat{s}J = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix} \right\} = \mathrm{SO}\left( \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix}, \mathbb{C} \right)$$

$$= \left\{ \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix};\ \left( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B \right) \in (\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}),\ \det A \cdot \det B = 1)/\mathbb{C}^\times \right\}.$$

Here $z \in \mathbb{C}^\times$ embeds as the central element $(z, z^{-1})$, and $\hat{s} = \mathrm{diag}(-1, 1, -1, 1)$ and $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus $\hat{C}$ is the $\hat{\theta}$-centralizer in $\hat{G}$ of the semisimple element $\hat{s}$, and $\hat{\theta}$ is defined on $\hat{G}$ by the same formula that defines $\theta$ on $G$.

Our result can be viewed as asserting that the $\theta$-invariant representation $\pi$ of $G = \mathbf{G}(F)$ is the endoscopic lift of the trivial representation of $C = \mathbf{C}(F)$. To state this we note that the embedding $\lambda_1 : \hat{C} \to \hat{G}$ defines a norm map. This norm map relates the stable $\theta$-conjugacy classes in $G$ with stable conjugacy classes in $C$. A stable conjugacy class is the intersection $G \cap (\mathrm{Int}(\mathbf{G}(\overline{F}))(\gamma))$, for some $\gamma \in G$. The crucial case of the character computation is that of $\theta$-elliptic elements. A stable $\theta$-conjugacy class consists of several $\theta$-conjugacy classes. The stable $\theta$-conjugacy classes of elements in $G$, and the $\theta$-conjugacy classes within the stable $\theta$-classes, have been described recently in [F], in analogy with the description of the conjugacy classes and the stable classes in the group of symplectic similitudes $\mathrm{Gp}(2, F)$ of Weissauer [W]. In fact in [F] we deal with $\theta$-classes in $\mathrm{GL}(4, F)$ while here we deal with the simpler case of $\mathrm{PGL}(4, F)$, so we give here full details of the description in our case.

There are four types of $\theta$-elliptic elements of $G$, named in [F], p. 16, and here (see the next section) I, II, III, IV, depending on their splitting behaviour. As in [F], our work relies on an explicit presentation of representatives of the $\theta$-conjugacy classes within the stable such classes in $G$, except that here we present a better looking set of such representatives.

The norm map, which we describe explicitly here, relates $\theta$-conjugacy classes of types I and III to conjugacy classes in $C$. It does not relate classes of types II, IV to classes in $C$.

*We prove that the $\theta$-character of $\pi$, $\chi_\pi(g\times\theta)$, vanishes on $\theta$-regular elements $g$ of type II and IV.* The stable $\theta$-conjugacy classes of types I and III come associated with a quadratic extension $E/F$ in type I and $E/E_3$ in type III (in this case $E_3/F$ is a quadratic extension, and $E/F$ is biquadratic). The two $\theta$-conjugacy classes $g_r$ within the stable $\theta$-classes are parametrized by $r$ in $F^\times/N_{E/F}E^\times$ in type I and by $E_3^\times/N_{E/E_3}E^\times$ in type III. *We show that the value of $\chi_\pi(g_r\times\theta)$, multiplied by a suitable Jacobian $\Delta(g_r\theta)/\Delta_C(Ng)$, is $2\kappa(r)$.* Here $\kappa$ is the nontrivial character of $F^\times/N_{E/F}E^\times$ in type I and of $E_3^\times/N_{E/E_3}E^\times$ in type III.

In particular the character $\chi_\pi(g\times\theta)$ is an unstable function, namely its value at one $\theta$-conjugacy class within a stable $\theta$-conjugacy class of type I or III is negative its value at the other $\theta$-conjugacy class.

Our result is a special case of the lifting with respect to $\lambda_1$ to the group $G=\mathrm{PGL}(4,F)$ of representations of the group $C=(\mathrm{GL}(2,F)\times\mathrm{GL}(2,F))'/F^\times$, where the prime indicates $\det g=\det g'$ for the two components $(g,g')$, and $F^\times$ embeds diagonally. This lifting is established in [F′] by means of a comparison of trace formulae, the fundamental lemma of [F], and character relations, for generic and nongeneric representations. It can be viewed as associating to a pair $\pi_1$, $\pi_2$ of representations of $\mathrm{GL}(2,F)$ (the product of whose central characters is 1) a product representation $\pi=\pi_1\boxtimes\pi_2$, or $\lambda_1(\pi_1\times\pi_2)$, of $\mathrm{PGL}(4,F)$.

The case that we consider here is that where $\pi_1$ and $\pi_2$ are the trivial representations of $\mathrm{GL}(2,F)$. Their product via the lifting $\lambda_1$, $1\boxtimes 1$ or $\lambda_1(1\times 1)$, is our $\pi=I_{(3,1)}(1_3)$.

This notion of multiplication should not be confused with that of induction from the standard parabolic subgroup of type $(2,2)$ and $\pi_1\otimes\pi_2$ on its Levi factor, thus: $I_{(2,2)}(\pi_1\otimes\pi_2)$, which plays the role of addition of $\pi_1$ and $\pi_2$. This multiplication is interesting in particular since conjecturally the notions of multiplication $\pi_1\boxtimes\pi_2=\lambda_1(\pi_1\times\pi_2)$ and addition $\pi_1\boxplus\pi_2=I_{(2,2)}(\pi_1\otimes\pi_2)$ – suitably extended to all $\mathrm{GL}(n)$ – give a Tannakian structure on the category of algebraic (= smooth) representations of all the $\mathrm{GL}(n,F)$, and its motivic Galois group (see Deligne-Milne [DM]) plays a key role in the principle of functoriality.

However, the proof of [F′], based on trace formulae comparison and the fundamental lemma of [F], is very involved. The current paper grew from an attempt to provide a purely local and self contained proof of a key initial case, where the representations involved are not generic, where all principal features can be explicitly viewed: the analysis of the $\theta$-conjugacy classes, the norm map, the Jacobian factors and the transfer factors, and the character relations can be computed directly to verify them without relying on long and complex theories.

This gives an independent verification of results obtainable by global techniques, by purely local and essentially elementary techniques.

Our method here is based on using a novel model of our representation $\pi=I_{(3,1)}(1_3)$, different from the standard model of a parabolically induced representation. It is a four dimensional analogue of a three dimensional model introduced and used with Kazhdan in [FK] to compute the twisted by transpose-inverse character of the representation $\pi_3 = I_{(2,1)}(1_2)$ of $\mathrm{PGL}(3,F)$ normalizedly induced from the trivial representation of the maximal parabolic subgroup. The original interest of [FK] was in a case of the fundamental lemma for the symmetric square lifting. But a purely local and simpler proof was given later in [Fsym; Unit elements]. In our case the fundamental lemma is established in [F].

The work of [FK] uses local arguments to compute the twisted character of $\pi_3$ on one of the two twisted conjugacy classes within the stable one (where the quadratic form is anisotropic), and global arguments to reduce the computation on the other class (where the quadratic form is isotropic) to that computed by local means. A purely local computation for the second class is given in [FZ]. Here we develop this local computation in our four dimensional case. A global type of argument as in [FK] is harder to apply as there are not enough anisotropic quadratic forms in our case. Anyway, here we give a simpler, local proof.

We believe that our method of computation is applicable in many cases of character computations, giving rise to a new theory of integration of functions on $p$-adic domains, and we plan to return to this topic in future work.

## Conjugacy classes

Let $F$ be a local nonarchimedean field, and $R$ its ring of integers. Put $\mathbf{G} = \mathrm{PGL}(4)$, $G = \mathbf{G}(F)$ and $K = \mathbf{G}(R)$. Put $\mathbf{C} = \{(g_1, g_2) \in \mathrm{GL}(2) \times \mathrm{GL}(2); \det(g_1) = \det(g_2)\}/\mathbb{G}_m$ ($\mathbb{G}_m$ embeds diagonally), $C = \mathbf{C}(F) = \{(g_1, g_2) \in \mathrm{GL}(2,F) \times \mathrm{GL}(2,F); \det(g_1) = \det(g_2)\}/F^\times$ and $K_C = \mathbf{C}(R)$. Put $J = (a_i \delta_{i,5-j})$, $a_1 = a_2 = 1$, $a_3 = a_4 = -1$, and set $\theta(\delta) = J^{-1\,t}\delta^{-1} J$ for $\delta$ in $G$. Fix a separable algebraic closure $\overline{F}$ of $F$. The elements $\delta$, $\delta'$ of $G$ are called (*stably*) *$\theta$-conjugate* if there is $g$ in $G$ (resp. $\mathrm{PGL}(4, \overline{F})$) with $\delta' = g^{-1}\delta\theta(g)$.

We recall some results of [F] concerning (stable) $\theta$-twisted regular conjugacy classes. There are four types of $\theta$-elliptic classes, but the norm map $N$ from $G$ to $C$ relates only the twisted classes in $G$ of type I and III to conjugacy classes in $C$. We should then expect the twisted character of the representation considered here to vanish on the twisted classes of type II and IV.

A set of representatives for the $\theta$-conjugacy classes within a stable semisimple $\theta$-conjugacy class of type I in $\mathrm{GL}(4,F)$ which splits over a quadratic extension $E = F(\sqrt{D})$ of $F$, $D \in F - F^2$, is parametrized by $(\mathbf{r}, \mathbf{s}) \in F^\times/N_{E/F}E^\times \times F^\times/N_{E/F}E^\times$ ([F], p. 16). Representatives for the $\theta$-regular (thus $t\theta(t)$ is regular) stable $\theta$-conjugacy classes of type I in $\mathrm{GL}(4,F)$ which split over $E$ can be found in a torus $T = \mathbf{T}(F)$, $\mathbf{T} = h^{-1}\mathbf{T}^* h$, $\mathbf{T}^*$ denoting the diagonal subgroup in $\mathbf{G}$, $h = \theta(h)$, and

$$T = \left\{ t = \begin{pmatrix} a_1 & 0 & 0 & a_2 D \\ 0 & b_1 & b_2 D & 0 \\ 0 & b_2 & b_1 & 0 \\ a_2 & 0 & 0 & a_1 \end{pmatrix} = h^{-1} t^* h; \quad t^* = \mathrm{diag}(a, b, \sigma b, \sigma a) \in T^* \right\}.$$

Here $a = a_1 + a_2\sqrt{D}$, $b = b_1 + b_2\sqrt{D} \in E^\times$, and $t$ is regular if $a/\sigma a$ and $b/\sigma b$ are distinct and not equal to $\pm 1$. Note that here $T^* = \mathbf{T}^*(F)$ where the Galois action is that obtained from the Galois action on $T$.

A set of representatives for the $\theta$-conjugacy classes within a stable $\theta$-conjugacy class can be chosen in $T$. Indeed, if $t = h^{-1}t^*h$ and $t_1 = h^{-1}t_1^*h$ in $T$ are stably $\theta$-conjugate, then there is $g = h^{-1}\mu h$ with $t_1 = gt\theta(g)^{-1}$, thus $t_1^* = \mu t^*\theta(\mu)^{-1}$ and $t_1^*\theta(t_1^*) = \mu t^*\theta(t^*)\mu^{-1}$. Since $t$ is $\theta$-regular, $\mu$ lies in the $\theta$-normalizer of $\mathbf{T}^*(\overline{F})$ in $\mathbf{G}(\overline{F})$. Since the group $W^\theta(\mathbf{T}^*, \mathbf{G}) = N^\theta(\mathbf{T}^*, \mathbf{G})/\mathbf{T}^*$, quotient by $\mathbf{T}^*(\overline{F})$ of the $\theta$-normalizer of $\mathbf{T}^*(\overline{F})$ in $\mathbf{G}(\overline{F})$, is represented by the group $W^\theta(T^*, G) = N^\theta(T^*, G)/T^*$, quotient by $T^*$ of the $\theta$-normalizer of $T^*$ in $G$, we may modify $\mu$ by an element of $W^\theta(T^*, G)$, that is replace $t_1$ by a $\theta$-conjugate element, and assume that $\mu$ lies in $\mathbf{T}^*(\overline{F})$. In this case $\mu\theta(\mu)^{-1} = \mathrm{diag}(u, u', \sigma u', \sigma u)$ (since $t$, $t_1$ lie in $T^*$), with $u = \sigma u$, $u' = \sigma u'$ in $F^\times$. Such $t$, $t_1$ are $\theta$-conjugate if $g \in G$, thus $g \in T$, so $\mu = \mathrm{diag}(v, v', \sigma v', \sigma v) \in T^*$ and $\mu\theta(\mu)^{-1} = \mathrm{diag}(v\sigma v, v'\sigma v', v'\sigma v', v\sigma v)$. Hence a set of representatives for the $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of the $\theta$-regular $t$ in $T$ is given by $t \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r})$, where $\mathbf{r}$, $\mathbf{s} \in F^\times/N_{E/F}E^\times$. Clearly in $\mathrm{PGL}(4, F)$ the $\theta$-classes within a stable class are parametrized only by $\mathbf{r}$, or equivalently only by $\mathbf{s}$.

A set of representatives for the $\theta$-conjugacy classes within a stable semisimple $\theta$-conjugacy class of type II in $\mathrm{GL}(4, F)$ which splits over the biquadratic extension $E = E_1E_2$ of $F$ with Galois group $\langle\sigma, \tau\rangle$, where $E_1 = F(\sqrt{D}) = E^\tau$, $E_2 = F(\sqrt{AD}) = E^{\sigma\tau}$, $E_3 = F(\sqrt{A}) = E^\sigma$ are quadratic extensions of $F$, thus $A, D \in F - F^2$, is parametrized by $\mathbf{r} \in F^\times/N_{E_1/F}E_1^\times$, $\mathbf{s} \in F^\times/N_{E_2/F}E_2^\times$ ([F], p. 16). It is given by

$$\begin{pmatrix} a_1\mathbf{r} & 0 & 0 & a_2D\mathbf{r} \\ 0 & b_1\mathbf{s} & b_2AD\mathbf{s} & 0 \\ 0 & b_2\mathbf{s} & b_1\mathbf{s} & 0 \\ a_2\mathbf{r} & 0 & 0 & a_1\mathbf{r} \end{pmatrix} = h^{-1}t^*h \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r}), \qquad t^* = \mathrm{diag}(a, b, \tau b, \sigma a).$$

Here $a = a_1 + a_2\sqrt{D} \in E_1^\times$, $b = b_1 + b_2\sqrt{AD} \in E_2^\times$, $\theta(h) = h$. In $\mathrm{PGL}(4, F)$ the $\theta$-classes within a stable class are parametrized only by $\mathbf{r}$, or equivalently only by $\mathbf{s}$.

