Canad. J. Math. Vol. **51** (4), 1999 pp. 771–791

Stable Bi-Period Summation Formula and Transfer Factors

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Abstract. This paper starts by introducing a bi-periodic summation formula for automorphic forms on a group $G(E)$, with periods by a subgroup $G(F)$, where E/F is a quadratic extension of number fields. The split case, where $E = F \oplus F$, is that of the standard trace formula. Then it introduces a notion of stable bi-conjugacy, and stabilizes the geometric side of the bi-period summation formula. Thus weighted sums in the stable bi-conjugacy class are expressed in terms of stable bi-orbital integrals. These stable integrals are on the same endoscopic groups H which occur in the case of standard conjugacy.

The spectral side of the bi-period summation formula involves periods, namely integrals over the group of F-adele points of G, of cusp forms on the group of E-adele points on the group G. Our stabilization suggests that such cusp forms—with non vanishing periods—and the resulting bi-period distributions associated to "periodic" automorphic forms, are related to analogous bi-period distributions associated to "periodic" automorphic forms on the endoscopic symmetric spaces $H(E)/H(F)$. This offers a sharpening of the theory of liftings, where periods play a key role.

The stabilization depends on the "fundamental lemma", which conjectures that the unit elements of the Hecke algebras on G and H have matching orbital integrals. Even in stating this conjecture, one needs to introduce a "transfer factor". A generalization of the standard transfer factor to the bi-periodic case is introduced. The generalization depends on a new definition of the factors even in the standard case.

Finally, the fundamental lemma is verified for $\mathrm{SL}(2)$.

The geometric side of the trace formula for a test function f' on the group of adele points of a reductive group G over a number field F, is a sum of orbital integrals of f' parametrized by rational conjugacy classes, in $G(F)$. It is obtained on integrating over the diagonal $x = y$ the kernel $K_{f'}(x, y)$ of a convolution operator $r(f')$. Each such orbital integral can be expressed as an average of weighted sums of such orbital integrals over the stable conjugacy class, which is the set of rational points in the conjugacy class under the points of the group over the algebraic closure. Each such weighted sum is conjecturally related to a stable (a sum where all coefficients are equal to 1) such sum on an endoscopic group H of the group G. This process of stabilization has been introduced by Langlands to establish lifting of automorphic and admissible representations from the endoscopic groups H to the original group G.

The purpose of this paper is to develop an analogue in the context of the symmetric space $G(E)/G(F)$, where E/F is a quadratic number field extension. Integrating the kernel $K_{f'}(x, y)$ of the convolution operator $r(f')$ for the test function f' on the group of E-adele points of the group G over two independent variables x and y in the subgroup of F-adele points of G, we obtain a sum of bi-orbital integrals of f' over rational bi-conjugacy classes. We introduce a notion of stable bi-conjugacy, and stabilize the geometric side of the bi-period summation formula. Thus we express the weighted sums in the stable bi-

Received by the editors September 18, 1998.
AMS subject classification: 11F72, 11F70, 14G27, 14L35.

conjugacy class in terms of stable bi-orbital integrals. These stable integrals are on the same endoscopic groups H which occur in the case of standard conjugacy.

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Our definitions and analysis closely follow those of Kottwitz [K2] (and [K1]), who dealt with the case of standard conjugacy, which can be viewed as the split case ($E = F \oplus F$, $G(F) \setminus G(E)/G(F) = \mathrm{Int}\big(G(F)\big) \setminus G(F)$) of our theory.

Our stabilization depends on the "fundamental lemma", which conjectures that the unit elements of the Hecke algebras on G and H have matching orbital integrals. Even in stating this conjecture, one needs to introduce a "transfer factor". For standard conjugacy, this factor is introduced in Langlands-Shelstad [LS], as a product of the cohomological factor Δ_1, corrected by Δ_I; a factor which depends only on stable conjugacy, not on conjugacy: Δ_{II}, corrected by Δ_2; and a Jacobian Δ_{IV}.

In Section 10 we introduce a generalization of the transfer factor of [LS] to our bi-periodic case. Our generalization depends on a new definition of the factors of [LS]. Thus we unite Δ_{II} and Δ_2 into a single factor $\chi_{G/H}$ which is obviously independent of the choices [LS] use in their definition. Further we note that Δ_1 and Δ_I depend only on the centralizers—not on the elements of the groups in question. This permits us to use the united ("cohomological") factor which we denote by Δ^{coh} in our generalization to the bi-periodic case.

In Section 11 the fundamental lemma for $\mathrm{SL}(2)$ is verified.

Much of this work was carried out at Mannheim, Germany, where I was supported by the Humboldt Stiftung and the hospitality of Rainer Weissauer.

1 Basic Sum

Let E/F be a quadratic extension of number fields. Denote by $\mathbb{A}_E$ (resp. $\mathbb{A}$) the ring of adeles of E (resp. F). Let G be a connected reductive linear algebraic group over F. We denote by $G(D)$ the group of D-valued points of G, for any commutative F-algebra D. Let Z be the center of G. We work with a compactly supported (subscript "c") smooth (superscript "∞") complex valued function $f' \in C_c^\infty\big(Z(\mathbb{A}_E) \setminus G(\mathbb{A}_E)\big)$. Fix a Haar measure on $Z(\mathbb{A}_E) \setminus G(\mathbb{A}_E)$, and consider the convolution algebra $L = L^2\big(G(E)Z(\mathbb{A}_E) \setminus G(\mathbb{A}_E)\big)$. Then L is a $G(\mathbb{A}_E)$-module by $\big(r(g)\phi\big)(h) = \phi(hg)$. The convolution operator $r(f') = \int_{Z(\mathbb{A}_E)\setminus G(\mathbb{A}_E)} f'(g)r(g)\,dg$ is an integral operator with kernel $K_{f'}(h,g) = \sum_\gamma f'(h^{-1}\gamma g)$, $\gamma \in Z(E) \setminus G(E)$. Indeed

$$\big(r(f')\phi\big)(h) = \int_{Z(\mathbb{A}_E)\setminus G(\mathbb{A}_E)} f'(g)\phi(hg)\,dg = \int_{G(E)Z(\mathbb{A}_E)\setminus G(\mathbb{A}_E)} \phi(g)K_{f'}(h,g)\,dg.$$

The bi-period summation formula for (f', G, E, F) is obtained on integrating $K_{f'}(h,g)$ over h, g in $G(F)Z(\mathbb{A}) \setminus G(\mathbb{A})$. We begin with

$$\int\int_{(G(F)Z(\mathbb{A})\setminus G(\mathbb{A}))^2} \sum_{\gamma \in Z(E)\setminus G(E)} f'(g^{-1}\gamma h)\, dg\, dh$$
$$= \int_{G(F)Z(\mathbb{A})\setminus G(\mathbb{A})} dg \sum_{\gamma \in Z(E)\setminus G(E)/G(F)} \int_{Z(\mathbb{A})\setminus G(\mathbb{A})} f'(g^{-1}\gamma h)\, dh$$
$$= \int_{G(F)Z(\mathbb{A})\setminus G(\mathbb{A})} dg \sum_{\gamma \in G(E)/G(F)Z(E)} f(g^{-1}xg)\, dg,$$

where $f(x) = \int_{Z(\mathbb{A})\setminus G(\mathbb{A})} f'(\gamma h)\, dh, x = \gamma\theta(\gamma)^{-1}$. Here θ denotes the action of $\mathrm{Gal}(E/F)$ on E (and $\mathbb{A}_E$, and $G(E)$ and $G(\mathbb{A}_E)$). We also write $X(F) = \{g\theta(g)^{-1}; g \in G(E)\} \subset G(E)$ and $X(\mathbb{A}) = \{g\theta(g)^{-1}; g \in G(\mathbb{A}_E)\}$. Note that $x\theta(x) = 1$ for any element x in $X(F)$ and $X(\mathbb{A})$. The function f lies in $C_c^\infty\big(X(\mathbb{A})/Z_X(\mathbb{A})\big)$, where $Z_X(F) = \{z\theta(z)^{-1}; z \in Z(E)\} \subset Z(E)$ and $Z_X(\mathbb{A}) = \{z\theta(z)^{-1}; z \in Z(\mathbb{A}_E)\}$.

An element $x \in X(F)$ is called *elliptic* if it is elliptic semi-simple in $G(E)$. Denote by X_{ell} the set of elliptic elements in $X(F)$. Since $x \in X(F)$ is an E-point of G, its centralizer $I = I_x = Z_G(x)$ is defined over E. But $I = Z_G(x) = Z_G(x^{-1}) = Z_G(\theta x) = \theta(I)$ implies that I is defined over F. Thus $I = I_x$ is a reductive F-subgroup of G, and x is an E-point (in $Z_I(E)$) of its center $Z_I = Z(I)$.

We shall consider only the subsum over x in $X_{ell}/Z_X(F)$, in our sum. It is denoted by $T_e(f)$. With $\Phi_f(x) = \int_{G(\mathbb{A})/I_X^0(\mathbb{A})} f(gxg^{-1})\, dg$, the integral $T_e(f)$ is equal to

$$\int_{G(\mathbb{A})/Z(\mathbb{A})G(F)} \sum_{x\in X_{ell}/Z_X(F)} \sum_{\delta \in G(F)/Z_G(x)(F)} f(g\delta x\delta^{-1}g^{-1})\, dg$$
$$= \sum_x \int_{G(\mathbb{A})/Z(\mathbb{A})Z_G(x)(F)} f(gxg^{-1})\, dg$$
$$= \sum_{x \in X_{ell}/Z_X(F)} [I_x(F) : I_x^0(F)]^{-1} \tau(I_x^0) \Phi_f(x).$$

The superscript 0 indicates the connected component of the identity, and we write $\tau(I_x^0)$ for the Tamagawa volume $|I_x^0(\mathbb{A})/Z(\mathbb{A})I_x^0(F)|$ of I_x^0 over F. Our aim is to stabilize this sum.

2 Stable Conjugacy

Let F be any field of characteristic 0, E a quadratic field extension of F, θ the non-trivial automorphism of E over F, $X(F) = \{x = g\theta(g)^{-1}; g \in G(E)\}$, and $I = I_x = Z_G(x)$. Since $\theta(x) = x^{-1}$, I is an F-group, reductive when x is semi-simple, and connected when the derived group G^{ss} of G is simply connected. Note that G^{ss} is semi-simple as G is connected and reductive. The elements x, x' of $X(F)$ are said to be *stably conjugate* if there is a $g \in G(\bar{F})$, where $\bar{F}$ is an algebraic closure of F, such that $x' = \mathrm{Int}(g)x (= gxg^{-1})$.

We write $\mathrm{inv}(x, x')$ for the principal I-homogeneous space over F defined by $Y = \{g \in G(\bar{F}); gxg^{-1} = x'\}$. It lies in $D(I_x/F) = \ker[H^1(F, I_x) \to H^1(F, G)]$. When I_x is connected, $D(I_x/F)$ is in bijection with the conjugacy classes (under $G(F)$) within the stable conjugacy class of x.

As in [K2], we write $A(G)$ for $\pi_0\big(Z(\hat{G})^\Gamma\big)^D$, where $\pi_0(\ )$ denotes the group of connected components, $(*)^D$ means the dual $\mathrm{Hom}(*, \mathbb{Q}/\mathbb{Z})$, $(\ )^\Gamma$ means invariants under the action of the absolute Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$, and $\hat{G}$ is a (unique up to a noncanonical isomorphism) connected Langlands dual group for G. When F is local, define $E(I/F)$ to be the finite abelian group $\ker[A(I) \to A(G)]$. By [K2, 4.3], there is a canonical map $D(I/F) \to E(I/F)$, which is bijective in the p-adic case.

When F is global, put $E(I/\mathbb{A}) = \oplus_v E(I/F_v)$, where v ranges over the places of F. By [K1, 2.3], the exact sequence

$$1 \to Z(\hat{G}) \to Z(\hat{I}) \to Z(\hat{I})/Z(\hat{G}) \to 1$$

yields a homomorphism $\pi_0\big([Z(\hat{I})/Z(\hat{G})]^\Gamma\big) \to H^1\big(F, Z(\hat{G})\big)$. Define $\mathfrak{K}(I/F)$ to be the subgroup of $\pi_0\big([Z(\hat{I})/Z(\hat{G})]^\Gamma\big)$ consisting of all elements whose image in $H^1\big(F, Z(\hat{G})\big)$ is trivial if F is local, locally trivial if F is global. If F is local, then $\mathfrak{K}(I/F) = \mathrm{coker}[A(G)^D \to A(I)^D] = E(I/F)^D$.

