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THE ADJOINT REPRESENTATION L-FUNCTION FOR $\mathrm{GL}(n)$

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Ideas underlying the proof of the "simple" trace formula are used to show the following. Let F be a global field, and $\mathbb{A}$ its ring of adeles. Let π be a cuspidal representation of $\mathrm{GL}(n, \mathbb{A})$ which has a supercuspidal component, and ω a unitary character of $\mathbb{A}^\times/F^\times$. Let s_0 be a complex number such that for every separable extension E of F of degree n, the L-function $L(s, \omega \circ \mathrm{Norm}_{E/F})$ over E vanishes at $s = s_0$ to the order $m \geq 0$. Then the product L-function $L(s, \pi \otimes \omega \times \check{\pi})$ vanishes at $s = s_0$ to the order m. This result is a reflection of the fact that the tensor product of a finite dimensional representation with its contragredient contains a copy of the trivial representation.

Let F be a global field, $\mathbb{A}$ its ring of adeles and $\mathbb{A}^\times$ its group of ideles. Denote by $\underline{G}$ the group scheme $\mathrm{GL}(n)$ over F, and put $G = \underline{G}(F)$, $\mathbb{G} = \underline{G}(\mathbb{A})$, and $Z \simeq F^\times$, $\mathbb{Z} \simeq \mathbb{A}^\times$ for the corresponding centers. Fix a unitary character ε of $\mathbb{Z}/Z$, and signify by π a cuspidal representation of $\mathbb{G}$ whose central character is ε. For almost all F-places v the component π_v of π at v is unramified and is determined by a semi-simple conjugacy class $t(\pi_v)$ in $\widehat{G} = \underline{G}(\mathbb{C})$ with eigenvalues $(z_i(\pi_v);\ 1 \leq i \leq n)$. Given a finite dimensional representation r of $\widehat{G}$, and a finite set V of F-places containing the archimedean places and those where π_v is ramified, one has the L-function

$$L^V(s, \pi, r) = \prod_{v \notin V} \det(I - q_v^{-s} r(t(\pi_v)))^{-1}$$

which converges absolutely in some right half plane $\mathrm{Re}(s) >> 1$. Here q_v is the cardinality of the residue field of the ring R_v of integers in the completion F_v of F at v.

In this paper we consider the representation r of $\widehat{G}$ on the (n^2-1)-dimensional space M of $n \times n$ complex matrices with trace zero, by the adjoint action $r(g)m = \mathrm{Ad}(g)m = gmg^{-1}$ $(m \in M,\ g \in \widehat{G})$. More generally we can introduce the representation Adj of $G \times \mathbb{C}^\times$ by $\mathrm{Adj}((g, z)) = zr(g)$, and hence for any character ω of $\mathbb{Z}/Z$ the

L-function

$$L^V(s, \pi, \omega, \mathrm{Adj}) = \prod_{v \notin V} \det(I - q_v^{-s} t(\omega_v) r(t(\pi_v)))^{-1}.$$

Here V contains all places v where π_v or the component ω_v of ω is ramified, and $t(\omega_v) = \omega_v(\underline{\pi}_v)$; $\underline{\pi}_v$ is a generator of the maximal ideal in R_v.

In fact the full L-function is defined as a product over all v of local L-functions. These are introduced in the p-adic case as (a quotient of) the "greatest common denominator" of a family of integrals whose definition is recalled from **[JPS]** after Proposition 3 below. The local L-functions in the archimedean case are introduced below as a quotient of the L-factors studied in **[JS1]**. We denote by $L(s, \pi, \dots)$ the full L-function.

More precisely, we have

$$L^V(s, \pi, \omega, \mathrm{Adj}) = L^V(s, \pi \otimes \omega \times \check{\pi}) / L^V(s, \omega),$$

where $L^V(s, \pi_1 \times \pi_2)$ denotes the partial L-function attached to the cuspidal $\mathrm{GL}(n_i, \mathbb{A})$-modules π_i $(i = 1, 2)$ and the tensor product of the standard representation of $\widehat{G}_1 = \mathrm{GL}(n_1, \mathbb{C})$ and $\widehat{G}_2 = \mathrm{GL}(n_2, \mathbb{C})$. This provides a natural definition for the complete function $L(s, \pi, \omega, \mathrm{Adj})$ globally, and also locally. This definition permits using the results of **[JPS]** and **[JS1]** mentioned above. In particular, for any cuspidal $\mathbb{G}$-module π, the L-function $L(s, \pi, \omega, \mathrm{Adj})$ has analytic continuation to the entire complex s-plane.

To simplify the notations we shall assume, when $\omega \neq 1$, that ω does not factorize through $z \mapsto \nu(z) = |z|$; this last case can easily be reduced to the case of $\omega = 1$. Indeed, $L(s, \pi, \omega \otimes \nu^{s'}, \mathrm{Adj}) = L(s + s', \pi, \omega, \mathrm{Adj})$. Our main result is the following.

1. Theorem. *Suppose that the cuspidal $\mathbb{G}$-module π has a supercuspidal component, and ω is a character of $\mathbb{Z}/Z$ of finite order for which the assumption* (Ass; E, ω) *below is satisfied for all separable field extensions E of F of degree n. Then the L-function* $L(s, \pi, \omega, \mathrm{Adj})$ *is entire, unless $\omega \neq 1$ and $\pi \otimes \omega \simeq \pi$. In this last case the L-function is holomorphic outside $s = 0$ and $s = 1$. There it has simple poles.*

To state (Ass; E, ω) note that given any separable field extension E of degree n of F there is a finite galois extension K of F, containing E, such that ω corresponds by class field theory to a character, denoted again by ω, of the galois group $J = \mathrm{Gal}(K/F)$.