A set of representatives for the $\theta$-conjugacy classes within a stable semisimple $\theta$-conjugacy class of type III in $\mathrm{GL}(4, F)$ which splits over the biquadratic extension $E = E_1E_2$ of $F$ with Galois group $\langle\sigma, \tau\rangle$, where $E_1 = F(\sqrt{D}) = E^\tau$, $E_2 = F(\sqrt{AD}) = E^{\sigma\tau}$, $E_3 = F(\sqrt{A}) = E^\sigma$ are quadratic extensions of $F$, thus $A, D \in F - F^2$, is parametrized by $r (= r_1 + r_2\sqrt{A}) \in E_3^\times/N_{E/E_3}E^\times$ ([F], p. 16). Representatives for the stable regular $\theta$-conjugacy classes can be taken in the torus $T = h^{-1}T^*h$, consisting of

$$t = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b}D \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix} = h^{-1}t^*h, \qquad t^* = \mathrm{diag}(\alpha, \tau\alpha, \sigma\tau\alpha, \sigma\alpha),$$

where $h = \theta(h)$ is described in [F], p. 16. This $t$ is $\theta$-regular when $\alpha/\sigma\alpha$, $\tau(\alpha/\sigma\alpha)$ are distinct and $\neq \pm 1$. Here

$$\boldsymbol{a} = \begin{pmatrix} a_1 & a_2A \\ a_2 & a_1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 & b_2A \\ b_2 & b_1 \end{pmatrix}; \qquad \text{put also} \qquad \boldsymbol{r} = \begin{pmatrix} r_1 & r_2A \\ r_2 & r_1 \end{pmatrix}.$$

Further $\alpha = a + b\sqrt{D} \in E^\times$, $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, $\sigma\alpha = a - b\sqrt{D}$, $\tau\alpha = \tau a + \tau b\sqrt{D}$. Representatives for all $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t$ can be taken in $T$. In fact if $t' = gt\theta(g)^{-1}$ lies in $T$ and $g = h^{-1}\mu h$, $\mu \in \mathbf{T}^*(\overline{F})$, then $\mu\theta(\mu)^{-1} = \operatorname{diag}(u, \tau u, \sigma\tau u, \sigma u)$ has $u = \sigma u$, thus $u \in E_3^\times$. If $g \in T$, thus $\mu \in T^*$, then $\mu = \operatorname{diag}(v, \tau v, \sigma\tau v, \sigma v)$ and $\mu\theta(\mu)^{-1} = \operatorname{diag}(v\sigma v, \tau v\sigma\tau v, \tau v\sigma\tau v, v\sigma v)$, with $v\sigma v \in N_{E/E_3}E^\times$. We conclude that a set of representatives for the $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t$ is given by $t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$, $r \in E_3^\times / N_{E/E_3}E^\times$.

Representatives for the stable regular $\theta$-conjugacy classes of type IV can be taken in the torus $T = h^{-1}T^*h$, consisting of

$$t = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{bD} \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix} = h^{-1}t^*h, \qquad t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha),$$

where $h = \theta(h)$ is described in [F], p. 18. Here $\alpha$ ranges over a quadratic extension $E = F(\sqrt{D}) = E_3(\sqrt{D})$ of a quadratic extension $E_3 = F(\sqrt{A})$ of $F$. Thus $A \in F - F^2$, $D = d_1 + d_2\sqrt{A}$ lies in $E_3 - E_3^2$ where $d_i \in F$. The normal closure $E'$ of $E$ over $F$ is $E$ if $E/F$ is cyclic with Galois group $\mathbb{Z}/4$, or a quadratic extension of $E$, generated by a fourth root of unity $\zeta$, in which case the Galois group is the dihedral group $D_4$. In both cases the Galois group contains an element $\sigma$ with $\sigma\sqrt{A} = -\sqrt{A}$, $\sigma\sqrt{D} = \sqrt{\sigma D}$, $\sigma^2\sqrt{D} = -\sqrt{D}$. In the $D_4$ case $\operatorname{Gal}(E'/F)$ contains also $\tau$ with $\tau\zeta = -\zeta$, we may choose $D = \sqrt{A}$, $\tau D = D$ and $\sigma\sqrt{D} = \zeta\sqrt{D}$.

In any case, $t$ is $\theta$-regular when $\alpha \neq \sigma^2\alpha$. We write $\alpha = a + b\sqrt{D} \in E^\times$, $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, $\sigma\alpha = \sigma a + \sigma b\sqrt{\sigma D}$, $\sigma^2\alpha = a - b\sqrt{D}$. Also

$$\boldsymbol{a} = \begin{pmatrix} a_1 & a_2A \\ a_2 & a_1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 & b_2A \\ b_2 & b_1 \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} d_1 & d_2A \\ d_2 & d_1 \end{pmatrix}.$$

Representatives for all $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t$ can be taken in $T$. In fact if $t' = gt\theta(g)^{-1}$ lies in $T$ and $g = h^{-1}\mu h$, $\mu \in \mathbf{T}^*(\overline{F})$, then $\mu\theta(\mu)^{-1} = \operatorname{diag}(u, \sigma u, \sigma^3 u, \sigma^2 u)$ has $u = \sigma^2 u$, thus $u \in E_3^\times$. If $g \in T$, thus $\mu \in T^*$, then $\mu = \operatorname{diag}(v, \sigma v, \sigma^3 v, \sigma^2 v)$ and $\mu\theta(\mu)^{-1} = \operatorname{diag}(v\sigma^2 v, \sigma(v\sigma^2 v), \sigma(v\sigma^2 v), v\sigma^2 v)$, with $v\sigma v \in N_{E/E_3}E^\times$. It follows that a set of representatives for the $\theta$-conjugacy classes within the stable $\theta$-conjugacy class of $t = h^{-1}t^*h = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{bD} \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix}$, where $t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$, is given by multiplying $\alpha$ by $r$, that is $t^*$ by $t_0^* = \operatorname{diag}(r, \sigma r, \sigma^3 r, \sigma^2 r)$, where $r = \sigma^2 r$ ranges over a set of representatives for $E_3^\times / N_{E/E_3}E^\times$. Now $t_0 = h^{-1}t_0^*h = \begin{pmatrix} \boldsymbol{r} & \\ & \boldsymbol{r} \end{pmatrix}$. Hence a set of representatives is given by $t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$, $r \in E_3^\times / N_{E/E_3}E^\times$.

## Norm map

The norm map $N : \mathbf{G} \to \mathbf{C}$ is defined on the diagonal torus $\mathbf{T}^*$ of $\mathbf{G}$ by

$$N(\operatorname{diag}(a, b, c, d)) = (\operatorname{diag}(ab, cd), \operatorname{diag}(ac, bd)).$$

Since both components have determinant $abcd$, the image of $N$ is indeed in $\mathbf{C}$.

In type I we have $a, b \in E^\times = F(\sqrt{D})^\times$ and the norm map becomes

$$N(\operatorname{diag}(a, b, \sigma b, \sigma a)) = (\operatorname{diag}(ab, \sigma(ab)), \operatorname{diag}(a\sigma b, b\sigma a)).$$

In type III we have $\alpha \in E^\times$, $\alpha\tau\alpha \in E_1^\times$, $\alpha\sigma\tau\alpha \in E_2^\times$, and the norm map becomes

$$N(\mathrm{diag}(\alpha, \tau\alpha, \sigma\tau\alpha, \sigma\alpha)) = (\mathrm{diag}(\alpha\tau\alpha, \sigma\alpha\sigma\tau\alpha), \mathrm{diag}(\alpha\sigma\tau\alpha, \tau\alpha\sigma\alpha)).$$

The $\mathrm{diag}(*,*)$ define conjugacy classes in $\mathrm{GL}(2,F)$, and since both components of $N(*)$ have equal determinants in $F^\times$, the norm map defines a conjugacy class in $C = \mathbf{C}(F)$ for each $\theta$-stable conjugacy class of type I or III in $G = \mathbf{G}(F)$.

In types II and IV no conjugacy class in $C$ corresponds to the image of the map $N$.

## Jacobians

The character relation that we study relates the product of the value at $t$ of the twisted character of our representation $\pi = I_{(3,1)}(\pi_1)$ by a factor $\Delta(t \times \theta)$, with the product by a factor $\Delta_C(Nt)$ of the value at $Nt$ of the character of the representation $\pi_C$ of $C$ which lifts to $\pi$.

The factor $\Delta(t \times \theta)$ is defined by

$$\Delta(t \times \theta)^2 = |\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}(G/T)|.$$

Here $t$ lies in the $\theta$-invariant torus $T$ which we take to have the form $T = h^{-1}T^*h$, $T^*$ is the diagonal subgroup and $h = \theta(h)$. Thus in the formula for $\Delta(t \times \theta)$ we may replace $t = h^{-1}t^*h$ and $T$ by the diagonal $t^*$ and $T^*$. Note that $\mathrm{Lie}(G/T^*) = \mathrm{Lie}\,U \oplus \mathrm{Lie}\,U^-$, and the upper and lower triangular subgroups $U$, $U^-$ are $\theta$-invariant. We have

$$|\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}\,U| = |\prod_{\Theta}(1 - \sum_{\beta\in\Theta}\beta(t))|,$$

where the product ranges over the orbits $\Theta$ of $\theta$ in the set of positive roots $\beta > 0$, and the sum ranges over the roots in the $\theta$-orbit. Thus for $t = \mathrm{diag}(a,b,c,d)$ we obtain

$$\left|\left(1 - \frac{a}{b}\frac{c}{d}\right)\left(1 - \frac{a}{c}\frac{b}{d}\right)\left(1 - \frac{a}{d}\right)\left(1 - \frac{b}{c}\right)\right|.$$

Further,

$$|\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}\,U^-| = \delta(a\theta)^{-1}|\det(1 - \mathrm{Ad}(t\theta))|\,\mathrm{Lie}\,U|$$

where

$$\delta(t\theta) = \prod_{\Theta}((\sum_{\beta\in\Theta}\beta)(t)) = (\frac{a}{b}\frac{c}{d})(\frac{a}{c}\frac{b}{d})(\frac{b}{c})(\frac{a}{d}) = \prod_{\beta>0}\beta(t) = \delta(t).$$

Altogether

$$\Delta(t\theta) = \left|\frac{(ac-bd)^2}{abcd}\cdot\frac{(ab-cd)^2}{abcd}\cdot\frac{(a-d)^2}{ad}\frac{(b-c)^2}{bc}\right|^{1/2}.$$

Similarly,

$$\Delta_C(Nt) = \delta_C^{-1/2}(Nt)|\det(1 - Nt)|\,\mathrm{Lie}\,U_C| = \left|\frac{ab}{cd}\cdot\frac{ac}{bd}\right|^{-1/2}\left|\left(1 - \frac{ab}{cd}\right)\left(1 - \frac{ac}{bd}\right)\right|,$$

and so

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left|\frac{(a-d)^2}{ad}\cdot\frac{(b-c)^2}{bc}\right|^{1/2}.$$

Then in case I if $t = \operatorname{diag}(a, b, \sigma b, \sigma a)$, $a = a_1 + a_2\sqrt{D}$, $b = b_1 + b_2\sqrt{D}$, we get

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left|\frac{(a-\sigma a)^2}{a\sigma a}\cdot\frac{(b-\sigma b)^2}{b\sigma b}\right|^{1/2} = \left|\frac{(2a_2\sqrt{D})^2}{a_1^2-a_2^2D}\cdot\frac{(2b_2\sqrt{D})^2}{b_1^2-b_2^2D}\right|^{1/2}.$$

In case III, if $t = \operatorname{diag}(\alpha, \tau\alpha, \sigma\tau\alpha, \sigma\alpha)$, $\alpha = a + b\sqrt{D}$, $a = a_1 + a_2\sqrt{A}$, $b = b_1 + b_2\sqrt{A}$, $\sigma\alpha = a - b\sqrt{D}$, $\tau\alpha = \tau a + \tau b\sqrt{D}$, $\alpha - \sigma\alpha = 2b\sqrt{D}$, $\tau(\alpha - \sigma\alpha) = 2\tau b\sqrt{D}$, and

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left|\frac{(\alpha-\sigma\alpha)^2}{\alpha\sigma\alpha}\cdot\frac{\tau(\alpha-\sigma\alpha)^2}{\tau\alpha\tau\sigma\alpha}\right|^{1/2} = \left|\frac{(4b\tau bD)^2}{(a^2-b^2D)(\tau a^2-\tau b^2D)}\right|^{1/2}.$$

## Characters

Denote by $f$ (resp. $f_C$) a complex-valued compactly-supported smooth (thus locally-constant since $F$ is nonarchimedean) function on $G$ (resp. $C$). Fix Haar measures on $G$ and on $C$.

By a $G$-module $\pi$ (resp. $C$-module $\pi_C$) we mean an admissible representation ([BZ]) of $G$ (resp. $C$) in a complex space. An irreducible $G$-module $\pi$ is called *$\theta$-invariant* if it is equivalent to the $G$-module ${}^\theta\pi$, defined by ${}^\theta\pi(g) = \pi(\theta(g))$. In this case there is an intertwining operator $A$ on the space of $\pi$ with $\pi(g)A = A\pi(\theta(g))$ for all $g$. Since $\theta^2 = 1$ we have $\pi(g)A^2 = A^2\pi(g)$ for all $g$, and since $\pi$ is irreducible $A^2$ is a scalar by Schur's lemma. We choose $A$ with $A^2 = 1$. This determines $A$ up to a sign. When $\pi$ has a Whittaker model, which happens for all components of cuspidal automorphic representations of the adele group $\mathrm{PGL}(4, \mathbb{A})$, we specify a normalization of $A$ which is compatible with a global normalization, as follows, and then put $\pi(g \times \theta) = \pi(g) \times A$.

Fix a nontrivial character $\boldsymbol{\psi}$ of $F$ in $\mathbb{C}^\times$, and a character $\psi(u) = \boldsymbol{\psi}(a_{1,2} + a_{2,3} - a_{3,4})$ of $u = (u_{i,j})$ in the upper triangular subgroup $U$ of $G$. Note that $\psi(\theta(u)) = \psi(u)$. Assume that $\pi$ is a *nondegenerate* $G$-module, namely it embeds in the space of "Whittaker" functions $W$ on $G$, which satisfy – by definition – $W(ugk) = \psi(u)W(g)$ for all $g \in G$, $u \in U$, $k$ in a compact open subgroup $K_W$ of $K$, as a $G$-module under right shifts: $(\pi(g)W)(h) = W(hg)$. Then ${}^\theta\pi$ is nondegenerate and can be realized in the space of functions ${}^\theta W(g) = W(\theta(g))$, $W$ in the space of $\pi$. We take $A$ to be the operator on the space of $\pi$ which maps $W$ to ${}^\theta W$.

A $G$-module $\pi$ is called *unramified* if the space of $\pi$ contains a nonzero $K$-fixed vector. The dimension of the space of $K$-fixed vectors is bounded by one if $\pi$ is irreducible. If $\pi$ is $\theta$-invariant and unramified, and $v_0 \neq 0$ is a $K$-fixed vector in the space of $\pi$, then $Av_0$ is a multiple of $v_0$ (since $\theta K = K$), namely $Av_0 = cv_0$, with $c = \pm 1$. Replace $A$ by $cA$ to have $Av_0 = v_0$, and put $\pi(\theta) = A$.

When $\pi$ is (irreducible) unramified and has a Whittaker model, both normalizations of the intertwining operator are equal. In this case $\boldsymbol{\psi}$ is unramified (trivial on $R$ but not on $\boldsymbol{\pi}^{-1}R$, where $\boldsymbol{\pi}$ is a generator of the maximal ideal of $R$), and there exists a unique Whittaker function $W_0$ in the space of $\pi$ with respect to $\psi$ with $W_0 = 1$ on $K$. It is mapped by $\pi(\theta) = A$ to ${}^\theta W_0$, which satisfies ${}^\theta W_0(k) = 1$ for all $k$ in $K$ since $K$ is $\theta$-invariant. Namely $A$ maps the unique normalized (by $W_0(K) = 1$) $K$-fixed vector $W_0$ in the space of $\pi$ to the unique normalized $K$-fixed vector ${}^\theta W_0$ in the space of ${}^\theta\pi$, and we have ${}^\theta W_0 = W_0$.

For any $\pi$ and $f$ the convolution operator $\pi(f) = \int_G f(g)\pi(g)dg$ has finite rank. If $\pi$ is $\theta$-invariant put $\pi(f \times \theta) = \int_G f(g)\pi(g)\pi(\theta)dg$. Denote by $\operatorname{tr} \pi(f \times \theta)$ the trace of the operator $\pi(f \times \theta)$. It depends on the choice of the Haar measure $dg$, but the (*twisted*) *character* $\chi_\pi$ of $\pi$ does not; $\chi_\pi$ is a locally-integrable (at least in characteristic zero) complex-valued function on $G \times \theta$ (see [C], [H]) which is $\theta$-conjugacy invariant and locally-constant on the $\theta$-regular set, with $\operatorname{tr} \pi(f \times \theta) = \int_G f(g)\chi_\pi(g \times \theta)dg$ for all $f$.

Local integrability is not used in this work; rather it is recovered for our twisted character.