3 (G, H)-Regular Elements

As in [K2], we need the notion of (G, H)-regular elements. Let F be a local or global field of characteristic zero, E a quadratic field extension, G a connected reductive group over F, (H, s, η) an endoscopic triple for G [K1, Section 7], and $X_H(F) = \{x_H = \gamma_H\theta(\gamma_H)^{-1}; \gamma_H \in H(E)\}$. Here θ denotes again the non-trivial automorphism of E which fixes F.

Let $x_H \in X_H(F) \subset H(E)$ be a semi-simple element of $X_H(F)$, by which we mean a semi-simple element of $H(E)$. Then $I_H = I_{H,x_H} = Z_H(x_H)$ is a reductive E-group, which is defined over F since $I_H = Z_H(x_H) = Z_H(x_H^{-1}) = Z_H(\theta x_H) = \theta(I_H)$. Moreover, $x_H \in Z(I_H)(E)$. Let T_H be a maximal F-torus in the group I_H. Then it is a maximal F-torus in H containing x_H. There is a canonical G-conjugacy class of embeddings $j\colon T_H \hookrightarrow G$. Choose one and let $x = j(x_H)$. The conjugacy class of x is independent of the choice of T_H and j. Thus $x_H \mapsto x$ induces a Γ-equivariant map from the set of semi-simple conjugacy classes in H to the set of conjugacy classes in G. In particular, $x = j(x_H) = j\big(\theta(x_H)^{-1}\big) = \theta\big(j(x_H)\big)^{-1} = \theta(x)^{-1}$, for $x_H = \gamma_H\theta(\gamma_H^{-1}) \in X_H(F)$. Then $x \in H^1\big(E/F, G(E)\big)$. This element represents the trivial class precisely when $x = \gamma\theta(\gamma^{-1}) \in X(F)$ for some $\gamma \in G(E)$.

Identify T_H and $T = j(T_H)$ via j. Let R be the set of roots of T in G, and R_H the set of roots of T in H. Then $R_H \subset R \subset X^*(T)$. We say that $x_H \in X_H(F)$ is (G, H)-regular if $\alpha(x_H) \neq 1$ for all $\alpha \in R - R_H$. The (G, H)- regularity of x_H depends only on x_H, not on the choice of T_H and j. Put $I = Z_G(x)$, $I_H = Z_H(x_H)$. The set $R(x)$ of roots of T in I is equal to $\{\alpha \in R; \alpha(x) = 1\}$. The set $R_H(x)$ of roots of T in I_H is equal to $\{\alpha \in R_H; \alpha(x) = 1\}$. Hence $R_H(x) \subset R(x)$ and the two sets are equal iff x_H is (G, H)-regular.

Assume that x_H is (G, H)-regular. Then $R_H(x) = R(x)$, and $j\colon T_H \tilde{\to} T$ extends to an isomorphism $j_1\colon I_H \tilde{\to} I$, unique up to an inner automorphism coming from T. If $x_H \in X_H(F)$ and $x \in X(F)$, then I_H, I are defined over F and j_1 is an inner twisting. Then $Z(\hat{I}_H) = Z(\hat{I})$. By [K2, 3.2], if I is connected, so is I_H.

4 Local Conjecture

Let E/F be a quadratic extension of local fields of characteristic 0, and G a connected reductive F-group. Put $X(F) = \{g\theta(g)^{-1}; g \in G(E)\}$. Assume that the derived group G^{ss} is simply connected. Then $I_x = Z_G(x)$ is connected for all semi-simple x in G. For any connected reductive F-group I, there is a sign $e(I) = \pm 1$ [K3]. Let x be a semi-simple element of $X(F)$, choose Haar measures dg, di on $G(F)$, $I_x(F)$, and consider the linear form $\Phi_f(x) = \int_{G(F)/I(F)} f\big(\mathrm{Int}(g)x\big)\, dg/di$ on $f \in C_c^\infty\big(X(F)\big)$. If $x' \in X(F)$ is stably conjugate to $x \in X(F)$, then $I' = Z_G(x')$ is an inner form of I, and the $\bar{F}$-isomorphism $I \simeq I'$ transfers the Haar measure di to di' on $I'(F)$. We use dg, di' to define $\Phi_f(x')$, and the linear form $\Phi_f^{\mathrm{st}}(x) = \sum_{x'} e(I')\Phi_f(x')$ on $C_c^\infty\big(X(F)\big)$, where x' ranges over a set of representatives for the conjugacy classes within the stable conjugacy class of x in $X(F)$. (For G such that G^{ss} is not simply connected, the factor $|\ker[H^1(F, I_{x'}^0) \to H^1(F, I_{x'})]|$ should multiply the summand indexed by x' in the definition of $\Phi_f^{\mathrm{st}}(x)$).

Let (H, s, η) be an endoscopic triple for G. Choose an extension of $\eta\colon \hat{H} \to \hat{G}$ to an L-homomorphism $\eta'\colon {}^L H \to {}^L G$.

Local Conjecture There are complex numbers $\Delta(x_H, x)$ such that there is a correspondence (f, f_H) between functions $f \in C_c^\infty\big(X(F)\big)$ and $f_H \in C_c^\infty\big(X_H(F)\big)$, defined by $\Phi_{f_H}^{\mathrm{st}}(x_H) = \sum_x \Delta(x_H, x)\Phi_f(x)$ for every G-regular semi-simple element x_H in $X_H(F)$. The sum ranges over a set of representatives for the $G(F)$-conjugacy classes within the stable conjugacy class of x in $X(F)$. The sum is empty and $\Phi_{f_H}^{\mathrm{st}}(x_H)$ is zero if the $G(\bar{F})$-conjugacy class of x contains no elements of $X(F)$. Further, we conjecture that the function $\Delta(x_H, x)$ can be extended (continuously?) to all pairs (x_H, x) consisting of a (G, H)-regular semi-simple element x_H of $X_H(F)$ and a corresponding element $x \in X(F)$, in such a way that $\Phi_{f_H}^{\mathrm{st}}(x_H) = \sum_x \Delta(x_H, x)e(I_x)\Phi_f(x)$. Here compatible measures on $I_x = Z_G(x)$ and $I_{x_H} = Z_H(x_H)$ are used, as the two groups are inner forms of each other, since x_H is (G, H)-regular.

Note that $e(T) = 1$ for any torus T, so that the (G, H)-regular case indeed extends the G-regular case.

If x' is stably conjugate to x (x', x in $X(F)$), we denoted by $\mathrm{inv}(x, x')$ the image under $D(I/F) \to E(I/F)$ of the element of $D(I/F)$ which measures the difference between x and x'. Via $Z(\hat{H}) \hookrightarrow Z(\hat{I}_H) \simeq Z(\hat{I})$, the element $s \in Z(\hat{H})$ defines $\kappa \in \mathfrak{K}(I/F) = E(I/F)^D$. The relation between $\Delta(x_H, x)$ and $\Delta(x_H, x')$ should be $\Delta(x_H, x') = \Delta(x_H, x)\kappa\big(\mathrm{inv}(x, x')\big)$. Then putting $\Phi_f^{\kappa}(x) = \sum_{x'} \kappa\big(\mathrm{inv}(x, x')\big)e(I_{x'})\Phi_f(x')$, our conjecture states that $\Phi_{f_H}^{\mathrm{st}}(x_H) = \Delta(x_H, x)\Phi_f^{\kappa}(x)$.

5 Global Obstruction

Let E/F be a quadratic extension of number fields and G a connected reductive group over F. Assume that the derived group G^{ss} is simply connected. Fix an inner twisting $\psi\colon G_{\mathrm{qs}} \to G$ with G_{qs} quasi-split. Let x_{qs} be a semi-simple element of $X_{\mathrm{qs}}(F) = \{\gamma_{\mathrm{qs}}\theta(\gamma_{\mathrm{qs}})^{-1}; \gamma_{\mathrm{qs}} \in G_{\mathrm{qs}}(E)\}$, where θ is the non-trivial automorphism of E over F. The centralizer $I_{\mathrm{qs}} = Z_{G_{\mathrm{qs}}}(x_{\mathrm{qs}})$ of x_{qs} in G_{qs} is connected. Let x be an element of $X(\mathbb{A}) = \{g\theta(g)^{-1}; g \in G(\mathbb{A}_E)\}$ which is conjugate to $\psi(x_{\mathrm{qs}})$ under $G(\bar{\mathbb{A}})$. We proceed to construct an element $\mathrm{obs}(x) \in$

$\mathfrak{K}(I_{qs}/F)^D$ which is trivial precisely when the $G(\mathbb{A})$-conjugacy class of x contains an element of $X(F)$. First, we construct an element $\mathrm{obs}_1(x)$ in $A(I_1)$ where $I_1 = Z_{G^{sc}_{qs}}(x_{qs})$ is the centralizer of x_{qs} in G^{sc}_{qs}, and $A(I_1) = \pi_0\big(Z(\hat{I}_1)^\Gamma\big)^D$, which is trivial precisely when the $G^{sc}(\mathbb{A})$-conjugacy class of x contains an element of $X(F)$.

Let Y_{qs} be the set of pairs (i, g), such that $i\colon I_{qs} \to G$ is conjugate to $\psi_{|_{I_{qs}}}$ under $G(\bar{F})$, $g \in G^{sc}(\bar{\mathbb{A}})$, and $i(x_{qs}) = gxg^{-1}$. Since there is $g \in G(\bar{\mathbb{A}})$ with $\psi(x_{qs}) = gxg^{-1}$ by assumption, $G(\bar{F}) = Z(\bar{F})G^{sc}(\bar{F})$, and $G_{qs}(\bar{\mathbb{A}}) = G^{sc}_{qs}(\bar{\mathbb{A}})I_{qs}(\bar{\mathbb{A}})$, there is $g \in G^{sc}(\bar{\mathbb{A}})$ with $(\psi_{|_{I_{qs}}}, g)$ in Y_{qs} (thus Y_{qs} is non-empty). The groups $\Gamma = \mathrm{Gal}(\bar{F}/F)$, $G^{sc}(\bar{F})$ and $I_1(\bar{\mathbb{A}})$ act on Y_{qs} as follows: For (i, g) in Y_{qs}, $\sigma(i, g) = (\sigma i, \sigma g)$ $(\sigma \in \Gamma)$, $h(i, g) = \big(\mathrm{Int}(h)i, hg\big)$ $\big(h \in G^{sc}(\bar{F})\big)$, $(i, g)t = \big(i, i(t^{-1})g\big)$ $\big(t \in I_1(\bar{\mathbb{A}})\big)$. Clearly $\sigma(hyt) = \sigma(h)\sigma(y)\sigma(t)$, and the actions of $G^{sc}(\bar{F})$ and $I_1(\bar{\mathbb{A}})$ commute (as $\big(h(i, g)\big)t = \big(\mathrm{Int}(h)i, hg\big)t = \big(\mathrm{Int}(h)i, hi(t^{-1})h^{-1}hg\big) = h\big(i, i(t^{-1})g\big) = h\big((i, g)t\big)$). Put $Y = Y_x = G^{sc}(\bar{F}) \setminus Y_{qs}$.