Denote by $H = \mathrm{Gal}(K/E)$ the subgroup of J corresponding to E, and by $\omega|E$ the restriction of ω to H. It corresponds to a character, denoted again by $\omega|E$, of the idele class group $\mathbb{A}_E^\times/E^\times$ of E. When E/F is galois, and $N_{E/F}$ is the norm map from E to F, then $\omega|E = \omega \circ N_{E/F}$. Our assumption is the following.

(Ass; E, ω) *The quotient* $L(s, \omega|E)/L(s, \omega)$ *of the Artin* (*or Hecke, by class field theory*) *L-functions attached to the characters* $\omega|E$ *of* $\mathrm{Gal}(K/E) = H$ *and* ω *of* $\mathrm{Gal}(K/F) = J$, *is entire, except at* $s = 0$ *and* $s = 1$ *when* $\omega \neq 1$ *and* $\omega|E = 1$.

If E/F is an abelian extension, (Ass; E, ω) follows by the product decomposition $L(s, \omega|E) = \prod_\zeta L(s, \omega\zeta)$, where ζ runs through the set of characters of $\mathrm{Gal}(E/F)$. More generally, (Ass; E, ω) is known when E/F is galois, and when the galois group of the galois closure of E over F is solvable, for $\omega = 1$ (see, e.g., **[CF]**, p. 225, and the survey article **[W]**). For a general E we have

$$L(s, \omega|E) = L(s, \mathrm{Ind}_H^J(\omega|E)) = L(s, \omega)L(s, \rho),$$

where the representation $\mathrm{Ind}_H^J(\omega|E)$ of $J = \mathrm{Gal}(K/F)$ induced from the character $\omega|E$ of H, contains the character ω with multiplicity one (by Frobenius reciprocity); ρ is the quotient by ω of $\mathrm{Ind}_H^J(\omega|E)$. Artin's conjecture for J now implies that $L(s, \rho)$ is entire, unless $\omega|E = 1$ and $\omega \neq 1$, in which case $L(s, \rho)$ is holomorphic except at $s = 0, 1$, where it has a simple pole. When $[E : F] = n$, $\omega = 1$ and K is a galois closure of E/F, then $J = \mathrm{Gal}(K/F)$ is a quotient of the symmetric group S_n. Artin's conjecture is known to hold for S_3 and S_4, hence (Ass; E, 1) holds for all E of degree 3 or 4 over F, and Theorem 1 holds unconditionally (when $\omega = 1$) for $\mathrm{GL}(3)$ and $\mathrm{GL}(4)$, as well as for $\mathrm{GL}(2)$.

The conclusion of Theorem 1 can be rephrased as asserting that $L(s, \omega)$ *divides* $L(s, \pi \otimes \omega \times \check{\pi})$ *when* $\pi \otimes \omega \not\simeq \pi$ *or* $\omega = 1$, *namely the quotient is entire, and that the quotient is holomorphic outside* $s = 0, 1$, *if* $\pi \otimes \omega \simeq \pi$ *and* $\omega \neq 1$; of course we assume (Ass; E, ω) for all separable extensions E of F of degree n. Note that the product L-function $L(s, \pi_1 \times \pi_2)$ has been shown in **[JS]**, **[JS1]**, **[JPS]** and (differently) in **[MW]** to be entire unless $\pi_2 \simeq \check{\pi}_1$. In this last case the L-function is holomorphic outside $s = 0, 1$, and has a simple pole at $s = 0$ and $s = 1$. This pole is matched by the simple pole of $L(s, \omega)$ when $\omega = 1$. Hence $L(s, \pi, 1, \mathrm{Adj})$ is also entire.

Another way to state the conclusion of Theorem 1 is that *if* $L(s, \omega)$ *vanishes at* $s = s_0$ *to the order* $m \geq 0$, *then so does* $L(s, \pi \otimes \omega \times \check{\pi})$, *provided that* (Ass; E, ω) *is satisfied for all separable extensions E of F of degree n.* Note that $L(s, \omega)$ does not vanish on $|\mathrm{Re}\, s - \frac{1}{2}| \geq \frac{1}{2}$.

Yet another restatement of the Theorem: *Let π be a cuspidal $\mathbb{G}$-module with a supercuspidal component, and ω a unitary character of $\mathbb{Z}/Z$. Let s_0 be a complex number such that for every separable extension E of F of degree n, the L-function $L(s, \omega|E)$ vanishes at $s = s_0$ to the order $m \geq 0$. Then $L(s, \pi \otimes \omega \times \check{\pi})$ vanishes at $s = s_0$ to the order m.* This is the statement which is proven below. Note that the assumption that ω is of finite order was put above only for convenience. Embedding $\mathbb{A}_E^\times$ as a torus in $\mathbb{G}$, the character $\omega|E$ can be defined also by $(\omega|E)(x) = \omega(\det x)$ on $x \in \mathbb{A}_E^\times \subset \mathbb{G}$. In general ω would be a character of a Weil group, and not a finite galois group.