## Small representation

To describe the $G$-module of interest in this paper, note that a Levi subgroup $M$ of a maximal parabolic subgroup $P$ of $G$ of type $(3,1)$ is isomorphic to $\mathrm{GL}(3,F)$. Hence a $GL(3,F)$-module $\pi_1$ extends to a $P$-module trivial on the unipotent radical $N(= F^3)$ of $P$. Let $\delta$ denote (as above) the character $\delta(p) = |\operatorname{Ad}(p)|\operatorname{Lie} N|$ of $P$; it is trivial on $N$. Take $P$ to be the upper triangular parabolic subgroup of type (3,1), and $M = \{m = \operatorname{diag}(ah, a)^*;\ h \in \mathrm{GL}(3,F),\ a \in F^\times\}$. Here $g^*$ denotes the image in $\mathrm{PGL}(4,F)$ of $g$ from $\mathrm{GL}(4,F)$. Then the value of $\delta$ at $p = mn$ is $|\det h|$. Denote by $I(\pi_1)$ the $G$-module $\pi = \operatorname{Ind}(\delta^{1/2}\pi_1;\ P, G)$ normalizedly induced from $\pi_1$ on $P$ to $G$. It is clear from [BZ] that when $\pi_1$ is self-contragredient and $I(\pi_1)$ is irreducible then it is $\theta$-invariant, and it is unramified if and only if $\pi_1$ is unramified.

*Our aim in this work is to compute the $\theta$-twisted character $\chi_\pi$ of the* $\mathrm{PGL}(4,F)$*-module* $\pi = I(1_3)$*, where* $1_3$ *is the trivial $P$-module, by purely local means.*

We begin by describing a useful model of our representation, in analogy with the model of [FK] of an analogous representation $I_{(2,1)}(1_2)$ of $\mathrm{PGL}(3,F)$. Indeed we shall express $\pi$ as an integral operator in a convenient model, and integrate the kernel over the diagonal to compute the character of $\pi$.

Denote by $\mu = \mu_s$ the character $\mu(x) = |x|^{(s+1)/2}$ of $F^\times$. It defines a character $\mu_P = \mu_{s,P}$ of $P$, trivial on $N$, by $\mu_P(p) = \mu((\det m_3)/m_1{}^3)$ if $p = mn$ and $m = \begin{pmatrix} m_3 & 0 \\ 0 & m_1 \end{pmatrix}^*$ with $m_3$ in $\mathrm{GL}(3,F)$, $m_1$ in $\mathrm{GL}(1,F)$. If $s = 0$, then $\mu_P = \delta^{1/2}$. Let $W_s$ be the space of complex-valued smooth functions $\psi$ on $G$ with $\psi(pg) = \mu_P(p)\psi(g)$ for all $p$ in $P$ and $g$ in $G$. The group $G$ acts on $W_s$ by right translation: $(\pi_s(g)\psi)(h) = \psi(hg)$. By definition, $I(1_3)$ is the $G$-module $W_s$ with $s = 0$. The parameter $s$ is introduced for purposes of analytic continuation.

We prefer to work in another model $V_s$ of the $G$-module $W_s$. Let $V$ denote the space of column 4-vectors over $F$. Let $V_s$ be the space of smooth complex-valued functions $\phi$ on $V - \{0\}$ with $\phi(\lambda\mathbf{v}) = \mu(\lambda)^{-4}\phi(\mathbf{v})$. The expression $\mu(\det g)\phi({}^t g\mathbf{v})$, which is initially defined for $g$ in $\mathrm{GL}(4,F)$, depends only on the image of $g$ in $G$. The group $G$ acts on $V_s$ by $(\tau_s(g)\phi)(\mathbf{v}) = \mu(\det g)\phi({}^t g\mathbf{v})$. Let $\mathbf{v}_0 \neq 0$ be a vector of $V$ such that the line $\{\lambda\mathbf{v}_0; \lambda$ in $F\}$ is fixed under the action of ${}^tP$. Explicitly, we take $\mathbf{v}_0 = {}^t(0,0,0,1)$. It is clear that the map $V_s \to W_s$, $\phi \mapsto \psi = \psi_\phi$, where $\psi(g) = (\tau_s(g)\phi)(\mathbf{v}_0) = \mu(\det g)\phi({}^tg\mathbf{v}_0)$, is a $G$-module isomorphism, with inverse $\psi \mapsto \phi = \phi_\psi$, $\phi(\mathbf{v}) = \mu(\det g)^{-1}\psi(g)$ if $\mathbf{v} = {}^tg\mathbf{v}_0$ ($G$ acts transitively on $V - \{0\}$).

For $\mathbf{v} = {}^t(x,y,z,t)$ in $V$ put $\|\mathbf{v}\| = \max(|x|,\ |y|,\ |z|,\ |t|)$. Let $V^0$ be the quotient of the set of $\mathbf{v}$ in $V$ with $\|\mathbf{v}\| = 1$ by the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ if $\alpha$ is a unit in $R$. Denote by $\mathbb{P}V$ the projective space of lines in $V - \{0\}$. If $\Phi$ is a function on $V - \{0\}$ with $\Phi(\lambda\mathbf{v}) = |\lambda|^{-4}\Phi(\mathbf{v})$ and $d\mathbf{v} = dx\ dy\ dz\ dt$, then $\Phi(\mathbf{v})d\mathbf{v}$ is homogeneous of degree zero. Define

$$\int_{\mathbb{P}V} \Phi(\mathbf{v})d\mathbf{v} \quad \text{to be} \quad \int_{V^0} \Phi(\mathbf{v})d\mathbf{v}.$$

Clearly we have

$$\int_{\mathbb{P}V} \Phi(\mathbf{v})d\mathbf{v} = \int_{\mathbb{P}V} \Phi(g\mathbf{v})d(g\mathbf{v}) = |\det g| \int_{\mathbb{P}V} \Phi(g\mathbf{v})d\mathbf{v}.$$

Put $\nu(x) = |x|$ and $m = 2(s-1)$. Note that $\nu/\mu_s = \mu_{-s}$. Put $\langle \mathbf{v}, \mathbf{w}\rangle = {}^t\mathbf{v}J\mathbf{w}$. Then $\langle g\mathbf{v}, \theta(g)\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$.

**1. Proposition.** *The operator $T_s : V_s \to V_{-s}$, $(T_s\phi)(\mathbf{v}) = \int_{\mathbb{P}V} \phi(\mathbf{w})|\langle \mathbf{w}, \mathbf{v}\rangle|^m d\mathbf{w}$, converges when* $\operatorname{Re} s > 1/2$. *It satisfies $T_s\tau_s(g) = \tau_{-s}(\theta(g))T_s$ for all $g$ in $G$ where it converges.*

*Proof.* We have

$$\begin{aligned}
(T_s(\tau_s(g)\phi))(\mathbf{v}) &= \int (\tau_s(g)\phi)(\mathbf{w})|{}^t\mathbf{w}J\mathbf{v}|^m d\mathbf{w} = \mu(\det g) \int \phi({}^tg\mathbf{w})|{}^t\mathbf{w}J\mathbf{v}|^m d\mathbf{w} \\
&= |\det g|^{-1}\mu(\det g) \int \phi(\mathbf{w})|{}^t({}^tg^{-1}\mathbf{w})J\mathbf{v}|^m d\mathbf{w} \\
&= (\mu/\nu)(\det g) \int \phi(\mathbf{w})|{}^t\mathbf{w}J \cdot J^{-1}g^{-1}J\mathbf{v}|^m d\mathbf{w} \\
&= (\mu/\nu)(\det g) \int \phi(\mathbf{w})|\langle \mathbf{w}, \theta({}^tg)\mathbf{v}\rangle|^m d\mathbf{w} = (\nu/\mu)(\det\theta(g)) \cdot (T_s\phi)({}^t\theta(g)\mathbf{v}) \\
&= [(\tau_{-s}(\theta(g)))(T_s\phi)](\mathbf{v})
\end{aligned}$$

for the functional equation.

For the convergence, we may assume that $\phi = 1$ and ${}^t\mathbf{v} = (0,0,0,1)$, so that the integral is $\int_R |x|^m dx$, which converges for $m > -1$. Our $m$ is $2s-2$, as required. □

The spaces $V_s$ are isomorphic to the space $W$ of locally-constant complex-valued functions on $V^0$, and $T_s$ is equivalent to an operator $T_s^0$ on $W$. The proof of Proposition 1 implies also

**1. Corollary.** *The operator $T_s^0 \circ \tau_s(g^{-1})$ is an integral operator with kernel*

$$(\mu/\nu)(\det\theta(g))|\langle \mathbf{w}, \theta({}^tg^{-1})\mathbf{v}\rangle|^m \qquad (\mathbf{v}, \mathbf{w} \textit{ in } V^0)$$

*and trace*

$$\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})] = (\nu/\mu)(\det g) \int_{V^0} |{}^t\mathbf{v}gJ\mathbf{v}|^m d\mathbf{v}.$$

*Remark.* (1) In the domain where the integral converges, it is clear that $\operatorname{tr}[\tau_{-s}({}^tg) \circ T_s^0]$, which is $\operatorname{tr}[T_s^0 \circ \tau_s(g^{-1})]$, depends only on the $\sigma$-conjugacy class of $g$ if (and only if) $s = 0$.

(2) To compute the trace of the analytic continuation of $T_s^0 \circ \tau_s(g^{-1})$ it suffices to compute this trace for $s$ in the domain of convergence, and then evaluate the resulting expression at the desired $s$. Indeed, for each compact open $\sigma$-invariant subgroup $K$ of $G$ the space $W_K$ of $K$-biinvariant functions in $W$ is finite dimensional. Denote by $p_K : W \to W_K$ the natural projection. Then $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ acts on $W_K$, and the trace of the analytic continuation of $p_K \circ T_s^0 \circ \tau_s(g^{-1})$ is the analytic continuation of the trace of $p_K \circ T_s^0 \circ \tau_s(g^{-1})$. Since $K$ can be taken to be arbitrarily small the claim follows.

Next we normalize the operator $T = T_0$ so that it acts trivially on the one-dimensional space of $K$-fixed vectors in $V_s$. This space is spanned by the function $\phi_0$ in $V_s$ with $\phi_0(\mathbf{v}) = 1$ for all $\mathbf{v}$ in $V^0$.

Denote again by $\boldsymbol{\pi}$ a generator of the maximal ideal of the ring $R$ of integers in our local nonarchimedean field $F$ of odd residual characteristic. Denote by $q$ the number of elements of the residue field $R/\boldsymbol{\pi}R$ of $R$. Normalize the absolute value by $|\boldsymbol{\pi}| = q^{-1}$, and the measures by $\mathrm{vol}\{|x| \leq 1\} = 1$. Then $\mathrm{vol}\{|x| = 1\} = 1 - q^{-1}$, and the volume of $V^0$ is $(1 - q^{-4})/(1 - q^{-1}) = 1 + q^{-1} + q^{-2} + q^{-3}$.

**2. Proposition.** *If* $\mathbf{v}_0 = {}^t(0,0,0,1)$, *then* $(T\phi_0)(\mathbf{v}_0) = (1 - q^{-2(s+1)})/(1 - q^{1-2s})\phi_0(\mathbf{v}_0)$. *When* $s = 0$, *the constant is* $-q^{-1}(1 + q^{-1})$.

*Proof.* Indeed,

$$(T\phi_0)(\mathbf{v}_0) = \int_{V^0} \phi_0(\mathbf{v})|{}^t\mathbf{v}J\mathbf{v}_0|^m d\mathbf{v} = \int_{V^0} |x|^m dxdydzdt$$

$$= \left[\int_{||\mathbf{v}|| \leq 1} - \int_{||\mathbf{v}||<1}\right] |x|^m dxdydzdt \Big/ \int_{|x|=1} dx$$

$$= (1 - q^{-m-4}) \int_{|x|\leq 1} |x|^m dx \Big/ \int_{|x|=1} dx = (1 - q^{-2(s+1)})/(1 - q^{1-2s}),$$

since $m = 2(s-1)$ and $\int_{|x|\leq 1} |x|^m dx = (1 - q^{-m-1})^{-1} \int_{|x|=1} dx$. □

## Character computation for type I

For the $\theta$-conjugacy class of type I, represented by $g = t \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r})$, the product

$${}^t\mathbf{v}gJ\mathbf{v} = (t, z, x, y) \begin{pmatrix} a_1\mathbf{r} & 0 & 0 & a_2D\mathbf{r} \\ 0 & b_1\mathbf{s} & b_2D\mathbf{s} & 0 \\ 0 & b_2\mathbf{s} & b_1\mathbf{s} & 0 \\ a_2\mathbf{r} & 0 & 0 & a_1\mathbf{r} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ z \\ x \\ y \end{pmatrix}$$

is equal to

$$-t^2a_2D\mathbf{r} - z^2b_2D\mathbf{s} + x^2b_2\mathbf{s} + y^2a_2\mathbf{r}.$$

Note that the trace is a function of $g$ in the projective group, and $\mathbf{r}$ and $\mathbf{s}$ range over a set of representatives for $F^\times/N_{E/F}E^\times$.

We need to compute

$$(\frac{\nu}{\mu})(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)}\int_{V^0}|{}^t\mathbf{v}gJ\mathbf{v}|^m d\mathbf{v}$$

$$=\frac{|\mathbf{rs}|^{1-s}|4a_2b_2D|}{|(a_1^2-a_2^2D)(b_1^2-b_2^2D)|^{s/2}}\int_{V^0}|x^2b_2\mathbf{s}+y^2a_2\mathbf{r}-t^2a_2D\mathbf{r}-z^2b_2D\mathbf{s}|^{2(s-1)}dxdydzdt.$$

This is equal to

$$|\frac{\mathbf{r}}{\mathbf{s}}|^{-s}|4D\mathbf{r}'||((\frac{a_1}{b_2})^2-(\frac{a_2}{b_2})^2D)((\frac{b_1}{b_2})^2-D)|^{-s/2}\int_{V^0}|x^2-y^2\mathbf{r}'+t^2D\mathbf{r}'-z^2D|^{2(s-1)}dxdydzdt.$$

Here $\mathbf{r}'=-\frac{a_2}{b_2}\frac{\mathbf{r}}{\mathbf{s}}$. As $\mathbf{r}$ ranges over $F^\times/N_{E/F}E^\times$, we may rename $\mathbf{r}'$ by $\mathbf{r}$. We get the product of a factor whose value at $s=0$ is 1, the factor $|4D\mathbf{r}|$, and the integral

$$I_s(\mathbf{r},D)=\int_{V^0}|x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2|^{2(s-1)}dxdydzdt.$$

*Remark.* If $F$ is the field $\mathbb{R}$ of real numbers, then its only algebraic extension is the field $E=\mathbb{C}=F(\sqrt{D})$ of complex numbers, and only case I occurs. We may take $D=-1$ and $\mathbf{r}$ to range over the group $\{\pm1\}$. If $\mathbf{r}=-1$ we get

$$I_s(-1,-1)=\int_{V^0}|x^2+y^2+z^2+t^2|^{2(s-1)}dxdydzdt,$$

which is 1 (and has to be multiplied by 4). Is

$$I_s(1,-1)=\int_{V^0}|x^2-y^2+z^2-t^2|^{2(s-1)}dxdydzdt$$

equal to $-1$ (at least at $s=0$)?