Lemma 5.1 *The F-space $Y = Y_x$ is a principal homogeneous space of $I_1(\bar{\mathbb{A}})/Z_{I_1}(\bar{F})$.*

Proof Since $G(\bar{F}) = G^{sc}(\bar{F})Z(\bar{F})$, we have $\mathrm{Int}\big(G(\bar{F})\big)(\psi_{|_{I_{qs}}}) = \mathrm{Int}\big(G^{sc}(\bar{F})\big)(\psi_{|_{I_{qs}}})$. Thus to show that Y is a homogeneous space under $I_1(\bar{\mathbb{A}})$ it suffices to take (i, g), (i, g_1) in Y_{qs} (same i). They satisfy $gxg^{-1} = i(x_{qs}) = g_1xg_1^{-1}$, thus $g_1g^{-1} \in Z_{i(G^{sc}_{qs})}\big(i(x_{qs})\big)(\bar{\mathbb{A}}) = i(I_1)(\bar{\mathbb{A}})$. The stabilizer of $(i, g) \in Y$ consists of $t \in I_1(\bar{\mathbb{A}})$ with $(i, g) = \big(i, i(t^{-1})g\big) = i(t^{-1})\Big(\mathrm{Int}\big(i(t)\big)i, g\Big)$, where for the last equality $i(t) \in G^{sc}(\bar{F})$, and $i(t)$ has to centralize the image of i, namely $t \in Z_{I_1}(\bar{\mathbb{A}})$, and $i(t) \in i(Z_{I_1})(\bar{\mathbb{A}}) \cap G^{sc}(\bar{F}) = i(Z_{I_1})(\bar{F})$. ■

Lemma 5.2 *The class $[Y]$ of Y in $H^1\big(F, I_1(\bar{\mathbb{A}})/Z_{I_1}(\bar{F})\big)$ lies in the image of $H^1\big(F, I_1(\bar{F})/Z_{I_1}(\bar{F})\big)$ precisely when $\big(Y/I_1(\bar{F})\big)^\Gamma$ is non-empty.*

Proof Consider the Γ-equivariant map $A \to B$ of Γ-modules A, B. Then b lies in $\mathrm{Im}[H^1(F, A) \to H^1(F, B)]$ precisely when the twist ${}_b(B/A)$ has Γ-invariant elements. Apply this observation with $A = I_1(\bar{F})/Z_{I_1}(\bar{F})$ and $B = I_1(\bar{\mathbb{A}})/Z_{I_1}(\bar{F})$ and b with ${}_b(B/A) = Y/I_1(\bar{F})$. ■

Lemma 5.3 *The element $x \in X(\mathbb{A})$ is $G^{sc}(\mathbb{A})$-conjugate to an element of $X(F)$ precisely when $Y_x/I_1(\bar{F})$ has a Γ-invariant element.*

Proof If $x \in X(\mathbb{A})$ is $G^{sc}(\mathbb{A})$-conjugate to an element of $X(F)$, then we may assume that $x \in X(F)$ and that $i = \psi$, on changing g. Thus $\psi(x_{qs}) = gxg^{-1}$ for g in $G^{sc}(\mathbb{A})$, but since $\psi(x_{qs})$ and x are in $G(\bar{F})$, we may take g in $G^{sc}(\bar{F})$. Since $\psi\colon G_{qs} \to G$ is an inner twist, for each $\sigma \in \Gamma$ there is a $g_\sigma \in G^{sc}(\bar{F})$ such that $\sigma\psi = \mathrm{Int}(g_\sigma)\psi$. Then

$$\sigma(\psi_{|_{I_{qs}}}, g) = \big(\mathrm{Int}(g_\sigma)\psi_{|_{I_{qs}}}, \sigma g\big) = g_\sigma(\psi_{|_{I_{qs}}}, g_\sigma^{-1}\sigma g \cdot g^{-1} \cdot g).$$

But $g_\sigma^{-1} \cdot \sigma g \cdot g^{-1} \in \psi(I_1)(\bar{F})$ since

$$g^{-1}\psi(x_{qs})g = x = \sigma g^{-1} \cdot \sigma\psi(x_{qs}) \cdot \sigma g = \sigma g^{-1} \cdot g_\sigma\psi(x_{qs})g_\sigma^{-1}\sigma g.$$

Hence $\sigma(\psi_{|_{I_{qs}}}, g) = g_\sigma(\psi_{|_{I_{qs}}}, g)t_\sigma$ for $t_\sigma \in I_1(\bar{F})$ defined by $\psi(t_\sigma) = g \cdot \sigma g^{-1} \cdot g_\sigma$, and $(\psi_{|_{I_{qs}}}, g)$ lies in $\big(Y/I_1(\bar{F})\big)^\Gamma$.

Conversely, if $\sigma(\psi, g) = \big(\mathrm{Int}(g_\sigma)\psi, \sigma g\big) = g_\sigma(\psi, g_\sigma^{-1}\cdot\sigma g\cdot g^{-1}g)$ is equal to $g_\sigma(\psi, g)t_\sigma$ for some $t_\sigma \in I_1(\bar{F})$, then $\psi(t_\sigma^{-1}) = g_\sigma^{-1} \cdot \sigma g \cdot g^{-1}$, and $\{\sigma \mapsto \sigma g \cdot g^{-1}\}$ lies in $\ker\big[H^1\big(F, G^{\mathrm{sc}}(\bar{F})\big) \to H^1\big(F, G^{\mathrm{sc}}(\bar{\mathbb{A}})\big)\big]$. The Hasse principle for the simply connected group G^{sc} implies that the kernel is trivial, namely g can be assumed to lie in $G^{\mathrm{sc}}(\bar{F})$. Hence $x = g^{-1}i(x_{\mathrm{qs}})g$ lies in $G(\bar{F})$ and in $X(\mathbb{A})$, namely in $X(F)$. ■

Definition Let $\mathrm{obs}_1(x)$ be the image of $[Y]$ under the map $\beta_{I_1}\colon H^1\big(F, I_1(\bar{\mathbb{A}})/Z_{I_1}(\bar{F})\big) \to A(I_1) = \pi_0\big(Z(\hat{I}_1)^\Gamma\big)^D$.

Since the kernel of β_{I_1} is the image of $H^1\big(F, I_1(\bar{F})/Z_{I_1}(\bar{F})\big) \to H^1\big(F, I_1(\bar{\mathbb{A}})/Z_{I_1}(\bar{F})\big)$ by [K2, 2.2], we conclude:

Lemma 5.4 *The element $x \in X(\mathbb{A}) \cap \mathrm{Int}\big(G(\bar{\mathbb{A}})\big)\psi(x_{\mathrm{qs}})$ is $G^{\mathrm{sc}}(\mathbb{A})$-conjugate to an element of $X(F)$ precisely when $\mathrm{obs}_1(x)$ is trivial.* ■

Suppose that $x' \in X(\mathbb{A})$ is $G(\bar{\mathbb{A}})$-conjugate to $x \in X(\mathbb{A})$. Since $G(\bar{\mathbb{A}}) = G^{\mathrm{sc}}(\bar{\mathbb{A}})I(\bar{\mathbb{A}})$, the F-set $S = \{h \in G^{\mathrm{sc}}(\bar{\mathbb{A}}); hxh^{-1} = x'\}$ is non-empty, hence it is a principal homogeneous space under $I_1(\bar{\mathbb{A}}) = Z_{G^{\mathrm{sc}}}(\bar{\mathbb{A}})(x)$. Recall that $Y = Y_x$ and $Y' = Y_{x'}$ denote the $I_1(\bar{\mathbb{A}})/Z_{I_1}(\bar{F})$-principal homogeneous F-spaces associated to x and x'. Using the map $S \times Y \to Y'$, $\big(h, (i, g)\big) \mapsto (i, gh^{-1})$, we obtain $\mathrm{obs}_1(x') = \mathrm{obs}_1(x) \cdot \mathrm{inv}(x, x')$, where $\mathrm{inv}_1(x, x')$ is the image of the class of S under the map $H^1\big(F, I_1(\bar{\mathbb{A}})\big) \to A(I_1)$ of [K2, 2.3.1].

Following Borovoi [B], we put $G^{\mathrm{tor}} = G/G^{ss}$. Then we have exact sequences

$$1 \to G^{\mathrm{sc}} \to G \to G^{\mathrm{tor}} \to 1 \quad \text{and} \quad 1 \to G^{\mathrm{sc}}_{\mathrm{qs}} \to G_{\mathrm{qs}} \to G^{\mathrm{tor}} \to 1.$$

Note that $G = G^{ss}Z$, $G_{\mathrm{qs}} = G^{ss}_{\mathrm{qs}}Z_{\mathrm{qs}}$ and $Z_{\mathrm{qs}} = Z$. We also have an exact sequence

$$1 \to I_1 \to I_{\mathrm{qs}} \to G^{\mathrm{tor}} \to 1.$$

As $Z(\hat{G}) = \mathrm{Hom}\big(X^*(G), \mathbb{C}^\times\big)$ and $X^*(G) = X^*(G^{\mathrm{tor}})$, the last exact sequence gives the exact sequence

$$1 \to Z(\hat{G}) \to Z(\hat{I}_{\mathrm{qs}}) \to Z(\hat{I}_1) \to 1.$$

Then $\mathfrak{K}(I_{\mathrm{qs}}/F) \subset \pi_0\big(Z(\hat{I}_1)^\Gamma\big)$ and by duality we get a homomorphism $A(I_1) \to \mathfrak{K}(I_{\mathrm{qs}}/F)^D$. Define $\mathrm{obs}(x)$ to be the image of $\mathrm{obs}_1(x)$.

Theorem 5.5 *The element x of $X(\mathbb{A}) \cap \mathrm{Int}\big(G(\bar{\mathbb{A}})\big)\psi(x_{\mathrm{qs}})$ is $G(\mathbb{A})$-conjugate to an element of $X(F)$ precisely when $\mathrm{obs}(x)$ is trivial.*

Proof This is the same as that of [K2, 6.6]. ■

With $x, x', \mathrm{inv}_1(x, x')$ as above, let $\mathrm{inv}(x, x')$ be the image of $\mathrm{inv}_1(x, x')$ under $A(I_1) \to \mathfrak{K}(I_{\mathrm{qs}}/F)^D$. Then $\mathrm{obs}(x') = \mathrm{obs}(x)\,\mathrm{inv}(x, x')$. If $x'_{\mathrm{qs}} \in X_{\mathrm{qs}}(F)$ is stably conjugate to x_{qs}, using x'_{qs} instead of x_{qs} in the definitions of Y_x, $\mathrm{obs}_1(x)$ and $\mathrm{obs}(x)$, we get $\mathrm{obs}(x)' \in \mathfrak{K}(I'_{\mathrm{qs}}/F)^D$, where $I'_{\mathrm{qs}} = Z_{G_{\mathrm{qs}}}(x'_{\mathrm{qs}})$. There is an inner twist $I'_{\mathrm{qs}} \to I_{\mathrm{qs}}$, canonical up to conjugation by an element of $I_{\mathrm{qs}}(\bar{F})$. Identifying $\mathfrak{K}(I'_{\mathrm{qs}}/F)^D$ with $\mathfrak{K}(I_{\mathrm{qs}}/F)^D$, we have $\mathrm{obs}(x)' = \mathrm{obs}(x)$ by [K2, 2.8].

6 Global Conjecture

Let (H, s, η) be an endoscopic triple for G. Choose an extension of $\eta\colon \hat{H} \to \hat{G}$ to an L-homomorphism $\eta'\colon {}^LH \to {}^LG$. Assume that the local conjecture holds at each place v of F. Write $\Delta_v(x_H, x)$ for the transfer factors at the place v; they can be multiplied by complex scalars.

Global conjecture For a suitable normalization of the local transfer factors one has:

(a) For any (G, H)-regular semi-simple $x_H \in X_H(F)$ and any $x \in X(\mathbb{A})$ coming from x_H, we conjecture that almost all factors $\Delta_v(x_H, x)$ are equal to 1. Put $\Delta(x_H, x) = \prod_v \Delta_v(x_H, x)$.

(b) Choose an inner twisting $\psi\colon G_{\mathrm{qs}} \to G$. Choose $x_{\mathrm{qs}} \in X_{\mathrm{qs}}(F)$ such that x_{qs} comes from x_H. Thus $x_H = \gamma_H\theta(\gamma_H)^{-1} \in X_H(F)$ ($\gamma_H \in H(E)$) lies in $T_H(E)$, where T_H is an F-torus in H, we fix a Γ-equivariant isomorphism $j\colon T_H \to T_{\mathrm{qs}}$, where T_{qs} is an F-torus in G_{qs}, and put $x_{\mathrm{qs}} = j(x_H)$; since G_{qs} is quasi-split over F, T_{qs} exists by Steinberg's theorem; see [K4, 4.4]). Then $x_{\mathrm{qs}} \in G_{\mathrm{qs}}(E)$ satisfies $x_{\mathrm{qs}}\theta(x_{\mathrm{qs}}) = 1$, namely it defines an element of $H^1\big(E/F, G_{\mathrm{qs}}(E)\big)$, whose image in $H^1\big(E_v/F_v, G_{\mathrm{qs}}(E_v)\big)$ is trivial if the local conjecture holds at v and $\Phi^{\mathrm{st}}_{f_{H_v}}(x_{\mathrm{qs}}) \neq 0$. To assure the existence of $\gamma_{\mathrm{qs}} \in G_{\mathrm{qs}}(E)$ with $x_{\mathrm{qs}} = \gamma_{\mathrm{qs}}\theta(\gamma_{\mathrm{qs}}^{-1})$, we assume that $\ker\big[H^1\big(E/F, G_{\mathrm{qs}}(E)\big) \to \prod_v H^1\big(E_v/F_v, G_{\mathrm{qs}}(E_v)\big)\big]$ is trivial. Note that at v which splits E/F, we have that $H^1\big(E_v/F_v, G_{\mathrm{qs}}(E_v)\big) = \{1\}$. The last assumption is implied for example by the assumption that $H^1\big(E/F, G_{\mathrm{qs}}(E)\big)$ is trivial.

