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UNRAMIFIED WHITTAKER FUNCTIONS ON THE METAPLECTIC GROUP

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ABSTRACT. Kazhdan (unpublished), Shintani [**Sh**] and Casselman and Shalika [**CS**] computed explicitly the unramified Whittaker function of a quasi-split p-adic group. This is the main local ingredient used in the Rankin-Selberg-Shimura method, which yielded interesting results in the study of Euler products such as $L(s,\pi\otimes\pi')$ by Jacquet and Shalika [**JS**] (here π,π' are cuspidal $GL(n,A_F)$-modules), and $L(s,\pi,r)$ by [**F**] (here π is a cuspidal $GL(n,A_E)$-module, E is a quadratic extension of the global field F, and r is the twisted tensor representation of the dual group of $\operatorname{Res}_{E/F}GL(n)$). Our purpose here is to generalize Shintani's computation [**Sh**] from the context of $GL(n)$ to that of the metaplectic r-fold covering group $\tilde{G}$ of $GL(n)$ (see [**F′**, **FK**]).

Notations. Let F be a nonarchimedean local field with a ring R of integers and a uniformizer u of the maximal ideal of R. Denote by q the cardinality of the residue field $R/(u)$ of F. Let r,n be positive integers. Put $G=GL(n,F), K=GL(n,R)$. Let μ_r be the cyclic group of order r. Denote by $\tilde{G}$ the r-fold central topological covering group of G (see [**FK**]). Then there is an exact sequence $1\to\mu_r\xrightarrow{i}\tilde{G}\xrightarrow{p}G\to 1$, with a preferred section $s\colon G\to\tilde{G}$ of p. We identify μ_r with its image via i. We also fix an embedding of μ_r in the field $\mathbf{C}$ of complex numbers. Suppose that r is a unit in R (its valuation is one). Then K embeds (see [**FK**]) as a subgroup of $\tilde{G}$; we identify K with its image. Fix a Haar measure on G by the requirement that the volume of K is one.

Let $L_c(\tilde{G}//K)$ denote the commutative convolution algebra (see [**FK**]) of complex-valued compactly-supported K-biinvariant anti-genuine functions on $\tilde{G}$. A function $f\colon\tilde{G}\to\mathbf{C}$ is called anti-genuine if $f(\varsigma g)=\varsigma^{-1}f(g)$ for all ς in μ_r and g in $\tilde{G}$. In writing ςg we used the embedding of μ_r in $\tilde{G}$; in writing $\varsigma^{-1}f(g)$ we used the embedding of μ_r in $\mathbf{C}^\times$. Let π be an irreducible representation of $\tilde{G}$ which is unramified (has a nonzero K-fixed vector) and genuine ($\pi(\varsigma g)=\varsigma\pi(g)$; ς in μ_r, g in $\tilde{G}$). By the theory of the Satake transform (see [**FK**]) it determines an algebra homomorphism, denoted again by π, of $L_c(\tilde{G}//K)$ into $\mathbf{C}$.

For any n-tuple $m=(m_1,\dots,m_n)$ of integers, denote by u^m the diagonal matrix whose ith diagonal entry is u^{m_i} $(1\le i\le n)$. Denote by N the group of upper triangular unipotent matrices in G. The section s injects N and $u^{r\mathbf{Z}}$ as subgroups of $\tilde{G}$ (see [**FK**]). We identify N and $u^{r\mathbf{Z}}$ with their images in $\tilde{G}$. Write $m(i)$ for

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$m = (1, \ldots, 1, 0, \ldots, 0)$ where 1 appears in the first i places. Let f_i $(1 \le i \le n)$ be the member of $L_c(\tilde{G}//K)$ which is supported on $\mu_r K u^{rm(i)} K$ and attains the value 1 at $u^{rm(i)}$. Then $L_c(\tilde{G}//K)$ is isomorphic to the polynomial ring generated by $f_1, \ldots, f_n$.

Choose n complex numbers $t_1, \ldots, t_n$ so that the ith elementary symmetric function $\mathrm{Sym}_i((t_j)) = \sum_{j_1 \le \cdots \le j_i} t_{j_1} \cdots t_{j_i}$ in the t_j's is equal to $q^{ir(i-1)/2}\pi(f_i)$ $(1 \le i \le n)$. Let t be the diagonal matrix whose ith diagonal entry is t_i $(1 \le i \le n)$. It lies in $GL(n, \mathbf{C})$ since $\det t = q^{nr(n-1)/2}\pi(f_n) \ne 0$.