**I. Theorem.** *The value of $|4\mathbf{r}D|I_s(\mathbf{r},D)/(T\phi_0)(\mathbf{v}_0)$ at $s=0$ is $2\kappa_E(\mathbf{r})$, where $\kappa_E$ is the nontrivial character of $F^\times/N_{E/F}E^\times$, $E=F(\sqrt{D})$.*

*Proof.* Consider the case when the quadratic form $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ does not represent zero (is anisotropic). Thus $D=\boldsymbol{\pi}$ and $\mathbf{r}\in R^\times-R^{\times2}$ (hence $|\mathbf{r}|=1$, $|D|=1/q$), or $D\in R^\times-R^{\times2}$ and $\mathbf{r}=\boldsymbol{\pi}$. The second case being equivalent to the first, it suffices to deal with the first case. The domain $\max\{|x|,|y|,|z|,|t|\}=1$ is the disjoint union of $\{|x|=1\}$, $\{|x|<1,|y|=1\}$, $\{|x|<1,|y|<1,|z|=1\}$ and $\{|x|<1,|y|<1,|z|<1,|t|=1\}$. Thus the integral $I_s(\mathbf{r},D)$ is the quotient by $\int_{|x|=1}dx$ of

$$\int_{|x|=1}dx+\int\int_{|x|<1,|y|=1}dxdy+q^{-m}\int\int\int_{|x|<1,|y|<1,|z|=1}dxdydz$$

$$+q^{-m}\int\int\int\int_{|x|<1,|y|<1,|z|<1,|t|=1} dxdydzdt$$
$$= 1+q^{-1}+q^{-m-2}+q^{-m-3} = 1+q^{-1}+q^{-2s}+q^{-2s-1}.$$

The value at $s=0$ is $2(1+q^{-1})$. Since $|\mathbf{r}D|=q^{-1}$, using Proposition 2 the value of the expression to be evaluated in the theorem is $-2$. Since $\kappa_E(\mathbf{r})=-1$, the theorem follows when the quadratic form is anisotropic.

We then turn to the case when the quadratic form is isotropic. Recall that $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E/F}E^\times$, $E=F(\sqrt{D})$. Thus $D\in F-F^2$, and we may assume that $|D|$ and $|\mathbf{r}|$ lie in $\{1,q^{-1}\}$.

**I.1. Proposition.** *When the quadratic form $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ is isotropic, $\mathbf{r}$ lies in $N_{E/F}E^\times$, and we may assume that the quadratic form takes one of three shapes:*

$$x^2-y^2-Dz^2+Dt^2,\ \ D\in R^\times-R^{\times 2};\quad x^2+\boldsymbol{\pi}y^2-\boldsymbol{\pi}z^2-\boldsymbol{\pi}^2t^2;\quad x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2.$$

*Proof.* (1) If $E/F$ is unramified, then $|D|=1$, thus $D\in R^\times-R^{\times 2}$. The norm group $N_{E/F}E^\times$ is $\pi^{2\mathbb{Z}}R^\times$. If $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ represents 0 then $\mathbf{r}\in R^\times$, so we may take $\mathbf{r}=1$.
(2) If $E/F$ is ramified then $|D|=q^{-1}$ and $N_{E/F}E^\times=(-D)^{\mathbb{Z}}R^{\times 2}$. The form $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ represents zero when $\mathbf{r}\in R^{\times 2}$ or $\mathbf{r}\in -DR^{\times 2}$. Then the form can be taken to be $x^2+Dy^2-Dz^2-D^2t^2$ with $\mathbf{r}=-D$ and $|D\mathbf{r}|=q^{-2}$, or $x^2-y^2-Dz^2+Dt^2$ with $\mathbf{r}=1$ and $|D\mathbf{r}|=q^{-1}$. The proposition follows. □

The set $V^0=V/\sim$, where $V=\{\mathbf{v}=(x,y,z,t)\in R^4;\max\{|x|,|y|,|z|,|t|\}=1\}$ and $\sim$ is the equivalence relation $\mathbf{v}\sim\alpha\mathbf{v}$ for $\alpha\in R^\times$, is the disjoint union of the subsets

$$V_n^0=V_n^0(\mathbf{r},D)=V_n(\mathbf{r},D)/\sim,$$

where

$$V_n=V_n(\mathbf{r},D)=\{\mathbf{v};\max\{|x|,|y|,|z|,|t|\}=1,|x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2|=1/q^n\},$$

over $n\geq 0$, and of $\{\mathbf{v};x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2=0\}/\sim$, a set of measure zero.

Thus the integral $I_s(\mathbf{r},D)$ coincides with the sum

$$\sum_{n=0}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r},D)).$$

When the quadratic form represents zero the problem is then to compute the volumes

$$\mathrm{vol}(V_n^0(\mathbf{r},D))=\mathrm{vol}(V_n(\mathbf{r},D))/(1-1/q)\qquad(n\geq 0).$$

We need the following Technical Lemma.

**I.0. Lemma.** *When $c \in R^{\times 2}$, $|c| = 1$, and $n \geq 1$, we have*

$$\int_{|c-x^2|=q^{-n}} dx = \frac{2}{q^n}\left(1 - \frac{1}{q}\right).$$

*Proof.* Recall that any $p$-adic number $a$ such that $|a| \leq 1$ can be written as a power series in $\boldsymbol{\pi}$:

$$a = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i = a_0 + a_1\boldsymbol{\pi} + a_2\boldsymbol{\pi}^2 + \dots,$$

where $a_i$ lies in a set $\mathbb{F}$ of representatives in $R$ for the finite field $R/\boldsymbol{\pi}$. In particular $|a| = 1/q^n$ implies that $a_0 = a_1 = \dots = a_{n-1} = 0$ and $a_n \neq 0$. We can write

$$x = \sum_{i=0}^{\infty} x_i\boldsymbol{\pi}^i, \qquad c = \sum_{i=0}^{\infty} c_i\boldsymbol{\pi}^i, \qquad x^2 = \sum_{i=0}^{\infty} a_i\boldsymbol{\pi}^i, \qquad a_i = \sum_{j=0}^{i} x_j x_{i-j} \qquad (x_i, c_i, a_i \in \mathbb{F}).$$

We have

$$c - x^2 = \sum_{i=0}^{\infty} f_i\boldsymbol{\pi}^i, \qquad f_i \in \mathbb{F},$$

where $f_i = c_i - a_i$ $(i \geq 0)$. Since $|c - x^2| = 1/q^n$ we have that $f_0 = f_1 = \dots = f_{n-1} = 0$ and $f_n \neq 0$. Thus we obtain the relations (for $a$, $c$ in the set $\mathbb{F}$ which (modulo $\boldsymbol{\pi}$) is the field $R/\boldsymbol{\pi}$)

$$c_i - a_i = 0 \qquad (i = 0, \dots, n-1), \qquad c_n - a_n \neq 0.$$

From $c_0 - a_0 = 0$ it follows that $x_0 = \pm c_0'$ (where $c \equiv (c_0')^2 \pmod{\boldsymbol{\pi}}$, $c_0' \in \mathbb{F}$, since $c \in R^{\times 2}$). From $c_i - a_i = 0$ $(i = 1, \dots, n-1)$ it follows that (since $x_0 \neq 0$)

$$x_i = (c_i - \sum_{j=1}^{i-1} x_j x_{i-j})/(2x_0), \qquad x_n \neq (c_n - \sum_{j=1}^{n-1} x_j x_{n-j})/(2x_0),$$

where in the case of $i = 1$ the sum over $j$ is empty. Thus we have

$$\int_{|c-x^2|=q^{-n}} dx = \frac{2}{q}\left(\frac{1}{q}\right)^{n-1}\left(1 - \frac{1}{q}\right) = \frac{2}{q^n}\left(1 - \frac{1}{q}\right).$$

The lemma follows. □

**I.1. Lemma.** *When $D = \boldsymbol{\pi}$ and $\mathbf{r} = 1$, thus $|\mathbf{r}D| = 1/q$, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} (1 - 1/q), & \text{if } n = 0, \\ q^{-1}(1 - 1/q)(2 + 1/q), & \text{if } n = 1, \\ 2q^{-n}(1 - 1/q)(1 + 1/q), & \text{if } n \geq 2. \end{cases}$$

*Proof.* In our case

$$V_0 = V_0(-1,\boldsymbol{\pi}) = \{(x,y,z,t);\ \max\{|x|,|y|,|z|,|t|\} = 1,\ |x^2 - y^2 - \boldsymbol{\pi} z^2 + \boldsymbol{\pi} t^2| = 1\}.$$

Since $|z| \leq 1$ and $|t| \leq 1$, we have $|\boldsymbol{\pi}(z^2 - t^2)| < 1$, and

$$1 = |x^2 - y^2 - \boldsymbol{\pi} z^2 + \boldsymbol{\pi} t^2| = |x^2 - y^2| = |x-y||x+y|.$$

Thus $|x-y| = |x+y| = 1$, and if $|x| \neq |y|$, $|x \pm y| = \max\{|x|,|y|\}$. We split $V_0$ into three distinct subsets, corresponding to the cases $|x| = |y| = 1$; $|x| = 1$, $|y| < 1$; and $|x| < 1$, $|y| = 1$. The volume is then

$$\begin{aligned}\mathrm{vol}(V_0) = &\int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|x|=1}\left[\int_{|y|=1,|x-y|=|x+y|=1}\right] dydxdzdt \\ &+ \int_{|t|\leq 1}\int_{|z|\leq 1}\left[\int_{|x|=1}\int_{|y|<1} + \int_{|x|<1}\int_{|y|=1}\right] dydxdzdt \\ = &\int_{|x|=1}\left[\int_{|y|=1,|x-y|=|x+y|=1}\right] dydx + \frac{2}{q}\left(1-\frac{1}{q}\right) = \left(1-\frac{1}{q}\right)^2.\end{aligned}$$

Let us consider the case of $V_n$ with $n \geq 2$. If $|x| = 1$, then put $c = c(x,t,z) = x^2 + \boldsymbol{\pi}(t^2 - z^2)$. Since $|\boldsymbol{\pi}(t^2 - z^2)| < 1$, we have $c(x,t,z) \in R^{\times 2}$, and we can apply Lemma I.0. Thus we obtain

$$\int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|x|=1}\int_{|c-y^2|=q^{-n}} dydxdzdt = \left(1-\frac{1}{q}\right)\frac{2}{q^n}\left(1-\frac{1}{q}\right) = \frac{2}{q^n}\left(1-\frac{1}{q}\right)^2.$$

If $|x| < 1$ it follows that $|y| < 1$. Since $\max\{|x|,|y|,|z|,|t|\} = 1$ and $n \geq 2$ it follows that $|t| = |z| = 1$. Indeed, if, say, $|t| = 1$ but $|z| < 1$, then $|x^2 - y^2 - \boldsymbol{\pi} z^2 + \boldsymbol{\pi} t^2| = |\boldsymbol{\pi} t^2| = 1/q$, which is a contradiction. Further, dividing by $\boldsymbol{\pi}$ we obtain $|z^2 + (y^2 - x^2)/\boldsymbol{\pi} - t^2| = q^{1-n}$. Put $c = c(x,y,z) = z^2 + (y^2 - x^2)/\boldsymbol{\pi}$. Since $|(y^2 - x^2)/\boldsymbol{\pi}| < 1$, we have $c \in R^{\times 2}$, and using Lemma I.0 we obtain

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|=1}\int_{|c-t^2|=q^{1-n}} dtdzdydx = \frac{1}{q^2}\left(1-\frac{1}{q}\right)\frac{2}{q^{n-1}}\left(1-\frac{1}{q}\right) = \frac{2}{q^{n+1}}\left(1-\frac{1}{q}\right)^2.$$

Adding the two subcases we have ($n \geq 2$)

$$\mathrm{vol}(V_n) = \frac{2}{q^n}\left(1-\frac{1}{q}\right)^2 + \frac{2}{q^{n+1}}\left(1-\frac{1}{q}\right) = \frac{2}{q^n}\left(1+\frac{1}{q}\right)\left(1-\frac{1}{q}\right)^2.$$

Let us consider the case $n = 1$. The subcase $|x| = 1$, is exactly the same as for $n \geq 2$. The contribution is $2/q(1-1/q)^2$. Now if $|x| < 1$ then $|y| < 1$ and $\max\{|z|,|t|\} = 1$. We have $|x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2| = |\boldsymbol{\pi}(z^2-t^2)| = q^{-1}$. Dividing by $\boldsymbol{\pi}$ gives $|z^2-t^2|=1$. The volume of this subset is

$$\frac{1}{q^2}\left[\int_{|z|=1}\int_{|z^2-t^2|=1} dtdz + \int_{|z|<1}\int_{|t|=1} dtdz\right]$$

$$=\frac{1}{q^2}\left[\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)+\frac{1}{q}\left(1-\frac{1}{q}\right)\right]=\frac{1}{q^2}\left(1-\frac{1}{q}\right)^2.$$

Adding the two subcases, we have

$$\operatorname{vol}(V_1)=\frac{2}{q}\left(1-\frac{1}{q}\right)^2+\frac{1}{q^2}\left(1-\frac{1}{q}\right)^2=\frac{1}{q}\left(2+\frac{1}{q}\right)\left(1-\frac{1}{q}\right)^2.$$

The lemma follows. □

**I.2. Lemma.** *When $D=\boldsymbol{\pi}$ and $\mathbf{r}=-\boldsymbol{\pi}$, thus $|\mathbf{r}D|=1/q^2$, we have*

$$\operatorname{vol}(V_n^0)=\begin{cases}1, & \text{if } n=0,\\ q^{-1}(1-1/q), & \text{if } n=1,\\ q^{-2}(2-1/q-2/q^2), & \text{if } n=2,\\ 2q^{-n}(1-1/q)(1+1/q), & \text{if } n\geq 3.\end{cases}$$

*Proof.* To compute $\operatorname{vol}(V_0)$, recall that in our case

$$V_0=\{(x,y,z,t);\max\{|x|,|y|,|z|,|t|\}=1, |x^2+\boldsymbol{\pi}(y^2-z^2)-\boldsymbol{\pi}^2t^2|=1\}.$$

Since $|y|\leq 1$, $|z|\leq 1$, $|t|\leq 1$, we have $|x^2+\boldsymbol{\pi}(y^2-z^2)-\boldsymbol{\pi}^2t^2|=|x^2|=1$, and so

$$\operatorname{vol}(V_0)=\int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|y|\leq 1}\int_{|x|=1} dxdydzdt = 1-\frac{1}{q}.$$

To compute $\operatorname{vol}(V_n)$, $n\geq 1$, recall that

$$V_n=\{(x,y,z,t);\max\{|x|,|y|,|z|,|t|\}=1, |x^2+\boldsymbol{\pi}(y^2-z^2)-\boldsymbol{\pi}^2t^2|=1/q^n\}.$$

Assume that $|x|=1$. Then

$$1=|x^2|=|x^2+\boldsymbol{\pi}(y^2-z^2)-\boldsymbol{\pi}^2t^2|=1/q^n<1.$$

Thus we have that $|x|<1$ and $\max\{|y|,|z|,|t|\}=1$.