When $n = 2$ the three dimensional representation Adj of $\mathrm{GL}(2, \mathbb{C})$ is the symmetric square Sym^2 representation, and the holomorphy of the L-function $L(s, \omega \otimes \mathrm{Sym}^2\pi)$ ($s \neq 0, 1$ if $\pi \otimes \omega \simeq \pi$, $\omega \neq 1$) is proven in **[GJ]** using the Rankin-Selberg technique of Shimura **[Sh]**, and in **[F1]** using a trace formula. Another proof was suggested by Zagier **[Z]** in the context of $\mathrm{SL}(2, \mathbb{R})$ and generalized by Jacquet-Zagier **[JZ]** to the context of π on $\mathrm{GL}(2, \mathbb{A})$. This last technique is the one extended to the context of cuspidal π with a supercuspidal component and arbitrary $n \geq 2$, in the present paper.

The path followed in **[Z]** and **[JZ]** is to compute the integral

$$\int K_\varphi(x, x) E(x, \Phi, \omega, s)\, dx$$

on x in $\mathbb{Z}G\backslash\mathbb{G}$, where $E(x, \Phi, \omega, s)$ is an Eisenstein series, and $K_\varphi(x, y)$ the kernel representing the cuspidal spectrum in the trace formula. The computation shows that the integral is a sum of multiples of $L(s, \omega|E)$ (with $[E : F] = 2$ in the case of **[Z]** and **[JZ]**), and on the other hand of (a sum of multiples of) $L(s, \pi \otimes \omega \times \check{\pi})$, from which the conclusion is readily deduced. However, **[Z]** and **[JZ]** computed all terms in the integral, and reported about the complexity of the formulae. To generalize their computations to $\mathrm{GL}(n)$, $n \geq 3$, considerable effort would be required.

To bypass these difficulties in this paper we use the ideas employed in **[FK]** and **[F2]** to establish various lifting theorems by means of a simple trace formula. In particular we use a special class of test functions φ, with one component supported on the elliptic regular set, and another component is chosen to be supercuspidal. The first choice reduces the conjugacy classes contributing to $K_\varphi(x, y)$ to elliptic ones only, while the second guarantees the vanishing of the non-cuspidal terms in the spectral kernel. The first choice does not restrict the applicability of our formulae. Thus our Theorem 1 is offered as another example of the power and usefulness of the ideas underlying the simple trace formula.

For a "twisted tensor" analogue of this paper see **[F4]**.

We shall work with the space $L(G)$ of smooth complex valued functions ϕ on $G\backslash\mathbb{G}$ which satisfy (1) $\phi(zg)=\varepsilon(z)\phi(g)$ $(z\in\mathbb{Z},\ g\in\mathbb{G})$, (2) ϕ is absolutely square integrable on $\mathbb{Z}G\backslash\mathbb{G}$. The group $\mathbb{G}$ acts on $L(G)$ by right translation: $(r(g)\phi)(h)=\phi(hg)$. The action is unitary since ε is. The function $\phi\in L(G)$ is called *cuspidal* if for each proper parabolic subgroup $\underline{P}$ of $\underline{G}$ over F with unipotent radical $\underline{N}$ we have $\int\phi(ng)dn=0$ $(n\in N\backslash\mathbb{N})$ for all $g\in\mathbb{G}$. Let r_0 be the restriction of r to the space $L_0(G)$ of cusp forms in $L(G)$. The space $L_0(G)$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $\mathbb{G}$-modules called *cuspidal* $\mathbb{G}$-modules.

Let φ be a complex valued function on $\mathbb{G}$ with $\varphi(g)=\varepsilon(z)\varphi(zg)$ $(z\in\mathbb{Z})$, compactly supported modulo $\mathbb{Z}$, smooth as a function on the archimedean part $G(F_\infty)$ of $\mathbb{G}$, and bi-invariant by an open compact subgroup of $G(\mathbb{A}_f)$; here $\mathbb{A}_f$ is the ring of adeles without archimedean components, and F_∞ is the product of F_v over the archimedean places. Fix Haar measures dg_v on G_v/Z_v $(G_v=\underline{G}(F_v),\ Z_v$ its center) for all v such that the product of the volumes $|K_v/Z_v\cap K_v|$ converges; K_v is a maximal compact subgroup of G_v, chosen to be $\mathrm{K}_v=\underline{G}(R_v)$ at the finite places. Then $dg=\bigotimes dg_v$ is a measure on $\mathbb{G}/\mathbb{Z}$. The convolution operator $r(\varphi)=\int_{\mathbb{G}/\mathbb{Z}}\varphi(g)r(g)dg$ is an integral operator on $L(G)$ with the kernel $K_\varphi(x,y)=\sum\varphi(x^{-1}\gamma y)$ $(\gamma\in G/Z)$. In this paper we work only with discrete functions φ.

DEFINITION. The function φ is called *discrete* if for every $x\in\mathbb{G}$ and $\gamma\in G$ we have $\varphi(x^{-1}\gamma x)=0$ unless γ is elliptic regular.

Recall that γ is called *regular* if its centralizer $Z_\gamma(\mathbb{G})$ is a torus, and *elliptic* if it is semi-simple and $Z_\gamma(\mathbb{G})/Z_\gamma(G)\mathbb{Z}$ has finite volume. The centralizer $Z_\gamma(G)$ of an elliptic regular $\gamma\in G$ is the multiplicative group of a field extension E of F of degree n. For a general elliptic γ, we have that $Z_\gamma(G)$ is $\mathrm{GL}(m,F')$ with $n=m[F':F]$.