Let $c(t)$ be the complex-valued function on $\mathbf{Z}^n$ which attains the value zero at $m = (m_1, \ldots, m_n)$ unless $m_1 \ge \cdots \ge m_n$, where

$$c(t; m) = \det(t_i^{m_j+n-j}; 1 \le i, j \le n) / \det(t_i^{n-j}; 1 \le i, j \le n).$$

Here the numerator is called a Schur function (see [**M**, p. 24]), and it is divisible in $\mathbf{Z}[t_1, \ldots, t_n]$ by the denominator, which is the Vandermonde determinant $\prod(t_i - t_j)$ $(1 \le i < j \le n)$. Note that for m with $m_1 \ge \cdots \ge m_n$, $c(t; m)$ is the value at t of the character of the irreducible representation of $GL(n, \mathbf{C})$ with highest weight m. We have $c(t; m(i)) = \mathrm{Sym}_i((t_j))$, and this is equal to $q^{ir(i-1)/2}\pi(f_i)$ by the definition of t.

Choose a character ψ of the additive group of F which is trivial on R but not on $u^{-1}R$. Denote by ψ also the character of N given by $\psi(x) = \prod_{i=1}^{n-1} \psi(x_{i,i+1})$, where $x_{i,i+1}$ is the $(i, i+1)$ entry of x. Given π and ψ, the function W on $\tilde{G}$ is called an *unramified Whittaker function* associated with π and ψ if it satisfies

$$W(\varsigma x g k) = \varsigma \psi(x) W(g) \quad (x \text{ in } N, \varsigma \text{ in } \mu_r, g \text{ in } \tilde{G}, k \text{ in } K), \tag{1}$$

and

$$\pi(f)W(g) = \int_G W(gh) f(h)\, dh \quad (f \text{ in } L_c(\tilde{G}//K), g \text{ in } \tilde{G}). \tag{2}$$

The integral is taken over $G \simeq \tilde{G}/\mu_r$; the integrand is invariant under μ_r. Let D be the set of m with $m_1 \ge \cdots \ge m_n \ge 0$ and $r + m_i < m_{i+1}$ $(1 \le i < n)$. For each d in D put $a(d; m) = W(s(u^d)u^{rm})$. Recall that we identify $u^{r\mathbf{Z}}$ with its image in $\tilde{G}$ via the section s. Since $G = \bigcup N u^m K$ (disjoint union over m in $\mathbf{Z}^n$), to determine W on $\tilde{G}$ it suffices (by (1)) to evaluate $a(d; m)$ for all d in D and m in $\mathbf{Z}^n$. Since the conductor of ψ is R, if follows from (1) that $a(d; m)$ is zero unless $m_1 \ge m_2 \ge \cdots \ge m_n$.

THEOREM. *For each d in D we have $a(d; m) = a(d; 0) q^{r\sum_{i=1}^n (i-n)m_i} c(t; m)$ for all m in $\mathbf{Z}^n$.*

Let $I(i)$ be the set of all n-tuples $e = (e_1, \ldots, e_n)$ with entries e_i equal to zero or one such that $e_1 + \cdots + e_n = i$. Put $N_R = N \cap K$ and $N_R(e) = N_R \cap u^e K u^{-e}$. Note that the cardinality $[N_R/N_r(re)]$ of $N_R/N_R(re)$ is q^{rw}, where

$$w = \sum_{j>k} \max(e_j - e_k, 0) = in - i(i-1)/2 - \sum_{j=1}^n j e_j.$$

Denote by $f[xu^{re}K]$ the right K-invariant anti-genuine complex-valued function on $\tilde{G}$ which is supported on $\mu_r x u^{re} K$ (e in $I(i)$, x in $N_R/N_R(re)$), and attains the value one at xu^{re}.

We first assume the validity of the following lemma.

LEMMA. *For each $1 \le i \le n$ we have $f_i = \sum_{e \in I(i)} \sum_{x \in N_R/N_R(re)} f[xu^{re}K]$.*

The Lemma implies that for $m = (m_1, \dots, m_n)$ in $\mathbf{Z}^n$ with $m_1 \ge \cdots \ge m_n$ we have

$$\begin{aligned}\pi(f_i)a(d;m) &= \int_G W(s(u^d)u^{rm}g)f_i(g)\,dg \\ &= \sum_e [N_R/N_R(re)]a(d;m+e) \qquad (e \text{ in } I(i)) \\ &= q^{irn-ir(i-1)/2}\sum_e q^{-r\sum_{j=1}^n je_j}a(d;m+e).\end{aligned}$$

Put

$$b(d;m) = q^{r\sum_{j=1}^n (n-j)m_j}a(d;m).$$

If $m_1 \ge \cdots \ge m_n$, then we have

$$q^{ir(i-1)/2}\pi(f_i)b(d;m) = \sum_e b(d;m+e) \qquad (1 \le i \le n).$$

Otherwise $b(d;m) = 0$. Namely $b(d;m)$ satisfies the equation

$$c(t;m(i))b(d;m) = \sum_e b(d;m+e) \quad \text{if } m_1 \ge \cdots \ge m_n.$$

On the other hand, the function $c(t;j)$ satisfies the equation

$$c(t;m(i))c(t;m) = \sum_e c(t;m+e) \quad \text{if } m_1 \ge \cdots \ge m_n.$$

Hence both $c(t;m)$ and $b(d;m)$ (for each d in D) are functions of m in $\mathbf{Z}^n$ which satisfy the same system of difference equations which has a unique solution up to a constant multiple. Since $c(t;0) = 1$ the theorem follows.