Now set $I_{\mathrm{qs}} = Z_{G_{\mathrm{qs}}}(x_{\mathrm{qs}})$. Then we conjecture that $\Delta(x_H, x) = \kappa\big(\mathrm{obs}(x)\big)$, where $\kappa \in \mathfrak{K}(I_{\mathrm{qs}}/F)$ is obtained from s via $Z(\hat{H}) \hookrightarrow Z(\hat{I}_H) \simeq Z(\hat{I}_{\mathrm{qs}})$ (where $I_H = Z_H(x_H)$ is connected). Note that $\mathrm{obs}(x) = \mathrm{obs}(x)'$ if x_{qs} is replaced by a stably conjugate x'_{qs} in the definition of obs. If x is replaced by a $G(\bar{\mathbb{A}})$-conjugate x', then both $\kappa\big(\mathrm{obs}(x)\big)$ and $\Delta(x_H, x)$ get multiplied by the same factor $\kappa\big(\mathrm{inv}(x, x')\big)$ (by the local conjectures in the case of Δ).

7 Stabilization

We can now return to the stabilization of the elliptic semi-simple part of the bi-period summation formula. All sums considered below are finite. But this we show only after we formally discuss the stabilization.

For a quasi-split reductive connected F-group G, define the stable analogue

$$\mathrm{ST}_e(f) = \sum_{x \in E_{\mathrm{st}}} |(I_x/I_x^0)(F)|^{-1}\tau(G)\Phi^{\mathrm{st}}_f(x).$$

Here E_{st} is a set of representatives for the elliptic semi-simple stable conjugacy classes in $X(F)$, and $\Phi^{\mathrm{st}}_f(x) = \sum_i e(x_i)\Phi_f(x_i)$. Here i ranges over $\ker\big[H^1\big(F, I_X^0(\bar{\mathbb{A}})\big) \to H^1\big(F, G(\bar{\mathbb{A}})\big)\big]$, and it determines a $G(\mathbb{A})$-conjugacy class x_i in $X(\mathbb{A})$, whose local components are all stably conjugate to x. The number $e(x_i)$ is defined to be $\prod_v e(I^0_{i,v})$, where $I_{i,v}$ is the centralizer in $G(F_v)$ of the component $x_{i,v}$ of x_i at the place v of F. The measures defining the orbital integrals are chosen in a compatible way—this is used in the definition of the stable orbital integrals. We show below that the sums which define $\Phi^{\mathrm{st}}_f(x)$ and $\mathrm{ST}_e(f)$ are finite, and the integral which defines $\Phi_f(x_i)$ is convergent. As $\Phi^{\mathrm{st}}_f(x)$ and $|(I_x/I_x^0)(F)|$ depend only on the stable conjugacy class of x in $X(F)$, $\mathrm{ST}_e(F)$ is well-defined.

For simplicity, assume that the derived group G^{ss} is simply connected. Choose an inner twisting $\psi\colon G_{qs} \to G$ with G_{qs} quasi-split over F. Choose a set $\mathbb{F}$ of representatives for the isomorphism classes of elliptic endoscopic triples (H,s,η) for G [K1]. For each $(H,s,\eta) \in \mathbb{F}$ choose an L-homomorphism $\eta'\colon {}^LH \to {}^LG$ extending η. Assume the local and global conjectures, and the "fundamental lemma", asserting that the unit elements $(f^0_{H,v}, f^0_v)$ in the Hecke algebras of H_v and G_v are matching. Then $f \in C_c^\infty\big(X(\mathbb{A})\big)$ defines $f_H \in C_c^\infty\big(X_H(\mathbb{A})\big)$ (depending on H, s, η'), satisfying $\Phi^{\text{st}}_{f_H}(x_H) = \sum_x \kappa\big(\text{obs}(x)\big)e(x)\Phi_f(x)$ for any (G,H)-regular semi-simple $x_H \in X_H(F)$. The sum ranges over x in a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ which come from x_H. The sign $e(x)$ is the product $\prod_v e(I_{x,v})$, where $I_{x,v}$ is the (connected) centralizer in $G(F_v)$ of the v component x_v of x.

Define $T_e^*(f)$ to be the sum which defines $T_e(f)$, omitting the terms indexed by the central elements x of $X(F)$. Define $\text{ST}_e^{**}(f_H)$ to be the sum which defines $\text{ST}_e(f_H)$, omitting the central elements of $X_H(F)$ if H is a quasi-split inner form of G, and omitting all terms indexed by the $x_H \in X_H(F)$ which are not (G,H)-regular when H is not a quasi-split inner form of G.

Theorem 7.1 *Assuming the local and global conjectures for (f_H, f), and the fundamental lemma, we have*

$$T_e^*(f) = \sum_{(H,s,\eta)\in\mathbb{F}} \big(\tau(G)/\tau(H)\big)\,[\text{Aut}(H,s,\eta)/H_{\text{ad}}(F)]^{-1}\, \text{ST}_e^{**}(f_H).$$

$(\text{Aut}(H,s,\eta)$ *is defined in [K1, 7.5]).*

Proof The assumption that G^{ss} is simply connected implies that the centralizer $I = Z_G(x)$ is connected, and by [K2, 3.2] that the centralizer $I_H = Z_H(x_H)$, of any (G,H)-regular semi-simple x_H in $X_H(F)$, is connected. Hence the numbers $|(I_{H,x}/I^0_{H,x})(F)|$ and $|I_x(F)/I^0_x(F)|$ which appear in the definitions of $\text{ST}_e(f_H)$ and $T_e(f)$ are 1.

Let E^*_{qs} be a set of representatives for the non-central elliptic semi-simple stable conjugacy classes in $X_{\text{qs}}(F)$. Put $I_{\text{qs}} = Z_{G_{\text{qs}}}(x_{\text{qs}})$. Then $T_e^*(f) = \sum_{x_{\text{qs}}\in E^*_{\text{qs}}} \tau(I_{\text{qs}}) \sum_x \Phi_f(x)$, where x ranges over a set of representatives for the $G(F)$-conjugacy classes in $X(F)$ contained in the $G(\bar{F})$-conjugacy class of $\psi(x_{\text{qs}})$. The orbital integral $\Phi_f(x)$ depends only on the $G(\mathbb{A})$-conjugacy class of x in $X(\mathbb{A})$. If x contributes to $T_e^*(f)$, put $I = Z_G(x)$, and note that the number of terms in the second sum in $T_e^*(f)$ which are indexed by $G(\mathbb{A})$-conjugates of x is equal to $|\ker[D(I/F) \to D(I/\mathbb{A})]| = |\ker[\ker^1(F,I) \to \ker^1(F,G)]|$. The equality follows from the commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1\\
 & & \downarrow & & \downarrow & & \downarrow\\
1 & \longrightarrow & \ker & \longrightarrow & \ker^1(F,I) & \longrightarrow & \ker^1(F,G)\\
 & & \downarrow & & \downarrow & & \downarrow\\
1 & \longrightarrow & D(I/F) & \longrightarrow & H^1(F,I) & \longrightarrow & H^1(F,G)\\
 & & \downarrow & & \downarrow & & \downarrow\\
1 & \longrightarrow & D(I/\mathbb{A}) & \longrightarrow & H^1\big(F, I(\bar{\mathbb{A}})\big) & \longrightarrow & H^1\big(F, G(\bar{\mathbb{A}})\big),
\end{array}$$

whose columns define $\ker^1(F, I)$ and $\ker^1(F, G)$.

Lemma 7.2 *The diagram (see [K1, Section 4])*

$$\begin{array}{ccccc} \ker^1(F,I) & \longrightarrow & \ker^1(F,G) & \xrightarrow{\sim} & \ker^1(F,G^{\mathrm{tor}}) \\ \downarrow & & \downarrow & & \downarrow \\ \ker^1\big(F,Z(\hat{I})\big)^D & \longrightarrow & \ker^1\big(F,Z(\hat{G})\big)^D & \xrightarrow{\sim} & \ker^1(F,\hat{G}^{\mathrm{tor}})^D, \end{array}$$

in which the vertical maps are bijections, and the bottom horizontal map is induced by the natural injection $Z(\hat{G}) \hookrightarrow Z(\hat{I})$, is commutative.

Proof The horizontal maps on the right are isomorphisms by [K2, 4.3.1]. Hence the commutativity follows form the functoriality of $\ker^1(F, G) \to \ker^1\big(F, Z(\hat{G})\big)^D$ for the normal homomorphisms $G \to G^{\mathrm{tor}}$ and $I \to G^{\mathrm{tor}}$. ∎

Corollary 7.3 *The number of terms in the second sum in $T_e^*(f)$ indexed by $G(\mathbb{A})$-conjugates of x is equal to*

$$|\ker\big[\ker^1\big(F, Z(\widehat{I_{\mathrm{qs}}})\big)^D \to \ker^1\big(F, Z(\hat{G})\big)^D\big]| = |\mathrm{coker}\big[\ker^1\big(F, Z(\hat{G})\big) \to \ker^1\big(F, Z(\widehat{I_{\mathrm{qs}}})\big)\big]|.$$

Lemma 7.4 *The quotient of the last number,* coker, *by $|\mathfrak{K}(I_{\mathrm{qs}}/F)|$, is equal to $\tau(G)/\tau(I_{\mathrm{qs}})$.*

Proof Since $\tau(G) = |\pi_0(Z(\hat{G})^\Gamma)|/|\ker^1\big(F, Z(\hat{G})\big)|$, this follows from the exact sequence

$$1 \to \pi_0\big(Z(\hat{G})^\Gamma\big) \to \pi_0\big(Z(\widehat{I_{\mathrm{qs}}})^\Gamma\big) \to \mathfrak{K}(I_{\mathrm{qs}}/F) \to$$
$$\ker^1\big(F, Z(\hat{G})\big) \to \ker^1\big(F, Z(\widehat{I_{\mathrm{qs}}})\big) \to \mathrm{coker} \to 1$$

(see [K1, 2.3 and 8.3.3]), which starts with 1 since $X_*\big(Z(\widehat{I_{\mathrm{qs}}})/Z(\hat{G})\big)^\Gamma$ is trivial for elliptic x_{qs}. ∎

Consider $x_{\mathrm{qs}} \in E^*_{\mathrm{qs}}$ and $x \in X(\mathbb{A})$ in the $G(\bar{\mathbb{A}})$-conjugacy class of $\psi(x_{\mathrm{qs}})$. We have $\mathrm{obs}(x) \in \mathfrak{K}(I_{\mathrm{qs}}/F)^D$, with the property that $|\mathfrak{K}(I_{\mathrm{qs}}/F)|^{-1} \sum_\kappa \kappa\big(\mathrm{obs}(x)\big)$, where κ ranges over $\mathfrak{K}(I_{\mathrm{qs}}/F)$, is 1 if the $G(\mathbb{A})$-conjugacy class of x contains an element of $G(F)$, and is equal to 0 otherwise. Note that $e(x) = \prod_v e(I_{x,v})$ is 1 if $x \in X(\mathbb{A})$ is $G(\mathbb{A})$-conjugate to an element of $X(F)$, by [K3]. Consequently, we can write

$$T_e^*(f) = \sum_{x_{\mathrm{qs}} \in E^*_{\mathrm{qs}}} \tau(G) \sum_x \sum_\kappa \kappa\big(\mathrm{obs}(x)\big) e(x) \Phi_f(x).$$

Here x ranges over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ contained in the $G(\bar{\mathbb{A}})$-conjugacy class of $\psi(x_{\mathrm{qs}})$, and κ ranges over $\mathfrak{K}(I_{\mathrm{qs}}/F)$. We show below that the triple sum has only finitely many non-zero terms, hence it can be rearranged at will.

The right side of the equality of the theorem is

$$\tau(G)\sum_{\mathbb{F}}[\mathrm{Aut}(H,s,\eta)/H_{\mathrm{ad}}(F)]^{-1}\sum\Phi^{\mathrm{st}}_{f_H}(x_H),$$

where the inner sum ranges over x_H in the set E_H^{**} which is a set of representatives for:

(1) the non-central elliptic semi-simple stable conjugacy classes in $X_H(F)$ if H is a quasi-split inner form of G, or

(2) the (G,H)-regular elliptic semi-simple stable conjugacy classes in $X_H(F)$ if H is not a quasi-split inner form of G.

Writing out the definition of $\Phi^{\mathrm{st}}_{f_H}(x_H)$, the right side of the equality of the theorem becomes

$$\tau(G)\sum_{\mathbb{F}}[\mathrm{Aut}(H,s,\eta)/H_{\mathrm{ad}}(F)]^{-1}\sum_{x_H\in E_H^{**}}\sum_{x}\kappa\big(\mathrm{obs}(x)\big)e(x)\Phi_f(x).$$

Here x ranges over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ which come from x_H.