The proof of Theorem 1 is based on integrating the kernel $K_\varphi(x,y)$ on $x=y$ against an Eisenstein series, as in **[Z]** and **[JZ]**.

Identify $\mathrm{GL}(n-1)$ with a subgroup of $\mathrm{GL}(n)$ via $g\mapsto\left(\begin{smallmatrix}g&0\\0&1\end{smallmatrix}\right)$. Let U be the unipotent radical of the upper triangular parabolic subgroup of type $(n-1,1)$. Put $Q=\mathrm{GL}(n-1)U$. Given a local field F, let $S(F^n)$ be the space of smooth and rapidly decreasing (if F is archimedean), or locally constant compactly supported (if F is non-archimedean) complex valued functions on F^n. Denote by Φ^0 the characteristic function of R^n in F^n if F is non-archimedean. For a global field F let $S(\mathbb{A}^n)$ be the linear span of the functions $\Phi = \bigotimes \Phi_v$, $\Phi_v \in S(F_v^n)$ for all v, Φ_v is Φ_v^0 for almost all v. Put $\underline{\varepsilon} = (0, \dots, 0, 1)$ $(\in \mathbb{A}^n)$. The integral of

$$f(g, s) = \omega(\det g)|\det g|^s \int_{\mathbb{A}^\times} \Phi(a\underline{\varepsilon}g)|a|^{ns}\omega^n(a)\, d^\times a \tag{1.1}$$

converges absolutely, uniformly in compact subsets of $\mathrm{Re}\, s \geq \frac{1}{n}$. The absolute value is normalized as usual, and ω is a character of $\mathbb{A}^\times/F^\times$.

It follows form Lemmas (11.5), (11.6) of **[GoJ]** that the Eisenstein series

$$E(g, \Phi, \omega, s) = \sum f(\gamma g, s) \qquad (\gamma \in ZQ\backslash G)$$

converges absolutely in $\mathrm{Re}\, s > 1$. In **[JS]**, (4.2), p. 545, and **[JS2]**, (3.5), p. 7, it is shown (with a slight modification caused by the presence of ω here) that $E(g, \Phi, \omega, s)$ extends to a meromorphic function on $\mathrm{Re}\, s > 0$, in fact to the entire complex s-plane with a functional equation $E(g, \Phi, \omega, s) = E({}^t g^{-1}, \widehat{\Phi}, \omega^{-1}, 1-s)$; here ${}^t g$ is the transpose of g and $\widehat{\Phi}$ is the Fourier transform of Φ. Moreover, $E(g, \Phi, \omega, s)$ is slowly increasing in $g \in G\backslash\mathbb{G}$, and it is holomorphic except for a possible simple pole at $s = 1$ and 0. Note that $f(g)$ and $E(g, s)$ are $\mathbb{Z}$-invariant.

2. Proposition. *For any character ω of $\mathbb{A}^\times/F^\times$, Schwartz function Φ in $S(\mathbb{A}^n)$, and discrete function φ on $\mathbb{G}$, for each extension E of degree n of F there is an entire holomorphic function $A(\Phi, \varphi, \omega, E, s)$ in s such that*

$$\int_{\mathbb{Z}G\backslash\mathbb{G}} K_\varphi(x, x) E(x, \Phi, \omega, s)\, dx = \sum_E A(\Phi, \varphi, \omega, E, s) L(s, \omega|E) \tag{2.1}$$

on $\mathrm{Re}\, s > 1$. The sum over E ranges over a finite set depending on (the support of) φ.

Proof. Since the function φ is discrete the sum in $K_\varphi(x, x) = \sum \varphi(x^{-1}\gamma x)$ ranges only over the elliptic regular elements γ in G/Z.

It can be expressed as

$$(2.2)\qquad K_\varphi(x, x) = \sum_T [W(T)]^{-1} \sum_{\gamma\in T/Z} \sum_{\delta\in G/T} \varphi(x^{-1}\delta^{-1}\gamma\delta x).$$

Here T ranges over a set of representatives for the conjugacy classes in G of elliptic tori (T is isomorphic over F to the multiplicative group of a field extension E of degree n of F; T is uniquely determined by such E, and each such E is so obtained). The cardinality of the Weyl group (normalizer/centralizer) $W(T)$ of T in G is denoted by $[W(T)]$. It is easy to check that for any elliptic T we have $G = TQ$, and $T\cap Q = \{1\}$. Hence the sum over δ can be taken to range over Q.

The left side of (2.1) is equal, in the domain of absolute convergence of the series which defines the Eisenstein series, to

$$\int_{\mathbb{Z}G\backslash\mathbb{G}} K_\varphi(x, x) \sum_{\gamma\in ZQ\backslash G} f(\gamma x, s)\, dx = \int_{\mathbb{Z}Q\backslash\mathbb{G}} K_\varphi(x, x) f(x, s)\, dx,$$

since $x \mapsto K_\varphi(x, x)$ is left G-invariant. Substituting (2.2) this is equal to

$$\int_{\mathbb{Z}Q\backslash\mathbb{G}} \sum_T [W(T)]^{-1} \sum_{\gamma\in T/Z} \sum_{\delta\in Q} \varphi(x^{-1}\delta^{-1}\gamma\delta x) f(x, s)\, dx$$
$$= \sum_T [W(T)]^{-1} \sum_{\gamma\in T/Z} \int_{\mathbb{Z}\backslash\mathbb{G}} \varphi(x^{-1}\gamma x) f(x, s)\, dx;$$

note that $x \mapsto f(x, s)$ is left Q-invariant.