It remains to prove the Lemma.

PROOF OF LEMMA. Let W be the Weyl group of permutation matrices in K, realized as a group of matrices with entries zero and one only. Let I be the Iwahori subgroup of K which consists of all matrices in K whose under diagonal entries are all in uR. Put $\overline{N}$ for the group of lower triangular unipotent matrices, N_I for $\overline{N} \cap I$, and A_R for the diagonal subgroup of K. We have the decompositions $I = N_R A_R \overline{N}_I = \overline{N}_I A_R N_R$ and $K = \bigcup N_R \overline{N}_I w N_R A_R$ (disjoint union over w in W). Put $e(i)$ for $m = (0, \dots, 0, 1, \dots, 1)$, where 1 appears in the last i entries. Since $u^{-re(i)} N_R u^{re(i)}$ lies in K, we have

$$Ku^{rm(i)}K = Ku^{re(i)}K \subset \bigcup_{w \in W} N_R A_R \overline{N}_I w u^{re(i)} w^{-1} K.$$

Put y for the element $wu^{re(i)}w^{-1}$ of $\tilde{G}$. For $1 \le j < k \le n$, and a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{jk}$ for the matrix (x_{uv}) in $GL(n)$ whose entries along the diagonal are one except that $x_{jj} = a$, $x_{kk} = d$, and its nondiagonal entries are zero except that $x_{jk} = b$ and $x_{kj} = c$.

Suppose that $\bar{n} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}_{jk}$ is a matrix in $\overline{N}_I$, and $y^{-1}\bar{n}y$ does not lie in K. Write $|\cdot|$ for the valuation on F normalized by $|u| = q^{-1}$. Then $q^{-r} < |x| < 1$, the

jth diagonal entry of $p(y)$ is one and the kth is u^r, and

$$y^{-1}\bar{n}y = \begin{pmatrix} 1 & 0 \\ u^{-r}x & 1 \end{pmatrix}_{jk}.$$

We have

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}_{jk} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}_{jk} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{jk} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}_{jk}.$$

Given the matrix $s(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)_{jk})$ in $\tilde{G}$, we have in $\tilde{G}$ (see [**FK**, (2.1)]) the relation

$$\begin{aligned} s\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk}\right)^{-1} \\ = s\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk}\right)\left(\frac{b}{a, x}\right). \end{aligned}$$

Here $(\cdot,\cdot)$ denotes the nondegenerate bimultiplicative rth Hilbert symbol

$$F^\times/F^{\times r} \times F^\times/F^{\times r} \to \mu_r.$$

Taking $b = 1$ and a unit a, namely $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)_{jk}$ in K, we conclude that the anti-genuine K-biinvariant function f_i attains the value zero at $\bar{n}y$, since

$$\begin{aligned} f_i(\bar{n}y) &= f_i\left(y\, s\left(\begin{pmatrix} 1 & x^{-1}u^r \\ 0 & 1 \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{jk}\right)\right. \\ &\qquad \left. \times\, s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} 1 & x^{-1}u^r \\ 0 & 1 \end{pmatrix}_{jk}\right)\right) \\ &= f_i\left(s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\right) \\ &= f_i\left(s\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\right) \\ &= (a,x) f_i\left(s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\, s\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk}\right)\right) \\ &= (a,x) f_i\left(s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\right) \\ &= (a,x) f_i(\bar{n}y). \end{aligned}$$

Since the Hilbert symbol $(\cdot,\cdot)$ is nondegenerate we can find a unit a with $(a, x) \neq 1$; indeed, $(a, b) = 1$ for any pair a, b of units, hence there exists some unit a with $(a, u) = \varsigma$, where ς is a primitive rth root of unity. It follows that f_i is supported on the subset

$$\mu_r \bigcup_{w \in W} N_R w u^{re(i)} w^{-1} K \quad \text{of } \tilde{G}.$$

Since this set is contained in $\mu_r K u^{m(i)} K$, the Lemma follows.

REMARK. If π has a Whittaker model, namely we have $\pi(g)W(h) = W(hg)$ for all h, g in $\tilde{G}$, then $a(d;m) = \pi(s(u^d)u^{rm})a(0;0)$, and in particular $a(d;0) = \pi(s(u^d))a(0;0)$ for all d in D. In this case there exists a unique (up to a scalar multiple) unramified Whittaker function associated with π, ψ.

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