Given $(H,s,\eta)\in\mathbb{F}$ and $x_H\in E_H^{**}$ we get $x_{\mathrm{qs}}\in X_{\mathrm{qs}}(F)$ up to stable conjugacy, and $\kappa\in\mathfrak{K}(I_{\mathrm{qs}}/F)$, where $I_{\mathrm{qs}}=Z_{G_{\mathrm{qs}}}(x_{\mathrm{qs}})$. A simple adaptation of [K2, 9.7] asserts that if x_{qs} is an elliptic semi-simple element of $X_{\mathrm{qs}}(F)$, and $\kappa\in\mathfrak{K}(I_{\mathrm{qs}}/F)$ where $I_{\mathrm{qs}}=Z_{G_{\mathrm{qs}}}(x_{\mathrm{qs}})$, then there exist $(H,s,\eta)\in\mathbb{F}$ and a (G,H)-regular semi-simple x_H in $X_H(F)$ such that (H,s,η,x_H) gives (x_{qs},κ). Moreover, (H_1,s_1,η_1,x_{H_1}) also gives (x_{qs},κ) if and only if there is an isomorphism $(H,s,\eta)\to(H_1,s_1,\eta_1)$ taking x_H to a stable conjugate of x_{H_1}, and such an isomorphism is unique up to composition with an element of $H_{\mathrm{ad}}(F)$. This shows that $T_e^*(f)$ is indeed equal to the expression asserted in the theorem. ■

8 Local Finiteness

It remains to establish the finiteness results asserted above.

Let E/F be a quadratic unramified extension of p-adic fields, R (resp. R_E) the ring of integers of F (resp. E), k (resp. k_E) the residue field of R (resp. R_E), $\bar{k}$ an algebraic closure of k (containing the quadratic extension k_E), and G an unramified connected reductive group over F. Let x_{qs} be a hyperspecial point in the building of G, and $\mathbf{G}$ the corresponding extension of G to a group scheme over R (see [T]). Write K for the hyperspecial maximal compact subgroup $\mathbf{G}(R)=\mathrm{Stab}_{G(F)}(x_{\mathrm{qs}})$ of $G(F)$, and $K_E=\mathbf{G}(R_E)=\mathrm{Stab}_{G(E)}(x_{\mathrm{qs}})$. Put $X(R)=\{g\theta(g^{-1});g\in K_E\}\subset K_E\subset G(E)$. Let $\bar{F}$ be an algebraic closure of F, and $R_{\bar{F}}$ its ring of integers.

Proposition 8.1 *Let x be a semi-simple element of $X(R)$ such that $1-\alpha(x)\in R_{\bar{F}}$ is 0 or a unit for every root α of G. Put $I=Z_G(x)$. Then I^0 is unramified, and $I^0(F)\cap K$ is a hyperspecial maximal compact subgroup of $I^0(F)$. Further,* $\ker[H^1(F,I^0)\to H^1(F,I)]$ *is trivial. Finally, if $x'\in X(R)$ is stably conjugate to $x\big(\in X(R)\big)$, then x' is conjugate to x under K.*

Proof Consider first the case where G^{ss} is simply connected. Then I is connected.

In the first part of the proof we assume that G is split over F, that $\mathbf{A}$ is a split maximal R-torus in G and $x=\gamma\theta(\gamma)^{-1}\in\mathbf{A}(R_E)\cap X(R)$ with $\gamma\in\mathbf{G}(R_E)$, and that $x'\in\mathbf{G}(R_E)$ is conjugate to x under $G(F)$.

Define the subgroup scheme $\mathbf{Z}_{\mathbf{G}}(x)$ of $\mathbf{G}$ by $\mathbf{Z}_{\mathbf{G}}(x)(D) = \{g \in \mathbf{G}(D); gxg^{-1} = x\}$ for any commutative R-algebra D. It is closed. The image $\bar{x}$ of x in $X(k) = \{g\theta(g^{-1}); g \in \mathbf{G}(k_E)\}$ is semi-simple, as it lies in $\mathbf{A}(k_E)$. The derived group of $\mathbf{G}_k = \mathbf{G} \times_R k$ is simply connected. Hence the special fiber $\mathbf{Z}_{\mathbf{G}}(x)_k$ of $\mathbf{Z}_{\mathbf{G}}(x)$ is a connected reductive group. The assumption that $1 - \alpha(x)$ be 0 or a unit implies that the special and generic fibers of $\mathbf{Z}_{\mathbf{G}}(x)$ have the same dimension. Hence $\mathbf{Z}_{\mathbf{G}}(x)$ is smooth over R [SGA3, VI_B, 4.4], with connected reductive fibers. In particular, $\mathbf{Z}_{\mathbf{G}}(x)$ is unramified, and $\mathbf{Z}_{\mathbf{G}}(x)(R) = K \cap \mathbf{Z}_{\mathbf{G}}(x)(F)$ is a hyperspecial maximal compact subgroup of $\mathbf{Z}_{\mathbf{G}}(x)(F)$.

If $x' \in \mathbf{G}(R_E)$ is conjugate to $x\ (\in X(R))$ under $G(F)$, choose a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ over R which contains $\mathbf{A}$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$. Then $G(F) = KN(F)A(F)$, and we may assume that $x' = nxn^{-1}$ for some $n \in N(F)$. It suffices to show that n lies in $\mathbf{N}(R)\mathbf{Z}_{\mathbf{N}}(x)(F)$, where $\mathbf{Z}_{\mathbf{N}}(x) = \mathbf{Z}_{\mathbf{G}}(x) \cap \mathbf{N}$.

Choose a total order $\alpha_1 < \cdots < \alpha_r$ with $\alpha_i < \alpha_i + \alpha_j$ on the set Δ of B-positive roots of A. This order can be used to fix an isomorphism $\prod_{1 \le i \le r} \mathbb{G}_a \simeq \mathbf{N}$ of varieties (not groups) over R. Under this isomorphism, $\prod_{j \le i \le r} \mathbb{G}_a$ corresponds to a subgroup $\mathbf{N}_j$ of $\mathbf{N}$, and the projection $\mathbf{N}_j \to \mathbb{G}_a$ on the j-th factor is a homomorphism (with kernel $\mathbf{N}_{j+1}$). Write n as $(n_1, \ldots, n_r)$ in $\prod_{1 \le i \le r} \mathbb{G}_a$. Then $nxn^{-1}x^{-1} = x'x^{-1}$ lies in $K_E \cap N(F) \subset \mathbf{N}(R_E)$, and its first coordinate is $\big(1 - \alpha_1(X)\big)n_1$. If $1 - \alpha_1(x) = 0$, then α_1 is a root of $Z_G(x)$. Multiplying n on the right by an element of $Z_N(x)(F)$, we may assume that $n_1 = 0$. If $1 - \alpha_1(x) \neq 0$, then $1 - \alpha_1(x)$ is a unit by assumption, hence $n_1 \in R$. Multiplying n on the left by an element of $\mathbf{N}(R)$, again we may assume that $x_1 = 0$. Then $n \in N_2(F)$. The same argument can be applied to n_2. Using all positive roots, we conclude that $n \in \mathbf{N}(R)Z_N(x)(F)$.

In the second part of the proof, we continue to assume that G^{ss} is simply connected, but drop the other assumptions. Choose a maximal F-torus T of G containing x (thus $x \in T(E)$), and choose a finite Galois extension F'/F which splits T and such that x and x' are conjugate under $G(F')$. Since G is split over F', we can choose a maximal split (over $R' = R_{F'}$) torus $\mathbf{A}$ of $\mathbf{G}$, and an element $x'' \in \mathbf{A}(R')$ which is conjugate under $G(F')$ to both x and x'. By the first part of the proof, x, x', x'' are all conjugate under $\mathbf{G}(R')$, and $\mathbf{Z}_{\mathbf{G}}(x'')$ is smooth over R'. The group $\mathbf{Z}_{\mathbf{G}}(x) \times_R R'$ is isomorphic over R' to $\mathbf{Z}_{\mathbf{G}}(x'')$, and R' is faithfully flat over R. Hence $\mathbf{Z}_{\mathbf{G}}(x)$ is smooth over R, and its fibers are connected reductive groups. In particular $\mathbf{Z}_{\mathbf{G}}(x)$ is unramified, and $\mathbf{Z}_{\mathbf{G}}(x)(R) = Z_G(x)(F) \cap \mathbf{G}(R)$ is a hyperspecial maximal compact subgroup of $\mathbf{Z}_{\mathbf{G}}(x)(F)$.

Let $\mathbf{S}$ be the closed subscheme of $\mathbf{G}$ with $\mathbf{S}(D) = \{g \in \mathbf{G}(D); gxg^{-1} = x'\}$ for any R-algebra D. Since $\mathbf{S}$ is isomorphic to $\mathbf{Z}_{\mathbf{G}}(x)$ over R', it is smooth over R. Let $\bar{x}, \bar{x}'$ be the images of x, x' in $X(k) \subset \mathbf{G}(k_E)$. Since x, x' are conjugate under $\mathbf{G}(R')$, $\bar{x}$ and $\bar{x}'$ are stably conjugate. The special fiber $\mathbf{Z}_{\mathbf{G}}(x)_k$ is connected. By Lang's lemma, $H^1\big(k, \mathbf{Z}_{\mathbf{G}}(x)_k\big)$ is trivial. Hence $\bar{x}, \bar{x}' \in X(k)$ are conjugate under $\mathbf{G}(k)$. Hence $\mathbf{S}(k)$ is non-empty, and so the smoothness of $\mathbf{S}$ over R implies that $\mathbf{S}(R)$ is non-empty. This completes the proof when the derived group G^{ss} is simply connected. The general case can be discussed along the lines of [K2, end of 7.1]. ■

Denote by f_K the characteristic function of the set $X(R) = \{k \cdot \theta(k^{-1}) \cdot z \cdot \theta(z^{-1}); k \in K_E, z \in Z(E)\}$ in $X(F) \subset G(E)$. Fix Haar measures dg, di on $G(F)$, $I^0(F)$ with $|K| = |I^0(F) \cap K|$. Here we fix $x \in X(R)$ satisfying the assumptions of the proposition, and put $I = Z_G(x)$. Let $x' \in X(R)$ be a stable conjugate of x, and define the orbital integral $\Phi_{f_K}(x')$

using dg/di', where di' is the measure on $I'^0(F)$ obtained from di via the isomorphism of $I' = Z_G(x')$ with I.

Corollary 8.2 *The orbital integral $\Phi_{f_K}(x')$ is 0 unless x' is conjugate to x, where it is 1.*

Proof If x' is not conjugate to x, then the orbit of x' under $G(F)$ does not meet $X(R)$, hence $\Phi_{f_K}(x') = 0$. On the other hand, $\Phi_{f_K}(x)$, for $x \in X(R)$, is the volume of $I^0(F) \setminus Y$, where $Y = \{g \in G(F); g^{-1}xg \in X(R)\}$. As G^{ss} is simply connected, the proposition shows that $Y = I(F)K$, and we are done. ■

Let (H, s, η) be an endoscopic triple for G, x_H a semi-simple (G, H)-regular element of $X_H(F)$, and x the corresponding element of $X(F)$ (note that $H^1(E/F, K_E) = 1$). Put $I = Z_G(x)$, assume that G^{ss} is simply connected, and let κ be the element of $\mathfrak{K}(I/F)$ obtained from s. We have the κ-orbital integral $\Phi_f^\kappa(x)$. Note that for $f' \in C_c^\infty\big(K_E \setminus G(E)/K_E\big)$, $f(y) = \int_{G(F)} f'(\gamma h)\, dh$, $y = \gamma\theta(\gamma)^{-1}$, satisfies $f\big(ky\theta(k^{-1})\big) = f(y)$ for all $k \in K_E$.

