UNRAMIFIED WHITTAKER FUNCTIONS ON
THE METAPLECTIC GROUP

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ABSTRACT. Kazhdan (unpublished), Shintani [Sh] and Casselman and Shalika [CS] computed explicitly the unramified Whittaker function of a quasi-split $p$-adic group. This is the main local ingredient used in the Rankin-Selberg-Shimura method, which yielded interesting results in the study of Euler products such as $L(s, \pi \otimes \pi')$ by Jacquet and Shalika [JS] (here $\pi, \pi'$ are cuspidal $GL(n, A_F)$-modules), and $L(s, \pi, \tau)$ by [F] (here $\pi$ is a cuspidal $GL(n, A_F)$-module, $E$ is a quadratic extension of the global field $F$, and $\tau$ is the twisted tensor representation of the dual group of $\text{Res}_{E/F} GL(n)$). Our purpose here is to generalize Shintani's computation [Sh] from the context of $GL(n)$ to that of the metaplectic $r$-fold covering group $\tilde{G}$ of $GL(n)$ (see [F'], [FK]).

Notations. Let $F$ be a nonarchimedean local field with a ring $R$ of integers and a uniformizer $u$ of the maximal ideal of $R$. Denote by $q$ the cardinality of the residue field $R/(u)$ of $F$. Let $r, n$ be positive integers. Put $G = GL(n, F), K = GL(n, R)$. Let $\mu_r$ be the cyclic group of order $r$. Denote by $\tilde{G}$ the $r$-fold central topological covering group of $G$ (see [FK]). Then there is an exact sequence $1 \to \mu_r \xrightarrow{i} \tilde{G} \xrightarrow{p} G \to 1$, with a preferred section $s: G \to \tilde{G}$ of $p$. We identify $\mu_r$ with its image via $i$. We also fix an embedding of $\mu_r$ in the field $C$ of complex numbers. Suppose that $r$ is a unit in $R$ (its valuation is one). Then $K$ embeds (see [FK]) as a subgroup of $\tilde{G}$; we identify $K$ with its image. Fix a Haar measure on $\tilde{G}$ by the requirement that the volume of $K$ is one.

Let $L_c(\tilde{G} // K)$ denote the commutative convolution algebra (see [FK]) of complex-valued compactly-supported $K$-invariant anti-genuine functions on $\tilde{G}$. A function $f: \tilde{G} \to C$ is called anti-genuine if $f(\zeta g) = \zeta^{-1} f(g)$ for all $\zeta$ in $\mu_r$ and $g$ in $\tilde{G}$. In writing $\zeta g$ we used the embedding of $\mu_r$ in $\tilde{G}$; in writing $\zeta^{-1} f(g)$ we used the embedding of $\mu_r$ in $C^\times$. Let $\pi$ be an irreducible representation of $G$ which is unramified (has a nonzero $K$-fixed vector) and genuine ($\pi(\zeta g) = \zeta \pi(g); \zeta$ in $\mu_r$, $g$ in $\tilde{G}$). By the theory of the Satake transform (see [FK]) it determines an algebra homomorphism, denoted again by $\pi$, of $L_c(\tilde{G} // K)$ into $C$.

For any $n$-tuple $m = (m_1, \ldots, m_n)$ of integers, denote by $u^m$ the diagonal matrix whose $i$th diagonal entry is $u^{m_i}$ ($1 \leq i \leq n$). Denote by $N$ the group of upper triangular unipotent matrices in $\tilde{G}$. The section $s$ injects $N$ and $u^Z$ as subgroups of $\tilde{G}$ (see [FK]). We identify $N$ and $u^Z$ with their images in $\tilde{G}$. Write $m(i)$ for
$m = (1, \ldots, 1, 0, \ldots, 0)$ where 1 appears in the first $i$ places. Let $f_i$ ($1 \leq i \leq n$) be the member of $L_c(\hat{G}/K)$ which is supported on $\mu_r K u^{-m(i)} K$ and attains the value 1 at $u^{-m(i)}$. Then $L_c(\hat{G}/K)$ is isomorphic to the polynomial ring generated by $f_1, \ldots, f_n$.

Choose $n$ complex numbers $t_1, \ldots, t_n$ so that the $i$th elementary symmetric function $\text{Sym}_i((t_j)) = \sum_{j_1 \leq \cdots \leq j_i} t_{j_1} \cdots t_{j_i}$ in the $t_j$'s is equal to $q^{ir(i-1)/2} \pi(f_i)$ ($1 \leq i \leq n$). Let $t$ be the diagonal matrix whose $i$th diagonal entry is $t_i$ ($1 \leq i \leq n$). It lies in $GL(n, \mathbb{C})$ since $\det t = q^{n(r-1)/2} \pi(f_n) \neq 0$.

