CYCLIC AUTOMORPHIC FORMS ON A UNITARY GROUP

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Contents.

1. Statement of Results.

Let $\mathbf{C}$ be an algebraic subgroup of an algebraic group $\mathbf{G}$ over a global field F, and denote by $\phi: \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$ a cusp form in the cuspidal representation π of the adele group $\mathbf{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of F-adeles. Denote by $\mathbf{Z}$ the center of $\mathbf{G}$ and by $\omega = \omega_\pi : \mathbf{Z}(\mathbb{A})/\mathbf{Z}(F) \to \mathbb{C}^\times$ the central character of π. Suppose that the homogeneous space $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ has finite volume; we call it a *cycle*. Fix a unitary character ξ of $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ in $\mathbb{C}^\times$. Signify by $P(\phi) = P_{\mathbf{C},\xi}(\phi) = \int_{\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})} \phi(c)\overline{\xi}(c)dc$ the ξ-*period* of the form ϕ on the cycle $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ of $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$. Here $\overline{\xi}(c)$ is the complex conjugate of $\xi(c)$. We simply say "period" when $\xi = 1$.

Motivated for example by classical questions on the L^2-cohomology of bounded symmetric spaces of the form , $\backslash G/C$, we wish to determine the ξ-*cyclic* cuspidal $\mathbf{G}(\mathbb{A})$-modules π; these are the cuspidal π for which there is a form $\phi \in \pi \subset L^2_\omega(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ with a non-zero ξ-period $P(\phi)$ (or $P_{\mathbf{C},\xi}(\phi)$ if the dependence on $\mathbf{C}$ and ξ needs to be made explicit). The interesting phenomenon which occurs in this context is, that in order to be cyclic, a global representation needs – in addition to being locally cyclic at all places – to overcome a purely global obstruction.

Such a question was studied first by Waldspurger [W1/2] when $\mathbf{G} = PGL(2) = SO(3)$ and $\mathbf{C} = SO(2)$ (=elliptic torus of $\mathbf{G}$ which splits over a quadratic extension of the base field) on using the Weil representation, then by Harder-Langlands-Rapoport [HLR] when $\mathbf{G}$ is $GL(2)$ over a quadratic extension E of F and $\mathbf{C}$ is $GL(2)$ over F, and then by Jacquet-Lai [JL] and Jacquet [J1/2] who introduced a "relative trace formula". The case of $\mathbf{G} = SO(4) \times SO(3)$ and

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$\mathbf{C} = SO(3)$ (and ξ is a form in a cuspidal, not one-dimensional, automorphic representation of $\mathbf{C}(\mathbb{A})$), was studied by (D. Prasad [P1/2] locally and) Harris-Kudla [HK], again on using the Weil representation. The analogue of the trace formula technique was later used in a series of cases, where $\mathbf{G} = GL(2,E)$ [F5], or where $\mathbf{G} = GL(n,E)$ [F6], and $\mathbf{C} = GL(n,F)$, with E/F being a quadratic extension of global (or local) fields. This last case is related to base-change for the unitary group. Local aspects of the dual case, where $\mathbf{G} = GL(n,E)$ and $\mathbf{C}$ is a unitary group – which is related to base change for $GL(n)$ – are discussed e.g. in Ye [Y]. For $\mathbf{G} = GSp(4)$ and $\mathbf{C} = SO(2,2)$ – a case related to counter examples to the Ramanujan conjecture – local (and global) aspects are studied in Zinoviev [Zi] (and in [FM2]).

The question of classifying the cyclic representations is too vast to even currently attempt proposing a general conjectural answer. Yet the examples of [W1/2] and [HK] suggest that for a suitable $\mathbf{C}$ the purely global obstruction involves the non-vanishing of an L-function at the middle of the critical strip when the representation ξ of $\mathbf{C}(\mathbb{A})$ is cuspidal, while the examples of [F5/6/7] suggest that this global obstruction should be stated in terms of liftings in the opposite extreme case where ξ is a one-dimensional automorphic non-cuspidal representation of $\mathbf{C}(\mathbb{A})$ (note that in the case of [W1/2], every irreducible representation of $\mathbf{C}(\mathbb{A})/\mathbf{C}(F)$ is both cuspidal and one-dimensional).

Our question is a global, arithmetic or automorphic, variant of the classical question concerning the constituents of the restriction of a representation of G to its subgroup C. The "classical" question – concerning the non-vanishing of $\operatorname{Hom}_C(\pi,\xi)$ – was treated in many cases, for example in Zhelobenko [Zh] ("the Gelfand-Cetlin basis") with $\xi = 1$ and compact real groups such as $G = U(n,\mathbb{R})$, $C = U(n-1,\mathbb{R})$, or $G = SO(n,\mathbb{R})$, $C = SO(n-1,\mathbb{R})$, in Thoma [Th] and Zelevinsky [Z2] (the "branching rule") with the finite groups $G = GL(n,q)$, $C = GL(n-1,q)$ ([Th] and [Z2] consider any representation ξ of C, not only one dimensional ξ), and in van Dijk, Poel [DP] with unitary representations of $G = GL(n,\mathbb{R})$ and $\xi = 1$ on $C = GL(n-1,\mathbb{R})$. The global question concerns not only the non-vanishing of $\operatorname{Hom}_{\mathbf{C}(\mathbb{A})}(\pi,\xi)$, but rather the non-vanishing of an explicitly constructed element in this space, namely $P = P_{\mathbf{C},\xi}$.

The purpose of this paper is to answer our question in a situation not involving just $GL(2)$ or base change, on using the promising "trace formula" type technique (suited to the case of $\dim\xi = 1$). This technique is based on an application of a global "Fourier summation formula", (as in [J], [F5], [F6]), which is a special case of the "relative trace formula". This formula is analogous to the standard trace formula, where the kernel $K_f(x,y)$ of the convolution operator $r(f)$ is integrated on the diagonal $x = y$. But it involves no traces. It is a summation formula. To obtain the Fourier summation formula one integrates $K_f(x,y)$ over x in a "cycle", and y over a unipotent subgroup, so that the formula involves both "periods" of the cusp form, and its Fourier coefficients. Another useful case of the "relative trace" formula is obtained when x and y both range over the cycle (see [JL], [FH], [F10]). We refer to this case as the bi-period summation formula.

This method is entirely different from the theta liftings techniques used by [W1/2] and [HK]. The method is perhaps even more interesting than the results, for its intrinsic simplicity and directness. The results follow from a natural comparison. They are not accidental. In the case of the particular example considered in this paper, perhaps stronger results can be obtained at present using the older technique of theta lifting (see, e.g., [GP]). But the potential of the Fourier summation formula is immense. It draws together and extends techniques from the theories of the trace formula, Whittaker functions, B-orbits on symmetric varieties, and it applies in cases where the theta lifting does not (see, e.g., [F6]).

In this paper we carefully study both the global and the local technical aspects of this method, treating general and spherical functions, at the split and non-split places. The technical computations concerning orbital integrals of spherical functions are relegated to [F8]. Interesting identities of "Fourier orbital integrals" and "Whittaker-Period" distributions are obtained. Applications concerning cyclicity of local representations are also obtained (from the global results). In an attempt to understand [J1], p. 211, we introduced (the standard) truncation in [F6] and [F7] to develop the summation formula. The formal discussion there is supplemented here with a detailed handling of convergence questions, analogous to that of [FM2].

We proceed now to describe the representation theoretic consequences of this work. For the possibly more interesting technical aspects of the Fourier summation formula, and the Fourier orbital integrals, the reader should consult the corresponding sections. The groups $\mathbf{G}$ and $\mathbf{C}$ will be taken here to be the quasi-split unitary groups $U(3,E/F)$ and $U(2,E/F)$ in three and two variables. To define them, let E/F be a fixed quadratic (separable) extension of global fields, of characteristic other than 2, and denote by $\mathrm{Gal}(\overline{F}/F)$ the galois group of a separable algebraic closure $\overline{F}$ of F with $\overline{F}\supset E$. The group $\mathbf{G}=U(3,E/F)$ over F is defined by its group $\mathbf{G}(\overline{F})=GL(3,\overline{F})$ of $\overline{F}$-valued points, and the galois action $\tau(a_{ij})=(\tau a_{ij})$, $a_{ij}\in\overline{F}$ $(1\le i,j\le 3)$, if $\tau\in\mathrm{Gal}(\overline{F}/E)\subset\mathrm{Gal}(\overline{F}/F)$, and $\tau(a_{ij})=\mathcal{J}{}^t(\tau a_{ij})^{-1}\mathcal{J}^{-1}$ if $\tau\in\mathrm{Gal}(\overline{F}/F)-\mathrm{Gal}(\overline{F}/E)$, where $\mathcal{J}=\begin{pmatrix}0&&1\\&-1&\\1&&0\end{pmatrix}$.

If $\mathbf{Z}$ denotes the center of $\mathbf{G}$ then $\mathbf{Z}(F)=U(1,E/F)=E^\bullet=\{a\in E^\times;a\overline{a}=1\}$, and $\mathbf{Z}(\mathbb{A})=U(1,E/F)_{\mathbb{A}}=\mathbb{A}_E^\bullet=\{a\in\mathbb{A}_E^\times;a\overline{a}=1\}$, where $a\mapsto\overline{a}$ indicates the action of the non-trivial element of $\mathrm{Gal}(E/F)$. Here ${}^tg$ indicates the transpose of $g\in\mathbf{G}(\overline{F})$. The F-subgroup $\mathbf{C}$ of $\mathbf{G}$ is taken to be the $U(2,E/F)$-factor in the centralizer $\mathbf{C}_{\mathcal{J}_0}$ of $\mathcal{J}_0=\begin{pmatrix}1&&0\\&-1&\\0&&1\end{pmatrix}$ in $\mathbf{G}$. The center of $\mathbf{C}$ is isomorphic to $\mathbf{Z}$, and the centralizer $\mathbf{C}_{\mathcal{J}_0}$ is $\simeq\mathbf{C}\times\mathbf{Z}$. Fix a unitary character $\boldsymbol{\xi}$ of $\mathbb{A}_E^\bullet/E^\bullet$, and put $\xi(c)=\boldsymbol{\xi}(\det c)$ for $c\in\mathbf{C}(\mathbb{A})$.

The global representation theoretic application of this paper is the determination of the cuspidal representations π of $\mathbf{G}(\mathbb{A})$ (with central character $\omega:\mathbf{Z}(\mathbb{A})/\mathbf{Z}(F)\to\mathbb{C}^\times$) which contain a form $\phi\in\pi\subset L^2_\omega(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ with a non-zero period $P(\phi)=P_{\mathbf{C},\xi}(\phi)$ on the cycle defined by the subgroup $\mathbf{C}$. Such $\mathbf{G}(\mathbb{A})$-modules π will be called here ξ-*cyclic*. In [F9] – using again the extensive computations of the Fourier orbital integrals developed here – we consider $\mathbf{G}(\mathbb{A})$-modules with a non-zero period with respect to anisotropic forms of $\mathbf{C}$, and compare them with the cyclic modules studied here.

We could equivalently consider $\mathbf{C}_{\mathcal{J}_0}$ and the character $(c,b)\mapsto\xi(c)(\omega/\boldsymbol{\xi}^2)(b)$ $(b\in\mathbb{A}_E^\bullet/E^\bullet$, $c\in\mathbf{C}(\mathbb{A})/\mathbf{C}(F))$, instead of ξ and $\mathbf{C}$. This will not change the value of $P(\phi)$, but will complicate the notations. Analogous objects are named "distinguished" in the situations considered in [F5] and [F6] – and we keep using this title in these cases – but the name ξ-cyclic has the advantage of indicating the property which distinguishes π, hence we use it in the situation studied in this paper. In fact, our proofs are carried out below only for $\xi = 1 = \omega$, to ease the notations. As suggested by J. Bernstein, we state the general case of any ξ, ω in this introduction, in order to have a clearer picture of the results.

The determination of the ξ-cyclic π will be stated in terms of a lifting to $\mathbf{G}$ from the F-group $\mathbf{H}_0 = U(2, E/F)$ (which is defined using a different form (w below) than $\mathbf{C} = U(2, E/F)$ above), defined – analogously to $\mathbf{G}$ – by $\mathbf{H}_0(\overline{F}) = GL(2, \overline{F})$, $\tau(a_{ij}) = (\tau a_{ij})$ if $\tau \in \operatorname{Gal}(\overline{F}/E)$, $a_{ij} \in \overline{F}$ $(1 \le i, j \le 2)$, and

$$\tau(a_{ij}) = w^t(\tau a_{ij})^{-1} w^{-1}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{if} \quad \tau \in \operatorname{Gal}(\overline{F}/F) - \operatorname{Gal}(\overline{F}/E).$$

The lifting of global representations is defined locally, in terms of almost all components, as in [F2]. Let v be a place of F which stays prime and is unramified in E. Denote by F_v the completion of F at v, and put $E_v = E \otimes_F F_v$. Any irreducible unramified representation of $H_{0v} = \mathbf{H}_0(F_v)$ is the unique such constituent in the composition series of an induced H_{0v}-module of the form $I_0(\mu_v)$, where $\mu_v : E_v^\times \to \mathbb{C}^\times$ is an unramified character, uniquely determined up to the equivalence relation $\mu_v \sim \overline{\mu}_v^{-1}$, where $\overline{\mu}_v(a) = \mu_v(\overline{a})$, $a \in E_v^\times$. The space of $I_0(\mu_v)$ consists of the smooth functions φ_v from H_{0v} to $\mathbb{C}$ with

$$\varphi_v\left(\begin{pmatrix} a & * \\ 0 & \overline{a}^{-1} \end{pmatrix} h\right) = |a|_{E_v}^{1/2} \mu_v(a) \varphi_v(h) \qquad (h \in H_{0v},\ a \in E_v^\times).$$

Similarly, any irreducible unramified $G_v = \mathbf{G}(F_v)$-module is the unique such constituent in the composition series of an induced G_v-module of the form $I(\mu_v)$, where the unramified character $\mu_v : E_v^\times \to \mathbb{C}^\times$ is uniquely determined up to the relation $\mu_v \sim \overline{\mu}_v^{-1}$. The space of $I(\mu_v)$ consists of the smooth $\varphi_v : G_v \to \mathbb{C}$ with

$$\varphi_v\left(\begin{pmatrix} a & & * \\ & b & \\ 0 & & \overline{a}^{-1} \end{pmatrix} g\right) = |a|_{E_v} \mu_v(a) \left(\frac{\omega_v}{\mu_v}\right)(b) \varphi_v(g) \qquad (g \in G_v, a \in E_v^\times, b \in E_v^\bullet).$$

Here ω_v is the component of ω at v.