To justify the change of summation and integration note that given φ, the sums over T and γ are finite. Indeed, the coefficients of the characteristic polynomial of γ are rational, and lie in a compact set depending on the support of φ (and a discrete subset of a compact is finite). This explains also the finiteness assertion at the end of the proposition.

Substituting now the expression (1.1) for $f(x, s)$ we obtain a sum over T and γ of

$$\int_{\mathbb{Z}\backslash\mathbb{G}} \varphi(x^{-1}\gamma x) f(x, s)\, dx = \int_{\mathbb{G}} \varphi(x^{-1}\gamma x)\omega(\det x)|\det x|^s \Phi(\underline{\varepsilon} x)\, dx$$
$$= \int_{\mathbb{T}\backslash\mathbb{G}} \varphi(x^{-1}\gamma x) \int_{\mathbb{T}} \Phi(\underline{\varepsilon} t x)\omega(\det tx)|\det tx|^s\, dt\, dx.$$

Here $\mathbb{T} = \underline{T}(\mathbb{A}) \simeq \mathbb{A}_E^\times$, where $\underline{T}$ is the centralizer of γ in $\underline{G}$, and $\underline{T}(F) = T$. The inner integral, over $\mathbb{T}$, is a “Tate integral” for

$L(s, \omega|E)$; it is a multiple of $L(s, \omega|E)$ by a function which is holomorphic in s in $\mathbb{C}$ and smooth in x, depending on Φ, ω and E. The integral over x ranges over a compact in $\mathbb{T}\backslash\mathbb{G}$, since φ is compactly supported modulo $\mathbb{Z}$. The proposition follows.

We now turn to the spectral expression for the kernel $K_\varphi(x, y)$.

Definition. The function φ on $\mathbb{G}$ is called *cuspidal* if for every x, y in $\mathbb{G}$ and every proper F-parabolic subgroup $\underline{P}$ of $\underline{G}$, we have $\int_{\mathbb{N}} \varphi(xny)dn = 0$, where $\mathbb{N} = \underline{N}(\mathbb{A})$ is the unipotent radical of $\mathbb{P} = \underline{P}(\mathbb{A})$.

When φ is cuspidal, the convolution operator $r(\varphi)$ factorizes through the projection on $L_0(G)$. Then $r(\varphi)$ is an integral operator whose kernel has the form

$$K_\varphi(x, y) = \sum_\pi K_\varphi^\pi(x, y), \quad \text{where } K_\varphi^\pi(x, y) = \sum_{\phi^\pi} (r(\varphi)\phi^\pi)(x)\overline{\phi}^\pi(y).$$

The sum over π ranges over all cuspidal $\mathbb{G}$-modules in $L_0(G)$. The ϕ^π range over an orthonormal basis consisting of $\mathbb{K} = \prod_v K_v$-finite vectors in π. The ϕ^π are rapidly decreasing functions and the sum over ϕ^π is finite for each φ (uniformly in x and y) since φ is $\mathbb{K}$-finite. The sum over π converges in L^2, and hence also in a space of rapidly decreasing functions. Hence $K_\varphi(x, y)$ is rapidly decreasing in x and y, and the product of $K_\varphi(x, x)$ with the slowly increasing functions $E(x, \Phi, \omega, s)$, is integrable over $\mathbb{Z}G\backslash\mathbb{G}$. The resulting integral, which is equal to (2.1), can also be expressed then in the form

$$\sum_\pi \sum_{\phi^\pi} \int_{\mathbb{Z}G\backslash\mathbb{G}} (r(\varphi)\phi^\pi)(x)\overline{\phi}^\pi(x)E(x, \Phi, \omega, s)\, dx.$$

To prove Theorem 1 *we now assume that* $L(s, \omega)$ is zero at $s = s_0$. It is well known then that $|\mathrm{Re}\, s_0 - \frac{1}{2}| < \frac{1}{2}$, hence $s_0 \neq 0, 1$. If s_0 is a zero of order m of $L(s, \omega)$, then by (Ass; E, ω) *the function* $L(s, \omega|E)$ *vanishes at* s_0 *to the order* m. Making this assumption for every separable field extension E of degree n of F we conclude that (2.1) vanishes at $s = s_0$ to the order m, and that for all j $(0 \leq j \leq m)$ we have

$$(2.3)_j \qquad \sum_\pi \sum_{\phi^\pi} \int_{\mathbb{Z}G\backslash\mathbb{G}} (\pi(\varphi)\phi^\pi)(x)\overline{\phi}^\pi(x)E^{(j)}(x, \Phi, \omega, s_0)\, dx = 0.$$

Here $E^{(j)}(*, s_0) = \frac{d^j}{ds^j}E(*, s)|_{s=s_0}$.