Consider the case $n=1$. Then $|y^2-z^2|=|y\pm z|=1$. We have

$$\operatorname{vol}(V_1)=\int_{|t|\leq 1}\int_{|x|<1}\left[\int_{|y|=1}\int_{|y\pm z|=1}+\int_{|y|<1}\int_{|z|=1}\right] dzdydxdt$$

$$= \frac{1}{q}\left[\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)+\frac{1}{q}\left(1-\frac{1}{q}\right)\right] = \frac{1}{q}\left(1-\frac{1}{q}\right)^2.$$

Consider the case $n \geq 3$. As in the analogous case of Lemma I.1, we consider the subcases of $|y| = 1$ and $|y| < 1$. Adding the two subcases we have ($n \geq 3$)

$$\mathrm{vol}(V_n) = \frac{2}{q^n}\left(1-\frac{1}{q}\right)^2 + \frac{2}{q^{n+1}}\left(1-\frac{1}{q}\right) = \frac{2}{q^n}\left(1+\frac{1}{q}\right)\left(1-\frac{1}{q}\right)^2.$$

Consider the case $n = 2$. If $|y| = 1$ we apply Lemma I.0 (as in the case $n \geq 3$) and the contribution is $2/q^2(1-1/q)^2$. Now if $|y| < 1$ then $|z| < 1$ and $|t| = |x/\boldsymbol{\pi}| = 1$. The contribution from this subset is

$$\int_{|y|<1}\int_{|z|<1}\int_{|t|=1}\int_{|t^2-(x/\pi)^2|=1} dxdtdzdy = \frac{1}{q^2}\left(1-\frac{1}{q}\right)\frac{1}{q}\left(1-\frac{2}{q}\right).$$

Adding the two subcases (see Lemma I.1 for details), we obtain

$$\mathrm{vol}(V_2) = \frac{2}{q^2}\left(1-\frac{1}{q}\right)^2 + \frac{1}{q^3}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right) = \frac{1}{q^2}\left(1-\frac{1}{q}\right)\left(2-\frac{1}{q}-\frac{2}{q^2}\right).$$

The lemma follows. □

**I.3. Lemma.** *When $E/F$ is unramified, thus $|\mathbf{r}D| = 1$, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1-1/q^2, & \text{if } n = 0, \\ q^{-n}(1-1/q)(1+2/q+1/q^2), & \text{if } n \geq 1. \end{cases}$$

*Proof.* First we compute $\mathrm{vol}(V_0)$. Recall that in our case

$$V_0 = \{(x,y,z,t); \max\{|x|,|y|,|z|,|t|\} = 1, |x^2-y^2-D(z^2-t^2)| = 1\}.$$

Since $|x^2-y^2-D(z^2-t^2)| \leq \max\{|x|,|y|,|z|,|t|\}$,

$$V_0 = \{(x,y,z,t) \in R^4; |x^2-y^2-D(z^2-t^2)| = 1\}.$$

Make the change of variables $x' = x+y$, $y' = x-y$, $z' = z+t$, $t' = z-t$. Renaming $x'$, $y'$, $z'$, $t'$ as $x$, $y$, $z$, $t$, we obtain

$$V_0 = \{(x,y,z,t) \in R^4; |xy - Dzt| = 1\}.$$

Assume that $|xy| < 1$. Since $|xy - Dzt| = 1$, it follows that $|zt| = |z| = |t| = 1$. The contribution from the set $|xy| < 1$ is

$$\int_{|t|=1}\int_{|z|=1}\left[\int_{|x|<1}\int_{|y|\leq 1} + \int_{|x|=1}\int_{|y|<1}\right] dydxdzdt$$

$$= \left(1-\frac{1}{q}\right)^2\left(\frac{1}{q}+\left(1-\frac{1}{q}\right)\frac{1}{q}\right) = \frac{1}{q}\left(1-\frac{1}{q}\right)^2\left(2-\frac{1}{q}\right).$$

Note that the contribution from $|xy|=1$, $|zt|<1$, is the same and equals

$$\frac{1}{q}\left(1-\frac{1}{q}\right)^2\left(2-\frac{1}{q}\right).$$

We are left with the case $|xy|=|zt|=1$, i.e. $|x|=|y|=|z|=|t|=1$. If $|x|=|y|=|z|=1$ we introduce $U(x,y,z)=\{t;\ |t|=1,\ |xy-Dzt|=1\}$, a set of volume $1-2/q$. The contribution from this subcase is

$$\int_{|x|=1}\int_{|y|=1|}\int_{|z|=1}\int_{U(x,y,z)} dtdzdydx = \left(1-\frac{1}{q}\right)^3\left(1-\frac{2}{q}\right).$$

Thus we obtain

$$\mathrm{vol}(V_0) = \frac{2}{q}\left(1-\frac{1}{q}\right)^2\left(2-\frac{1}{q}\right)+\left(1-\frac{1}{q}\right)^3\left(1-\frac{2}{q}\right) = \left(1-\frac{1}{q}\right)^2\left(1+\frac{1}{q}\right).$$

Next we compute $\mathrm{vol}(V_n)$, $n\geq 1$. Recall that in our case

$$V_n = \{(x,y,z,t); \max\{|x|,|y|,|z|,|t|\}=1, |x^2-y^2-D(z^2-t^2)|=1/q^n\}.$$

Making the change of variables $u=x+y$, $v=x-y$, we obtain

$$V_n = \{(u,v,z,t); \max\{|u+v|,|u-v|,|z|,|t|\}=1, |uv-D(z^2-t^2)|=1/q^n\}.$$

Since the set $\{v=0\}$ is of measure zero, we assume that $v\neq 0$. Then $|uv-D(z^2-t^2)|=1/q^n$ implies that $u=D(z^2-t^2)v^{-1}+wv^{-1}\boldsymbol{\pi}^n$, where $|w|=1$. There are two cases.

Assume that $|v|=1$. Note that if $|z^2-t^2|=1$, then $\max\{|z|,|t|\}=1$, and if $|z^2-t^2|<1$, then (since $n\geq 1$)

$$|u| = |D(z^2-t^2)v^{-1}+wv^{-1}\boldsymbol{\pi}^n| \leq \max\{|z^2-t^2|, q^{-n}\} < 1,$$

and consequently $|u+v|=|v|=1$. So $|v|=1$ implies that $\max\{|u+v|,|u-v|,|z|,|t|\}=1$. Further, since $|v|=1$, we have $du=q^{-n}dw$. Thus the contribution from the set with $|v|=1$ is

$$\int_{|t|\leq 1}\int_{|z|\leq 1}\int_{|v|=1}\int_{|uv-D(z^2-t^2)|=1/q^n} dudvdzdt = \int_{|v|=1}\int_{|w|=1}\frac{dw}{q^n}dv = \frac{1}{q^n}\left(1-\frac{1}{q}\right)^2.$$

Assume that $|v|<1$ and $|u|=1$. Thus $\max\{|u+v|,|u-v|,|z|,|t|\}=1$. We write $v=D(z^2-t^2)u^{-1}+wu^{-1}\boldsymbol{\pi}^n$ where $|w|=1$ and $dv=q^{-n}dw$. Since

$$|z^2-t^2| \leq \max\{|v|, q^{-n}\} < 1,$$

it follows that $|z^2 - t^2| < 1$. Note that

$$\int\int_{|z^2-t^2|<1} dzdt = \int_{|z|=1}\int_{|z^2-t^2|<1} dtdz + \int_{|z|<1}\int_{|t|<1} dt = \left(1-\frac{1}{q}\right)\frac{2}{q} + \frac{1}{q^2} = \frac{1}{q}\left(2-\frac{1}{q}\right).$$

The volume of this subset equals

$$\int\int_{|z^2-t^2|<1}\int_{|u|=1}\int_{|uv-D(z^2-t^2)|=1/q^n} dvdudzdt = \frac{1}{q}\left(2-\frac{1}{q}\right)\frac{1}{q^n}\int_{|u|=1}\int_{|w|=1} dwdu$$

$$= \frac{1}{q}\frac{1}{q^n}\left(1-\frac{1}{q}\right)^2\left(2-\frac{1}{q}\right).$$

Assume that $|v| < 1$ and $|u| < 1$. Then we have $|u \pm v| < 1$ and thus $\max\{|z|, |t|\} = 1$. Since $|uv - D(z^2 - t^2)| < 1$ it follows that $|z^2 - t^2| < 1$. So we have $|z| = |t| = 1$. Put $c = c(z, u, v) = z^2 - uvD^{-1}$. Then $c \in R^{\times 2}$ (since $|uvD^{-1}| < 1$). Dividing by $D$, we have

$$\frac{1}{q^n} = |uv - D(z^2 - t^2)| = |c - t^2|.$$

Applying Lemma I.0, the contribution from this subset is equal to

$$\int_{|u|<1}\int_{|v|<1}\int_{|z|=1}\int_{|c-t^2|=q^{-n}} dtdzdvdu = \frac{1}{q^2}\frac{2}{q^n}\left(1-\frac{1}{q}\right)^2.$$

Adding the contributions from $|v| = 1$ and $|v| < 1$ we obtain

$$\mathrm{vol}(V_n) = \frac{1}{q^n}\left(1-\frac{1}{q}\right)^2 + \frac{1}{q}\frac{1}{q^n}\left(1-\frac{1}{q}\right)^2\left(2-\frac{1}{q}\right) + \frac{1}{q^2}\frac{2}{q^n}\left(1-\frac{1}{q}\right)^2$$

$$= \frac{1}{q^n}\left(1-\frac{1}{q}\right)^2\left(1+\frac{2}{q}+\frac{1}{q^2}\right).$$

The lemma follows. □

*Proof of Theorem I.* We are now ready to complete the proof of Theorem I in the isotropic case. Recall that we need to compute the value at $s = 0$ $(m = -2)$ of the product $|\mathbf{r}D|I_s(\mathbf{r}, D)$. Here $I_s(\mathbf{r}, D)$ coincides with the sum

$$\sum_{n=0}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r}, D))$$

which converges for $m > -1$ by Proposition 1 or alternatively by Lemmas I.1-I.3. The value at $m = -2$ is obtained then by analytic continuation of this sum.

*Case of Lemma* I.1. We have $|\mathbf{r}D| = 1/q$, and $I_s(\mathbf{r}, D)$ is equal to

$$\mathrm{vol}(V_0^0) + q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 - \frac{1}{q} + \frac{1}{q}\left(1-\frac{1}{q}\right)\left(2+\frac{1}{q}\right)\frac{1}{q^m} + 2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-2(m+1)}\left(1-\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 - \frac{1}{q} + q\left(2 - \frac{1}{q} - \frac{1}{q^2}\right) + 2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^2}{1-q} = -2(1+q^{-1}).$$

Multiplied by $|\mathbf{r}D| = 1/q$, we obtain $-2q^{-1}(1+q^{-1})$.
*Case of Lemma* I.2. We have $|\mathbf{r}D| = 1/q^2$, and $I_s(\mathbf{r}, D)$ is equal to

$$\mathrm{vol}(V_0^0) + q^{-m}\,\mathrm{vol}(V_1^0) + q^{-2m}\,\mathrm{vol}(V_2^0) + \sum_{n=3}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 + \frac{1}{q}\left(1-\frac{1}{q}\right)q^{-m} + \frac{1}{q^2}\left(2-\frac{1}{q}-\frac{2}{q^2}\right)q^{-2m}$$

$$+2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-3(m+1)}\left(1-\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 + \frac{1}{q}\left(1-\frac{1}{q}\right)q^2 + \frac{1}{q^2}\left(2-\frac{1}{q}-\frac{2}{q^2}\right)q^4 + 2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^3}{1-q}.$$

Once simplified and multiplied by $|\mathbf{r}D| = 1/q^2$, we obtain $-2q^{-1}(1+q^{-1})$.
*Case of Lemma* I.3. We have $|\mathbf{r}D| = 1$, and $I_s(\mathbf{r}, D)$ is equal to

$$\mathrm{vol}(V_0^0) + \sum_{n=1}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 - \frac{1}{q^2} + \left(1-\frac{1}{q}\right)\left(1+\frac{2}{q}+\frac{1}{q^2}\right)q^{-(m+1)}\left(1-\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$= 1 - \frac{1}{q^2} + \left(1-\frac{1}{q}\right)\left(1+\frac{2}{q}+\frac{1}{q^2}\right)\frac{q}{1-q} = -2q^{-1}(1+q^{-1}).$$

In all 3 cases the value of the expression of the theorem is 2 by virtue of Proposition 2, and indeed $\kappa_E(\mathbf{r}) = 1$ in these cases as $\mathbf{r}$ lies in $N_{E/F}E^{\times}$ by Proposition I.1. The theorem follows. $\square$

## CHARACTER COMPUTATION FOR TYPE II

For the $\theta$-conjugacy class of type II, represented by $g = t \cdot \mathrm{diag}(\mathbf{r}, \mathbf{s}, \mathbf{s}, \mathbf{r})$, the product

$$ {}^t\mathbf{v} gJ\mathbf{v} = (t, z, x, y) \begin{pmatrix} a_1\mathbf{r} & 0 & 0 & a_2D\mathbf{r} \\ 0 & b_1\mathbf{s} & b_2AD\mathbf{s} & 0 \\ 0 & b_2\mathbf{s} & b_1\mathbf{s} & 0 \\ a_2\mathbf{r} & 0 & 0 & a_1\mathbf{r} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ z \\ x \\ y \end{pmatrix} $$

is equal to

$$ -t^2a_2D\mathbf{r} - z^2b_2AD\mathbf{s} + x^2b_2\mathbf{s} + y^2a_2\mathbf{r}, $$

where $a_1 + a_2\sqrt{D} \in E_1^\times$ ($E_1 = F(\sqrt{D})$) and $b_1 + b_2\sqrt{AD} \in E_2^\times$ ($E_2 = F(\sqrt{AD})$). The trace being a function of $g$ in the projective group, and $\mathbf{r}$ ranging over a set of representatives for $F^\times/N_{E_1/F}E_1^\times$ (and $\mathbf{s}$ for $F^\times/N_{E_2/F}E_2^\times$), we may divide the quadratic form by $b_2\mathbf{s}$ and rename $-a_2\mathbf{r}/b_2\mathbf{s}$ by $\mathbf{r}$. The quadratic form becomes $x^2 - y^2\mathbf{r} - z^2AD + t^2D\mathbf{r}$.

Thus we need to compute the integral

$$ I_s(\mathbf{r}, A, D) = \int_{V^0} |x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2|^{2(s-1)} dxdydzdt. $$

The property of the numbers $A$, $D$ and $AD$ that we need is that their square roots generate the three distinct quadratic extensions of $F$. Thus we may assume that $\{A, D, AD\} = \{u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$, where $u \in R^\times - R^{\times 2}$. Of course with this normalization $AD$ is no longer the product of $A$ and $D$, but its representative in the set $\{1, u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$ mod $F^{\times 2}$. Since $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E_1/F}E_1^\times$, it can be assumed to range over $\{1, \boldsymbol{\pi}\}$ if $D = u$, and over $\{1, u\}$ if $|D| = |\boldsymbol{\pi}|$.

In this section we prove

**II. Theorem.** *The value of $I_s(\mathbf{r}, A, D)$ at $s = 0$ is $0$.*

To prove this theorem we need some lemmas.

**II.1. Proposition.** *The quadratic form $x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2$ takes one of six forms: $x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)$, $x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)$, $x^2 - y^2 + ut^2 - u\boldsymbol{\pi}z^2$, $x^2 - y^2 - uz^2 + \boldsymbol{\pi}t^2$, $x^2 - \boldsymbol{\pi}y^2 + u\boldsymbol{\pi}(t^2 - z^2)$, $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi}t^2$, where $u \in R^\times - R^{\times 2}$. It is always isotropic.*

*Proof.* (1) If $E_1/F$ is unramified then $D = u$ where $u \in R^\times - R^{\times 2}$. The norm group $N_{E_1/F}E_1^\times$ is $\boldsymbol{\pi}^{2\mathbb{Z}}R^\times$. So $\mathbf{r} = 1$ or $\boldsymbol{\pi}$ and $A = \boldsymbol{\pi}$. We obtain two quadratic forms: $x^2 - y^2 - u\boldsymbol{\pi}z^2 + ut^2$, and $x^2 - \boldsymbol{\pi}y^2 - u\boldsymbol{\pi}(z^2 - t^2)$.
(2) If $E_1/F$ is ramified then $D = \boldsymbol{\pi}$ and $N_{E_1/F}E_1^\times = (-D)^{\mathbb{Z}}R^{\times 2}$. Then $\mathbf{r} = 1$ or $u$, and $A = u$ or $u\boldsymbol{\pi}$. Note that if $A = u\boldsymbol{\pi}$ we take $AD = u$. We obtain the following quadratic forms: if $\mathbf{r} = 1$, $A = u$ we have $x^2 - y^2 - \boldsymbol{\pi}(uz^2 - t^2)$; if $\mathbf{r} = 1$, $A = u\boldsymbol{\pi}$ we have $x^2 - y^2 - uz^2 + \boldsymbol{\pi}t^2$; if $\mathbf{r} = u$, $A = u$ we have $x^2 - uy^2 - u\boldsymbol{\pi}(z^2 - t^2)$; if $\mathbf{r} = u$, $A = u\boldsymbol{\pi}$ we have $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi}t^2$. The proposition follows. □

The set $V^0 = V/\sim$, where $V = \{\mathbf{v} = (x, y, z, t) \in R^4; \max\{|x|, |y|, |z|, |t|\} = 1\}$ and $\sim$ is the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ for $\alpha \in R^\times$, is the disjoint union of the subsets

$$ V_n^0 = V_n^0(\mathbf{r}, A, D) = V_n(\mathbf{r}, A, D)/\sim, $$

where

$$V_n = V_n(\mathbf{r}, A, D) = \{\mathbf{v}; \max\{|x|, |y|, |z|, |t|\} = 1, |x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2| = 1/q^n\},$$

over $n \geq 0$, and of $\{\mathbf{v}; x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2 = 0\}/\sim$, a set of measure zero.