Let $\kappa_v : E_v^\times \to \mathbb{C}^\times$ be the unramified character of $E_v^\times$ whose value at a local uniformizer $\boldsymbol{\pi}_v$ (of the maximal ideal in the ring of integers R_v of F_v, or R'_v of E_v) is -1. The *local κ_v-endoscopic lifting* in the non-split unramified case is defined as in [F2] to map the irreducible unramified constituent of $I_0(\mu_v)$ to the unramified irreducible constituent of $I(\mu_v \kappa_v)$.

In the case when the place v of F splits in E, we have that $E_v = E \otimes_F F_v$ is $F_v \oplus F_v$, and $H_{0v} = \mathbf{H}_0(F_v) = GL(2, F_v)$, $G_v = \mathbf{G}(F_v) = GL(3, F_v)$. The κ_v-endoscopic lifting is defined as in [F2] in this case simply by mapping ρ_v to $I(\rho_v \otimes \kappa_v \times \omega_v/\kappa_v^2 \omega_{\rho_v})$, the $GL(3, F_v)$-module normalizedly induced from the representation of a maximal parabolic, defined by $\rho_v \otimes \kappa_v$ on the 2×2 block of the Levi factor, by $\omega_v/\omega_{\rho_v}\kappa_v^2$ on the 1×1 factor, and by 1 on the unipotent radical. Of course the lifting depends on a choice of a character κ_v of $F_v^\times$. Since ρ_v is unitarizable and κ_v, ω_v are unitary, the induced $GL(3, F_v)$-module is irreducible.

For the global lifting, we say that the cuspidal representation $\rho = \otimes\rho_v$ of $\mathbb{H}_0 = \mathbf{H}_0(\mathbb{A})$ *κ-endo-lifts* to the cuspidal representation $\pi = \otimes\pi_v$ of $\mathbb{G} = \mathbf{G}(\mathbb{A})$ if ρ_v lifts to π_v for almost all places v of F. The lifting depends on a choice of a character $\kappa : \mathbb{A}_E^\times/E^\times N_{E/F}\mathbb{A}_E^\times \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}_F^\times/F^\times$ is non-trivial.

Very detailed results about this "endoscopic" lifting were obtained in [F2/3/3′], on developing simultaneously the theory of base change from $\mathbf{G} = U(3, E/F)$ to $GL(3, E)$ and using the (twisted) trace formula. The representation theoretic applications of this paper include establishing the existence of the endoscopic lifting for generic (or non-one-dimensional) representations independently of [F3], proving in particular the existence of a generic element in the packet – a notion introduced in [F3] – obtained from this endoscopic lifting. Then we use an early result of [F3] to characterize the image as the set of packets of generic cyclic representations, and to derive the local representation theoretic results.

To dissipate some misconceptions, note that the problem of studying the endoscopic lifting from $U(2)$ to $U(3)$ was raised by R. Langlands [L]. An attempt at this problem was made in reference [25] of [L] (= [Rogawski] in [GP]), but as explained in [F2], §4.6, p. 562/3, this attempt – based on stabilizing the trace formula for $U(3)$ alone – was conceptually insufficient for that purpose. The preprint "L-packets and liftings for $U(3)$" (reference [Flicker] in [GP], and [2] in [A3], and p. −2 in [L]) proposed studying the endoscopic lifting from $U(2)$ to $U(3)$ simultaneously with base-change from $U(3)$ to $GL(3, E)$ by means of the twisted trace formula. It introduced a definition of packets, and reduced a complete description of these packets – as well as the lifting from $U(2)$ to $U(3)$ and $U(3)$ to $GL(3, E)$ – to important technical assumptions, later proven by Langlands and others (twisted trace formula, transfer of orbital integrals). Moreover, rigidity and multiplicity one theorem for $U(3)$ were reduced to the assertions of [GP], which was written later than our preprint. The papers [F2/3] contain a much improved exposition of the preliminary preprint. The paper [F3′] contains a new technique, based on the usage of Iwahori-regular functions. It affords a proof of a trace formula identity for *all* test functions – thus extending the results of [F2/3] to all representations of $U(3)$ – by simple means. Later, an alternative exposition for these results – but not for [F3′] – was published by Rogawski (Ann. of Math. Studies (1990)), who later corrected a mistake in the computation of the multiplicities of the non-tempered discrete series representations in [F3]. The final, representation theoretic part, of the present paper, uses results of [F2/3/3′] in the proof of Proposition 22, but it relies on a Fourier summation formula rather than on a trace formula.

Our main global results are the identification of the cyclic generic modules, a new proof of the endoscopic lifting from $U(2)$ to $U(3)$, and a new characterization of the image of this lifting in terms of cyclicity, and not in terms of base change to $GL(3, E)$ as in [F3]. Our work involves a panoply of techniques. We develop and apply the "Fourier" summation formula. We develop a theory of asymptotic expansion, and of matching "Fourier" orbital integrals. A very detailed and careful analysis of these integrals for spherical functions is carried out in [F8] in the non-split and split cases.

Since we work only with generic representations of $\mathbf{H}_0(\mathbb{A})$ and $\mathbf{G}(\mathbb{A})$, let us recall the definition. Let $\boldsymbol{\psi} : \mathbb{A}/F \to \mathbb{C}^\times$ be a non-trivial additive character. Denote by ψ the character of $\mathbf{U}(\mathbb{A})/\mathbf{U}(F)$ ($\mathbf{U}$ is the unipotent upper triangular subgroup of $\mathbf{G}$) whose value is $\boldsymbol{\psi}(a + \overline{a})$

at the element

$$u = \begin{pmatrix} 1 & a & \frac{1}{2}a\overline{a}+b \\ 0 & 1 & \overline{a} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{of} \quad \mathbf{U}(\mathbb{A}) \qquad (b \in \mathbb{A}_E,\ b+\overline{b}=0;\ a \in \mathbb{A}_E).$$

Then the cuspidal $\mathbf{G}(\mathbb{A})$-module π is called *generic* if there is a form $\phi \in \pi$ with non-zero

$$W_\psi(\phi) = \int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} \phi(u)\overline{\psi}(u)du.$$

Similarly, let $\mathbf{U}_{H_0}$ denote the unipotent upper triangular subgroup of $\mathbf{H}_0$. Then $\mathbf{U}_{H_0}(F)$ $= \{u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}; x \in F\}$. Denote by ψ the character of $\mathbf{U}_{H_0}(\mathbb{A})/\mathbf{U}_{H_0}(F)$ whose value at $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{A}$, is $\psi(x)$. The cuspidal $\mathbf{H}_0(\mathbb{A})$-module ρ is called *generic* if there is a form $\phi \in \rho$ with non-zero

$$W_\psi^{H_0}(\phi) = \int_{\mathbf{U}_{H_0}(F)\backslash\mathbf{U}_{H_0}(\mathbb{A})} \phi(u)\overline{\psi}(u)du.$$

Our qualitative global lifting result, characterizing the cyclic modules, is the following (see Proposition(s 21 and 22, and) 27). It uses the local "multiplicity one" theorem of J. Bernstein, which asserts that on an irreducible admissible $PGL(3,F_v)$-module (resp. $PU(3,E_v/F_v)$-module) there exists at most one (up to a scalar multiple) non-zero $GL(2,F_v)$-invariant (resp. $U(2,E_v/F_v)$-invariant) $\mathbb{C}$-valued linear form. A proof of this is recorded in the Appendix at the end of this paper. Our quantitative results concern identities of Fourier orbital integrals, Fourier summation formulae, and Whittaker-Period distributions. Recall that the notion of a packet is introduced in [F3], locally and globally.

Global Theorem. *Let E/F be a quadratic extension of global fields of characteristic not equal to two, ξ, ω unitary characters of $\mathbb{A}_E^\bullet/E^\bullet$, and κ a unitary character of $\mathbb{A}_E^\times/E^\times N_{E/F}\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}^\times/F^\times N\mathbb{A}_E^\times$ is non-trivial. Suppose that π is a ξ-cyclic generic cuspidal representation of the quasi-split unitary group $G(\mathbb{A}) = U(3,E/F)_\mathbb{A}$ with central character ω. Then (the packet of) π is the endoscopic κ-lift of (the packet of) a generic cuspidal representation ρ of the quasi-split unitary group $\mathbf{H}_0(\mathbb{A}) = U(2,E/F)_\mathbb{A}$ in two variables, whose central character is $\omega/\xi\kappa$. Conversely, any generic cuspidal representation ρ of $\mathbf{H}_0(\mathbb{A})$ with central character $\omega/\xi\kappa$, lifts (κ-endoscopically) to (the packet of) a generic cuspidal ξ-cyclic $\mathbf{G}(\mathbb{A})$-module π with central character ω.*

Our proof that a generic cuspidal ρ lifts to a generic cuspidal π implies in particular that the packet of the π which are lifted from such ρ on $U(2,E/F)_\mathbb{A}$ contains a generic element. Our proof here that a cyclic generic π corresponds to a ρ does not use the work of [GP]. But most of our qualitative results can also be derived from the theta lifting studied in [GP] on using some of the earlier results of [F3] (which do not depend on [GP]).

As explained in [F3], each packet of representations of the unitary group in question, contains a (single, provided a twisted analogue of Rodier [Ro] is assumed) generic element. But we do not use this result here. The lifting studied in this paper relates (bijectively, under this assumption) these generic elements.

Using the results of [F3] we also derive a local analogue of the Global Theorem. Let v be a non-archimedean place of F which stays prime in E, and ξ_v a unitary character of $E_v^{\bullet}$. An irreducible admissible G_v-module π_v is called *ξ_v-cyclic* if $\operatorname{Hom}_{C_v}(\pi_v, \xi_v) \neq 0$, namely there exists a non-zero linear form $\ell : \pi_v \to \mathbb{C}$ on the space of π_v with $\ell(\pi_v(c)w) = \xi_v(c)\ell(w)$ for all $c \in C_v$ and $w \in \pi_v$, or ξ_v is a quotient of the restriction of π_v to C_v. By Bernstein's multiplicity one theorem of the Appendix, if ℓ exists then it is unique up to a scalar multiple.

The local lifting from H_{0v} to G_v is defined and studied in [F3], in terms of packets of representations of H_{0v} and G_v. These packets are finite sets, and we shall not reproduce here the definition which can be found in [F3]. As in [F3] we shall say that the irreducible $H_{0,v}$-module ρ_v lifts to the irreducible G_v-module π_v if the packet $\{\rho_v\}$ lifts to the packet $\{\pi_v\}$. The Global Theorem has the following consequence. Put $G_v = U(3, E_v/F_v)$, $H_{0v} = U(2, E_v/F_v)$.

Local Theorem. *((a) Non-split case). Every ξ_v-cyclic generic G_v-module π_v with central character ω_v is an endoscopic κ_v-lift of a generic H_{0v}-module ρ_v with central character $\omega_v/\xi_v\kappa_v$. Every generic H_{0v}-module ρ_v with central character $\omega_v/\xi_v\kappa_v$ endo κ_v-lifts to a ξ_v-cyclic generic G_v-module π_v with central character ω_v (see Proposition 26).*
((b) Split case). Every ξ_v-cyclic generic $GL(3, F_v)$-module with central character ω_v is a κ_v-lift of a generic $GL(2, F_v)$-module ρ_v with central character $\omega_v/\xi_v\kappa_v^2$. Every generic $GL(2, F_v)$-module ρ_v with central character $\omega_v/\kappa_v^2\xi_v$ endo κ_v-lifts to a ξ_v-cyclic generic G_v-module with central character ω_v.

In part (a), the cases where π_v or ρ_v are not square-integrable can be handled directly locally (see Proposition 29), and only the case of square-integrable π_v and ρ_v requires using the Global Theorem (see Proposition 26). It is not hard to see that if π_v is cyclic and unramified, then the H_{0v}-invariant form ℓ on π_v is non-zero at the K_v-fixed vector, where K_v is the standard maximal compact subgroup of G_v.

In the split case, where v splits in E, we have $G_v = GL(3, F_v)$ and $H_{0v} = GL(2, F_v)$, $C_v = GL(2, F_v)$ and $K_v = GL(3, R_v)$, where R_v is the ring of integers in F_v; ξ_v, κ_v and ω_v are unitary characters of $F_v^\times$. As in the non-split case, an irreducible admissible G_v-module π_v is called *ξ_v-cyclic* if there exists a non-zero linear form $\ell : \pi_v \to \mathbb{C}$ on the space of π_v with $\ell(\pi_v(c)w) = \xi_v(c)\ell(w)$ for all $c \in C_v$ and $w \in \pi_v$. By the Theorem of the Appendix (with $n = 3$), if ℓ exists then it is unique up to a scalar multiple. Part (b) of the Local Theorem is proved – using [BZ] – in Propositions 0 and 0.1 of [F7], and Prasad [P3]. Proposition 25 here gives a global proof in the case of a square-integrable ρ_v. The case of principal series ρ_v is treated by purely local means also in Proposition 30, whose proof shows in addition that if π_v is unramified then ℓ takes a non-zero value at the K_v-fixed vector.

Combining the Global and Local Theorems, we obtain the following interesting obstruction for a cuspidal generic locally-cyclic representation to be cyclic.

Corollary. *A cuspidal generic representation π of $\mathbf{G}(\mathbb{A}) = U(3, E/F)_{\mathbb{A}}$ with central character ω such that each of its components π_v is ξ_v-cyclic, will not be ξ-cyclic unless it is the endoscopic*

κ-lift of a cuspidal $\mathbf{H}_0(\mathbb{A})$-module ρ (necessarily generic, with central character $\omega/\xi\kappa$).

It follows from [F3] that a cuspidal generic $\mathbf{G}(\mathbb{A})$-module whose components are all κ_v-lifts from H_{0v} need not be a global κ-lift from $\mathbf{H}_0(\mathbb{A})$. For example all of its components may be unramified, yet π may base-change to a cuspidal representation of $GL(3, \mathbb{A}_E)$ by the base-change lifting of [F3]. Similar obstructions occur in the cases studied in [F5] and [F6], and in [W1/2].