Let $c(t)$ be the complex-valued function on $\mathbb{Z}^n$ which attains the value zero at $m = (m_1, \ldots, m_n)$ unless $m_1 \geq \cdots \geq m_n$, where

$$c(t; m) = \det(t_i^{m_i + n - j}; 1 \leq i, j \leq n) / \det(t_i^{n-j}; 1 \leq i, j \leq n).$$

Here the numerator is called a Schur function (see [M, p. 24]), and it is divisible in $\mathbb{Z}[t_1, \ldots, t_n]$ by the denominator, which is the Vandermonde determinant $\prod(t_i - t_j)$ ($1 \leq i < j \leq n$). Note that for $m$ with $m_1 \geq \cdots \geq m_n$, $c(t; m)$ is the value at $t$ of the character of the irreducible representation of $GL(n, \mathbb{C})$ with highest weight $m$. We have $c(t; m(i)) = \text{Sym}_i((t_j))$, and this is equal to $q^{ir(i-1)/2} \pi(f_i)$ by the definition of $t$.

Choose a character $\psi$ of the additive group of $F$ which is trivial on $R$ but not on $u^{-1}R$. Denote by $\psi$ also the character of $N$ given by $\psi(x) = \prod_{i=1}^{r-1} \psi(x_{i,i+1})$, where $x_{i,i+1}$ is the $(i, i+1)$ entry of $x$. Given $\pi$ and $\psi$, the function $W$ on $\hat{G}$ is called an unramified Whittaker function associated with $\pi$ and $\psi$ if it satisfies

(1) \quad $W(\zeta x g k) = \zeta \psi(x) W(g)$ \quad ($x$ in $N$, $\zeta$ in $\mu_r$, $g$ in $\hat{G}$, $k$ in $K$),

and

(2) \quad $\pi(f) W(g) = \int_G W(gh) f(h) \, dh$ \quad ($f$ in $L_c(\hat{G}/K)$, $g$ in $\hat{G}$).

The integral is taken over $G \simeq \hat{G}/\mu_r$; the integrand is invariant under $\mu_r$. Let $D$ be the set of $m$ with $m_1 \geq \cdots \geq m_n \geq 0$ and $r + m_i < m_{i+1}$ ($1 \leq i < n$). For each $d$ in $D$ put $a(d; m) = W(s(u^d) u^{m})$. Recall that we identify $u^* \mathbb{Z}$ with its image in $\hat{G}$ via the section $s$. Since $G = \bigcup N u^m K$ (disjoint union over $m$ in $\mathbb{Z}^n$), to determine $W$ on $\hat{G}$ it suffices (by (1)) to evaluate $a(d; m)$ for all $d$ in $D$ and $m$ in $\mathbb{Z}^n$. Since the conductor of $\psi$ is $R$, if follows from (1) that $a(d; m)$ is zero unless $m_1 \geq m_2 \geq \cdots \geq m_n$.

**Theorem.** For each $d$ in $D$ we have $a(d; m) = a(d; 0) q^{\sum_{i=1}^{n} (i-n) m_i} c(t; m)$ for all $m$ in $\mathbb{Z}^n$.

Let $I(i)$ be the set of all $n$-tuples $e = (e_1, \ldots, e_n)$ with entries $e_i$ equal to zero or one such that $e_1 + \cdots + e_n = i$. Put $N_R = N \cap K$ and $N_R(e) = N_R \cap u^e K u^{-e}$. Note that the cardinality $|N_R/N_R(re)|$ of $N_R/N_R(re)$ is $q^{rw}$, where

$$w = \sum_{j > k} \max(e_j - e_k, 0) = in - i(i-1)/2 - \sum_{j=1}^{n} je_j.$$

Denote by $f[zu u^e K]$ the right $K$-invariant anti-genuine complex-valued function on $\hat{G}$ which is supported on $\mu_r zu u^e K$ ($e$ in $I(i)$, $x$ in $N_R/N_R(re)$), and attains the value one at $zu u^e$.

We first assume the validity of the following lemma.
LEMMA. For each $1 \leq i \leq n$ we have $f_i = \sum_{e \in I(i)} \sum_{x \in N_R/N_R(re) F[xu^{re}K]}$. The Lemma implies that for $m = (m_1, \ldots, m_n)$ in $\mathbb{Z}^n$ with $m_1 \geq \cdots \geq m_n$ we have

$$
\pi(f_i)a(d; m) = \int_G W(s(u^d)u^{rm}g)f_i(g) dg
= \sum_e [N_R/N_R(re)]a(d; m + e) \quad (e \text{ in } I(i))
= q^{irn-ir(i-1)/2} \sum_e q^{-r} \sum_{j=1}^{n-j} a(d; m + e).
$$

Put

$$
b(d; m) = q^r \sum_{j=1}^{n-j} a(d; m + e).
$$

If $m_1 \geq \cdots \geq m_n$, then we have

$$
q^{ir(i-1)/2} \pi(f_i)b(d; m) = \sum_e b(d; m + e) \quad (1 \leq i \leq n).
$$

Otherwise $b(d; m) = 0$. Namely $b(d; m)$ satisfies the equation

$$
c(t; m(i))b(d; m) = \sum_e b(d; m + e) \quad \text{if } m_1 \geq \cdots \geq m_n.
$$

On the other hand, the function $c(t; j)$ satisfies the equation

$$
c(t; m(i))c(t; m) = \sum_e c(t; m + e) \quad \text{if } m_1 \geq \cdots \geq m_n.
$$

Hence both $c(t; m)$ and $b(d; m)$ (for each $d$ in $D$) are functions of $m$ in $\mathbb{Z}^n$ which satisfy the same system of difference equations which has a unique solution up to a constant multiple. Since $c(t; 0) = 1$ the theorem follows.