At our disposal we have all cuspidal discrete functions φ on $\mathbb{G}$, and our aim is to show the vanishing of some summands in the last double sum over π and ϕ^π. In fact, fix a π for which Theorem 1 will now be proven. Let V be a finite set of F-primes, containing the archimedean primes and those where π or ω ramify. Consider $\varphi = \bigotimes_v \varphi_v$ (product over all F-places v) where each φ_v is a smooth compactly supported modulo Z_v function on G_v which transforms under Z_v via ε_v^{-1}. For almost all v the function φ_v is the unit element φ_v^0 in the Hecke algebra $\mathbb{H}_v$ of K_v-biinvariant (compactly supported modulo Z_v transforming under Z_v via ε_v^{-1}) functions on G_v. For all $v \notin V$ the component φ_v is taken to be spherical, namely in $\mathbb{H}_v$.

Each of the operators $\pi_v(\varphi_v)$ for $v \notin V$ factorizes through the projection on the subspace $\pi_v^{K_v}$ of K_v-fixed vectors in π_v. This subspace is zero unless π_v is unramified, in which case $\pi_v^{K_v}$ is one-dimensional. On this K_v-fixed vector, the operator $\pi_v(\varphi_v)$ acts as the scalar $\varphi_v^\vee(t(\pi_v))$, where $\varphi_v^\vee$ denotes the Satake transform of φ_v. Put $\varphi^\vee(t(\pi^V))$ for the product over $v \notin V$ of $\varphi_v^\vee(t(\pi_v))$, and $\pi_V(\varphi_V) = \bigotimes_{v\in V} \pi_v(\varphi_v)$. Then $(2.3)_j$ takes the form

$$(2.4)_j \qquad \sum_{\{\pi\,;\,\pi^{\mathbb{K},V}\neq 0\}} \varphi^\vee(t(\pi^V))a(\pi, \varphi_V, j, \Phi, \omega, s_0) = 0,$$

where

$$(2.5)_j \quad a(\pi, \varphi_V, j, \Phi, \omega, s) = \sum_{\phi^\pi} \int_{\mathbb{Z}G\backslash\mathbb{G}} (\pi_V(\varphi_V)\phi^\pi)(x)\overline{\phi}^\pi(x)E^{(j)}(x, \Phi, \omega, s)\,dx.$$

The sum over π ranges over the cuspidal $\mathbb{G}$-modules $\pi = \bigotimes \pi_v$ with $\pi_v^{K_v} \neq \{0\}$ for all $v \notin V$; $\pi^{\mathbb{K},V}$ denotes the space of $\prod_{v\notin V} K_v$-fixed vectors in π. The sum over ϕ^π ranges over those elements in the orthonormal basis of π which appears in $(2.3)_j$, which, for any $v \notin V$, as functions in $x \in G_v$, are K_v-invariant and eigenfunctions of $\pi_v(\varphi_v)$, $\varphi_v \in \mathbb{H}_v$, with eigenvalues $t(\pi_v)$. In particular $\phi^\pi(x) = \phi_V^\pi(x_V)\prod_{v\notin V}\phi_v^\pi(x_v)$, for such ϕ_v^π $(v \notin V)$.

A standard argument (see, e.g., Theorem 2 in **[FK]** in a more elaborate situation), based on the absolute convergence of the sum over π in $(2.4)_j$, standard estimates on the Hecke parameter $t(\pi_v)$ of the unitary unramified π_v $(v \notin V)$, and the Stone-Weierstrass theorem, implies the following.

3. Proposition. *Let π be a cuspidal $\mathbb{G}$-module which has a supercuspidal component. Let ω be a character of $\mathbb{Z}/Z$. Suppose that*

$L(s, \omega|E)$ vanishes at $s = s_0$ to the order m for every separable extension E of F of degree n. Then for any Φ and a function φ_V such that φ is cuspidal and discrete with any choice of $\bigotimes \varphi_v$ $(v \notin V)$, we have that $a(\pi, \varphi_V, j, \Phi, \omega, s_0)$ is zero.

We shall now recall the relation between the summands in $(2.5)_j$ and the L-function $L(s, \pi\otimes\omega\times\check{\pi})$. Let ψ be an additive non-trivial character of $\mathbb{A}$ modulo F (into the unit circle in $\mathbb{C}$), and denote by ψ_v its component at v. An irreducible admissible G_v-module π_v is called *generic* if $\operatorname{Hom}_{N_v}(\pi_v, \psi_v) \neq \{0\}$. By **[GK]**, or Corollary 5.17 of **[BZ]**, such π_v embeds in the G_v-module $\operatorname{Ind}(\psi_v; G_v, N_v)$ induced from the character $n = (n_{ij}) \mapsto \psi(n) = \psi(\sum_{1\leq i<n} n_{i,i+1})$ of the unipotent upper triangular subgroup N_v of G_v. Moreover, this embedding is unique, equivalently the dimension of $\operatorname{Hom}_{N_v}(\pi_v, \psi_v)$ is at most one. The embedding is given by $\pi_v \ni \xi \mapsto W_\xi$, where $W_\xi(g) = \lambda(\pi(g)\xi)$ $(g \in G)$ and $\lambda \neq 0$ is a fixed element in $\operatorname{Hom}_{N_v}(\pi_v, \psi_v)$. Since π_v is admissible, each of the functions W_ξ is smooth (under right action by G_v). If π_v is generic, denote by $W(\pi_v)$ its realization in $\operatorname{Ind}(\psi_v)$; $W(\pi_v)$ is called the *Whittaker model* of π_v. It is well-known that any component of a cuspidal $\mathbb{G}$-module is generic.