To explain the obstruction, consider $\pi = \otimes\pi_v$ as in the Corollary. Since π_v is ξ_v-cyclic, there is a non-zero form ℓ_v on π_v, in $\mathrm{Hom}_{C_v}(\pi_v, \xi_v)(\simeq \mathbb{C})$, unique up to a scalar, by Bernstein's multiplicity one theorem of the Appendix. This form is uniquely determined when π_v is unramified by the requirement that $\ell_v(\xi_v^0) = 1$. Here ξ_v^0 is the chosen K_v-fixed vector in π_v. The product $\ell = \otimes\ell_v$ over all places is a non-zero form on the cuspidal generic π, in $\mathrm{Hom}_{\mathbf{C}(\mathbb{A})}(\pi, \xi)(\simeq \mathbb{C})$. By the local uniqueness property of ℓ_v, the linear form P is a multiple of ℓ. Thus $P = c\ell$ for some $c \in \mathbb{C}$. The Corollary asserts that c is zero unless π is a global lift. Being ξ-cyclic is a global property, more than the sum of its local parts.

It will in fact be more natural to compare the ξ-cyclic $\mathbf{G}(\mathbb{A})$-modules not with the $\mathbf{H}_0(\mathbb{A})$-modules as described above, but rather with the distinguished $GL(2, \mathbb{A}_E)$-modules of [F5]. We shall proceed to recall their definition, then state the Main (global) Theorem, relating the ξ-cyclic $\mathbf{G}(\mathbb{A})$-modules with the distinguished representations of [F5]. The Global Theorem follows from this parametrization when combined with the main global result of [F5], which relates the generic $\mathbf{H}_0(\mathbb{A})$-modules with the distinguished $GL(2, \mathbb{A}_E)$-modules, via the unstable κ-base-change lifting of [F1].

Let $\mathbf{H}$ denote $GL(2)$ as an algebraic group over the global field F, and put $\mathbf{H}'$ for the F-group $\mathrm{Res}_{E/F}\mathbf{H}$ obtained on restricting scalars from E to F. Thus $\mathbf{H}'(F) = \mathbf{H}(E) = GL(2, E)$, while $\mathbf{H}(F) = GL(2, F)$. A cuspidal representation ρ of $\mathbf{H}'(\mathbb{A})$ is called (in Jacquet [J], and in [F5]) *distinguished* if there is a form ϕ in its space such that $P_H(\phi) = \int_{\mathbf{Z}(\mathbb{A})\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} \phi(h)dh$ is non-zero. We denote here by $\mathbf{Z}$ also the center of $\mathbf{H}$. The global result of [F5] asserts that the distinguished $\mathbf{H}'(\mathbb{A})$-modules are precisely those obtained by the unstable base change lifting of [F1] as unstable κ-lifts of the cuspidal $\mathbf{H}(\mathbb{A})$-modules. This unstable base-change lifting depends on a choice of a character $\kappa : \mathbb{A}_E^\times/E^\times N\mathbb{A}_E^\times \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}^\times/F^\times$ is non-trivial.

We proceed to define a correspondence (depending on ξ, but not on κ) from the set of the distinguished $\mathbf{H}'(\mathbb{A}) = GL(2, \mathbb{A}_E)$-modules to the set of automorphic $\mathbf{G}(\mathbb{A}) = U(3, E/F)_{\mathbb{A}}$-modules. Our Main Global Theorem below would assert that the image of this correspondence consists of the ξ-cyclic generic modules, and each generic ξ-cyclic module is so obtained, for all ξ.

The correspondence is defined – as in the case of the lifting from $\mathbf{H}_0 = U(2)$ to $\mathbf{G} = U(3)$ – in terms of almost all local components. It suffices to consider the unramified components. Let v be a place of F which stays prime in E. An irreducible admissible generic unramified $H'_v = \mathbf{H}'(F_v) = GL(2, E_v)$-module is necessarily of the form $I'(\mu_{1v}, \mu_{2v})$, normalizedly induced from the character $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mapsto \mu_{1v}(a)\mu_{2v}(b)$ of the upper triangular subgroup of H'_v; $\mu_{iv} : E_v^\times \to \mathbb{C}^\times$ $(i = 1, 2)$ are unramified characters. It is distinguished precisely when $\mu_{2v} = \overline{\mu}_{1v}^{-1}$, thus $\mu_{2v}(a) = \mu_{1v}(\overline{a})^{-1}$. We write $I'(\mu_v)$ for $I'(\mu_v, \overline{\mu}_v^{-1})$. The *local ξ_v-correspondence* in the non-split unramified case can be defined to associate to the unramified H'_v-module $I'(\mu_v)$ the unramified (irreducible constituent of the) G_v-module $I(\mu_v)$. Note that the unstable κ_v-base change lifting of [F1] associates the H_{0v}-module $I_0(\mu_v)$ to the H'_v-module $I'(\mu_v\kappa_v) = \kappa_v \otimes I'(\mu_v)$ (since $\kappa_v = \overline{\kappa}_v^{-1}$). The local endoscopic κ_v-lifting defined above maps $I_0(\mu_v)$ to $I(\mu_v\kappa_v)$, and the correspondence relates $I'(\mu_v\kappa_v)$ to $I(\mu_v\kappa_v)$, making a commutative diagram.

At a place v of F which splits in E, the group $H'_v = GL(2, E_v)$ is isomorphic to $GL(2, F_v) \times GL(2, F_v)$. An H'_v-module $\rho_v = \rho_{1v} \times \rho_{2v}$ is distinguished, namely there is a $GL(2, F_v)$-invariant non-zero form on its space, precisely when ρ_{2v} is the contragredient $\check{\rho}_{1v}$ of ρ_{1v}. We define the *local correspondence* in the split case to associate to $\rho_v = \rho_{1v} \times \check{\rho}_{1v}$ the G_v-module $I(\rho_{1v}) = I(\rho_{1v} \times \frac{\omega_v}{\omega_{\rho_{1v}}})$ normalizedly induced from the representation of a maximal parabolic subgroup of $G_v = GL(3, F_v)$ which is ρ_{1v} on the 2×2 part of the Levi factor, and $\omega_v/\omega_{\rho_{1v}}$ on the 1×1 part, extended trivially on the unipotent radical.

As in the case of the lifting from $\mathbf{H}_0(\mathbb{A})$ to $\mathbf{G}(\mathbb{A})$, we say that the automorphic representation $\rho = \otimes \rho_v$ of $\mathbf{H}'(\mathbb{A}) = GL(2, \mathbb{A}_E)$ *corresponds* to the cuspidal representation $\pi = \otimes \pi_v$ of $\mathbf{G}(\mathbb{A}) = U(3, E/F)_{\mathbb{A}}$ if ρ_v corresponds to π_v for almost all places v of F. Let ω, ξ be unitary characters of $\mathbb{A}_E^{\bullet}/E^{\bullet}$, and put $\omega'(z) = \omega(z/\overline{z})$, $\xi'(z) = \xi(z/\overline{z})$ $(z \in \mathbb{A}_E^{\times})$.

The global result which we actually prove in this paper is as follows.

Main Global Theorem. *Let E/F be a quadratic extension of global fields of characteristic $\neq 2$. Suppose that π is a generic ξ-cyclic cuspidal representation of the quasi-split unitary group $\mathbf{G}(\mathbb{A}) = U(3, E/F)_{\mathbb{A}}$ whose central character is ω. Then it is obtained by the ξ-correspondence from a distinguished representation π' of $\mathbf{H}'(\mathbb{A}) = GL(2, \mathbb{A}_E)$ whose central character is ω'/ξ', or from an induced representation $\pi' = I'(\mu_1, \mu_2)$, $\mu_i : \mathbb{A}_E^{\times}/E^{\times}\mathbb{A}_F^{\times} \to \mathbb{C}^{\times}$, $\mu_1 \neq \mu_2$, $\mu_1\mu_2 = \omega'/\xi'$. Conversely, any distinguished $\mathbf{H}'(\mathbb{A})$-module π' with central character ω'/ξ', and any induced $\pi' = I'(\mu_1, \mu_2)$ as above, ξ-corresponds to a generic cuspidal ξ-cyclic $\mathbf{G}(\mathbb{A})$-module π with central character ω.*

From this global result we deduce the

Main Local Theorem. *(a) (Non-split case). Every ξ-cyclic unitarizable generic $G_v = U(3, E_v/F_v)$-module π_v with central character ω_v corresponds to a generic distinguished $H'_v = GL(2, E_v)$-module π_v with central character ω'_v/ξ'_v, or to an induced representation $\pi'_v = I'(\mu_{1v}, \mu_{2v})$, $\mu_{iv} : E_v^{\times}/F_v^{\times} \to \mathbb{C}^{\times}$, $\mu_{1v} \neq \mu_{2v}$, $\mu_{1v}\mu_{2v} = \omega'_v/\xi'_v$. Any unitarizable generic distinguished H'_v-module π'_v with central character ω'_v/ξ'_v, and any $\pi'_v = I'(\mu_{1v}, \mu_{2v})$ as in the previous sentence, corresponds to a ξ_v-cyclic generic G_v-module π_v with central character ω_v.*
(b) (Split case). Every generic unitarizable $GL(3, F_v)$-module π_v with central character ω_v which admits a non-zero linear form $\ell : \pi_v \to \mathbb{C}$ which transforms under $GL(2, F_v)$ via ξ_v, is of the form $I(\rho_v) = I(\rho_v \times \xi_v)$, normalizedly induced from a generic unitarizable representation of a maximal parabolic subgroup, which is ρ_v on $GL(2, F_v)$, $\xi_v = \omega_v/\omega_{\rho_v}$ on the 1×1 factor of the Levi subgroup, and extended trivially on the unipotent radical. Conversely, for every generic $GL(2, F_v)$-module ρ_v with central character ω_v/ξ_v, the $GL(3, F_v)$-module $\pi_v = I(\rho_v)$ whose central character is ω_v admits a non-zero form $\ell : \pi_v \to \mathbb{C}$ which transform according to ξ_v under $GL(2, F_v)$.

The Main Theorems follow from Propositions 26 (π_v square-integrable) and 29 (π_v principal series) in the local non-split case, and 27 in the global case. The Main Local Theorem in the split case is proven in [F7], Propositions 0 and 0.1 (in the context of $GL(n)$, by a purely local technique, based on [BZ2]). Proposition 25 is an alternative, global proof of the "conversely" part, but only for square-integrable representations ρ_v. Proposition 29 is an analogue – in the context of $U(3)$ – of Proposition 28 (= B17 of [FH]) which concerns $GL(2,E)$.

It will be interesting to determine whether there is only one cyclic representation in each generic packet of $U(3,E_v/F_v)$ which is in the image of the endo-lifting from $U(2,E_v/F_v)$, and it is the generic element of this packet, and that every cyclic unitarizable infinite-dimensional representation of $U(3,E_v/F_v)$ is generic and hence lies in one of these packets (an endo-lift of a generic $U(2,E_v/F_v)$-packet). But these questions are not considered in this paper.

In this introduction we stated a general form of the results. But from now on, to simplify the notations we restrict our attention only to the case where $\omega = 1$ and $\xi = 1$. Thus ω and ξ do not appear below, and we replace $\mathbf{G} = U(3,E/F)$ by its projective form, so $\mathbf{G} = PU(3,E/F)$. Also we replace $\mathbf{H'} = GL(2,E)$ by its projective form, so $\mathbf{H'} = PGL(2,E)$ below. The extension to the general ω and $\boldsymbol{\xi}$ will be left to the reader, the difficulty is merely notational. Moreover, as we do not deal with the archimedean components, our results are proven here only for positive characteristics, but the extension to number fields is immediate once the archimedean computations are written out.

Here is a schematic listing of the diagrams of liftings mentioned above.

$$\begin{array}{ccccccc} GL(2,E) & \rightarrow & U(3,E/F) & & \pi' = \kappa \otimes I'(\mu \times \overline{\mu}^{-1}) & \rightarrow & \pi = I(\mu\kappa) \\ \nwarrow & & \nearrow & & \nwarrow & & \nearrow \\ & U(2,E/F) & & & & \pi_0 = I_0(\mu) & \end{array}$$

$$\begin{array}{ccc} \omega_{\pi'}(z) = (\kappa\mu)(z/\overline{z}) = \kappa^2(z)(\mu/\overline{\mu})(z)[= (\omega/\xi)(z/\overline{z})] & \rightarrow & \omega_\pi(z/\overline{z}) = \omega(z/\overline{z}) \\ \nwarrow & & \nearrow \\ \multicolumn{3}{c}{\omega_{\pi_0}(z/\overline{z}) = \mu(z/\overline{z})[= (\omega/\xi\kappa)(z/\overline{z})] \quad (z \in E^\times, z/\overline{z} \in E^\bullet)} \end{array}$$

Here E/F is a quadratic extension of local fields, or it can indicate $\mathbb{A}_E/\mathbb{A}_F$ in the global case. The top horizontal arrow is the correspondence studied in this paper. The arrow on the left is the unstable κ-base-change lifting of [F1] and [F5]. The arrow on the right is the endoscopic κ-lifting of [F2] and [F3]. The first triangle indicates the groups involved, the second traces the induced representations, and the third traces their central characters. Note that the left arrow takes some (super) cuspidal representations to induced ones of the form $I(\mu_1,\mu_2)$, $\mu_i : E^\times/F^\times \rightarrow \mathbb{C}^\times$, $\mu_1 \neq \mu_2$, which are distinguished; the top arrow takes them back to (super) cuspidal representations. The top arrow relates distinguished and cyclic representations. It is not defined for other representations. The fact that it is well-defined for the distinguished and cyclic representations other than those indicated is the assertion of our main Theorems.

In the split case, when $E = F \oplus F$, the diagrams take the following form

$$\pi' \times \check{\pi}'(\pi' = \rho \otimes \kappa_1), \omega_{\pi'} = \omega_1/\xi_1, GL(2,F) \times GL(2,F)$$

$$\to \qquad \pi = I(\rho \otimes \kappa_1 \times \xi_1), \omega_\pi = \omega_1, GL(3,F)$$

$$\nwarrow \qquad\qquad\qquad\qquad \nearrow$$

$$\rho, \omega_\rho = \omega_1/\kappa_1^2\xi_1, GL(2,F)$$

We listed the representation, its central character, and the group. The diagram in the split case is a special instance of the previous diagram, at least when ρ is the induced $\pi_0 = I_0(\mu)$. To see this note that on $z = (z_1, z_2) \in E^\times = F^\times \times F^\times$, the character $\mu : E^\times \to \mathbb{C}^\times$ is given by $\mu(z) = \mu_1(z_1)\mu_2(z_2)$; ω and ξ are defined on $E^\bullet = \{(z, z^{-1}); z \in F^\times\}$ by $\omega(z, z^{-1}) = \omega_1(z)$, $\xi(z, z^{-1}) = \xi_1(z)$. Since $\kappa(z, z) = 1$ on $NE^\times = \{(z, z); z \in F^\times\}$, we have $\kappa = \kappa_1 \times \kappa_1^{-1}$, and hence $\kappa(z, z^{-1}) = \kappa_1(z^2)$. Consequently $(\omega/\xi\kappa)(z, z^{-1}) = (\omega_1/\xi_1\kappa_1^2)(z)$, $z \in F^\times$.