It remains to prove the Lemma.

PROOF OF LEMMA. Let $W$ be the Weyl group of permutation matrices in $K$, realized as a group of matrices with entries zero and one only. Let $I$ be the Iwahori subgroup of $K$ which consists of all matrices in $K$ whose lower diagonal entries are all in $uR$. Put $\overline{N}$ for the group of lower triangular unipotent matrices, $N_I$ for $\overline{N} \cap I$, and $A_R$ for the diagonal subgroup of $K$. We have the decompositions $I = N_R A_R \overline{N}_I = \overline{N}_I A_R N_R$ and $K = \bigcup N_R \overline{N}_I wN_R A_R$ (disjoint union over $w$ in $W$). Put $c(i)$ for $m = (0, \ldots, 0, 1, \ldots, 1)$, where 1 appears in the last $i$ entries. Since $u^{-re(i)} N_R u^{re(i)}$ lies in $K$, we have

$$
K u^{rm(i)} K = K u^{re(i)} K \subset \bigcup_{w \in W} N_R A_R \overline{N}_I w u^{re(i)} w^{-1} K.
$$

Put $y$ for the element $wu^{re(i)}w^{-1}$ of $\tilde{G}$. For $1 \leq j < k \leq n$, and a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{jk}$ for the matrix $(x_{uv})$ in $GL(n)$ whose entries along the diagonal are one except that $x_{jj} = a$, $x_{kk} = d$, and its nondiagonal entries are zero except that $x_{jk} = b$ and $x_{kj} = c$.

Suppose that $\tilde{n} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, is a matrix in $\overline{N}_I$, and $y^{-1} \tilde{n} y$ does not lie in $K$.

Write $| \cdot |$ for the valuation on $F$ normalized by $|u| = q^{-1}$. Then $q^{-r} < |x| < 1$, the
\( j \)th diagonal entry of \( p(y) \) is one and the \( k \)th is \( u^r \), and

\[
y^{-1} \tilde{ny} = \begin{pmatrix} 1 & 0 \\ u^{-r} & 1 \end{pmatrix}_{jk}.
\]

We have

\[
\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}_{jk} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}_{jk} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{jk} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}_{jk}.
\]

Given the matrix \( s \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk} \) in \( \tilde{G} \), we have in \( \tilde{G} \) (see [FK, (2.1)]) the relation

\[
s \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}_{jk} s \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk} s \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk}^{-1} = s \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk} \left( \frac{b}{a, x} \right).
\]

Here \((\cdot, \cdot)\) denotes the nondegenerate bimultiplicative \( r \)th Hilbert symbol

\[
F^\times / F^\times \times F^\times / F^\times \to \mu_r.
\]

Taking \( b = 1 \) and a unit \( a \), namely \( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk} \) in \( K \), we conclude that the anti-genuine \( K \)-bi-invariant function \( f_t \) attains the value zero at \( \tilde{ny} \), since

\[
f_t(\tilde{ny}) = f_t \left( \begin{pmatrix} 1 & 0 \\ x^{-1}u^r & 1 \end{pmatrix}_{jk} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{jk} \right) \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk} \begin{pmatrix} 1 & x^{-1}u^r \\ 0 & 1 \end{pmatrix}_{jk})
\]

\[
= f_t \left( s \begin{pmatrix} x^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk} \right) y
\]

\[
= f_t \left( s \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk} \right) s \begin{pmatrix} x^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk} \right) y
\]

\[
= (a, x)f_t \left( s \begin{pmatrix} x^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk} \right) y s \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk}
\]

\[
= (a, x)f_t(\tilde{ny}).
\]

Since the Hilbert symbol \((\cdot, \cdot)\) is nondegenerate we can find a unit \( a \) with \((a, x) \neq 1\); indeed, \((a, b) = 1\) for any pair \( a, b \) of units, hence there exists some unit \( a \) with \((a, u) = \zeta\), where \( \zeta \) is a primitive \( r \)th root of unity. It follows that \( f_t \) is supported on the subset

\[
\mu_r \bigcup_{w \in W} N_{Rwu^{r\epsilon(i)}w^{-1}K} \text{ of } \tilde{G}.
\]

Since this set is contained in \( \mu_r Ku^{m(i)}K \), the Lemma follows.
REMARK. If $\pi$ has a Whittaker model, namely we have $\pi(g)W(h) = W(hg)$ for all $h, g$ in $\hat{G}$, then $a(d; m) = \pi(s(u^d)u^m)a(0; 0)$, and in particular $a(d; 0) = \pi(s(u^d))a(0; 0)$ for all $d$ in $D$. In this case there exists a unique (up to a scalar multiple) unramified Whittaker function associated with $\pi, \psi$.

REFERENCES


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