Given π, consider $W_v' \neq 0$ in $W(\pi_v)$ for all v, such that W_v' is the normalized unramified vector W_v^0 (it is K_v-invariant and $W_v^0(1) = 1$) for all $v \notin V$. The function $\phi'(x) = \sum_{p\in N\backslash Q} W'(px)$, where $W'(x) = \prod_v W_v'(x_v)$, is a cuspidal function in the space of $\pi \subset L_0(G)$. Substituting the series definition of $E(x, \Phi, \omega, s) = \sum_{ZQ\backslash G} f(\gamma x, s)$ in

$$\int_{\mathbb{Z}G\backslash\mathbb{G}} \phi''(x)\overline{\phi}'(x)E(x, \Phi, \omega, s)\,dx \qquad (\phi'' \in \pi \subset L_0(G))$$

one obtains

$$\int_{\mathbb{Z}Q\backslash\mathbb{G}} \phi''(x)\overline{\phi}'(x)f(x, s)\,dx = \int_{\mathbb{Z}N\backslash\mathbb{G}} \phi''(x)\overline{W}'(x)f(x, s)\,dx.$$

Since $W'(nx) = \psi(n)W'(x)$, and $\int_{N\backslash\mathbb{N}} \phi''(nx)\overline{\psi}(n)\,dn = W_{\phi''}(x)$ is the Whittaker function associated to the cusp form ϕ'', the integral is equal to

$$\begin{aligned}&\int_{\mathbb{Z}\mathbb{N}\backslash\mathbb{G}} W_{\phi''}(x)\overline{W}'(x)f(x, s)\,dx \\ &\quad = \int_{\mathbb{N}\backslash\mathbb{G}} W_{\phi''}(x)\overline{W}'(x)\Phi(\underline{\varepsilon}x)\omega(\det x)|\det x|^s\,dx.\end{aligned}$$

If ϕ'' is also of the form $\phi''(x) = \sum_{p\in N\backslash Q} W''(px)$, where $W''(x) = \prod_v W_v''(x_v)$ is factorizable, then $W_{\phi''} = W''$ and the integral factorizes as a product over all v of the local integrals

$$(3.1)\qquad \int_{N_v\backslash G_v} W_v''(x)\overline{W}'_v(x)\Phi_v(\underline{\varepsilon}x)\omega_v(\det x)|\det x|_v^s\,dx,$$

provided that $\Phi(x) = \prod_v \Phi_v(x_v)$.

When $W_v' = W_v^0 = W_v''$, and Φ_v is the characteristic function Φ_v^0 of R_v^n (and $v \notin V$), the integral (3.1) is easily seen (on using Schur function computations; see **[F3]**, p. 305) to be equal to $L(s, \pi_v \otimes \omega_v \times \check{\pi}_v)$. For a non-archimedean $v \in V$ the L-factor is defined in **[JPS]**, Theorem 2.7, as a "g.c.d" of the integrals (3.1) for all $W_{1v}, W_{2v} \in W(\pi_v)$ and Φ_v. In the archimedean case the L-factor is defined in **[JS1]**, Theorem 5.1. It is shown in **[JPS]** and **[JS1]** that the L-factor lies in the span of the integrals (3.1). The product of the L-factors, as well as the various manipulations above, converges absolutely for s in some right half plane.

4. Lemma. *The functions $W_v' \in W(\pi_v)$ (and so $\phi' \in \pi$) can be chosen to have the property that ϕ' factorizes as $\bigotimes_v \phi_v'$.*

Proof. Since W_v' is K_v-invariant for $v \notin V$, so is ϕ', and we have

$$\phi'(x) = \phi_V'(x_V)\prod_{v\notin V}\phi_v^0(x_v),$$

where ϕ_v^0 is the K_v-invariant function on G_v which takes the value 1 at 1 and is the eigenfunction of the operators $\pi_v(\varphi_v)$, $\varphi_v \in \mathbb{H}_v$, with the eigenvalue $t(\pi_v)$.

The space $\pi \subset L_0(G)$ is spanned by factorizable functions, namely ϕ' is a finite sum over j $(1 \le j \le J)$ of products $\bigotimes_v \phi_{jv}'$ of functions ϕ_{jv}' on G_v (which are smooth, compactly supported modulo Z_v, transform under Z_v via ε_v), with $\phi_{jv}' = \phi_v^0$ for all $v \notin V$. Each of the functions ϕ_{1v}' $(v \in V)$ is (right) invariant under a congruence subgroup K_v' of the standard compact subgroup K_v of G_v. Namely ϕ_{1v}' is a non-zero vector in the finite dimensional space ${\pi_v}^{K_v'}$ of K_v'-fixed vectors in π_v. The Hecke algebra $\mathbb{H}(K_v')$ of K_v'-biinvariant compactly supported modulo Z_v functions on G_v which transform under Z_v via ε_v^{-1} generate the algebra of endomorphisms of the finite dimensional space ${\pi_v}^{K_v'}$. Consider $\varphi_v \in \mathbb{H}(K_v')$ such that $\pi_v(\varphi_v)$ acts as an orthogonal projection on ϕ'_{1v}. Then $(\bigotimes_{v\in V}\pi_v(\varphi_v))\phi'$ lies in π, is of the form $\bigotimes_v \phi'_{1v}$, and is defined by the Whittaker functions $\pi_v(\varphi_v)W'_v$, as required.