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2. Fourier Summation Formula.

The global tool used in the proof of the Main Global Theorem is a comparison of two "Fourier" summation formulae. We first recall this formula for distinguished $\mathbf{H}'(\mathbb{A}) = PGL(2, \mathbb{A}_E)$-modules from [F5], Proposition 5, p. 157. The formula of [F5] was formally generalized in [F6] to the context of distinguished representations of $GL(n, \mathbb{A}_E)$. Later we develop an analogous formula for cyclic $\mathbf{G}(\mathbb{A})$-modules.

The Fourier summation formula for $\mathbf{H}'$ is stated for a test function $f' = \otimes f'_v$ on $\mathbf{H}'(\mathbb{A})$, where f'_v is a C_c^∞ (compactly supported, smooth) function on H'_v for all v, which is equal to the unit element $f_v'^0 = |K_{H'_v}|^{-1} \operatorname{ch}_{K_{H'_v}}$ in the Hecke convolution algebra $\mathbb{H}'_v$ of the $K_{H'_v}$-biinvariant functions of compact support on H'_v, for almost all v. Here $|K_{H'_v}|$ is the volume of the standard maximal compact subgroup $\mathbf{H}'(R'_v)$ of H'_v (it is $PGL(2, R'_v)$), and $\operatorname{ch}_{K_{H'_v}}$ is the characteristic function of $K_{H'_v}$. A choice of a Haar measure is implicit.

Let $L^2(H')$ be the space of smooth functions $\phi : \mathbf{H}'(\mathbb{A}) \to \mathbb{C}$ with $\phi(\gamma h) = \phi(h)$ ($h \in \mathbf{H}'(\mathbb{A})$, $\gamma \in \mathbf{H}'(F)$) and $\int_{\mathbf{H}'(F)\backslash\mathbf{H}'(\mathbb{A})} |\phi(h)|^2 dh < \infty$. The convolution operator

$$(\rho(f')\phi)(g) = \int_{\mathbf{H}'(\mathbb{A})} f'(h)\phi(gh)dh = \int_{\mathbf{H}'(F)\backslash\mathbf{H}'(\mathbb{A})} K_{f'}(g,h)\phi(h)dh$$

is an integral operator with kernel $K_{f'}(g,h) = \sum_{\gamma \in \mathbf{H}'(F)} f'(g^{-1}\gamma h)$.

The theory of Eisenstein series decomposes $L^2(H')$ as the direct sum of three mutually orthogonal invariant subspaces: the space $L_0^2(H')$ of cusp forms, the space $L_1^2(H')$ of functions $\phi(g) = \chi(\det g)$ with $\chi^2 = 1$, and the continuous spectrum $L_c^2(H')$. The corresponding kernels are denoted by $K_0(g,h)$, $K_1(g,h)$, $K_c(g,h)$. The Fourier summation formula is the equality obtained on integrating $K(n,h)\overline{\psi}(n)$ on $h \in \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ and on $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in $\mathbf{N}(\mathbb{A}_E)/\mathbf{N}(E)$ (i.e. x in $\mathbb{A}_E/E$); here $\psi(n) = \boldsymbol{\psi}(x + \overline{x})$, where $\boldsymbol{\psi} \neq 1$ is the character on $\mathbb{A}/F$ fixed above. The convergence of this double integral is easily established.

The geometric expression for the integral of $K(n,h)\overline{\psi}(n)$ is computed in [F6], Propositions 4 and 9 (in the context of $GL(n)$; the discussion of the general case leads to an expression clearer than that obtained by the special discussion of the case of $n = 2$ in [F5]). It is the sum of

$$\Psi(0, f', \psi) = \int_{\mathbf{N}(\mathbb{A}_E)/\mathbf{N}(\mathbb{A}_F)} \int_{PGL(2,\mathbb{A}_F)} f'(nh)\psi(n)dn\,dh,$$

and of

$$\Psi(b, f', \psi) = \int_{\mathbf{N}(\mathbb{A}_E)} \int_{PGL(2,\mathbb{A}_F)} f'(n \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \eta h)\psi(n)dndh, \qquad \eta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

over all b in $E^\times/E^\bullet(\xrightarrow{\sim} N_{E/F}E^\times$ by $b \mapsto N_{E/F}b$, where $E^\bullet = \{z/\overline{z}; z \in E^\times\})$. Here i is a non-zero element of E with $\overline{i} = -i$, and η satisfies $\eta\overline{\eta}^{-1} = w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The integral of $K_1(n,h)\overline{\psi}(n)$ is zero since ψ is non-trivial. The cuspidal kernel takes the form $K_0(g,h) = \sum_{\pi'} \sum_{\phi \in \pi'} (\pi'(f')\phi)(g)\overline{\phi}(h)$, where π' ranges over the cuspidal $PGL(2,\mathbb{A}_E)$-modules, and ϕ over an orthonormal basis of π' (with standard finiteness properties). The integral of $K_0(n,h)\overline{\psi}(n)$ is equal to $\sum_{\pi'}(W_\psi \overline{P}_H)_{\pi'}(f')$, where

$$(W_\psi \overline{P}_H)_{\pi'}(f') = \sum_{\phi \in \pi'} W_\psi(\pi'(f')\phi)\overline{P}_H(\phi)$$

is independent of the choice of the basis $\{\phi\}$ of π'.

The integral of $K_c(n,h)\overline{\psi}(n)$ is computed in [F5], Proposition 5. It is the longest part of the summation formula. To record it, we need some notations. Let μ be a unitary character of $\mathbb{A}_E^\times/E^\times$. Write $\mu = \overline{\mu}$ if $\mu(a) = \mu(\overline{a})$ for all $a \in \mathbb{A}_E^\times$. For any complex number s consider the Hilbert space $H'(\mu, s)$ of functions $\phi : PGL(2,\mathbb{A}_E) \to \mathbb{C}$ which satisfy

$$\phi\left(\begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix} h\right) = |a|_E^{s+\frac{1}{2}}\mu(a)\phi(h) \qquad (a \in \mathbb{A}_E^\times,\ h \in PGL(2,\mathbb{A}_E))$$

and $\int_{\mathbb{K}'} |\phi(k)|^2 dk < \infty$. Here $\mathbb{K}' = \prod_v K_v'$, where K_v' is the standard maximal compact subgroup of H_v'. The restriction-to-$\mathbb{K}'$ map $\phi \mapsto \phi|\mathbb{K}'$ defines an isomorphism from $H'(\mu, s)$ to $H'(\mu) = H'(\mu, 0)$. Identify $H'(\mu, s)$ with $H'(\mu)$, and denote by $\phi(\mu, s)$ the element of $H'(\mu, s)$

corresponding to $\phi(\mu)$ in $H'(\mu)$. Let $I'(\mu,s)$ denote the representation of $\mathbf{H}'(\mathbb{A})$ on $H'(\mu,s)$ by right translation, and

$$E(h,\phi,\mu,s)=\sum_{\gamma\in\mathbf{B}(E)\backslash\mathbf{H}(E)}\phi(\gamma h,\mu,s)\qquad(\phi=\phi(\mu)\in H'(\mu))$$

the Eisenstein series. The kernel on the continuous spectrum is given by

$$K_c(g,h)=\frac{1}{4\pi}\sum_{\mu}\int_{-\infty}^{\infty}\sum_{\phi}E(g,I'(\mu,it;f')\phi,\mu,it)\overline{E}(h,\phi,\mu,it)dt.$$

(In the function field case the integral ranges from 0 to $2\pi/\ln q$ – q is the cardinality of the field of constants; this notation will be so understood below too). Here ϕ ranges over an orthonormal basis $\{\phi_\alpha\}$ of $\mathbb{K}'$-finite functions in $H'(\mu)$, and μ ranges over a set of representatives of the unitary characters μ on $\mathbb{A}_E^\times/E^\times$ under the equivalence relation $\mu'\sim\mu$ if $\mu'(a)=\mu(a)|a|^{it}$ ($t\in\mathbb{R}$) for all $a\in\mathbb{A}_E^\times$, and $\mu\sim\mu^{-1}$. Also note that $\mathbf{N}(E)\backslash\mathbf{N}(\mathbb{A}_E)$ is compact, and put $E_\psi(\phi,\mu)=\int_{\mathbf{N}(E)\backslash\mathbf{N}(\mathbb{A}_E)}E(n,\phi,\mu,0)\overline{\psi}(n)dn$. Denote by $\mathbb{A}_F^1$ the group of $a\in\mathbb{A}_F^\times$ with absolute value $|a|=\prod_v|a_v|_v$ equal to 1, and by $|\mathbb{A}_F^1/F^\times|$ the volume of the compact $\mathbb{A}_F^1/F^\times$.

Proposition 5 of [F5] asserts – for a function $f'=\otimes f'_v$ such that f'_v is the unit element $f_v'^0$ for v outside a finite set V, and where $f'_V=\bigotimes_{v\in V}f'_v$ – that there exists an integrable function $d(\mu\nu^{it},f'_V)$ on $t\in\mathbb{R}$ for each $\mu=\overline{\mu}$, with

$$\int_{-\infty}^{\infty}|d(\mu\nu^{it},f'_V)|dt<\infty \tag{1.1}$$

(we write $\nu(a)$ for $|a|$), such that the integral of $K_c(n,h)\overline{\psi}(n)$ is the sum of:

$$\sum_{\mu=\overline{\mu}}\int_{-\infty}^{\infty}d(\mu\nu^{it},f'_V)[\prod_{v\notin V}\operatorname{tr}\ I'(\mu_v\nu_v^{it},f'_v)]dt \tag{1.2}$$

and of (1.3), defined by the following expression (in which ϕ_α,ϕ_β appear twice):

$$\frac{|\mathbb{A}_F^1/F^\times|}{4\pi}\sum_{\substack{\mu^2\neq1\\ \mu|\mathbb{A}_F^\times=1}}[\prod_{v\notin V}\operatorname{tr}\ I'(\mu_v,f'_v)]\sum_{\alpha,\beta}\int_{\mathbb{K}}\phi_\alpha(k)dk\cdot\overline{E}_\psi(\phi_\beta,\mu)$$

$$\cdot\int_{\mathbb{K}'}\int_{\mathbb{K}'}\phi_\beta(k')\overline{\phi}_\alpha(k)\prod_{v\in V}\int_{E_v^\times}\mu_v(a)|a|_{E_v}^{1/2}\int_{N(E_v)}f'_v(k_v^{-1}\begin{pmatrix}a&0\\0&1\end{pmatrix}nk'_v)d^\times a dn\,dk\,dk'.$$

In summary, the Fourier summation formula is the equality in the following:

1. Proposition. *With the above notations, we have the equality*

$$\Psi(0,f',\psi)+\sum_{b\in E^\times/E^\bullet}\Psi(b,f',\psi)=\sum_{\pi'}(W_\psi\overline{P}_H)_{\pi'}(f')+(1.2)+(1.3)$$

of distributions in $f'=\bigotimes f'_v$, $f'_v=f'^0_v$ *for almost all* v, $f'_v\in C_c^\infty(H'_v)$ *for all* v. □

Next we develop an analogous Fourier summation formula in the context of $\mathbf{G}=U(3,E/F)$. In fact it will put our problem in the correct perspective if we enlarge our horizons and work in this initial stage with a more general unitary group, and its subgroup. Suppose then that $n\geq 3$, and put

$$\mathcal{J}=\begin{pmatrix}0&&1\\&-I&\\1&&0\end{pmatrix}\in GL(n)\quad(\text{thus } I \text{ is the identity in } GL(n-2)).$$

Introduce the algebraic group $\mathbf{G}$ over F whose group G of F-points consists of the $g\in GL(n,E)$ with $g\mathcal{J}{}^t\overline{g}=\mathcal{J}$, where $\overline{g}=(\overline{g}_{ij})$ if $g=(g_{ij})$, $g_{ij}\in E$. Let $\mathbb{P}_E^{n-1}$ denote the projective $(n-1)$-space over E, and let Y be the subvariety of x in $\mathbb{P}_E^{n-1}$ with $x\mathcal{J}{}^t\overline{x}=0$. Then G acts transitively on Y by $g:x=(x_1,\dots,x_n)\mapsto xg^{-1}$. If $x_0=(0,\dots,0,1)$ then its stabilizer $B=\operatorname{stab}_G x_0$ consists of the matrices g in G whose entries on the first column and last row are zero except the entries at $(1,1)$ and (n,n), which are related by $g_{11}\overline{g}_{nn}=1$. The subgroup B is a maximal proper parabolic subgroup of G, which is also minimal when $n=3$.

Put $\mathcal{J}_0=\operatorname{diag}(1,-1,I)$,

$$g_0=\begin{pmatrix}1&-1&0&\frac12\\1&0&0&-\frac12\\0&0&I&0\\\frac12&\frac12&0&\frac14\end{pmatrix}=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&I&0\\\frac12&1&0&1\end{pmatrix}\begin{pmatrix}1&-1&0&\frac12\\0&1&0&-1\\0&0&I&0\\0&0&0&1\end{pmatrix}.$$

Note that $\frac12(I-\mathcal{J}_0)={}^t\varepsilon_0\varepsilon_0$, $\varepsilon_0=(0,1,0,\dots,0)$. Define C to be the centralizer of $\mathcal{J}_0$, or of ${}^t\varepsilon_0\varepsilon_0$, in G. Note that $\mathcal{J}_0g_0\mathcal{J}_0={}^tg_0$, and $g_0^{-1}=\mathcal{J}{}^tg_0\mathcal{J}=\mathcal{J}\mathcal{J}_0g_0\mathcal{J}_0\mathcal{J}$.