Thus the integral $I_s(\mathbf{r}, A, D)$ coincides with the sum

$$\sum_{n=0}^{\infty} q^{-nm} \operatorname{vol}(V_n^0(\mathbf{r}, A, D)).$$

The problem is then to compute the volumes

$$\operatorname{vol}(V_n^0(\mathbf{r}, A, D)) = \operatorname{vol}(V_n(\mathbf{r}, A, D))/(1 - 1/q) \qquad (n \geq 0).$$

In the following lemmas, $u$ is a nonsquare unit.

**II.1. Lemma.** *When the quadratic form is $x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)$, we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1 - 1/q, & \text{if } n = 0, \\ 2/q - 1/q^2 + 1/q^3, & \text{if } n = 1, \\ 2q^{-n}(1 - 1/q), & \text{if } n \geq 2. \end{cases}$$

*Proof.* In our case

$$V_0 = \{(x, y, z, t); \max\{|x|, |y|, |z|, |t|\} = 1, \ |x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)| = 1\}.$$

This is the same case as that of Lemma I.1. The volume is then

$$\operatorname{vol}(V_0) = \left(1 - \frac{1}{q}\right)^2.$$

Let us consider the case of $V_n$ with $n \geq 2$. If $|x| = 1$, then put $c = c(x, t, z) = x^2 + \boldsymbol{\pi}(t^2 - uz^2)$. Since $|\boldsymbol{\pi}(t^2 - uz^2)| < 1$, we have $c(x, t, z) \in R^{\times 2}$, and we can apply Lemma I.0. Thus we obtain

$$\int_{|t|\leq 1} \int_{|z|\leq 1} \int_{|x|=1} \int_{|c-y^2|=q^{-n}} dy dx dz dt = \left(1 - \frac{1}{q}\right) \frac{2}{q^n} \left(1 - \frac{1}{q}\right) = \frac{2}{q^n} \left(1 - \frac{1}{q}\right)^2.$$

If $|x| < 1$ it follows that $|y| < 1$. Thus $\max\{|z|, |t|\} = 1$. Since $t^2 - uz^2$ does not represent zero non trivially, we have $|t^2 - uz^2| = 1$, which is a contradiction. Thus

$$\operatorname{vol}(V_n) = \frac{2}{q^n} \left(1 - \frac{1}{q}\right)^2.$$

Let us consider the case $n=1$. Note that in this case $|x^2-y^2|<1$.
(A) Subcase of $|t^2-uz^2|<1$. Since $t^2-uz^2$ does not represent zero non trivially, it follows that $|z|<1$, $|t|<1$. Thus $|\boldsymbol{\pi}(t^2-uz^2)|<1/q^2$, and

$$V_1=\{(x,y,z,t);\,\max\{|x|,|y|\}=1,\,|x^2-y^2|=1/q\}.$$

Applying Lemma I.0, the contribution of this subcase is

$$\int_{|z|<1}\int_{|t|<1}\int_{|x|=1}\int_{|x^2-y^2|=q^{-1}} dydxdtdz=\frac{1}{q^2}\left(1-\frac{1}{q}\right)\frac{2}{q}\left(1-\frac{1}{q}\right).$$

(B1) Subcase of $|t^2-uz^2|=1$ and $|x^2-y^2|<1/q$. Since $t^2-uz^2$ does not represent zero non trivially, we have

$$\int\int_{|t^2-uz^2|=1} dzdt=\int_{|t|=1}\int_{|z|\leq 1} dzdt+\int_{|t|<1}\int_{|z|=1} dzdt=\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right).$$

Furthermore,

$$\int\int_{|x^2-y^2|\leq 1/q^2} dxdy=\int_{|x|=1}\int_{|x^2-y^2|\leq 1/q^2} dydx+\int_{|x|<1}\int_{|y|<1} dydx=\left(1-\frac{1}{q}\right)\frac{2}{q^2}+\frac{1}{q^2}.$$

Thus the contribution of this subcase is equal to

$$\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\left[\left(1-\frac{1}{q}\right)\frac{2}{q^2}+\frac{1}{q^2}\right].$$

(B2) Subcase of $|t^2-uz^2|=1$ and $|x^2-y^2|=1/q$. Set $w=x-y$, $v=x+y$ ($dwdv=dxdy$). Then $|wv|=1/q$ and $|wv+\boldsymbol{\pi}(t^2-uz^2)|=1/q$, and the contribution of this subcase is (there are two integrals that correspond to $|w|=1$, $|v|=1/q$ and $|w|=1/q$, $|v|=1$):

$$2\int\int_{|t^2-uz^2|=1}\int_{|v|=1/q}\int_{|w|=1,|wv+\pi(t^2-uz^2)|=1/q} dwdvdtdz.$$

Since $|(t^2-uz^2)(v/\boldsymbol{\pi})^{-1}|=1$, we have that $w\neq 0,-(t^2-uz^2)(v/\boldsymbol{\pi})^{-1} \pmod{\boldsymbol{\pi}}$. So, the above integral is equal to

$$2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{1}{q}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right).$$

Adding the contributions from Subcases (A), (B1), and (B2) (divided by $(1-1/q)$), we obtain

$$\operatorname{vol}(V_1^0)=\frac{2}{q^3}\left(1-\frac{1}{q}\right)+\frac{1}{q^2}\left(1+\frac{1}{q}\right)\left(3-\frac{2}{q}\right)+\frac{2}{q}\left(1-\frac{1}{q^2}\right)\left(1-\frac{2}{q}\right).$$

Once simplified this is equal to $2/q-1/q^2+1/q^3$. The lemma follows. □

**II.2. Lemma.** *When the quadratic form is $x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)$, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1 + 1/q, & \text{if } n = 0, \\ q^{-2}(1 - 1/q), & \text{if } n = 1, \\ 2q^{-(n+1)}(1 - 1/q), & \text{if } n \geq 2. \end{cases}$$

*Proof.* Consider the case of $n = 0$. Then

$$V_0 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = 1\}.$$

Since $x^2 - uy^2$ does not represent zero non trivially, we have

$$1 = |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = |x^2 - uy^2| = \max\{|x|, |y|\}.$$

We obtain

$$\mathrm{vol}(V_0) = \int_{|x|=1}\int_{|y|\leq 1} dydx + \int_{|x|<1}\int_{|y|=1} dydx = 1 - \frac{1}{q} + \frac{1}{q}\left(1 - \frac{1}{q}\right) = \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right).$$

Let us consider the case of $n = 1$. Then

$$V_1 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = 1/q\}.$$

If $\max\{|x|, |y|\} = 1$, then $|x^2 - uy^2| = 1$. It implies that $|x| < 1$, $|y| < 1$, and $|x^2 - uy^2| \leq 1/q^2$. The contribution from this subset is equal to

$$\int_{|x|<1}\int_{|y|<1}\int\int_{|t^2-z^2|=1} dtdzdydx = \frac{1}{q^2}\left[\int_{|t|=1}\int_{|t^2-z^2|=1} dzdt + \int_{|t|<1}\int_{|z|=1} dzdt\right]$$

$$= \frac{1}{q^2}\left[\left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right) + \frac{1}{q}\left(1 - \frac{1}{q}\right)\right] = \frac{1}{q^2}\left(1 - \frac{1}{q}\right)^2.$$

Let us consider the case of $n \geq 2$. Then

$$V_n = \{(x, y, z, t);\ \max\{|y|, |z|, |t|\} = 1,\ |x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)| = q^{-n}\}.$$

If $\max\{|x|, |y|\} = 1$ then $|x^2 - uy^2| = 1$, which is a contradiction. Hence $|x| < 1$, $|y| < 1$, and $\max\{|z|, |t|\} = 1$. The latter implies that $|z| = |t| = 1$. Dividing by $u\boldsymbol{\pi}$, we have

$$V_n = \{(x, y, z, t);\ |x| < 1,\ |y| < 1,\ |z| = 1,\ |t| = 1,\ |z^2 - t^2 + \boldsymbol{\pi}((y/\boldsymbol{\pi})^2 - u^{-1}(x/\boldsymbol{\pi})^2)| = q^{1-n}\}.$$

Applying Lemma I.0 (with $c = z^2 + \boldsymbol{\pi}((y/\boldsymbol{\pi})^2 - u^{-1}(x/\boldsymbol{\pi})^2)$), its volume is equal to

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|=1}\int_{|c-t^2|=q^{1-n}} dtdzdydx = \frac{1}{q^2}\frac{2}{q^{n-1}}\left(1 - \frac{1}{q}\right)^2.$$

Dividing by $(1 - 1/q)$ we obtain the $\mathrm{vol}(V_n^0)$. The lemma follows. □

In all other cased we get the same result for the volumes. Since the proofs are different, we state the remaining cases separately as Lemmas II.3, II.4, II.5.

**II.3. Lemma.** *When the quadratic form is $x^2 - y^2 + ut^2 - u\boldsymbol{\pi} z^2$ or $x^2 - y^2 - uz^2 + \boldsymbol{\pi} t^2$, we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ 1/q, & \text{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \text{if } n \geq 2. \end{cases}$$

*Proof.* Since the quadratic form $x^2 - y^2 - uz^2 + \boldsymbol{\pi} t^2$ is equal to $-(y^2 - x^2 + uz^2 - \boldsymbol{\pi} t^2)$, the computations for this form are identical to those of $x^2 - y^2 + ut^2 - u\boldsymbol{\pi} z^2$.

Consider the case of $n = 0$. Then

$$V_0 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - y^2 + ut^2 - u\boldsymbol{\pi} z^2| = 1\}.$$

We have the following three subcases.

(A) Subcase of $|t| < 1$. It follows that $|x^2 - y^2| = 1$. The contribution of this subcase is

$$\int_{|z|\leq 1} \int_{|t|<1} \int \int_{|x^2-y^2|=1} dydxdtdz = \frac{1}{q}\left[\int_{|x|=1} \int_{|x^2-y^2|=1} dydx + \int_{|x|<1} \int_{|y|=1} dydx\right]$$

$$= \frac{1}{q}\left[\left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right) + \frac{1}{q}\left(1 - \frac{1}{q}\right)\right] = \frac{1}{q}\left(1 - \frac{1}{q}\right)^2.$$

(B1) Subcase of $|t| = 1$, $|x^2 - y^2| < 1$. The contribution from this subcase is equal to (we apply Lemma I.0):

$$\int_{|t|=1} \int_{|x|=1} \int_{|x^2-y^2|\leq q^{-1}} dydxdt + \int_{|t|=1} \int_{|x|<1} \int_{|y|<1} dydxdt$$

$$= \frac{2}{q}\left(1 - \frac{1}{q}\right)^2 + \frac{1}{q^2}\left(1 - \frac{1}{q}\right) = \left(1 - \frac{1}{q}\right)\left(\frac{2}{q} - \frac{1}{q^2}\right).$$

(B2) Subcase of $|t| = 1$, $|x^2 - y^2| = 1$. Set $w = x - y$, $v = x + y$. Then $|w| = |v| = 1$ and also $|ut^2 w^{-1}| = 1$. Thus the contribution from this subcase is given by the integral

$$\int_{|t|=1} \int_{|w|=1} \int_{|v|=1, |wv+ut^2|=1} dvdwdt = \left(1 - \frac{1}{q}\right)^2 \left(1 - \frac{2}{q}\right).$$

Adding the contributions from Subcases (A), (B1), and (B2) (divided by $(1 - 1/q)$), we obtain

$$\operatorname{vol}(V_0^0) = \frac{1}{q}\left(1 - \frac{1}{q}\right) + \frac{2}{q} - \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right) = 1.$$

Let us consider the case of $n \geq 2$. We have the following three subcases.

(A) Subcase of $|x|<1$. Since the quadratic form $y^2-ut^2$ does not represent zero non trivially, we have that $|y^2-ut^2|=1$ if and only if $\max\{|y|,|t|\}=1$. It implies that $|y|<1$, $|t|<1$, and thus $|x^2-y^2+ut^2-u\boldsymbol{\pi}z^2|=|\boldsymbol{\pi}z^2|=1/q$, which is a contradiction.

(B1) Subcase of $|x|=1$, $|t|<1$. The contribution from this subcase is given by the following integral (we apply Lemma I.0):

$$\int_{|t|<1}\int_{|z|\le 1}\int_{|x|=1}\int_{|x^2-y^2|=q^{-n}} dydxdzdt=\frac{1}{q}\left(1-\frac{1}{q}\right)\frac{2}{q^n}\left(1-\frac{1}{q}\right).$$

(B2) Subcase of $|x|=1$, $|t|=1$. Set $w=x-y$, $v=x+y$. Then we have that $|w|=|v|=1$, and from $|wv+ut^2-u\boldsymbol{\pi}z^2|=q^{-n}$, we have

$$w=u(\boldsymbol{\pi}z^2-t^2)v^{-1}+\varepsilon v^{-1}\boldsymbol{\pi}^n,\qquad dw=\frac{1}{q^n}d\varepsilon,\qquad |\varepsilon|=1.$$

The volume of this subset is given by

$$\int_{|z|\le 1}\int_{|t|=1}\int_{|v|=1}\int_{|w|=1,|vw+ut^2-u\pi z^2|=q^{-n}} dwdvdtdz$$
$$=\left(1-\frac{1}{q}\right)^2\int_{|\varepsilon|=1}\frac{d\varepsilon}{q^n}=\frac{1}{q^n}\left(1-\frac{1}{q}\right)^3.$$

Adding the contributions from subcases (A), (B1), and (B2) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_n^0)=\frac{2}{q}\left(1-\frac{1}{q}\right)\frac{1}{q^n}+\left(1-\frac{1}{q}\right)^2\frac{1}{q^n}=\left(1-\frac{1}{q^2}\right)\frac{1}{q^n}.$$

Let us consider the case of $n=1$. We have the following three subcases.