Proposition 8.3 *Suppose that H is a ramified group, and $f \in C_c^\infty\big(X(F)\big)$ satisfies $f\big(ky\theta(k^{-1})\big) = f(y)$ for all $y \in X(F)$, $k \in K_E$. Then $\Phi_f^\kappa(x) = 0$.*

Proof Denote by $\Gamma_{in} = \mathrm{Gal}(\bar{F}/F^{ur})$ the inertia subgroup of $\Gamma = \mathrm{Gal}(\bar{F}/F)$. Since G is unramified, Γ_{in} acts trivially on $X^*\big(Z(G)\big)$. Hence $Z(G)$ can be embedded in an unramified F-torus C'. Put $C = C'/Z(G)$. Embed $Z(G)$ diagonally in $G \times C'$; the group $G_1 = (G \times C')/Z(G)$ is unramified. The center of G_1 is C'; it is connected. The exact sequence

$$1 \to G \to G_1 \to C \to 1$$

yields a dual exact sequence

$$1 \to \hat{C} \to \hat{G}_1 \to \hat{G} \to 1.$$

Defining $\hat{H}_1$ to be the fiber product of $\hat{G}_1(\to \hat{G})$ and $\hat{H}(\xrightarrow{\eta} \hat{G})$ over $\hat{G}$, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{H}_1 & \longrightarrow & \hat{H} & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta} & & \\ 1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{G}_1 & \longrightarrow & \hat{G} & \longrightarrow & 1. \end{array}$$

Since η need not be a Γ-map, the fiber product construction does not immediately give an action of Γ on $\hat{H}_1$. To define an action of Γ on $\hat{H}_1$, note that the $\hat{G}$-conjugacy class of η is fixed by Γ, namely for each $\sigma \in \Gamma$ there exists $g_\sigma \in \hat{G}$ such that $\eta \circ \sigma = \mathrm{Int}(g_\sigma) \circ \sigma \circ \eta$. For each $\sigma \in \Gamma$ choose $x_\sigma \in \hat{G}_1$ with $x_\sigma \mapsto g_\sigma$. The restriction of $\mathrm{Int}(x_\sigma)$ to $\sigma\big(\eta_1(\hat{H}_1)\big)$ is independent of the choices of g_σ and x_σ. Let Γ act on $\hat{H}_1$ in the unique way for which $\eta_1 \circ \sigma = \mathrm{Int}(x_\sigma) \circ \sigma \circ \eta_1$.

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{G}_1) & \longrightarrow & Z(\hat{G}) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{H}_1) & \longrightarrow & Z(\hat{H}) & \longrightarrow & 1. \end{array}$$

Lemma 8.4 *The element $s \in Z(\hat{H})$ does not belong to the image of $Z(\hat{H}_1)^{\Gamma_{in}}$.*

Proof Let $s_1 \in Z(\hat{H}_1)$ map to s. Since H is ramified, there is a $\sigma \in \Gamma_{in}$ which acts non-trivially on $\hat{H}$. As G is unramified, σ acts trivially on $\hat{G}$. Now $\eta \circ \sigma = \mathrm{Int}(g_\sigma) \circ \sigma \circ \eta$ implies that $g_\sigma \notin \eta(\hat{H})$, otherwise σ would act on $\hat{H}$ by an inner automorphism, which would be trivial as σ preserves some splitting of $\hat{H}$. But then $x_\sigma \notin \eta_1(\hat{H}_1)$. Now $\eta_1(\hat{H}_1)$ is the identity component of $Z_{\hat{G}_1}(s_1)$. As G^{ss} is simply connected, so is G_1^{ss}. Hence $\eta_1(\hat{H}_1) = Z_{\hat{G}_1}(s_1)$. Hence x_σ does not centralize s_1, and $\sigma s_1 \neq s_1$. ■

Since F is local, we may assume that $s \in Z(\hat{H})^\Gamma$. From the exact sequence

$$1 \to \hat{C} \to Z(\hat{H}_1) \to Z(\hat{H}) \to 1$$

we get $Z(\hat{H})^\Gamma \to H^1(F, \hat{C}) \simeq \mathrm{Hom}_{cts}\big(C(F), \mathbb{C}^\times\big)$, then s is mapped to $\alpha \in H^1(F, \hat{C})$, which is the Langlands parameter for a character χ of $C(F)$. By Lemma 8.4, s does not lie in the image of $Z(\hat{H}_1)^{\Gamma_{in}}$. Hence α is a ramified Langlands parameter for the unramified torus C, so χ is non-trivial on $\mathbf{C}(R)$ for the unique extension $\mathbf{C}$ of C to a torus over R.

Let H_1 be a quasi-split connected reductive group over F whose dual is $\hat{H}_1$. Choose an embedding $H \hookrightarrow H_1$ over F, dual to $\hat{H}_1 \to \hat{H}$. Since $s_1 \notin Z(\hat{G}_1)Z(\hat{H}_1)^\Gamma$, (H_1, s_1, η_1) is not an endoscopic triple for G_1; yet the results of [K2, Section 3] extend to H_1, G_1. Put $I_1 = Z_{G_1}(x)$ and $I_{H_1} = Z_{H_1}(x_H)$; both are connected. Then x_H is (G_1, H_1)-regular, and we have $Z(\hat{H}_1) \hookrightarrow Z(\hat{I}_{H_1}) \simeq Z(\hat{I}_1)$. We also have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{H}_1) & \longrightarrow & Z(\hat{H}) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \hat{C} & \longrightarrow & Z(\hat{I}_1) & \longrightarrow & Z(\hat{I}) & \longrightarrow & 1, \end{array}$$

where $I = Z_G(x)$. Let $c \in C(F)$ be the image of $g_1 \in G_1(F)$. Put $x' = g_1 x g_1^{-1}$. Since $G_1/C' = G/Z(G)$, x' and x are conjugate under $G(\bar{F})$. Since G is normal in G_1, if $x = \gamma\theta(\gamma)^{-1}$ with $\gamma \in G(E)$, $g_1\gamma g_1^{-1}$ also lies in $G(E)$, and from $x' = g_1\gamma g_1^{-1}\theta(g_1\gamma g_1^{-1})^{-1}$ we see that $x' \in X(F)$. Then $x', x \in X(F)$ are stably conjugate.

Lemma 8.5 *We have* $\kappa\big(\mathrm{inv}(x, x')\big) = \chi(c)^{-1}$.

Proof Let $\kappa_0 \in Z(\hat{I})^\Gamma$ be the image of $s \in Z(\hat{H})^\Gamma$. From the exact sequence

$$1 \to \hat{C} \to Z(\hat{I}_1) \to Z(\hat{I}) \to 1$$

we get $Z(\hat{I})^\Gamma \to H^1(F, \hat{C})$, which—using the commutative diagram above—maps κ_0 to α. From the exact sequence $1 \to I \to I_1 \to C \to 1$ we get $C(F) \to H^1(F, I)$. Denote by $d \in H^1(F, I)$ the image of $c^{-1} \in C(F)$. The commutative square

$$\begin{array}{cccc} C(F) & \longrightarrow & H^1(F, I) & c^{-1} \mapsto d \\ \downarrow & & \downarrow & \\ H^1(F, \hat{C})^D & \longrightarrow & \pi_0\big(Z(\hat{I})^\Gamma\big)^D & x \mapsto \kappa_0 \end{array}$$

of [K2, 1.6] (with $1 \to G \to H \to I \to 1$ of [K2] denoted by $1 \to I \to I_1 \to C \to 1$ here) implies that $\chi(c^{-1}) = \kappa_0(d)$. Hence it suffices to show that $d = \mathrm{inv}(x, x')$. Choose $g_2 \in I_1(\bar{F})$ with $g_2 \mapsto c^{-1}$. Since $g_1, g_2^{-1} \in G_1(\bar{F})$ have image $c \in C(\bar{F})$, the exact sequence $1 \to G \to G_1 \to C \to 1$ produces $g \in G(\bar{F})$ with $g_1 = gg_2^{-1}$. Then $g^{-1}\tau(g) = g_2^{-1}\tau(g_2)$ ($\tau \in \Gamma$) is a 1-cocycle which represents $\mathrm{inv}(x, x')$ (using the left side) and also d (using the right side).

Using $g_1 \in G_1(F)$ and $f \in C_c^\infty\big(X(F)\big)$ we define $f_1(y) = f(g_1 y g_1^{-1})$. Then $f_1 \in C_c^\infty\big(X(F)\big)$, and $\Phi_{f_1}^\kappa(x) = \chi(c)\Phi_f^\kappa(x)$, where $c \in C(F)$ is the image of g_1, since $\chi(c^{-1}) = \kappa\big(\mathrm{inv}(x, x')\big)$. Note that $e\big(Z_G(x)\big) = e\big(Z_G(g_1 x g_1^{-1})\big)$, since $\mathrm{Int}(g_1)$ defines an F-isomorphism of these two groups.

Choose a maximal F-torus T in G with maximal split component, whose apartment contains x_{qs}. If T_1 is the centralizer of T in G, we have an exact sequence of unramified tori:

$$1 \to T \to T_1 \to C \to 1.$$

Denote by $\mathbf{T}$, $\mathbf{T}_1$ the unique extension of T, T_1 to tori over R. Then

$$1 \to \mathbf{T}(R) \to \mathbf{T}_1(R) \to \mathbf{C}(R) \to 1$$

is exact. Since χ is non-trivial on $\mathbf{C}(R)$, we may choose $g_1 \in \mathbf{T}_1(R)$ with image c in $\mathbf{C}(R)$ such that $\chi(c) \neq 1$. As above, $\Phi_{f_1}^\kappa(x) = \chi(c)\Phi_f^\kappa(x)$. This will be 0 once we show that $f_1 = f$. It suffices to consider f on $X(F)$ defined by $f\big(y\theta(y^{-1})\big) = \int u(yh)\,dh$ ($h \in G(F)/Z(F)$), where u is the characteristic function of $Z(E)K_E a K_E$, $a \in T(E)$. Thus f is the characteristic function of the set $\{zk_1 ak\theta(k_1 akz)^{-1}; k, k_1 \in K_E, z \in Z(E)\}$. Since $g_1 \in T_1(F)$ normalizes K_E and commutes with a, this set is the same as its conjugate under g_1. Hence $f_1 = f$ and the proposition follows. ■

9 Global Finiteness

Let E/F be a quadratic extension of number fields, G a connected reductive group over F, and $\psi\colon G_{\mathrm{qs}} \to G$ an inner twisting with a quasi-split G_{qs} over F. Under the assumption that $H^1\big(E_v/F_v, G_{\mathrm{qs}}(E_v)\big)$ is trivial for all v, the inner twisting ψ induces maps from the stable semi-simple classes in $X(F_v)$ to the stable semi-simple classes in $X_{\mathrm{qs}}(F_v)$, for each place v. Indeed, $x_v \in X(F_v)$ lies in an F_v-torus T_v in G_v, there is a unique (up to $G(\bar{F}_v)$-conjugacy) Γ_v-equivariant embedding $j_v\colon T_v \to T_{\mathrm{qs},v} \subset G_{\mathrm{qs},v}$, thus $x_{\mathrm{qs},v} = j_v(x_v) = j_v\big(\theta_v(x_v^{-1})\big) = \theta_v(x_{\mathrm{qs},v})^{-1} \in G_{\mathrm{qs}}(E_v)$ lies in $X_{\mathrm{qs}}(F_v)$ since $H^1\big(E_v/F_v, G_{\mathrm{qs}}(E_v)\big) = \{0\}$. We say that the $G(\mathbb{A})$-conjugacy class of a semi-simple $x \in X(\mathbb{A})$ comes from $x_{\mathrm{qs}} \in X_{\mathrm{qs}}(F)$ if every local component of x maps to the stable class of x_{qs}.

Proposition 9.1 *Let C be a compact subset of $X(\mathbb{A})$. Then there are only finitely many $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ which intersect C non-trivially and come from some semi-simple element of $X_{\mathrm{qs}}(F)$.*

Proof Suppose that G^{ss} is simply connected. Fix an injection $G \hookrightarrow \mathrm{GL}(n)$. Hence $X(F) \hookrightarrow X_n(F) = \{g\theta(g)^{-1}; g \in \mathrm{GL}(n, E)\}$ and $X(\mathbb{A}) \hookrightarrow X_n(\mathbb{A}) = \{g\theta(g)^{-1}; g \in \mathrm{GL}(n, \mathbb{A}_E)\}$. Denote by $\mathbb{A}_n$ the set of $(a_{n-1}, \dots, a_0)$ in $\mathbb{A}_E^{n-1} \times \mathbb{A}_E^\times$ with $a_i = \theta(a_{n-i})a_0$, $a_0 \neq 0$, $a_n = 1$

$(0 \leq i \leq n)$. Similarly introduce F_n. Consider the natural continuous map $X_n(\mathbb{A}) \to \mathbb{A}_n$ defined by the coefficients of the characteristic polynomial of an $n \times n$ matrix, $x \mapsto p_x(t) = \det(t - x) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, $a_0 = \det(-x)$, $a_n = 1$. If $x = \theta(x)^{-1}$, then $a_i = \theta(a_{n-i})a_0$.