2. Proposition. *We have the disjoint decomposition* $G=BC\cup Bg_0C=BC\cup Bg_0^{-1}C$.

Proof. Denote the ith row of $h\in C$ by $r_i=(r_{i1},\dots,r_{in})$. Then $r_{i2}=0$ and $r_{2i}=0$ if $i\neq 2$. Put r'_i for r_i with r_{i2} deleted, and h' for the $(n-1)\times(n-1)$ matrix with rows $r'_1,r'_3,\dots,r'_n$. Put $\mathcal{J}'=\begin{pmatrix}0&&1\\&-I&\\1&&0\end{pmatrix}\in GL(n-1)$. As $h\in(C\subset)G$ we have

$$r'_i\mathcal{J}'{}^t\overline{r}'_j=\begin{cases}-1, & 3\leq i=j<n,\\ 1, & \{i,j\}=\{1,n\},\\ 0, & \text{otherwise},\end{cases}\tag{2.1}$$

and $r_{22}\overline{r}_{22}=1$. Conversely, given a row $(n-1)$-vector r'_n with $r'_n\mathcal{J}'{}^t\overline{r}'_n=0$, there are row $(n-1)$-vectors $r'_1,r'_3,\dots,r'_{n-1}$ such that (2.1) holds. Namely there is some $h\in C$ whose last row is r_n. Consequently the orbit $\{r_n=x_0h\}$ of $x_0\in Y$ under C consists of all $x\in Y$, $x=(r_n=x_0h=)\ (x_1,\dots,x_n)$ with $x_2=0$.

On the other hand, $x_0g_0=(\frac{1}{2},\frac{1}{2},\mathbf{0},\frac{1}{4})$, where $\mathbf{0}$ is the zero $(n-3)$-vector. The orbit $\{x=x_0g_0h=\frac{1}{4}(r_n+2r_1+2r_2)\}$ of x_0g_0 under C consists of all $x=(x_1,\dots,x_n)\in Y$ with $x_2\neq 0$ (then $x_2\overline{x}_2=1/4$). Indeed $r_1\mathcal{J}{}^t\overline{r}_n=1$ and $r_2\mathcal{J}\overline{r}_2=-1$, hence $x=\frac{1}{4}(r_n+2r_1+2r_2)$ satisfies $x\mathcal{J}{}^t\overline{x}=0$. Conversely, given a vector x which projects to Y, thus $x\mathcal{J}{}^t\overline{x}=0$, with $x_2\overline{x}_2=1/4$, there are $(n-1)$-vectors $r'_1,r'_3,\dots,r'_{n-1}$ with $r'_i\mathcal{J}'{}^t\overline{x}'=\frac{1}{4}\delta_{i1}$ and $r'_i\mathcal{J}'{}^t\overline{r}'_j=\delta_{i1}\delta_{j1}-\delta_{ij}$, namely an $h\in C$ with x_0g_0h equal to the given x. □

The Fourier summation formula is obtained on integrating (for $f\in C_c^\infty(\mathbf{G}(\mathbb{A}))$) the kernel $K_f(g,h)=\sum_{\gamma\in\mathbf{G}(F)}f(g^{-1}\gamma h)$ of the convolution operator $r(f)=\int_{\mathbf{G}(\mathbb{A})}f(g)r(g)dg$ on the space $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ of automorphic forms. More precisely we integrate $K_f(u,h)\overline{\psi}(u)$ over h in $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$, and over u in $\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})$, where $\mathbf{U}$ is the unipotent radical of the maximal parabolic subgroup $\mathbf{B}$. The E-points of $\mathbf{U}$ are

$$\mathbf{U}(E)=\left\{\begin{pmatrix}1&p&z\\0&I&{}^tq\\0&0&1\end{pmatrix};p=(p_2,\dots,p_{n-1})\in E^{n-2},\ q\in E^{n-2},\ z\in E\right\},$$

and

$$\mathbf{U}(F)=\mathbf{U}(E)\cap\mathbf{G}(F)=\left\{u=\begin{pmatrix}1&p&\frac{1}{2}p{}^t\overline{p}+if\\0&I&{}^t\overline{p}\\0&0&1\end{pmatrix};f\in F,\ p=(p_2,\dots,p_{n-1})\in E^{n-2}\right\},$$

where ${}^tq$ is the transpose of the row vector q. The character $\psi:\mathbf{U}(\mathbb{A})/\mathbf{U}(F)\to\mathbb{C}$ is defined by $\psi(u)=\boldsymbol{\psi}(p_2+\overline{p}_2)$.

The absolute convergence of this integral is immediate. Let $||g||$ denote the usual norm function on the group $\mathbf{G}(\mathbb{A})$ ([HCM], p. 6). Then $\sum_{\gamma\in\mathbf{G}(F)}|f(g^{-1}\gamma h)|\leq c||g||^N$ (for some $c=c(f)>0$, $N=N(f)>0$) for all g, h. Integrating the last sum over h in the space $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$, which has finite volume, and over g in the compact $\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})$, where $||g||$ is bounded, we obtain a finite number.

To compute this double integral we need to know which double cosets $\mathbf{U}(F)\gamma\mathbf{C}(F)$, $\gamma\in\mathbf{G}(F)$, may contribute a non-zero term. We then introduce the integral

$$\Psi(\gamma,f,\psi)=\iint_{Z\backslash(\mathbf{U}(\mathbb{A})\times\mathbf{C}(\mathbb{A}))}f(u^{-1}\gamma h)\overline{\psi}(u)du dh,$$

where Z denotes the centralizer, consisting of the (u,h) with $u^{-1}\gamma h=\gamma$. It vanishes precisely when ψ is non trivial on $\mathbf{Z}(\mathbb{A})$. The same comment applies to the local analogues

$$\Psi(\gamma,f_v,\psi_v)=\iint_{Z_v\backslash(U_v\times C_v)}f_v(u^{-1}\gamma h)\overline{\psi}_v(u)du\,dh.$$

When v splits in E then $G_v = GL(n,F_v)$, C_v is the centralizer of $\mathcal{J}_0$, or of $\frac{1}{2}(I-\mathcal{J}_0) = {}^t\varepsilon_0\varepsilon_0$, $\varepsilon_0 = (0,1,0,\dots,0)$, in G_v; U_v consists of $u = \begin{pmatrix} 1 & p & z \\ 0 & I & {}^tq \\ 0 & 0 & 1 \end{pmatrix}$, $p = (p_2,\dots,p_{n-1})$, $q = (q_2,\dots,q_{n-1})$ in F_v^{n-2}. Lemma 2 of [F7] takes $\psi_v(u) = \boldsymbol{\psi}_v(p_2+q_{n-1})$ and asserts that $\Psi(\gamma,f_v,\psi_v)$ vanishes unless γ lies in the (U_v,C_v)-double coset of g_bg_0, where $g_b = \operatorname{diag}(1,\dots,1,1/b)$, $b\in F_v^\times$, or of I (when $n\geq 3$; the case of $n=2$ is also considered in [F7], Lemma 2). A similar proof applies in the non-split case to establish:

3. Lemma. *If v stays prime in E, and $\gamma\in G_v$ satisfies $\Psi(\gamma,f_v,\psi_v)\neq 0$ for some f_v, ψ_v, then γ lies in the (U_v,C_v)-double coset of g_bg_0, where $g_b = \operatorname{diag}(b,1,\dots,1,\overline{b}^{-1})$, $b\in E_v^\times$, or of I. Moreover, $U_vg_bg_0C_v = U_vg_{b'}g_0C_v$ implies that $b'/b\in E^\bullet$.*

Proof. Put $F(g) = \int_{C_v} f_v(gh)dh$. This F is a function on G_v/C_v, and this symmetric space injects into the space of $n\times n$ matrices x with entries in E_v, rank $=1$, trace $=1$, and with $x = \mathcal{J}\,{}^t\overline{x}\mathcal{J}^{-1}$, by the map $G_v/C_v\ni g\mapsto g\,{}^t\varepsilon_0\varepsilon_0g^{-1} = -g\,{}^t\varepsilon_0\,{}^t(\overline{g}\,{}^t\varepsilon_0)\mathcal{J}^{-1}$. The image X consists of the x of the form $x = -{}^tv\overline{v}\mathcal{J}^{-1}$, with $v = (v_1,\dots,v_\ell,0,\dots,0)$, $v_\ell\neq 0$, $1\leq\ell\leq n$, and $\ell\geq 2$ if $n\geq 3$. All elements $x=(x_{ij})$ of X with $x_{n1}\neq 0$ (then $\ell = n$ and $x_{n1} = v_n\overline{v}_n\in N_{E_v/F_v}E_v^\times$) are obtained as the image of the double cosets

$$U_vg_bg_0C_v/C_v \ni \begin{pmatrix} 1 & p & \frac{1}{2}p\,{}^t\overline{p}+if \\ 0 & I & {}^t\overline{p} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b & & 0 \\ & I & \\ 0 & & \overline{b}^{-1} \end{pmatrix}\cdot g_0$$

$$\mapsto \begin{pmatrix} -b+\overline{b}^{-1}(\frac{1}{4}p\,{}^t\overline{p}+\frac{1}{2}if) \\ {}^t\overline{p}/2\overline{b} \\ 1/2\overline{b} \end{pmatrix} \left(\frac{-1}{2b},\frac{p}{2b},\overline{b}-\frac{p\,{}^t\overline{p}-i2f}{4b}\right).$$

If $3\leq\ell<n$, and $u = \begin{pmatrix} 1 & p & * \\ 0 & I & * \\ 0 & 0 & 1 \end{pmatrix}$, then $u\,{}^tv = {}^tv'$, where $v' = (v_1+p_2v_2+\cdots+p_\ell v_\ell, v_2,\dots,v_\ell,0.\dots,0)$. Take $p_3=\cdots=p_{\ell-1}=0$, and $p_\ell = -p_2v_2/v_\ell$, to get $u\in U_v$ with arbitrary $p_2\neq 0$ which satisfies $uxu^{-1}=x$ (x is ${}^tv\overline{v}\mathcal{J}^{-1}$). Consequently $\int_{U_v} F(ug)\psi(u)du = 0$ for $g\in G_v/C_v$ with $g\,{}^t\varepsilon\varepsilon g^{-1} = x$.

If $\ell=2$ and $u = \begin{pmatrix} 1 & p & * \\ 0 & I & * \\ 0 & 0 & 1 \end{pmatrix}$, then $v = (v_1,v_2,0,\dots,0)$, and $u\,{}^tv = {}^tv'$, where $v' = (v_1+p_2v_2,v_2,0,\dots,0)$. Hence U_vA_v, where $A_v = \left\{\begin{pmatrix} b & & 0 \\ & I & \\ 0 & & \overline{b}^{-1} \end{pmatrix}\right\}$, acts transitively on the unique orbit of the $x = {}^tv\overline{v}\mathcal{J}^{-1}$, $v = (v_1,v_2,0,\dots,0)$, $v_2\overline{v}_2 = 1$. This orbit is obtained by $I\in G_v$, since $I\,{}^t\varepsilon_0 = {}^t(0,1,0,\dots,0)$, and $-{}^t\varepsilon_0\varepsilon_0\mathcal{J} = {}^t\varepsilon_0\varepsilon_0$. $\square$

We are now in a position to obtain the geometric part of the Fourier summation formula.

3.1 Corollary. *The integral of* $K_f(u,h)\overline{\psi}(u)$ *over* u *in* $\mathbf{U}(\mathbb{A})/\mathbf{U}(F)$ *and over* h *in* $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ *is absolutely convergent and equal to the finite sum* $\Psi(0,f,\psi)+\sum_{b\in E^\times/E^\bullet}\Psi(b,f,\psi)$*, where*

$$\Psi(0,f,\psi)=\int_{\mathbf{U}(\mathbb{A})/\mathbf{U}_0(\mathbb{A})}\int_{\mathbf{C}(\mathbb{A})}f(uh)\psi(u)du\,dh$$

and

$$\Psi(b,f,\psi)=\int_{\mathbf{U}(\mathbb{A})}\int_{\mathbf{C}(\mathbb{A})}f(ug_bg_0h)\psi(u)du\,dh.$$

Here $g_b=\operatorname{diag}(b,I,\overline{b}^{-1})$*, and* $\mathbf{U}_0$ *is the group of* $u=\begin{pmatrix}1&p&*\\0&I&{}^t\overline{p}\\0&0&1\end{pmatrix}$ *in* $\mathbf{U}$ *with* $p_2=0$*.*

Proof. This follows from Lemma 3 above and Lemma 2 of [F7]. □

We shall not use the following observation. Put $n=3$.

3.2 Lemma. *For any* $f\in C_c(\mathbf{G}(\mathbb{A}))$*, the function* K_f *is compactly supported on* $U\backslash\mathbf{U}(\mathbb{A})\times C\backslash\mathbf{C}(\mathbb{A}),U=\mathbf{U}(F),C=\mathbf{C}(F),G=\mathbf{G}(F)$*.*

Proof. If $\mathbf{G}$ is a connected linear algebraic group over F, and $\mathbf{C}$ is a reductive closed subgroup over F, then $\mathbf{G}/\mathbf{C}$ is an affine variety $\mathbf{V}$ over F ([Bo], Proposition 7.7). Then $V=\mathbf{V}(F)$ is discrete and closed in $\mathbb{V}=V(\mathbb{A})$. Put $\mathbb{G}=\mathbf{G}(\mathbb{A})$, etc. The natural map $\mathbb{G}/\mathbb{C}\to\mathbb{V}$ is continuous, and it maps $G/C\subset\mathbb{G}/\mathbb{C}$ to $V\subset\mathbb{V}$. Hence G/C is closed in $\mathbb{G}/\mathbb{C}$, namely $G\mathbb{C}$ is closed in $\mathbb{G}$ and so $\mathbb{C}/C$ is closed in $\mathbb{G}/G$. Moreover, for G over F as above, for any closed F-subgroup H of G, $\mathbb{H}^1/H$ is closed in $\mathbb{G}/G$ ([G], (2.1)). Now for our function f, since $U\backslash\mathbb{U}$ is compact, $K_f(u,h)=\sum_{Z\backslash G}f(u^{-1}\gamma h)$ has compact support on $U\backslash\mathbb{U}\times G\backslash\mathbb{G}$, hence also on its closed subset $U\backslash\mathbb{U}\times C\backslash\mathbb{C}$, by either of these results.