(A) Subcase of $|x|<1$. Since the quadratic form $y^2-ut^2$ does not represent zero non trivially, we have that $|y^2-ut^2|=1$ if and only if $\max\{|y|,|t|\}=1$. Hence $|y|<1$, $|t|<1$, and thus $|z|=1$. The volume of this subset is equal to

$$\int_{|x|<1}\int_{|y|<1}\int_{|t|<1}\int_{|z|=1} dzdtdydx=\frac{1}{q^3}\left(1-\frac{1}{q}\right).$$

(B1) Subcase of $|x|=1$, $|t|<1$. Applying Lemma I.0, we have

$$\int_{|t|<1}\int_{|z|\le 1}\int_{|x|=1}\int_{|x^2-y^2|=q^{-1}} dydxdzdt=\frac{1}{q}\left(1-\frac{1}{q}\right)\frac{2}{q}\left(1-\frac{1}{q}\right).$$

(B2) Subcase of $|x|=1$, $|t|=1$. Set $w=x-y$, $v=x+y$. We have that $|w|=|v|=1$, and we arrive to the same case as that of $n\ge 2$ (with $n=1$). The contribution is

$$\frac{1}{q}\left(1-\frac{1}{q}\right)^3.$$

Adding the contributions from subcases (A), (B1), and (B2) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_1^0)=\frac{1}{q^3}+\frac{2}{q}\left(1-\frac{1}{q}\right)+\frac{1}{q}\left(1-\frac{1}{q}\right)^2=\frac{1}{q}.$$

The lemma follows. $\square$

**II.4. Lemma.** *When the quadratic form is* $x^2 - \boldsymbol{\pi} y^2 + u\boldsymbol{\pi}(t^2 - z^2)$, *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ 1/q, & \text{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \text{if } n \geq 2. \end{cases}$$

*Proof.* Consider the case of $n = 0$. Then

$$V_0 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - \boldsymbol{\pi} y^2 + u\boldsymbol{\pi}(z^2 - t^2)| = 1\}.$$

Obviously we have

$$\mathrm{vol}(V_0) = \int_{|x|=1} dx = 1 - \frac{1}{q}.$$

Let us consider the case of $n \geq 2$. It follows that $|x| < 1$, and dividing by $\boldsymbol{\pi}$, we have

$$V_n = \{(x, y, z, t);\ \max\{|y|, |z|, |t|\} = 1,\ |x| < 1,\ |z^2 - t^2 + uy^2 - u\boldsymbol{\pi}(x/\boldsymbol{\pi})^2| = q^{1-n}\}.$$

This case is the same as that of Lemma II.3. We have that $\mathrm{vol}(V_n^0)$ is the product of $1/q$ and the $\mathrm{vol}(V_{n-1}^0)$ of Lemma II.3, which is equal to $q^{-1}(1 - 1/q^2)q^{-(n-1)} = (1 - 1/q^2)q^{-n}$.

Let us consider the case of $n = 1$. Then

$$V_1 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 - \boldsymbol{\pi} y^2 + u\boldsymbol{\pi}(z^2 - t^2)| = 1/q\}.$$

It follows that $|x| < 1$, and dividing by $\boldsymbol{\pi}$, we have

$$V_1 = \{(x, y, z, t);\ \max\{|y|, |z|, |t|\} = 1,\ |x| < 1,\ |z^2 - t^2 + uy^2 + u\boldsymbol{\pi}(x/\boldsymbol{\pi})^2| = 1/q\}.$$

The volume of this subset is the volume of $V_0$ of Lemma II.3 multiplied by $1/q$. The lemma follows. $\square$

**II.5. Lemma.** *When the quadratic form is* $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi} t^2$, *we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ 1/q, & \text{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \text{if } n \geq 2. \end{cases}$$

*Proof.* If $-1 \in R^{\times 2}$, the form is $-u(y^2 + z^2 - u^{-1}x^2 - \boldsymbol{\pi} t^2)$, and its integral has already been considered in Lemma II.3. Thus we can take $u = -1$, so the form is $x^2 + y^2 + z^2 - \boldsymbol{\pi} t^2$.

In the proof of this lemma we will use Theorem 6.27 of the book "Finite Fields" [LN] by Lidl and Niederreiter. This Theorem 6.27 asserts that if $f$ is a quadratic form in odd number $n$ of variables over the finite field $\mathbb{F}_q$ of $q$ elements, then the number of solutions in $\mathbb{F}_q$ of the quadratic equation $f(x_1, x_2, ..., x_n) = b$, $b \in \mathbb{F}_q$, is $q^{n-1} + q^{(n-1)/2}\eta((-1)^{(n-1)/2} b \det(f))$.

Here $\det(f)$ is the determinant of the symmetric matrix representing the quadratic form $f$, and $\eta$ is the quadratic character of $\mathbb{F}_q$: its value on $\mathbb{F}_q^{\times 2}$ is 1, on $\mathbb{F}_q^{\times} - \mathbb{F}_q^{\times 2}$ its value is $-1$, and $\eta(0) = 0$. The case of even $n$ is dealt with in [LN], Theorem 6.26. It asserts that – putting $v(b) = -1$ if $b \neq 0$, and $v(0) = q - 1$ – the number of solutions of $f = b$ is $q^{n-1} + v(b)q^{(n-2)/2}\eta((-1)^{n/2}\det(f))$; but it is not used here. The Theorem 6.27 implies that the equation $f = b$, $b = 0$, where $f$ is the form $x_1^2 + x_2^2 + x_3^2$ in $n = 3$ variables, has $q^2$ solutions over $\mathbb{F}_q$.

Consider the case of $n = 0$. Then

$$V_0 = \{(x, y, z, t);\ \max\{|x|, |y|, |z|, |t|\} = 1,\ |x^2 + y^2 + z^2 - \boldsymbol{\pi} t^2| = 1\},$$

namely $V_0 = \{(x, y, z, t);\ |x^2 + y^2 + z^2| = 1\}$. By Theorem 6.27 of [LN], we have

$$\int\int\int_{|x^2+y^2+z^2|<1} dxdydz = \frac{1}{q}.$$

Hence

$$\mathrm{vol}(V_0) = \int_{|t|\leq 1}\int\int\int_{|x^2+y^2+z^2|=1} dxdydzdt = 1 - \int\int\int_{|x^2+y^2+z^2|<1} dxdydz = 1 - \frac{1}{q}.$$

Let us consider the case of $n \geq 2$. As in Lemma I.0, recall that any $p$-adic number $a$ such that $|a| \leq 1$ can be written as a power series in $\boldsymbol{\pi}$:

$$a = \sum_{i=0}^{\infty} a_i \boldsymbol{\pi}^i = a_0 + a_1\boldsymbol{\pi} + a_2\boldsymbol{\pi}^2 + \dots,$$

where $a_i$ lies in a set $\mathbb{F}$ of representatives in $R$ for the finite field $R/\boldsymbol{\pi}$. In particular $|a| = 1/q^n$ implies that $a_0 = a_1 = \dots = a_{n-1} = 0$ and $a_n \neq 0$. We can write

$$x = \sum_{i=0}^{\infty} x_i\boldsymbol{\pi}^i, \qquad y = \sum_{i=0}^{\infty} y_i\boldsymbol{\pi}^i, \qquad z = \sum_{i=0}^{\infty} z_i\boldsymbol{\pi}^i, \qquad t = \sum_{i=0}^{\infty} t_i\boldsymbol{\pi}^i.$$

Their squares are

$$x^2 = \sum_{i=0}^{\infty} a_i\boldsymbol{\pi}^i, \qquad y^2 = \sum_{i=0}^{\infty} b_i\boldsymbol{\pi}^i, \qquad z^2 = \sum_{i=0}^{\infty} c_i\boldsymbol{\pi}^i, \qquad t^2 = \sum_{i=0}^{\infty} d_i\boldsymbol{\pi}^i,$$

where

$$a_i = \sum_{j=0}^{i} x_j x_{i-j}, \qquad b_i = \sum_{j=0}^{i} y_j y_{i-j}, \qquad c_i = \sum_{j=0}^{i} z_j z_{i-j}, \qquad d_i = \sum_{j=0}^{i} t_j t_{i-j},$$

and $x_i$, $y_i$, $z_i$, $t_i$, $a_i$, $b_i$, $c_i$, $d_i \in \mathbb{F}$.

We have

$$x^2+y^2+z^2-\boldsymbol{\pi}t^2=\sum_{i=0}^{\infty} f_i\boldsymbol{\pi}^i, \qquad f_i\in\mathbb{F},$$

where $f_0=a_0+b_0+c_0$, $f_i=a_i+b_i+c_i-d_{i-1}$ $(i\geq 1)$. Since $|x^2+y^2+z^2-\boldsymbol{\pi}t^2|=1/q^n$ we have that $f_0=f_1=...=f_{n-1}=0$ and $f_n\neq 0$. Thus we obtain the relations (for $a$, $c$ in the set $\mathbb{F}$ which (modulo $\boldsymbol{\pi}$) is the field $R/\boldsymbol{\pi}$)

$$a_0+b_0+c_0=0, \quad a_i+b_i+c_i-d_{i-1}=0\,(i=1,...,n-1), \quad a_n+b_n+c_n-d_{n-1}\neq 0.$$

If $a_0=b_0=c_0=0$, it follows that $x_0=y_0=z_0=t_0=0$ (i.e. $|x|<1$, $|y|<1$, $|z|<1$). Then $a_1=2x_0x_1=0$, $b_1=2y_0y_1=0$, $c_1=2z_0z_1=0$, and thus $d_0=a_1+b_1+c_1=0$, i.e. $|t|<1$. This is a contradiction, since $\max\{|x|,|y|,|z|,|t|\}=1$. Assume that $a_0\neq 0$ (i.e. $x_0\neq 0$). From $a_i+b_i+c_i-d_{i-1}=0$ $(i=1,...,n-1)$ it follows that (since $x_0\neq 0$)

$$x_i=(d_{i-1}-b_i-c_i-\sum_{j=1}^{i-1}x_jx_{i-j})/(2x_0), \qquad x_n\neq(d_{n-1}-b_n-c_n-\sum_{j=1}^{n-1}x_jx_{n-j})/(2x_0),$$

where in the case of $i=1$ the sum over $j$ is empty. Thus, we have

$$\int_{|x|=1}\int_{|t|\leq 1}\int\int_{|x^2+y^2+z^2-\pi t^2|=q^{-n}} dydzdtdx=\left(\frac{1}{q}\right)^{n-1}\left(1-\frac{1}{q}\right).$$

By Theorem 6.27 of [LN], we have

$$\int\int\int_{|x^2+y^2+z^2|<1,\,\max\{|x|,|y|,|z|\}=1} dxdydz=\frac{1}{q}\left(1-\frac{1}{q^2}\right).$$

Thus

$$\mathrm{vol}(V_n)=\frac{1}{q^{n-1}}\left(1-\frac{1}{q}\right)\times\frac{1}{q^3}\left(q^2-1\right)=\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)\frac{1}{q^n}.$$

Let us consider the case of $n=1$. Recall that

$$V_1=\{(x,y,z,t);\,\max\{|x|,|y|,|z|,|t|\}=1,\ |x^2+y^2+z^2-\boldsymbol{\pi}t^2|=1/q\}.$$

We consider two subcases.

(A) Subcase of $\max\{|x|,|y|,|z|\}<1$, i.e. $|x|<1$, $|y|<1$, $|z|<1$, and, consequently, $|t|=1$. The contribution from this subcase is

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|<1}\int_{|t|=1} dtdzdydx=\frac{1}{q^3}\left(1-\frac{1}{q}\right).$$

(B) Subcase of $\max\{|x|,|y|,|z|\}=1$. This is the same as case $n \geq 2$ (with $n=1$). It contributes $(1-q^{-1})q^{-1}(1-q^{-2})$.

Adding the contributions from Subcases (A) and (B) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_1^0)=\frac{1}{q^3}+\frac{1}{q}\left(1-\frac{1}{q^2}\right)=\frac{1}{q}.$$

The lemma follows. □

*Proof of Theorem II.* We are now ready to complete the proof of Theorem II. Recall that we need to compute the value at $s=0$ ($m=-2$) of $I_s(\mathbf{r},A,D)$. Here $I_s(\mathbf{r},A,D)$ coincides with the sum

$$\sum_{n=0}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r},A,D))$$

which converges for $m>-1$ by Proposition 1 or alternatively by Lemmas II.1-II.5. The value at $m=-2$ is obtained then by analytic continuation of this sum.

*Case of Lemma* II.1. The integral $I_s(\mathbf{r},A,D)$ is equal to

$$\mathrm{vol}(V_0^0)+q^{-m}\,\mathrm{vol}(V_1^0)+\sum_{n=2}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$=1-\frac{1}{q}+\left(\frac{2}{q}-\frac{1}{q^2}+\frac{1}{q^3}\right)\frac{1}{q^m}+2\left(1-\frac{1}{q}\right)q^{-2(m+1)}\left(1-\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1-\frac{1}{q}+q^2\left(\frac{2}{q}-\frac{1}{q^2}+\frac{1}{q^3}\right)+2\left(1-\frac{1}{q}\right)\frac{q^2}{1-q}=0.$$

*Case of Lemma* II.2. The integral $I_s(\mathbf{r},A,D)$ is equal to

$$\mathrm{vol}(V_0^0)+q^{-m}\,\mathrm{vol}(V_1^0)+\sum_{n=2}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$=1+\frac{1}{q}+\frac{1}{q^2}\left(1-\frac{1}{q}\right)\frac{1}{q^m}+\frac{2}{q}\left(1-\frac{1}{q}\right)q^{-2(m+1)}\left(1-\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1+\frac{1}{q}+1-\frac{1}{q}+\frac{2}{q}\left(1-\frac{1}{q}\right)\frac{q^2}{1-q}=0.$$

*Case of Lemmas* II.3, II.4, II.5. The integral $I_s(\mathbf{r}, A, D)$ is equal to

$$\mathrm{vol}(V_0^0) + q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 + \frac{1}{q}\frac{1}{q^m} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right) q^{-2(m+1)} \left(1 - \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 + \frac{1}{q}q^2 + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)\frac{q^2}{1-q} = 0.$$

The theorem follows. □

## Character computation for type III

For the $\theta$-conjugacy class of type III we write out the representative $g = t \cdot \mathrm{diag}(\boldsymbol{r}, \boldsymbol{r})$ as

$$\begin{pmatrix} a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A & (b_1r_1 + b_2r_2A)D & (b_1r_2 + b_2r_1)AD \\ a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A & (b_1r_2 + b_2r_1)D & (b_1r_1 + b_2r_2A)D \\ b_1r_1 + b_2r_2A & (b_1r_2 + b_2r_1)A & a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A \\ b_1r_2 + b_2r_1 & b_1r_1 + b_2r_2A & a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A \end{pmatrix}.$$

The product ${}^t\mathbf{v}gJ\mathbf{v}$ (where ${}^t\mathbf{v} = (x, y, z, t)$) is equal to

$$(b_1r_2 + b_2r_1)(t^2 + z^2A - y^2D - x^2AD) + 2(b_1r_1 + b_2r_2A)(zt - xyD),$$

where $a_1 + a_2\sqrt{A} \in E_3^\times$ and $b_1 + b_2\sqrt{A} \in E_3^\times$. The trace is a function of $g$ in the projective group, and $r = r_1 + r_2\sqrt{A}$ ranges over a set of representatives in $E_3^\times$ ($E_3 = F(\sqrt{A})$) for $E_3^\times / N_{E/E_3}E^\times$.

By definition, the quadratic form can be written as

$$\frac{rb - \tau(rb)}{2\sqrt{A}}(t^2 + z^2A - y^2D - x^2AD) + (rb + \tau(rb))(zt - xyD).$$

Set $I_s(r, A, D)$ to be equal to

$$\int_{V^0} \left| \frac{rb - \tau(rb)}{2\sqrt{A}}(t^2 + z^2A - y^2D - x^2AD) + (rb + \tau(rb))(zt - xyD) \right|^{2(s-1)} dxdydzdt.$$

The property of the numbers $A$, $D$ and $AD$ that we need is that their square roots generate the three distinct quadratic extensions of $F$. Thus we may assume that $\{A, D, AD\} = \{u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$, where $u \in R^\times - R^{\times 2}$. Of course with this normalization $AD$ is no longer the product of $A$ and $D$, but its representative in the set $\{1, u, \boldsymbol{\pi}, u\boldsymbol{\pi}\}$ mod $F^{\times 2}$.