Proof of Theorem 1. For π as in the theorem, and s_0 as in $(2.3)_j$, we shall choose $W'_v \in W(\pi_v)$ with factorizable $\phi'(x) = \bigotimes_v \phi'_v(x_v) = \sum_{p\in N\backslash Q} W'(px)$ and proceed to show the vanishing of the corresponding summand in $(2.5)_j$. Recall that by the assumption of Theorem 1 there is an F-place v_2 such that π_{v_2} is supercuspidal. Let v_1 be another F-place in V, say where π and ω are unramified. Put $V'' = V - \{v_2\}$ and V' for $V'' - \{v_1\}$.

Consider the matrix coefficient $\varphi'_{v_2}(x) = \langle \pi_{v_2}(x^{-1})\phi'_{v_2}, \phi'_{v_2}\rangle$ of the supercuspidal G_{v_2}-module π_{v_2}. Note that ϕ'_{v_2} is a C_c^∞-function on G_{v_2} modulo Z_{v_2}, and $\langle\cdot,\cdot\rangle$ denotes the natural inner product. The function φ'_{v_2} is smooth and compactly supported on G_{v_2} modulo Z_{v_2}, and it is a supercusp form ($\int \varphi'_{v_2}(xny)\,dn = 0$, $n \in N_{v_2}$ = unipotent radical of any parabolic subgroup of G_{v_2}). It is well-known that a function $\varphi = \bigotimes \varphi_v$ whose component at v_2 is a supercusp form is cuspidal. By the Schur orthogonality relations, the convolution operator $\pi_{v_2}(\varphi'_{v_2})$ acts as an orthogonal projection on the subspace generated by ϕ'_{v_2}. Working with $\varphi = \bigotimes \varphi_v$ whose component at v_2 is φ'_{v_2} we then have that φ is cuspidal and that the sum in $(2.5)_j$ ranges only over the ϕ $(= \phi^\pi)$ whose component at v_2 is ϕ'_{v_2} (up to a scalar multiple).

As in the proof of Lemma 4, for each $v \in V'$ we may choose φ'_v in $\mathbb{H}(K'_v)$ such that $\pi_v(\varphi'_v)$ acts as an orthogonal projection to the subspace of π'_v spanned by ϕ'_v. Choosing the components φ_v of φ at $v \in V'$ to be of the form $\varphi''_v * \varphi'_v$, with any φ''_v, the sum in $(2.5)_j$ for our π extends only over those ϕ in the orthonormal basis of the chosen $\pi \subset L_0(G)$ whose component at $v \neq v_1$ is ϕ'_v. But ϕ is left G-invariant, being a cusp form, and $\mathbb{G} = G\prod_{v\neq v_1} G_v$. Hence the only ϕ which contributes to the sum in $(2.5)_j$ is ϕ', whatever φ_{v_1} is.

We still need to choose φ_{v_1} such that $\varphi = \bigotimes \varphi_v$ be discrete. It suffices to choose φ_{v_1} to be supported on the regular elliptic set in G_{v_1}. Moreover, since ϕ'_{v_1} is right invariant under a compact open subgroup K'_{v_1} of $K_{v_1} \subset G_{v_1}$, we can choose the support of φ_{v_1} to be contained in $Z_{v_1}K'_{v_1}$. Then $\pi_{v_1}(\varphi_{v_1})$ acts as a scalar on ϕ'_1, and we normalize φ_{v_1} so that this scalar be one.

In conclusion, for any choice of $W'_v \in W(\pi_v)$ for all v, with $W'_v =$

W_v^0 for $v \notin V$, and any choice of φ_v $(v \in V')$, we have that

$$\begin{aligned}\int_{\mathbb{Z}G\backslash\mathbb{G}} &(\pi_{V'}(\varphi_{V'})\phi')(x)\overline{\phi}'(x)E(x, \Phi, \omega, s)\, dx \\ &= \prod_{v\in V}\int_{N_v\backslash G_v}(\pi_v(\varphi_v)W_v')(x)\overline{W}'_v(x)\Phi_v(\underline{\varepsilon}x)\omega_v(\det x)|\det x|_v^s\, dx \\ &\quad\cdot\prod_{v\notin V} L(s, \pi_v\otimes\omega_v\times\check{\pi}_v)\end{aligned}$$

vanishes at s_0 to the order m. Here $\pi_{v_1}(\varphi_{v_1})W'_{v_1} = W'_{v_1}$. In fact we may choose W'_{v_1} to be $W^0_{v_1} \in W(\pi_{v_1})$, and Φ_{v_1} to be $\Phi^0_{v_1}$. Since π_{v_1} and ω_{v_1} are unramified, the corresponding integral is then equal to the L-factor, so v_1 can be deleted from the set V.

To complete the proof of Theorem 1, note that the L-function $L(s, \pi_v\otimes\omega_v\times\check{\pi}_v)$ lies in the span of the integrals (3.1). Hence the assumption for every separable extension E of F of degree n that $L(s, \omega|E)$ vanishes at $s = s_0$ to the order m, implies the vanishing of $\prod L(s, \pi_v\otimes\omega_v\times\check{\pi}_v)$ to the order m. This completes the proof of Theorem 1.

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