A computational proof is as follows. We have $K_f(u^{-1},h)=\sum_{\gamma\in G}f(u\gamma h)=\sum_\gamma\sum_{\eta\in C}\sum_\nu f(u\nu\gamma\eta h)$ $(\gamma\in U\backslash G/C,\nu\in U/U\cap\gamma C\gamma^{-1})$. If u lies in a compact subset of $\mathbf{U}(\mathbb{A})$ and $f(u\nu\gamma\eta h)\neq 0$, then $\operatorname{Int}(\nu\gamma)\mathcal{J}_0$ lies in a compact of $\mathbf{G}(\mathbb{A})$. A set of representatives γ for $U\backslash G/C$ is given (in Proposition 2) by I and $d(b)g_0$, $d(b)=\operatorname{diag}(b,1,\overline{b}^{-1}),b\in E^\times/E^\bullet$. Also write $\nu=\nu(p,f)$ for an element of U. Then $\{\nu(p,0);p\in E\}$ is a set of representatives in U for $U/U\cap C$. Multiplying out $\operatorname{Int}(\nu\gamma)\mathcal{J}_0$ for $\gamma=I$ and for $\gamma=d(b)g_0$, we see that the rational γ (namely b) and ν (namely p (and f if $\gamma\neq I$)) lie in a compact, hence in a finite set. Hence ηh lies in a compact of $\mathbf{C}(\mathbb{A})$, and h lies in a compact of $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ if $K_f(u^{-1},h)\neq 0$ for $u\in\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})$. □

As in the case of $\mathbf{H}'(F)=PGL(2,E)$ discussed above, the kernel $K_0(g,h)$ of the convolution operator $r(f)$ on the space of cusp forms is $\sum_\pi n(\pi)\sum_{\phi\in\pi}(\pi(f)\phi)(g)\overline{\phi}(h)$. Here π ranges over a set of representatives for the set of equivalence classes of cuspidal representations of $\mathbf{G}(\mathbb{A})$, and $n(\pi)$ denotes the multiplicity of π in the space of cusp forms. In the case of $n=3$ it is shown in [F3/4] that $n(\pi)=1$, and this is conjectured to be true for all $n\geq 3$, but we make no use of this remark. The ϕ range over an orthonormal basis of cusp forms with standard $\mathbb{K}$-finiteness properties, in the space of π. The sum is convergent to a rapidly decreasing function in g and h in $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$, for every test function f. In fact the sum is finite in the function field case, since $\pi(f)\phi \neq 0$ only for ϕ with fixed ramification depending on f, and the space of such cusp forms is finite dimensional.

3.3 Proposition. *The integral of $K_0(u,h)\overline{\psi}(u)$ over h in $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ and u in $\mathbf{U}(\mathbb{A})/\mathbf{U}(F)$ is equal to $\sum_\pi n(\pi)(W_\psi\overline{P})_\pi(f)$, where $(W_\psi\overline{P})_\pi(f) = \sum_{\phi\in\pi} W_\psi(\pi(f)\phi)\overline{P}(\phi)$, and $W_\psi(\phi)$ is $\int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} \phi(u)\overline{\psi}(u)du$.* □

Remark. The global considerations simplify if we assume that at some place v_0 of F the component f_{v_0} of the test function f is a supercusp form. Of course v_0 has to be non-archimedean, and since it is easy to see that a supercuspidal $GL(3,F_v)$-module cannot have a non-zero $GL(2,F_v)$-invariant linear form on its space (see Proposition 0 of [F7]), we assume that v_0 stays prime in E. A well-known observation of D. Kazhdan implies that the convolution operator $r(f)$ will then factorize through the orthonormal projection to the space of cusp forms. Consequently $K_f(g,h) = K_{0,f}(g,h)$, and the Fourier summation formula in this case is simply the following identity, obtained from Corollary 4 and Proposition 5. *If the component f_{v_0} of f at some place v_0 of F which stays prime in E is a supercusp form, then we have*

$$\Psi(0,f,\psi) + \sum_{b\in E^\times/E^\bullet} \Psi(b,f,\psi) = \sum_\pi n(\pi)(W_\psi\overline{P})_\pi(f).$$

This form of the Fourier summation formula is too restrictive to derive our global results. In this paper we use the general Fourier summation formula which we proceed to develop (when $n=3$).

To deal with convergence questions, we briefly recall some consequences of Arthur's work [A1/2], mostly in his (standard) notations. This is best done in the context of a general reductive group $\mathbf{G}$ over F. Let $\mathbb{G}^1$ denote the subgroup of the g in $\mathbb{G} = \mathbf{G}(\mathbb{A})$ with $|\chi(g)| = 1$ for every rational character χ of $\mathbf{G}$ ([A1], p. 917). Put $\mathbb{K} = \prod_v K_v$, product over all places v in F, of hyperspecial maximal compact subgroups K_v of $\mathbf{G}(F_v)$. Let $f \in C_c^\infty(\mathbf{G}(\mathbb{A}))$ be a $\mathbb{K}$-*finite* (the space spanned by its left and right $\mathbb{K}$-translates is finite dimensional) smooth compactly supported function on $\mathbb{G}$. Denote by Λ_2^T truncation ([A2], p. 89) with respect to the second variable, and by χ any cuspidal data ([A1], p. 924/6). Denote by $\mathbf{U}$ a closed F-subgroup of $\mathbf{G}$ such that $U\backslash\mathbb{U}$ is compact, and by ψ a character of $U\backslash\mathbb{U}$ with $|\psi| = 1$. Let $\mathbf{C}$ be a closed reductive F-subgroup of $\mathbf{G}$, such that $C\backslash\mathbb{C}$ has finite volume, and such that for any Siegel domain S ([HCM], [PR]) in $\mathbb{G}^1$, $S_C = S\cap\mathbb{C}$ is a Siegel domain in $\mathbb{C}$. We put $C = \mathbf{C}(F)$, $\mathbb{C} = \mathbf{C}(\mathbb{A})$, etc.

4. Proposition. *Let ω be a compact set in $\mathbb{G}^1$. Then for any sufficiently regular ([A2], p. 89) T in $\mathfrak{A}_0^+$ we have $K_f(u,h) = \Lambda_2^T K_f(u,h)$ and $K_{f,\chi}(u,h) = \Lambda_2^T K_{f,\chi}(u,h)$ ([A1], p. 935), for all $u\in\omega$, $h\in\mathbb{G}$. For any Siegel domain S in $\mathbb{G}^1$ and $N>0$, there is $c>0$ such that $\sum_\chi |K_{f,\chi}(u,h)| \leq c||h||^{-N}$ for all $u\in\omega$ and $h\in S$. Consequently*

$$\int_{C\backslash\mathbb{C}}\int_{U\backslash\mathbb{U}} K_f(u,h)\overline{\psi}(u)dudh = \sum_\chi \iint K_{f,\chi}(u,h)\overline{\psi}(u)dudh.$$

Each side is finite even if $K\overline{\psi}$ is replaced by its absolute value. The Eisenstein series being defined in [A1], p. 926, put $E_\psi(\phi,\pi) = \int_{U\backslash\mathbb{U}} E(u,\phi,\pi)\overline{\psi}(u)du$. Then

$$\sum_P n(A_P)^{-1} \int_{\Pi^G(M)} |\sum_{\phi\in\mathfrak{B}_P(\pi)_\chi} E_\psi(I_P(\pi,f)\phi,\pi)\int_{C\backslash\mathbb{C}} \Lambda^T\overline{E}(h,\phi,\pi)dh|d\pi$$

is finite. The expression obtained on erasing the absolute values is equal to

$$\int_{C\backslash\mathbb{C}}\int_{U\backslash\mathbb{U}} K_{f,\chi}(u,h)\overline{\psi}(u)dudh.$$

Proof. The truncation operator Λ^T is defined in [A2], p. 89, to be (we put $|A/Z|$ for $\dim(A/Z)$)

$$\Lambda^T\phi(h) = \sum_P (-1)^{|A/Z|} \sum_{\delta\in P\backslash G} \hat{\tau}_P(H(\delta h)-T)\int_{N\backslash\mathbb{N}} \phi(n\delta h)dn.$$

Then

$$\Lambda_2^T K_f(u,h) = \sum_P (-1)^{|A/Z|} \sum_{\delta\in P\backslash G} \hat{\tau}_P(H(\delta h)-T)\int_{N\backslash\mathbb{N}} K_f(u,n\delta h)dn.$$

Put $K_{P,f}(u,h) = \sum_{\mu\in M}\int_{\mathbb{N}} f(u^{-1}\mu nh)dn$, as in [A1], p. 923. Then

$$\int_{N\backslash\mathbb{N}} K_f(u,nh)dn = \sum_{\gamma\in P\backslash G} K_{P,f}(\gamma u,h).$$

By [A2], p. 101, sentence including (2.4), if $K_{P,f}(\gamma u,\delta h)\neq 0$, then there exists $T_0\in\mathfrak{A}_0$, depending only on the compact support $\mathrm{supp}(f)$ of f, such that $\hat{\tau}_P(H(\gamma u)-H(\delta h)-T_0)=1$. By [A1], (5.2), p. 936, there is $c>0$ such that $\varpi(H(\gamma u))\le c(1+\ell n||u||)$ for all $u\in\mathbb{G}^1$, $\gamma\in G$, $\varpi\in\hat{\Delta}_0$. Our u lies in the compact ω, hence there is some $c>0$ with $\varpi(H(\gamma u))\le c\,(u\in\omega,\,\gamma\in G)$, for all $\varpi\in\hat{\Delta}_P$. Hence $\varpi(H(\delta h))<c-\varpi(T_0)$, and $\hat{\tau}_P(H(\delta h)-T)$ is zero for a sufficiently regular T. Then the term indexed by $P\neq G$ vanishes, and $\Lambda^T K_f = K_f$. But the sentence including (2.4) on p. 101 of [A2] is valid also for $K_{P,f,\chi}$, for all χ. Hence $\Lambda^T K_{f,\chi} = K_{f,\chi}$.

The kernel $K_{f,\chi}$ is defined in [A1], p. 935, to be

$$K_{f,\chi}(u,h) = \sum_P n(A_P)^{-1}\int_{\Pi^G(M)} \sum_{\phi\in\mathfrak{B}_P(\pi)_\chi} E(u,I_P(\pi,f)\phi,\pi)\overline{E}(h,\phi,\pi)d\pi.$$

By [A1], Lemma 4.4, there is $N>0$ and a semi-norm $||\cdot||_0$ on $C_c^\infty(\mathbf{G}(\mathbb{A}))$ such that

$$\sum_\chi\sum_P n(A_P)^{-1}\int_{\Pi^G(M)} |\sum_{\phi\in\mathfrak{B}_P(\pi)_\chi} E(u,I_P(\pi,f)\phi,\pi)\overline{E}(h,\phi,\pi)|d\pi \le ||f||_0\cdot||u||^N\cdot||h||^N.$$

In particular $\sum_\chi |K_{f,\chi}(u,h)| \le ||f||_0 \cdot ||u||^N \cdot ||h||^N$. By [A1], Corollary 5.2 (see also [A2], mid page 89), we can truncate $K_f(u,h) = \sum_\chi K_{f,\chi}(u,h)$ term by term: $\Lambda_2^T K_f(u,h) = \sum_\chi \Lambda_2^T K_{f,\chi}(u,h)$. Moreover, by [A1], Lemma 4.4, and [A2], Lemma 1.4, there exists some $N' > 0$ such that for any $N > 0$ there is $c > 0$ such that for all $u \in \mathbb{G}^1$ and h in a Siegel domain S, we have

$$\sum_\chi \sum_P n(A_P)^{-1} \int_{\Pi^G(M)} |\sum_{\phi\in\mathfrak{B}_P(\pi)_\chi} E(u, I_P(\pi,f)\phi,\pi)\Lambda^T \overline{E}(h,\phi,\pi)| d\pi \le c||u||^{N'}||h||^{-N}.$$

Hence

$$\sum_\chi |K_{f,\chi}(u,h)| = \sum_\chi |\Lambda_2^T K_{f,\chi}(u,h)| \le c||h||^{-N}$$

for $u \in \omega$ and $h \in S$, and the proposition follows. □

By [A1], (3.1), p. 928, the Eisenstein series $E(x,\phi,\zeta)$, and each of its derivatives in x, is bounded by $c(\zeta)||x||^N (x \in \mathbb{G})$, where $c(\zeta)$ is a locally bounded function on the set of $\zeta \in \mathfrak{A}^*_{\mathbb{C}}$ where $E(x,\phi,\zeta)$ is regular. Let us review the well known fact that on $i\mathfrak{A}^*$, where $E(x,\phi,\zeta)$ is regular, it has polynomial growth in ζ. For this purpose, embed $\mathbb{R}^\times_{>0}$ in $\mathbb{A}^\times_E$ via $x \mapsto (x,\dots,x,1,\dots)$ (x in the archimedean components, 1 in the finite components). Put (as in [A1], p. 925) $\Pi = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}^\times_E/E^\times\mathbb{R}^\times_{>0}, S^1)$, where S^1 is the unit circle in the complex plane, and $\Pi_0 = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}^\times_E/E^\times\mathbb{R}^\times_{>0}U, S^1)$, where $U = \prod_v U_v$, and U_v is the maximal compact subgroup of $E^\times_v$. If $v_j\,(1 \le j \le r)$ are the archimedean places of E, for $\mu \in \Pi_0$ we have $\mu(z_{v_j}) = |z_{v_j}|^{\mu_j}_{v_j}$, $\mu_j \in i\mathbb{R}$, with $\sum_j \mu_j[E_{v_j} : \mathbb{R}] = 0$. These $\mu_j\,(\mu \in \Pi_0)$ form a discrete subgroup of rank $r-1$ in these hyperplane. Denote by $C_0(\mu)$ a function on Π_0 of the form $C_0(\mu) = c\prod_j(1-\mu_j^2)^{c_j}$ with $c > 0$, $c_j > 0$. In fact it depends only on the restriction of μ to $E^\times U E^\times_\infty$, where $E^\times_\infty = \prod_j E^\times_j$.