Lemma 9.2 *It suffices to consider only those $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ which map to a fixed element of $\mathbb{A}_n$.*

Proof Let C' be the image of C under the composite map $X(\mathbb{A}) \to X_n(\mathbb{A}) \to \mathbb{A}_n$. Then $C' \cap F_n$ is both discrete and compact, hence finite. If x is a semi-simple element of $X(\mathbb{A})$ whose $G(\mathbb{A})$-conjugacy class intersects C and comes from some semisimple element of $X_{\text{qs}}(F)$, then the image of x in $\mathbb{A}_n$ lies in C' and in the image of $G(\bar{F}) \cap X(\mathbb{A}) = X(F)$, thus in $C' \cap F$. ■

Lemma 9.3 *It suffices to consider $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ which come from a fixed semisimple $x_{\text{qs}} \in X_{\text{qs}}(F)$.*

Proof There are only finitely many semisimple $G_{\text{qs}}(\bar{F})$-conjugacy classes in $G_{\text{qs}}(\bar{F})$ whose image under the composition $G_{\text{qs}}(\bar{F}) \stackrel{\psi}{\to} G(\bar{F}) \to \mathrm{GL}(n, \bar{F}) \to \bar{F}^n$ is a fixed element of $\bar{F}^n$. ■

Assume now that there is $x \in X(\mathbb{A})$ whose conjugacy class comes from x_{qs}; otherwise there is nothing to prove. Choose a subset K_X of $X(\mathbb{A})$ of the form $\{k\theta(k)^{-1}; k \in K\}$, where K is an open compact subgroup of $G(\mathbb{A}_{E,f})$. Choose a finite set V of places of F, including the infinite places such that:

(a) for all $v \notin V$, the group G is unramified at v, and K can be written as $K_v K^v$, where K_v is a hyperspecial maximal compact subgroup of $G(E_v)$, and K^v is a compact open subgroup of $G(\mathbb{A}^v_{E,f})$, where $\mathbb{A}^v_{E,f}$ denotes the ring of finite E adeles without v-component;

(b) for all $v \notin V$ the v-component x_v of x lies in $K_{X,v} = \{k\theta(k^{-1}); k \in K_v\}$, and $1-\alpha(x_v)$ is zero or a unit in $R_{\bar{F}}$ for every root α of G;

(c) C is contained in $\prod_{v \in V} X(F_v) \cdot \prod_{v \notin V} K_{X,v}$.

Note that x comes from $x_{\text{qs}} \in X_{\text{qs}}(F)$. Since $1 - \alpha(x_{\text{qs}}) \in \bar{F}$ is zero or a unit locally almost everywhere, we have that $1 - \alpha(x) \in \bar{F}$ is zero or a unit locally almost everywhere, hence there is V for which (b) is satisfied.

Let Y denote the set of $G(\mathbb{A})$-conjugacy classes in $X(\mathbb{A})$ which intersect C and come from $x_{\text{qs}} \in X_{\text{qs}}(F)$. At each place v of F, the stable conjugacy class of x_v contains only finitely many conjugacy classes. Any conjugacy class in Y contains an element x' with $x'_v = x_v$ for all $v \notin V$, by Proposition 8.1. Hence Y is finite, as required. ■

10 Transfer Factors

We shall now define the transfer factors whose existence is conjectured in Section 4 (and 6). Our definition coincides with that of [LS] when E/F is split. However, to make our generalized definition we have to redefine the transfer factor of [LS]. Our presentation of the transfer factor of [LS] is new and makes it more transparent to see its independence of the various auxiliary data used in its definition in [LS]. In particular we combine the factors Δ_{II} and Δ_2 of [LS] to a factor which we denote by $\chi_{G/H}$, whose definition does not involve the three choices (of $T_H \to T$, of a-data and of χ-data) made in [LS].

Further we observe that the cohomological factor Δ_1, which is the only factor which depends on the conjugacy class of x and not only on its stable conjugacy class as is the case with all other factors, in fact depends on the centralizer T of x in G, and not on x itself. The same is true for the factor Δ_I of [LS], whose role is to make the product $\Delta^{\mathrm{coh}}_{G/H} = \Delta_1\Delta_I$ independent of the three choices, especially of $T_H \to T$.

Let F be a local or global field and E a quadratic separable extension of F. Denote by θ a generator of $\mathrm{Gal}(E/F)$. Let G be a connected reductive F-group, and put $X(F) = \{x = g\theta(g)^{-1}; g \in G(E)\}$. Let (H, s, η) be an endoscopic triple for G (see [LS, (1.2)]), and define $X_H(F)$ using H instead of G. Let x_H and ${}_!x_H$ be strongly G-regular elements in $X_H(F)$ which are images of the elements x and ${}_!x$ in $X(F)$. Let T_H and ${}_!T_H$ be the centralizers of x_H and ${}_!x_H$ in H; these are F-tori. Fix admissible embeddings $T_H \to T_{\mathrm{qs}}$ and ${}_!T \to {}_!T_{\mathrm{qs}}$ in the quasi-split form G_{qs} of G. Denote by x_{qs} and ${}_!x_{\mathrm{qs}}$ the images of x_H and ${}_!x_H$ under these embeddings. Notations for tori and elements in G_{qs} could include the subscript qs. But to simplify the notations, in the definitions of $\Delta^{\mathrm{abs}}_{G/H}(x_H, x)$ and $\chi_{G/H}(x_H, x)$ below, T, x, *etc.*, signify T_{qs}, x_{qs}, *etc.*, but in the definition of $\Delta^{\mathrm{coh}}_{G/H}(x_H, x)$ we work with T, x, *etc.*, and not with the quasi-split image.

Denote by R the root system of T in G and choose a subset R_+ of positive roots, denote by $R^\vee$ the coroots, by Ω the Weyl group, by $R^\vee_H$ the subsystem of coroots from H, by R_H the subset of roots from H, by Ω_H the Weyl group generated by R_H and $R^\vee_H$. The analogous objects for ${}_!T$ will be denoted by ${}_!R$, ${}_!R^\vee$, ${}_!\Omega$, *etc.* Since $x \in T(E) \cap X(F)$ satisfies $\theta(x) = x^{-1}$, the product $\prod_{\alpha\in R}(\alpha(x) - 1)$ lies in F. We put

$$\Delta^{\mathrm{abs}}_{G/H}(x_H, x) = \Delta^{\mathrm{abs}}_G(x)/\Delta^{\mathrm{abs}}_H(x_H), \quad \Delta^{\mathrm{abs}}_G(x) = \Big|\prod_{\alpha\in R}\big(\alpha(x) - 1\big)\Big|^{1/2}.$$

Then $\Delta^{\mathrm{abs}}_G(x)$ depends only on the stable conjugacy class of x ($\Delta^{\mathrm{abs}}_{G/H}$ is Δ_{IV} of [LS]).

Let L be a Galois extension of F which contains E such that $\mathrm{Gal}(\bar{F}/L)$ fixes each α in R. Denote by R^{sym} the set of symmetric $\Gamma = \mathrm{Gal}(L/F)$-orbits in R; a Γ-orbit $\mathcal{O}$ is called *symmetric* if $\mathcal{O} = -\mathcal{O}$. Put $\mathcal{O}_+ = \mathcal{O} \cap R_+$. The free abelian group $X^{\mathcal{O}} = \mathbb{Z}[\mathcal{O}_+]$ on $\mathcal{O}_+$ is a Γ-module. Fix $\alpha \in \mathcal{O}_+$ and put $\Gamma_{+\alpha} = \{\sigma \in \Gamma; \sigma\alpha = \alpha\}$ and $\Gamma_{\pm\alpha} = \{\sigma \in \Gamma; \sigma\alpha = \pm\alpha\}$. Then $[\Gamma_{\pm\alpha} : \Gamma_{+\alpha}] = 2$, and $[F_{+\alpha} : F_{\pm\alpha}] = 2$, where $F_{+\alpha}$ is the subfield of L fixed by $\Gamma_{+\alpha}$, and $F_{\pm\alpha}$ is the subfield of L fixed by $\Gamma_{\pm\alpha}$.

Let χ_α be a character of $(EF_{\pm\alpha})^\times$ whose restriction to the norm subgroup $N_{F_{+\alpha}/F_{\pm\alpha}}F^\times_{+\alpha}$ is trivial, but its restriction to $F^\times_{\pm\alpha}$ is not (in the global case, χ_α is a character of $\mathbb{A}^\times_{EF_{\pm\alpha}}/(EF_{\pm\alpha})^\times N_{F_{+\alpha}/F_{\pm\alpha}}\mathbb{A}^\times_{+\alpha}$ which is non trivial on $\mathbb{A}^\times_{\pm\alpha}$).

Define $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$ on the orbit $\mathcal{O} = \Gamma \cdot \alpha$ of α. In particular, if $\sigma_\alpha \in \Gamma$ has $\sigma_\alpha\alpha = -\alpha$, then its restriction to $F_{+\alpha}$ generates $\mathrm{Gal}(F_{+\alpha}/F_{\pm\alpha})$, and hence $\chi_\alpha(x)\chi_{\sigma_\alpha\alpha}(x) = \chi_\alpha\big(x\sigma_\alpha^{-1}(x)\big) = 1$ on $x \in (EF_{\pm\alpha})^\times$, thus $\chi_{-\alpha} = \chi_\alpha^{-1}$ if $E = F_{\pm\alpha}$.

Put $X^\alpha = \mathbb{Z} \cdot \alpha$. It is a $\Gamma_{\pm\alpha}$-submodule of $X^{\mathcal{O}}$, and $X^{\mathcal{O}} = \mathrm{Ind}^{\Gamma}_{\Gamma_{\pm\alpha}} X^\alpha$. Define the torus $T^{\mathcal{O}}$ over F by $X^*(T^{\mathcal{O}}) = X^{\mathcal{O}}$, and the torus T^α over $F_{\pm\alpha}$ by $X^*(T^\alpha) = X^\alpha$. Then T^α is a one dimensional torus, anisotropic over $F_{\pm\alpha}$, split over F_α, and $T^{\mathcal{O}} = \mathrm{Res}_{F_{\pm\alpha}/F}\, T^\alpha$.

The inclusion $X^{\mathcal{O}} \to X^*(T)$ yields a homomorphism $T \to T^{\mathcal{O}}$ over F, hence a homomorphism $T(F) \to T^{\mathcal{O}}(F) = (\mathrm{Res}_{F_{\pm\alpha}/F}\, T^\alpha)(F) \simeq T^\alpha(F_{\pm\alpha})$ and $T(E) \to T^\alpha(F_{\pm\alpha}E)$. If x^α is the image in $T^\alpha(F_{\pm\alpha}E)$ of x in $T(E)$ then $\alpha(x) = \alpha(x^\alpha)$.

Denote by τ_α conjugation in $T^\alpha(F_{+\alpha})$ with respect to $T^\alpha(F_{\pm\alpha})$. Since the norm map $T^\alpha(F_{+\alpha}) \to T^\alpha(F_{\pm\alpha})$ is onto, we may write $x^\alpha = y^\alpha\tau_\alpha(y^\alpha)$, where y^α lies in $T^\alpha(F_{+\alpha}E)$. If σ_α denotes conjugation of $F_{+\alpha}$ over $F_{\pm\alpha}$, we have $\alpha(x) = \alpha(x^\alpha) = \alpha\big(y^\alpha\tau_\alpha(y^\alpha)\big) = \alpha(y^\alpha)/\sigma_\alpha\big(\alpha(y^\alpha)\big)$. Note that $\sigma_\alpha\alpha = -\alpha$. If $F_{+\alpha}E$ is a quadratic field extension of $F_{+\alpha}$ then θ is the automorphism of $F_{+\alpha}E$ over $F_{+\alpha}$, hence θ fixes α, namely $\theta\alpha = \alpha$, and $\chi_\alpha = \chi_{\theta\alpha} = \chi_\alpha \circ \theta^{-1}$. If $F_{\pm\alpha}E = F_{+\alpha}$ then $\theta = \sigma_\alpha$ and $\chi_{\theta\alpha} = \chi_{-\alpha} = \chi_\alpha^{-1}$. Put $\chi_{G/H}(x_H, x) = \chi_G(x)/\chi_H(x_H)$,

$$\chi_G(x) = \prod_{\mathcal{O}\subset R^{sym}} \chi_\alpha\Big(\big[\alpha(y^\alpha) - \sigma_\alpha\big(\alpha(y^\alpha)\big)\big]/\big[\alpha({}_!y^\alpha) - \sigma_\alpha\big(\alpha({}_!y^\alpha)\big)\big]\Big);$$

$\chi_H(x_H)$ is defined similarly, with R replaced by R_H and x by x_H. The product ranges over the symmetric Γ-orbits in R, and since $\chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}$, each factor $\chi_\alpha(*)$ is independent of the choice of a representative α in $\mathcal{O}_+ \subset \mathcal{O}$. Note that the fraction

$$\big[\alpha(y^\alpha) - \sigma_\alpha\big(\alpha(y^\alpha)\big)\big]/\big[\alpha({}_!y^\alpha) - \sigma_\alpha\big(\alpha({}_!y^\alpha)\big)\big]$$

is fixed by σ_α, hence it lies in $F_{\pm\alpha}E$, and χ_α can be evaluated at this element.