**III.1. Proposition.** *(i) If $D = u$ and $A = \boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then $\sqrt{A} \notin N_{E/E_3}E^\times = A^{\mathbb{Z}}R_3^\times$.*
*(ii) If $A = u$ and $-1 \in R^{\times 2}$, and $D = \boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then $\sqrt{A} \notin N_{E/E_3}E^\times = (-D)^{\mathbb{Z}}R_3^{\times 2}$.*
*(iii) If $A = u = -1 \notin R^{\times 2}$ and $D = \boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then there is $d \in R^\times$ with $d^2+1 \in -R^{\times 2} = R^\times - R^{\times 2}$, hence $d + i \in R_3^\times - R_3^{\times 2}$ $(i = \sqrt{A})$ and so $d + i \in E_3^\times - N_{E/E_3}E^\times$.*

*Proof.* For (iii) note that $R^\times/\{1+\boldsymbol{\pi}R\}$ is the multiplicative group of a finite field $\mathbb{F}$ of $q$ elements. There are $1+\frac{1}{2}(q-1)$ elements in each of the sets $\{1+x^2; x \in \mathbb{F}\}$ and $\{-y^2; y \in \mathbb{F}\}$. As $2(1+\frac{1}{2}(q-1)) > q$, there are $x$, $y$ with $1+x^2 = -y^2$. But $y \neq 0$ as $-1 \notin \mathbb{F}^{\times 2}$. Hence there is $x$ with $1+x^2 \notin \mathbb{F}^{\times 2}$, and our $d$ exists. □

Since $r$ ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, by Proposition III.1 we can choose $br$ to be 1 or $\sqrt{A}$ or $d+i$. Correspondingly the quadratic form takes one of the three shapes

$$t^2 + z^2A - y^2D - x^2AD, \qquad \text{or} \qquad zt - xyD, \qquad t^2 - z^2 - y^2D + x^2D + 2d(zt - xyD).$$

Recall that we need to compute

$$\left(\frac{\nu}{\mu}\right)(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)}\int_{V^0} |{}^t\mathbf{v}gJ\mathbf{v}|^m d\mathbf{v}. \tag{$*$}$$

Since $\det g = \alpha r \cdot \sigma(\alpha r) \cdot \tau(\alpha r) \cdot \tau\sigma(\alpha r)$, we have

$$\begin{aligned}\left(\frac{\nu}{\mu}\right)(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)} &= |\det g|^{(1-s)/2}\left|\frac{(\alpha r - \sigma(\alpha r))^2}{\alpha r\sigma(\alpha r)} \cdot \frac{\tau(\alpha r - \sigma(\alpha r))^2}{\tau(\alpha r)\tau\sigma(\alpha r)}\right|^{1/2} \\ &= \frac{|4br\tau(br)D|}{|r^2\tau r^2(a^2 - b^2D)(\tau a^2 - \tau b^2 D)|^{s/2}}.\end{aligned}$$

When $s = 0$, this is $|br\tau(br)D|$, and $(*)$ is independent of $b$. So we may assume that $b = 1$.

**III. Theorem.** *The value of $|br\tau(br)D|I_s(r, A, D)/(T\phi_0)(\mathbf{v}_0)$ at $s = 0$ is $2\kappa_{E/E_3}(r)$, where $\kappa_{E/E_3}$ is the nontrivial character of $E_3^\times/N_{E/E_3}E^\times$, $E = E_3(\sqrt{D})$.*

*Proof.* Assume that $br = \sqrt{A} \notin N_{E/E_3}E^\times$, thus $|br\tau(br)D| = |AD|$, and the quadratic form is $t^2 + z^2A - y^2D - x^2AD$. If $|A| = 1/q$ or $-1$ is a square, we can replace $A$ with $-A$. The quadratic form then becomes the same as that of type I. The result of the computation is $-2$, see proof of Theorem I, case of anisotropic quadratic forms. Since $\kappa_{E/E_3}(\sqrt{A}) = -1$ we are done in this case.

If $A = -1$, $br = d+i \notin N_{E/E_3}E^\times$, the quadratic form is $t^2 - z^2 - y^2D + x^2D + 2d(zt - xyD)$. It is equal to $X^2 - uY^2 - D(Z^2 - uT^2)$ with $X = t + dz$, $Y = z$, $Z = y + dx$, $T = x$ and $u = d^2 + 1 \in R^\times - R^{\times 2}$. Since $|D| = 1/q$ the quadratic form is anisotropic and the result of the computation is $-2$ by the proof of Theorem I, case of anisotropic quadratic forms.

Assume that $br = 1$, thus $|br\tau(br)D| = |D|$ and the quadratic form is $zt - xyD$. Then it is $\frac{1}{4}$ times $(z+t)^2 - (z-t)^2 - D[(x+y)^2 - (x-y)^2]$. Since $\max\{|x|, |y|, |z|, |t|\} = 1$ implies $\max\{|x+y|, |x-y|, |z+t|, |z-t|\} = 1$, the result of the computation is 2 by the proof of Theorem I, cases of Lemmas I.1 and I.3. The theorem follows. □

## Character computation for type IV

For the $\theta$-conjugacy class of type IV we write the representative $g = t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$ (where $t = h^{-1}t^*h$, $t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$) as

$$\begin{pmatrix} a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A & (b_1'r_1 + b_2'r_2A)D & (b_1'r_2 + b_2'r_1)AD \\ a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A & (b_1'r_2 + b_2'r_1)D & (b_1'r_1 + b_2'r_2A)D \\ b_1r_1 + b_2r_2A & (b_1r_2 + b_2r_1)A & a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A \\ b_1r_2 + b_2r_1 & b_1r_1 + b_2r_2A & a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A \end{pmatrix}.$$

Here $E_3 = F(\sqrt{A})$ is a quadratic extension of $F$ and $E = E_3(\sqrt{D})$ is a quadratic extension of $E_3$, thus $A \in F - F^2$ and $D = d_1 + d_2\sqrt{A} \in E_3 - E_3^2$, $d_i \in F$.

If $-1 \in F^{\times 2}$ we can and do take $D = \sqrt{A}$, where $A$ is a nonsquare unit $u$ if $E_3/E$ is unramified, or a uniformizer $\boldsymbol{\pi}$ if $E_3/F$ is ramified. If $-1 \notin F^{\times 2}$ and $E_3/F$ is ramified, once again we may and do take $A = \boldsymbol{\pi}$ and $D = \sqrt{A}$.

If $-1 \notin F^{\times 2}$ and $E_3/F$ is unramified, take $A = -1$ and note that a primitive 4th root $\zeta = i$ of 1 lies in $E_3$ (and generates it over $F$). Then $E/E_3$ is unramified, generated by $\sqrt{D}$, $D = d_1 + id_2$, and we can (and do) take $d_2 = 1$ and a unit $d_1 = d$ in $F^\times$ such that $d^2 + 1 \notin F^{\times 2}$. Then $D = d + i \notin E_3^{\times 2}$. The existence of $d$ is shown as in the proof of Proposition III.1.

Further $\alpha = a + b\sqrt{D} \in E^\times$, where $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, and $r = r_1 + r_2\sqrt{A} \in E_3^\times/N_{E/E_3}E^\times$. The relation $bD = b_1' + b_2'\sqrt{A}$ defines $b_1' = b_1d_1 + b_2d_2A$ and $b_2' = b_2d_1 + b_1d_2$.

The product ${}^t\mathbf{v}gJ\mathbf{v}$ (where ${}^t\mathbf{v} = (x, y, z, t)$) is then equal to

$$(b_1r_2 + b_2r_1)(t^2 + z^2A) - (b_1'r_2 + b_2'r_1)(y^2 + x^2A) + 2(b_1r_1 + b_2r_2A)zt - 2(b_1'r_1 + b_2'r_2A)xy.$$

Since $bD = b_1' + b_2'\sqrt{A}$, this is

$$\frac{br - \sigma(br)}{2\sqrt{A}}(t^2 + z^2A) + (br + \sigma(br))zt$$
$$-\frac{brD - \sigma(brD)}{2\sqrt{A}}(y^2 + x^2A) - (brD + \sigma(brD))xy.$$

Note that $r$ ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, and $b$ lies in $E_3^\times$. As $b$ is fixed, we may take $br$ to range over $E_3^\times/N_{E/E_3}E^\times$. Thus we may assume that $b = 1$.

Further, note that $E_3/F$ is unramified if and only if $E/E_3$ is unramified. Hence $r$ can be taken to range over $\{1, \boldsymbol{\pi}\}$ if $E_3/F$ is unramified, and over $\{1, u\}$ if $E_3/F$ is ramified, where $\boldsymbol{\pi}$ is a uniformizer in $F$ and $u$ is a nonsquare unit in $F$, in these two cases. Thus in both cases we have that $\sigma(r) = r$, and the quadratic form is equal to the product of $r$ and

$$2zt - \frac{D - \sigma(D)}{2\sqrt{A}}(y^2 + x^2A) - (D + \sigma(D))xy.$$

Our aim is to compute the value at $s = 0$ of the integral $I_s(A, D)$ defined by

$$\int_{V^0} \left| 2zt - \frac{D - \sigma(D)}{2\sqrt{A}}(y^2 + x^2A) - (D + \sigma(D))xy \right|^{2(s-1)} dxdydzdt.$$

**IV. Theorem.** *The value of $I_s(A,D)$ at $s=0$ is 0.*

To prove this theorem we need some lemmas.

**IV.1. Proposition.** *Up to a change of coordinates, the quadratic form*

$$2zt - \frac{D-\sigma(D)}{2\sqrt{A}}(y^2+x^2A) - (D+\sigma(D))xy$$

*is equal to either $x^2+\boldsymbol{\pi}y^2-2zt$ or $x^2-uy^2-2zt$ with $u \in R^\times - R^{\times 2}$. It is always isotropic.*

*Proof.* In the cases when $D=\sqrt{A}$, we have $\sigma(D)=-D$. When $D=d+i$, $\sigma D = d-i$. Thus the quadratic form takes one of the following three shapes

$$2zt-(y^2+\boldsymbol{\pi}x^2), \qquad 2zt-(y^2-ux^2), \qquad 2zt-(y^2-x^2)-2dxy.$$

For the third quadratic form we have

$$2zt-(y^2-x^2)-2dxy \;=\; (x-dy)^2-(d^2+1)y^2+2zt.$$

Recall that $u=d^2+1 \in R^\times - R^{\times 2}$. After the change of variables $x'=x-dy$, followed by $x' \mapsto x$, the quadratic form is $x^2-uy^2+2zt$. Change $z \mapsto -z$ to get $x^2-uy^2-2zt$. □

**IV.2. Lemma.** *When the quadratic form is $x^2-uy^2-2zt$, we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1+1/q^2, & \text{if } n=0, \\ q^{-n}(1-1/q)(1+1/q^2), & \text{if } n \geq 1. \end{cases}$$

*Proof.* Consider the case of $n=0$. Then

$$V_0 = \{(x,y,z,t);\, \max\{|x|,|y|,|z|,|t|\}=1,\ |x^2-uy^2-2zt|=1\}.$$

(A) Subcase of $|x^2-uy^2|=1$ and $|zt|<1$. The contribution is the product of

$$\int\int_{|x^2-uy^2|=1} dxdy = \int_{|x|=1}\int_{|y|\leq 1} dydx + \int_{|x|<1}\int_{|y|=1} dydx = \left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)$$

and

$$\int\int_{|zt|<1} dzdt = \int_{|z|<1}\int_{|t|\leq 1} dtdz + \int_{|z|=1}\int_{|t|<1} dzdt = \frac{1}{q}+\frac{1}{q}\left(1-\frac{1}{q}\right) = \frac{1}{q}\left(2-\frac{1}{q}\right).$$

(B) Subcase of $|x^2-uy^2|<1$ and $|zt|=1$ (i.e. $|z|=1$, $|t|=1$). Since $x^2-uy^2$ does not represent zero non trivially, the condition implies that $|x|<1$, $|y|<1$. Thus we obtain

$$\int_{|x|<1}\int_{|y|<1}\int_{|z|=1}\int_{|t|=1} dtdzdydx = \frac{1}{q^2}\left(1-\frac{1}{q}\right)^2.$$

(C) Subcase of $|x^2-uy^2|=1$ and $|zt|=1$. In this case, once $x$, $y$, and $z$ are chosen, we have that $|t|=1$, and the condition $|x^2-uy^2-2zt|=1$ implies $t\not\equiv (x^2-uy^2)/(2z) \pmod{\boldsymbol{\pi}}$. Thus, we obtain

$$\int\int_{|x^2-uy^2|=1}\int_{|z|=1}\int_{|t|=1,|x^2-uy^2+zt|=1} dtdzdxdy=\left(1-\frac{1}{q}\right)^2\left(1+\frac{1}{q}\right)\left(1-\frac{2}{q}\right).$$

Adding the contributions from Subcases (A), (B), and (C) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_0^0)=\frac{1}{q}\left(1+\frac{1}{q}\right)\left(2-\frac{1}{q}\right)+\frac{1}{q^2}\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\left(1-\frac{2}{q}\right).$$

Once simplified this is equal to $1+1/q^2$.

Consider the case $n\geq 1$. We have the following two subcases.

(A) Subcase of $|z|=1$. Then $x^2-uy^2-2zt=\varepsilon\boldsymbol{\pi}^n$, where $|\varepsilon|=1$, and $t=(x^2-uy^2-\varepsilon\boldsymbol{\pi}^n)/(2z)$. Further, $dt=q^{-n}d\varepsilon$, and the contribution from this subcase is

$$\int_{|x|\leq 1}\int_{|y|\leq 1}\int_{|z|=1}\int_{|\varepsilon|=1}\frac{1}{q^n}d\varepsilon dzdydx=\left(1-\frac{1}{q}\right)\frac{1}{q^n}\int_{|\varepsilon|=1}d\varepsilon=\frac{1}{q^n}\left(1-\frac{1}{q}\right)^2.$$

(B) Subcase of $|z|<1$. If $|t|<1$, then $\max\{|x|,|y|\}=1$, and since $x^2-uy^2$ does not represent zero non trivially, we have that $|x^2-uy^2-2zt|=|x^2-uy^2|=1$, which is a contradiction, since $n\geq 1$. Hence $|t|=1$. We have $x^2-uy^2-2zt=\varepsilon\boldsymbol{\pi}^n$, where $|\varepsilon|=1$. Further, from

$$|z|=\left|\frac{x^2-uy^2}{2t}-\frac{\varepsilon}{2t}\boldsymbol{\pi}^n\right|=|x^2-uy^2-\varepsilon\boldsymbol{\pi}^n|<1,$$

it follows that $|x^2-uy^2|<1$, and thus $|x|<1$, $|y|<1$. The contribution from this subcase is

$$\int_{|x|<1}\int_{|y|<1}\int_{|t|=1}\int_{|\varepsilon|=1}\frac{1}{q^n}d\varepsilon dtdydx=\frac{1}{q^n}\left(1-\frac{1}{q}\right)^2\frac{1}{q^2}.$$

Adding the contributions from Subcases (A) and (B) (divided by $(1-1/q)$), we obtain

$$\mathrm{vol}(V_n^0)=\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q^2}\right)\frac{1}{q^n}.$$

The lemma follows. □

*Proof of Theorem IV.* We are now ready to complete the proof of Theorem IV. Recall that we need to compute the value at $s=0$ $(m=-2)$ of $I_s(A,D)$. Here $I_s(A,D)$ coincides with the sum

$$\sum_{n=0}^{\infty}q^{-nm}\,\mathrm{vol}(V_n^0(A,D))$$

which converges for $m > -1$. The value at $m = -2$ is obtained then by analytic continuation of this sum.

*Case of* $x^2 + \boldsymbol{\pi} y^2 - 2zt$. Make a change of variables $z \mapsto 2u^{-1}z'$, followed by $z' \mapsto z$. Thus the quadratic form is equal to

$$-u^{-1}((z-t)^2 - (z+t)^2 - ux^2 - u\boldsymbol{\pi} y^2).$$

Note that up to a multiple by a unit, this is a form of Lemma II.3. Since $\max\{|z|,|t|\} = 1$ implies $\max\{|z+t|,|z-t|\} = 1$, the result of that lemma holds for our quadratic form as well.

*Case of* $x^2 - uy^2 - 2zt$. By Lemma IV.2, the integral

$$I_s(A,D) = \mathrm{vol}(V_0^0) + \sum_{n=1}^{\infty} q^{-nm}\,\mathrm{vol}(V_n^0)$$

is equal to

$$1 + \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right) q^{-(m+1)} \left(1 - \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 + \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\frac{q}{1-q} = 0.$$

The theorem follows. $\square$

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