Choose a set of representatives $\tilde\mu$ for Π/Π_0, and a function $C_{\tilde\mu}$ on Π_0 of the above type for each $\tilde\mu$. Denote by $C(\mu)$ the function on Π defined by $C(\mu) = C_{\tilde\mu}(\mu/\tilde\mu)$ if $\mu = \tilde\mu$ on U; then $C(\mu)$ depends only on the restriction of μ to $E^\times U E^\times_\infty$. Denote by $c(\mu)$ a non negative valued function on Π which depends only on the restriction of μ to U. Using the existence of zero free regions of L-functions about $\mathrm{Re}(\zeta) = 1$, we have:

4.1 Lemma. *There are functions $C_1(\mu)$, $C_2(\mu)$, $c_1(\mu)$, $c_2(\mu)$ as above, such that for complex ζ with $|\mathrm{Re}\,\zeta| \le C_1(\mu)^{-1}(1+(\mathrm{Im}\,\zeta)^2)^{-c_1(\mu)}$ we have that $|L(\zeta,\mu)/L(1+\zeta,\mu)|$ is bounded by $C_2(\mu)(1+(\mathrm{Im}\,\zeta)^2)^{c_2(\mu)}$ (a bound of the same type holds for any derivative of the quotient, by Cauchy's integral formula).*

Proof. For a complex number $s = \sigma + it$, put $L_f(s,\mu) = \prod_{v<\infty} L(s,\mu_v)$. This $L_f(s,\mu)$ converges absolutely on $\sigma \ge 1+\delta$, $\delta > 0$, by [La], p. 158. It has analytic continuation to the entire complex plane, and it has no zeroes on $\sigma = 1$. For any vertical strip of finite width there are $C(\mu)$ and $c(\mu)$ such that for all μ, and s with σ in the strip, $|L_f(s,\mu)|$ is bounded by $C(\mu)(1+s\overline{s})^{c(\mu)}$. In fact, by [La], p. 334, for any $t_0 > 0$ there is $m > 0$ such that $s(s-1)L_f(s,\mu)$ is $O(|t|^m)$ in the vertical strip $-1 < \sigma < 2$, $|t| > t_0$. Then $|L_f(s,\mu)| < C_1^{-1}|t|^m$

in this strip, and by Cauchy's integral formula we also have $|L'_f(s,\mu)| < C_1^{-1}|t|^m$ there. Take $\varepsilon_0 > 0$ such that $|L_f(s,\mu)| > C_2$ in $|t| \leq 1$, $|\sigma - 1| \leq \varepsilon_0$. Here C_i are positive constants. As in [La], p. 313, one has $|L_f(\sigma,1)^3 L_f(\sigma+it,\mu)^4 L_f(\sigma+2it,\mu^2)| \geq 1$ on $\sigma > 1$. Hence $|L_f(s,\mu)| \geq |L_f(\sigma,1)|^{-3/4}|L_f(\sigma+2it,\mu^2)|^{-1/4} > C_3|\sigma-1|^{3/4}|t|^{-m/4}$ on $\sigma > 1$, $|t| \geq 1$. Put $C_4 = (C_1C_3/3)^4$, and $m' = 6m$. Given ζ with $1 - C_4|t|^{-m'} <$ Re $\zeta \leq 1$, put $s = 1 + C_4|t|^{-m'} + it$. Then $|L_f(s,\mu) - L_f(\sigma,\mu)| = |\int_{\text{Re }\zeta}^{\text{Re }s} L'_f(u+it,\mu)du|$ is bounded by $C_1^{-1}|t|^m(\text{Re }s - \text{Re }\zeta) \leq 2(C_4/C_1)|t|^{m-m'}$. By the triangle inequality,

$$|L_f(\zeta,\mu)| \geq |L_f(s,\mu)| - |L_f(s,\mu) - L_f(\zeta,\mu)| \geq C_3C_4^{3/4}|t|^{-(3m'+m)/4} - 2(C_4/C_1)|t|^{m-m'}$$

on $|\text{ Im }\zeta| \geq 1$. Since $|L_f(\zeta,\mu)| > C_2$ in $|\text{ Im }\zeta| \leq 1$, $|\text{ Re }\zeta - 1| \leq \varepsilon_0$, we are done (replacing C_i by $C_i(\mu)$ and m by $c(\mu)$, and using Stirling's formula to bound the ratio of the gamma factors at infinity).

Note that for characters μ of finite order, much better estimates are known: Im ζ can be replaced by ℓn Im ζ in our estimates. But we need here only our crude estimates. □

From now on $n = 3$, namely $\mathbf{G}$ is the projective quasi-split unitary group $PU(3,E/F)$ specified above. Let μ be a unitary character of $\mathbb{A}_E^\times/E^\times\mathbb{R}_{>0}^\times$, and $\mathbf{B} = \mathbf{AU}$ the upper triangular subgroup, where $\mathbf{A}$ is the diagonal subgroup ($\mathbf{A}(\mathbb{A})$ consisting of $a = \text{ diag }(b, m, \bar{b}^{-1})$, $b \in \mathbb{A}_E^\times$, $m \in \mathbb{A}_E^\bullet$), and $\mathbf{U}$ is the unipotent upper triangular subgroup. The character μ defines a character of $\mathbb{B} = \mathbf{B}(\mathbb{A})$ (by $\mu(au) = \mu(b)$). As in [A1], p. 925, let $H^0(\mu)$ be the space of right $\mathbb{K}$-finite functions ϕ on $\mathbb{G} = \mathbf{G}(\mathbb{A})$ such that $\phi(uak) = \mu(b)\phi(k)$. Here $\mathbb{K} = \prod_v K_v$, where K_v is the standard maximal compact subgroup in $\mathbf{G}(F_v)$. For $\zeta \in \mathfrak{A}_\mathbb{C}^*$ (= the complex plane in our case), put $\phi(g,\mu,\zeta) = \phi(g)\delta(g)^{\frac{1}{2}+\zeta}$, where $\delta(g) = |b|_E^2$ for g in $\mathbf{G}(\mathbb{A})$ with Iwasawa decomposition $g = uak$, $u \in \mathbb{U}$, $k \in \mathbb{K}$, $a \in \mathbf{A}(\mathbb{A})$. The Eisenstein series is defined as the analytic continuation of $E(g,\phi,\mu,\zeta) = \sum_{\gamma \in B\backslash G}\phi(\gamma g,\mu,\zeta)$ ($B = \mathbf{B}(F)$, $G = \mathbf{G}(F)$), a series which converges in some right half plane. Denote by $I(\mu,\zeta)$ the $\mathbb{G}$-module normalizedly induced from the $\mathbb{B}$-module $\mu_\zeta = \mu \otimes \delta^\zeta$. The restriction of $I(\mu,\zeta)$ to $\mathbb{K}$ depends only on the restriction of μ to U, and its space is contained in $L^2(\mathbb{K})$.

By $M(\zeta)\phi$ we denote the image of ϕ under the action of the intertwining operator $M(\zeta) = M(\mu,\zeta) = M(\mathcal{J},\mu,\zeta)$, associated with the reflection $\mathcal{J}$ ([A1], p. 926). As $\phi(g,\mu,\zeta)$ lies in the induced $I(\mu,\zeta)$, the function $(M(\zeta)\phi)(g, {}^{\mathcal{J}}\mu, \mathcal{J}\zeta)$ lies in the induced $I({}^{\mathcal{J}}\mu, \mathcal{J}\zeta)$. The operator $M(\zeta)$ has no singularity on the imaginary axis.

A $\mathbb{K}$-*type* κ is a finite set of equivalence classes of irreducible $\mathbb{K}$-modules. The norm of the intertwining operator $M(\mu,\zeta)$ on the κ-component of $I(\mu,\zeta)$ is denoted by $||M(\mu,\zeta)||_\kappa$. This component is zero unless the restriction of μ to U lies in a finite set depending on κ.

4.2 Proposition. *(1) Fix a $\mathbb{K}$-type κ. There are functions $C_j(\mu)$, $c_j(\mu)$, such that for any complex ζ in the set Ω defined by $|\text{ Re}(\zeta)| \leq C_1(\mu)^{-1}(1 + (\text{Im }\zeta)^2)^{-c_1(\mu)}$, we have that $||M(\mu,\zeta)||_\kappa$ is bounded by $C_2(\mu)(1 + (\text{ Im }\zeta)^2)^{c_2(\mu)}$. A bound of the same type holds for any derivative of the intertwining operator.*
(2) Given κ, there are $C_j(\mu)$, $c_j(\mu)$, such that for any ζ in Ω, and for any ϕ in the κ-component of $I(\mu,\zeta)$, the integral $\int_{G\backslash\mathbb{G}}|\Lambda^T E(g,\phi,\mu,\zeta)|^2 dg$ is bounded by the product of $||\phi||$,

$C_2(\mu)(1+(\operatorname{Im}\zeta)^2)^{c_2(\mu)}$, *and* $\exp(c_3(\mu)||T||)$.
(3) For any $\mathbb{K}$*-finite* $f \in C_c^\infty(\mathbf{G}(\mathbb{A}))$ *there are* $C_j(\mu)$, $c_j(\mu)$, *such that for any* ζ *in* Ω, $x \in \mathbb{G}^1$, *we have that* $|E(g, I(\mu,\zeta;f)\phi,\mu,\zeta)|$ *is bounded by the product of* $||\phi||$, $C_2(\mu)(1+(\operatorname{Im}\zeta)^2)^{c_2(\mu)}$, *and* $||g||^{c_3(\mu)}$. *The same holds for any derivative in* ζ *of this function.*

Proof. By [Sh], end of §2, M is the product of a normalized intertwining operator, which is easily majorized, a factor of absolute value one, and a product of two quotients of L-functions of the type which appears in Lemma 4.1. (1) follows. (2) follows from this, via the scalar product formula of [A2], Lemma 4.2, p. 119 (see also J. Arthur, On the inner product of truncated Eisenstein series, Duke Math. J. 49 (1982), 35-70).

For (3), note that in general, given a compact ω_1 in $\mathbb{G}^1$, we have $\Lambda^T\phi(g) = \phi(g)$ for any $g \in \omega_1$ and any function ϕ, provided that T is sufficiently regular with respect to ω_1. Indeed, [A1], (5.2), p. 936, asserts that there is a constant $c > 0$ such that for any $\varpi \in \hat{\Delta}$, $\gamma \in G$, and $g \in \mathbb{G}^1$, we have $\varpi(H(\gamma g)) \le c(1+\ell n||g||)$. It suffices to take T with $\varpi(T) \ge c(1+\ell n||g||)$ for all $\varpi \in \hat{\Delta}$ and $g \in \omega_1$. In fact we take T_1 with $\varpi(T_1) \ge c$ for all $\varpi \in \hat{\Delta}$, and $T = T_1 \cdot \max\{1+\ell n||g||; g \in \omega_1 \cdot \operatorname{supp}(f)\}$. Then $\Lambda^T\phi(g) = \phi(g)$ for all g in the compact $\omega_1 \cdot \operatorname{supp}(f)$, and $||T|| \le c_1 \max\{1+\ell n||g||; g \in \omega_1\}$ for some $c_1 = c_1(f) > 0$. For these f, ω_1, and T, we have for all $g \in \omega_1$,

$$E(g, I(\mu,\zeta;f)\phi,\mu,\zeta) = \int_{\mathbb{G}} E(gh,\phi,\mu,\zeta)f(h)dh$$
$$= \int_{\mathbb{G}} \Lambda^T E(gh,\phi,\mu,\zeta)f(h)dh = \int_{G\backslash\mathbb{G}} \Lambda^T E(h,\phi,\mu,\zeta)K_f(g,h)dh,$$

where $K_f(g,h) = \sum_{\gamma\in G} f(g^{-1}\gamma h)$. But $|K_f(g,h)| \le c_2||g||^N$, and (2) gives an L^2-bound for $\Lambda^T E$. Hence the expression to be estimated is bounded by the product of $||\phi||$, $C(\mu)(1+(\operatorname{Im}\zeta)^2)^{c_1(\mu)}$, and $\max ||g||^{c_3(\mu)}$. The maxima are taken over x in the compact ω_1. Finally, taking ω to be a compact neighborhood of the identity, we observe that for any $x \in \mathbb{G}^1$, $\max ||g||$ is bounded on $\omega_1 = x\omega$ by a multiple of $\max ||x||$, and (3) follows. □

Put $\mathbf{C}_1 = g_0^{-1}\mathbf{C}g_0$. This is the centralizer of $\mathcal{J}_1 = \begin{pmatrix} 0 & & \frac{1}{2} \\ & 1 & \\ 2 & & 0 \end{pmatrix} = g_0^{-1}\mathcal{J}_0 g_0$ in $\mathbf{G}$. Note that $g_0\mathcal{J}_0 g_0^{-1} = {}^t\mathcal{J}_1$. Put $\mathbb{K}_{\mathbf{C}} = \mathbb{K} \cap \mathbf{C}(\mathbb{A})$.

5. Proposition. *For a sufficiently large* T *and* $\operatorname{Re}(\zeta)$, *we have that* $\int_{\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})} \Lambda^T E(h,\phi,\mu,\zeta)dh$ *is absolutely convergent and equal to the sum of*

$$\text{(5.1.1)} \qquad \frac{\delta(\mu)}{2\zeta}\left[T^\zeta \int \phi(k)dk - T^{-\zeta}\int (M\phi)(k)dk\right]$$

(k ranges over $\mathbb{K}_{\mathbf{C}}$*), and* $\mathcal{J}(\mu,\phi,\zeta) = \varepsilon(\mu)\int_{\mathbf{B}(\mathbb{A})\cap\mathbf{C}_1(\mathbb{A})\backslash\mathbf{C}_1(\mathbb{A})} \phi(hg_0^{-1},\mu,\zeta)dh$. *Here* $\mu : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ *indicates also the representation* $\rho : \operatorname{diag}(b,m,\overline{b}^{-1}) \to \mu(b)$ *of* $\mathbf{B}(\mathbb{A})$*;* $\delta(\mu)$ *is* 0 *unless* μ *factorizes through* $\nu(b) = |b|$, *where* $\delta(\mu) = |\mathbb{A}_E^1/E^\times|$, *and* $\mathbb{A}_E^1 = \{b \in \mathbb{A}_E^\times; |b| = 1\}$*; and* $\varepsilon(\mu)$ *is*

0 *unless* μ *is* 1 *on* $\mathbb{A}_E^\bullet = \{b \in \mathbb{A}_E^\times; b\overline{b} = 1\}$, *where* $\varepsilon(\mu) = |\mathbb{A}_E^\bullet/E^\bullet|$. *Moreover, on* $\zeta \in i\mathbb{R}$, *the function* $\mathcal{J}(\mu, \phi, \zeta)$ *is holomorphic.*

Proof. By virtue of Proposition 2, for ζ with large $\mathrm{Re}\,(\zeta)$ the Eisenstein series $E(h, \phi, \mu, \zeta)$ can be written as

$$\sum_{\gamma \in \mathbf{B}_0(F)\backslash \mathbf{C}(F)} \phi(\gamma h, \mu, \zeta) + \sum_{\gamma \in g_0 \mathbf{B}_0(F) g_0^{-1} \cap \mathbf{C}(F) \backslash \mathbf{C}(F)} \phi(g_0^{-1}\gamma h, \mu, \zeta).$$

The constant term formula $E_{\mathbf{U}}(g, \zeta) = \phi(g, \zeta) + (M\phi)(g, \mathcal{J}\zeta)$ implies that $\Lambda^T E(h, \zeta)$ is the sum of

$$\sum_{\gamma \in B_0 \backslash C} \chi(\delta(\gamma h) < T)\delta(\gamma h)^{\zeta + \frac{1}{2}}\phi(\gamma h) - \sum_{\gamma \in B_0 \backslash C} \chi(\delta(\gamma h) > T)\delta(\gamma h)^{\frac{1}{2} - \zeta}(M\phi)(\gamma h)$$

and

$$\sum \phi(g_0^{-1}\gamma h, \mu, \zeta)\chi(\delta(g_0^{-1}\gamma h) < T) - \sum (M\phi)(g_0^{-1}\gamma h, \mathcal{J}\mu, \mathcal{J}\zeta)\chi(\delta(g_0^{-1}\gamma h) > T).$$

Here $B_0 = \mathbf{B}_0(F)$ and $C = \mathbf{C}(F)$. The last two sums range over $\gamma \in g_0 B_0 g_0^{-1} \cap C\backslash C$.