When E is contained in $F_{+\alpha}$ the fraction lies in $F_{\pm\alpha}^\times$ and χ_α is simply the non trivial character on $F_{\pm\alpha}^\times/N_{F_{+\alpha}/F_{\pm\alpha}}F_{+\alpha}^\times$. This is the case of [LS], where E splits over F (in which case we take E to be F in our definitions above).

Let us consider a global situation, and a place where $F_{+\alpha}/F_{\pm\alpha}$ splits. Dropping the place from the notations, we have $F_{+\alpha} = F_{\pm\alpha} \oplus F_{\pm\alpha}$, where $F_{\pm\alpha}$ is a local field. The character χ_α is trivial on $N_{F_{+\alpha}/F_{\pm\alpha}}F_{+\alpha}^\times$, which is $F_{\pm\alpha}^\times$ embedded diagonally in $F_{+\alpha}^\times$. Then $\chi_\alpha(u,v) = \chi'_\alpha(u)\chi'_{-\alpha}(v)$ $(u, v \in F_{\pm\alpha}^\times)$, where $\chi'_{-\alpha} = 1/\chi'_\alpha$. The conjugation τ_α of $T^\alpha(F_{+\alpha})$ over $T^\alpha(F_{\pm\alpha})$ takes $y = (y_1, y_2) \in T^\alpha(F_{\pm\alpha}) \times T^\alpha(F_{\pm\alpha})$ to $\tau_\alpha y = (\tau'_\alpha y_2, \tau'_\alpha y_1)$, where τ'_α is the action of the Weyl group on $T^\alpha(F_{\pm\alpha})$. The norm map $T^\alpha(F_{+\alpha}) \to T^\alpha(F_{\pm\alpha})$ is onto and it takes the form $y^\alpha = (y_1^\alpha, y_2^\alpha) \mapsto x^\alpha = (y_1^\alpha\tau'_\alpha y_2^\alpha, y_2^\alpha\tau'_\alpha y_1^\alpha)$, with $y_i^\alpha \in T^\alpha(F_{\pm\alpha})$.

Recall that $T(E) \to T^\alpha(F_{\pm\alpha}E)$, $x \mapsto x^\alpha = y^\alpha\tau_\alpha(y^\alpha)$, has

$$\alpha(x) = \alpha(x^\alpha) = \alpha\big(y^\alpha\tau_\alpha(y^\alpha)\big) = \alpha(y^\alpha)/\sigma_\alpha\big(\alpha(y^\alpha)\big).$$

The factor $\chi_\alpha\Big(\alpha(y^\alpha) - \sigma_\alpha\big(\alpha(y^\alpha)\big)\Big)$ is the product of $\chi'_\alpha\big(\alpha(y_1^\alpha) - \alpha(y_2^\alpha)\big)$ and $\chi'_{-\alpha}\big(\alpha(y_2^\alpha) - \alpha(y_1^\alpha)\big)$, which is $\chi'_\alpha(-1)$. It is equal of course to the same factor with y^α replaced by ${}_!y^\alpha$. For this reason only symmetric Γ-orbits occur in $\chi_{G/H}(x_H, x)$. The term $\chi_{G/H}(x_H, x)$ is the product of the terms Δ_{II} and $\Delta_2 = \Delta_{III_2}$ of [LS].

The only term in Δ of [LS] which depends on the conjugacy class of x and not only on its stable conjugacy class is $\Delta_1 = \Delta_{III_1}$, but in fact this factor does not depend on x as much as on its centralizer T. It is $\langle \mathrm{inv}(T_H, T; {}_!T_H, {}_!T), \mathbf{s}_U\rangle$ in the notations of [LS, p. 246], where we replace γ of [LS] by T, and $\bar\gamma$ by ${}_!T$. However, replacing $T_H \to T$ and ${}_!T_H \to {}_!T$ by g and ${}_!g$ conjugates, this factor is multiplied by $\langle g_T, s_T\rangle/\langle g_{!T}, s_{!T}\rangle$ (see [LS, Lemma 3.4.A]). Consequently [LS] multiply Δ_1 by Δ_I, which is $\langle \lambda_{\{a\}}(T_{\mathrm{sc}}), s_T\rangle/\langle \lambda_{\{a\}}({}_!T_{\mathrm{sc}}), s_{!T}\rangle$ (see [LS, (3.2)]); by [LS, Lemma 3.2.B] the product is independent of a replacement of $T_H \to T$ by a g-conjugate. Moreover, the definition of $\lambda_{\{a\}}$ depends on a choice of an "a-data" (see [LS, line before (2.3.1)]), although $\{a\}$ no longer appears in the notations of [LS, (3.2)]. However, [LS, Lemma 3.2.C] guarantees that the quotient $\langle \lambda_{\{a\}}(T_{\rm sc}), s_T\rangle / \langle \lambda_{\{a\}}({}_!T_{\rm sc}), s_{!T}\rangle$ is independent of the choice of the a-data $\{a\}$. We put

$$\Delta^{\rm coh}_{G/H}(x_H, x) = \langle \mathrm{inv}(T_H, T; {}_!T_H, {}_!T), \mathbf{s}_U\rangle \cdot \langle \lambda_{\{a\}}(T_{\rm sc}), s_T\rangle / \langle \lambda_{\{a\}}({}_!T_{\rm sc}), s_{!T}\rangle$$

for the cohomological transfer factor ($\Delta_1\Delta_I$ in [LS]). It is obviously invariant under the replacements (A), (B) of [LS, p. 241], but depends on the conjugacy class of T, not only on its stable conjugacy class. Our transfer factor is then

$$\Delta_{G/H}(x_H, x) = \Delta^{\rm coh}_{G/H}(x_H, x)\chi_{G/H}(x_H, x)\Delta^{\rm abs}_{G/H}(x_H, x).$$

11 Fundamental Lemma For SL(2)

Let us verify the fundamental lemma when $G = \mathrm{SL}(2)$. Put $X(F) = \{x = g\theta(g)^{-1}; g \in G(E)\}$, where θ is the non trivial automorphism of the two dimensional commutative semi simple algebra E over the local field F. We consider an element x in $T(E) \cap X(F)$, where T is an F-torus in G. Up to stable conjugacy, T is T_D, where $T_D(A) = \left\{ \left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right) \in \mathrm{SL}(2, A)\right\}$ for any commutative F-algebra A, and $D \in F^\times$ (the torus T_D is split if $D \in F^{\times 2}$, non split otherwise). The torus T_D splits over $F_D = F(\sqrt{D})$. Then

$$x = g\theta(g)^{-1} = \begin{pmatrix} A & BD \\ B & A \end{pmatrix} = \begin{pmatrix} \theta A & -\theta BD \\ -\theta B & \theta A \end{pmatrix} \quad (A, B \in E; \det x = 1)$$

has the eigenvalues $x_\pm = A \pm B\sqrt{D}$. So $x_+ - x_- = 2B\sqrt{D}$, $\theta B = -B$, and

$$(x_+ - x_-)/({}_!x_+ - {}_!x_-) = B/{}_!B \in F^\times,$$

where ${}_!x$ is a fixed regular element, and

$$\left(\frac{x_+}{x_-} - 1\right) \cdot \left(\frac{x_-}{x_+} - 1\right) = \frac{2B\sqrt{D} \cdot (-2B\sqrt{D})}{x_+ x_-} = -4B^2 D \in F^\times.$$

Let χ_D be the character of $F^\times$ whose kernel is the norm subgroup $NF_D^\times = N_{F_D/F}F_D^\times$. Then the transfer factor $\Delta^{\rm abs}\chi$ is

$$\chi_D\Big(\frac{x_+ - x_-}{{}_!x_+ - {}_!x_-}\Big)\Big|\Big(\frac{x_+}{x_-} - 1\Big) \cdot \Big(\frac{x_-}{x_+} - 1\Big)\Big|^{1/2} = \chi_D(B/{}_!B)|4B^2D|^{1/2}.$$

The case where $E = F \oplus F$ reduces to that of standard conjugacy. Here $g = (g_1, g_2)$, and

$$x = (g_1 g_2^{-1}, g_2 g_1^{-1}) = \left(\begin{pmatrix} a_1 & b_1 D \\ b_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & -b_1 D \\ -b_1 & a_1 \end{pmatrix}\right) \quad a_1, b_1 \in F,$$

$x_\pm = a_1 \pm b_1\sqrt{D}$, and the transfer factor is $\chi_D(b_1/_!b_1)|4b_1^2D|^{1/2}$, since $B = (b_1, -b_1)$, $B^2 = (b_1^2, b_1^2) \in F^\times$, $B/_!B = (b_1/_!b_1, b_1/_!b_1) = b_1/_!b_1 \in F^\times$.

We then restrict attention to the case where E is a quadratic field extension of F. The case where $D \in F^{\times 2}$, namely the torus T_D/F is split over F, has $\chi_D = 1$, and the stable bi-conjugacy class of x consists of a single bi-conjugacy class, which can in fact be represented by a diagonal element. This case is easily handled by a standard change of variables.

Suppose then that both E and F_D are fields, and E/F is an unramified quadratic extension. The field F_D is a ramified quadratic extension of F, or $F_D = E$. In both cases, the κ-orbital integral is a weighted (by $\chi_D(\rho)$) sum over $\{1, \rho\} \in F^\times/NF_D^\times$ (take $\rho \in R^\times - R^{\times 2}$ if F_D/F is ramified, ρ a uniformizer $\boldsymbol{\pi}$ of the maximal ideal in the ring R of integers of F if $F_D = E$), of the form

$$\int_{T_D(F)\backslash G(F)} 1_{K_E}(g^{-1}xg)\,dg - \int_{T_D^\rho(F)\backslash G(F)} 1_{K_E}(g^{-1}x^\rho g)\,dg$$
$$= \int_{T_D'(F)\backslash \mathrm{GL}(2,F)} \chi_D(\det g) 1_{K_E'}(g^{-1}xg)\,dg$$

where $x^\rho = \begin{pmatrix} A & BD\rho \\ B/\rho & A \end{pmatrix}$, $T_D' = \left\{\begin{pmatrix} a & bD \\ b & a \end{pmatrix} \in \mathrm{GL}(2)\right\}$, $K_E = \mathrm{SL}(2, R_E)$, $K_E' = \mathrm{GL}(2, R_E)$. The last integral is clearly zero when F_D/F is ramified, as $\chi_D(\rho) = -1$ for a unit ρ. There remains the case of $F_D = E$, unramified over F.

Put $r = r_m = \mathrm{diag}(1, \boldsymbol{\pi}^m)$. Then $\mathrm{GL}(2, F) = \cup_{m \geq 0} Tr_mK$, where $T = T_D'(F)$. Further $K \cap r^{-1}Tr \simeq R_D(m)^\times$, where $R_D(m) = \{a + b\sqrt{D}; |a| \leq 1, |b| \leq |\boldsymbol{\pi}|^m\}$, $R_D = R_D(0)$. Putting $q_D = q^2$ in the unramified case, where $q|\boldsymbol{\pi}| = 1$, we have

$$|T \setminus TrK| = |K \cap r^{-1}Tr \setminus K| = [R_D^\times : R_D(m)^\times] = \frac{[R_D^\times : 1 + \boldsymbol{\pi}^m R_D]}{[R_D(m)^\times : 1 + \boldsymbol{\pi}^m R_D]}$$
$$= \frac{(q_D - 1)q_D^{m-1}}{(q-1)q^{m-1}} = (1 + q^{-1})q^m$$

if $m \geq 1$, and $= 1$ if $m = 0$. If $B = u\boldsymbol{\pi}^\beta$, $u \in R_D^\times$, then our integral is

$$\sum_{m \geq 0} [R_D^\times : R_D(m)^\times](-1)^m 1_K\left(r_m^{-1}\begin{pmatrix} A & BD \\ B & A \end{pmatrix} r_m\right)$$
$$= 1 + \sum_{1 \leq m \leq \beta} (-1)^m (1 + q^{-1}) q^m$$
$$= 1 - (q+1) \sum_{0 \leq m < \beta} (-q)^m (-q)^\beta = \chi_D(B)|B|^{-1},$$

and our verification of the fundamental lemma for $\mathrm{SL}(2, E)/\mathrm{SL}(2, F)$ is complete.

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