We claim that $\delta(g_0^{-1}h)$ is bounded on $(h \in)\mathbf{C}(\mathbb{A})$. Indeed, the proof of Proposition 29(a) below shows that $\delta_v(g_0^{-1}h)$ is bounded (by 1 if v is finite) when v stays prime in E. If the place v of F splits in E then note that $Z_v\backslash C_{1v}$ consists of $g_0^{-1}dmkg_0$ with a diagonal $d(= \mathrm{diag}\,(a, 1, b), a, b \in F_v^\times)$, unipotent $m = \begin{pmatrix} 1 & & c \\ & 1 & \\ 0 & & 1 \end{pmatrix}$, and $k \in K_v \cap C_v$. Note that

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} g_0^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 2 \\ 1/2 & -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$Ng_0^{-1}dmK = N \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/x \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} K$$
$$= N\ \mathrm{diag}\,(z/(1, z, y - xz), (1, z, y - xz)/(1, x, y), (1, x, y)/x)K.$$

Here $x = 2/a, z = 2b, y = c + 2b/a$, and (x, y, z) is an element of F_v with absolute value $\max(|x|_v, |y|_v, |z|_v)$ if v is finite, and $(x\overline{x} + y\overline{y} + z\overline{z})^{1/2}$ if not. Hence $\delta_v(g_0^{-1}dmkg_0) = |xz/(1, x, y)(1, z, y - xz)|_v$ is bounded (by 1 if v is finite), and so is $\delta(g_0^{-1}h) = \prod_v \delta_v(g_0^{-1}h)$ for h in $\mathbf{C}(\mathbb{A})$.

The function $\Lambda^T E(h,\zeta)$ is rapidly decreasing as a function of h, by [A2], p. 108, ℓ. 8. Its integral over h in $C\backslash\mathbf{C}(\mathbb{A})$ converges to a meromorphic function in ζ, whose poles are at most those of $E(h,\zeta)$. For a large $\mathrm{Re}(\zeta)$, the integral is the sum of

$$\int\limits_{B_0\backslash\mathbf{C}(\mathbb{A})} \chi(\delta(h)<T)\delta(h)^{\zeta+\frac{1}{2}}\phi(h)dh = \delta(\mu)\int\limits_{\mathbb{K}_{\mathbf{C}}}\phi(k)dk\cdot\int\limits_{|b|_E^{n-1}<T}|b|_E^{(n-1)(\zeta+\frac{1}{2})-(n-2)}d^\times b$$

$$=\delta(\mu)\int\limits_{\mathbb{K}_{\mathbf{C}}}\phi(k)dk\cdot\int\limits_{|b|_E<T^{1/2}}|b|_E^{2\zeta}d^\times b = \frac{\delta(\mu)}{2\zeta}T^\zeta\int\limits_{\mathbb{K}_{\mathbf{C}}}\phi(k)dk,$$

(for the first equality we used $dh=\delta_{\mathbf{C}}^{-1}(b)dndbdk$, in the second we recalled that $n=3$), and

$$-\int\limits_{B_0\backslash\mathbf{C}(\mathbb{A})}\chi(\delta(h)>T)\delta(h)^{\frac{1}{2}-\zeta}(M\phi)(h)dh$$

$$=-\delta(\mu)\int\limits_{\mathbb{K}_{\mathbf{C}}}(M\phi)(k)dk\cdot\int\limits_{|b|_E^{n-1}>T}|b|_E^{(n-1)(\frac{1}{2}-\zeta)-(n-2)}d^\times b$$

$$=-\delta(\mu)\int\limits_{\mathbb{K}_{\mathbf{C}}}(M\phi)(k)dk\cdot\int\limits_{|b|_E>T^{1/2}}|b|_E^{-2\zeta}d^\times b=-\frac{\delta(\mu)}{2\zeta}T^{-\zeta}\int\limits_{\mathbb{K}_{\mathbf{C}}}(M\phi)(k)dk,$$

and

$$\int\limits_{g_0Bg_0^{-1}\cap C\backslash\mathbf{C}(\mathbb{A})}\phi(g_0^{-1}h,\mu,\zeta)dh=\varepsilon(\mu)\int\limits_{\mathbf{B}(\mathbb{A})\cap\mathbf{C}_1(\mathbb{A})\backslash\mathbf{C}_1(\mathbb{A})}\phi(hg_0^{-1},\mu,\zeta)dh.$$

Here $\mathbf{B}(\mathbb{A})\cap\mathbf{C}_1(\mathbb{A})=\mathbb{A}_E^\bullet\times\mathbb{A}_E^\bullet$. The local factors in the last integral are computed in Proposition 29 when v stays prime in E, but we do not need here this computation.

Let us note that $\int_{\mathbb{K}_{\mathbf{C}}}\phi(k)dk$ and $\int_{\mathbb{K}_{\mathbf{C}}}(M\phi)(k)dk$ are equal when $\delta(\mu)\neq 0$ (we may and will take $\mu=1$ to represent the class of the characters $\nu^s, s\in i\mathbb{R}$). Indeed, the intertwining operator M can be written as a product of a scalar $m(\mu,\zeta)$ and the product $\otimes R_v(\mu_v,\zeta)$ of local normalized intertwining operators. At $\mu=1$, the induced $I(\mu_v,0)$ is irreducible, and $R_v(\mu_v,0)$ acts as the identity. The normalizing factor $m(\mu,\zeta)$ is computed in Shahidi [Sh], last line in Section 2, to have the value 1 at $\mu=1,\zeta=0$. It follows that (5.1.1) is holomorphic on $\zeta\in i\mathbb{R}$, and it is slowly increasing there. Since $E(h,\zeta)$ is holomorphic and slowly increasing on $\zeta\in i\mathbb{R}$, so is $\int\Lambda^T E(h,\zeta)dh$, and consequently so is $\mathcal{J}(\mu,\phi,\zeta)$. □

5.1 Lemma. *Let f_1, f_2 be Schwartz (smooth, rapidly decreasing as $|\zeta|\to\infty$) functions on $i\mathbb{R}$ with $f_1(0)=f_2(0)$. Then* $\lim\limits_{T\to\infty}\int_{i\mathbb{R}}[f_1(\zeta)\zeta^{-1}T^\zeta - f_2(\zeta)\zeta^{-1}T^{-\zeta}]d\zeta = 2\pi f_1(0)$.

Proof. An elementary proof of this is given at [FM1], end of proof of Lemma 1. □

5.2 Lemma. *Let π be a unitary $\mathbf{G}(\mathbb{A})$-module on a Hilbert space H, and let H^0 be the subspace of $\mathbb{K}$-finite vectors. Suppose that each $\mathbb{K}$-type has finite multiplicity, and let L_1, L_2 be linear forms on H^0. Let f be $\mathbb{K}$-finite in $C_c^\infty(\mathbf{G}(\mathbb{A}))$, and $\{\phi\}$ an orthonormal basis of H^0. Then the sum $\sum_{\{\phi\}} L_1(\pi(f)\phi)\overline{L_2(\phi)}$ is independent of the choice of the orthonormal basis $\{\phi\}$. In particular, if $f = f_1 * f_2^*$, $f_2^*(g) = \overline{f_2(g^{-1})}$, f_1 and f_2 are $\mathbb{K}$-finite elements of $C_c^\infty(\mathbf{G}(\mathbb{A}))$, then $\sum_{\{\phi\}} L_1(\pi(f_1)\phi)\overline{L_2(\pi(f_2)\phi)} = \sum_{\{\phi\}} L_1(\pi(f)\phi)\overline{L_2(\phi)}$.* □

The group $\mathbf{G}$ over F has rank one, namely $\mathbf{B}$ is the unique (up to conjugation) proper parabolic subgroup of $\mathbf{G}$ over F. The non-cuspidal spectrum consists of residual spectrum, which does not contribute to the Fourier summation formula, as the residual spectrum is non-generic, and of the continuous spectrum. The kernel $K_{f,c}(g,h)$ over the continuous spectrum has the form

$$\frac{1}{4\pi}\sum_\mu \int_{-\infty}^{\infty} \sum_\phi E(g, I(\mu, i\zeta, f)\phi, \mu, i\zeta)\overline{E}(h, \phi, \mu, i\zeta)d\zeta.$$

6. Proposition. *Suppose that $f = f_1 * f_2^*$, where $f_2^*(g) = \overline{f}_2(g^{-1})$, and f_1, f_2 are $\mathbb{K}$-finite elements of $C_c^\infty(\mathbf{G}(\mathbb{A}))$. Then the contribution $\iint K_{f,c}(u,h)\overline{\psi}(u)dudh$ from the continuous spectrum is the sum of*

$$\frac{1}{2}\sum_\phi E_\psi(I(1,0,f_1)\phi, 1, 0)\cdot \int_{\mathbb{K}_{\mathbf{C}}} (\overline{I(1,0,f_2)\phi})(k)dk \tag{6.1}$$

and

$$\frac{1}{4\pi}\sum_\mu \int_{-\infty}^{\infty} \sum_\phi E_\psi(I(\mu, i\zeta, f_1)\phi, \mu, i\zeta)\overline{\mathcal{J}}(\mu, I(\mu, i\zeta, f_2)\phi, i\zeta)d\zeta. \tag{6.2}$$

The first sum ranges over the characters $\mu : \mathbb{A}_E^\times/E^\times\mathbb{A}_E^\bullet\mathbb{R}_{>0}^\times \to S^1$. The second ranges over a smooth orthonormal basis $\{\phi\}$ of the space $I(\mu, i\zeta)$ ($\mu = 1$ in (6.1)). Moreover,

$$\frac{1}{4\pi}\sum_\mu \int_{-\infty}^{\infty} |\sum_\phi E_\psi(I(\mu, i\zeta, f_1)\phi, \mu, i\zeta)\overline{\mathcal{J}}(\mu, I(\mu, i\zeta, f_2)\phi, i\zeta)|d\zeta$$

is finite.

Proof. By Proposition 4 the sum

$$\frac{1}{4\pi}\sum_\mu \int_{\mathbb{R}} |\sum_\phi E_\psi(I(\mu, i\zeta; f_1)\phi, \mu, i\zeta) \int_{C\backslash\mathbb{C}} \Lambda^T\overline{E}(h, I(\mu, i\zeta; f_2)\phi, \mu, i\zeta)dh|d\zeta$$

is finite. Using Proposition 5, the last claim of the proposition would follow once we show that for $\mu = 1$, the integral

$$\int_{\mathbb{R}} |\frac{T^{i\zeta} - T^{-i\zeta}}{2i\zeta}A_\mu(i\zeta) - \frac{T^{-i\zeta}}{2i\zeta}B_\mu(i\zeta)|d\zeta$$

is finite, where

$$\overline{A}_\mu(\zeta) = \sum_\phi \overline{E}_\psi(I(\mu,\zeta;f_1)\phi,\mu,\zeta)\int_{\mathbb{K}_{\mathbf{C}}}(I(\mu,\zeta;f_2)\phi)(k)dk,$$

and

$$\overline{B}_\mu(\zeta) = \sum_\phi \overline{E}_\psi(I(\mu,\zeta;f_1)\phi,\mu,\zeta)[\int\limits_{\mathbb{K}_{\mathbf{C}}}(I(\mu,\zeta;f_2)\phi)(k)dk - \int\limits_{\mathbb{K}_{\mathbf{C}}}(M(\mu,\zeta)I(\mu,\zeta;f_2)\phi)(k)dk].$$

In fact, even the sum over all $\mu : \mathbb{A}_E^\times/E^\times\mathbb{R}_{>0}^\times \to S^1$ is finite. Indeed, by Proposition 4.2(3), for a given f_1 with a fixed $\mathbb{K}$-type, $|E_\psi(I(\mu,i\zeta;f_1)\phi,\mu,i\zeta)|$ is bounded by some $C(\mu)(1+\zeta^2)^{c(\mu)}$. Moreover,

$$|\int_{\mathbb{K}_{\mathbf{C}}}(M(\mu,i\zeta)I(\mu,i\zeta;f_2)\phi)(k)dk| \le ||I(\mu,i\zeta;f_2)||,$$

where the last norm is the operator norm on the finite dimensional space of vectors with a given $\mathbb{K}$-type. This norm is bounded by the norm of some matrix of the form

$$(\int_{\mathbf{A}(\mathbb{A})\mathbf{U}(\mathbb{A})} f_2(k_i^{-1}auk_j)du\cdot\mu(a)\delta(a)^\zeta da), \qquad (k_i\in\mathbb{K}).$$

It follows that the functions $A_\mu(i\zeta)$ and $B_\mu(i\zeta)$ are Schwartz functions on the imaginary axis $i\mathbb{R}$ (the sum over ϕ ranges only over vectors with the given $\mathbb{K}$-type). The absolute convergence follows, and so does the proposition, on using Lemma 5.1. □

Remark. An integral analogous to that of Lemma 5.1 is considered in [F6], but in the Remark on p. 431 of [F6], "Consequently we also have" on ℓ. -10 should read: "However we do not have", and "However we prefer to" on ℓ. -8 should be: "We shall".

Corollary. *For any test function $f=\otimes f_v$, $f_v\in C_c^\infty(G_v)$ for all v and $f_v=f_v^0$ for almost all v, the Fourier summation formula takes the form*

$$\Psi(0,f,\psi) + \sum_{b\in E^\times/E^\bullet}\Psi(b,f,\psi) = \sum_\pi n(\pi)(W_\psi\overline{P})_\pi(f) + (6.1) + (6.2),$$

where π ranges over a set of representatives for the equivalence classes of cuspidal generic cyclic representations of $\mathbf{G}(\mathbb{A}) = PU(3,E/F)_{\mathbb{A}}$. □

3. Matching Fourier Orbital Integrals.

The proof of the global representation theoretic results relies on a comparison of the two summation formulae obtained in Proposition 1 and Corollary 6 (here $n = 3$). Our technique will be to show that given $f = \otimes f_v$ on $\mathbf{G}(\mathbb{A})$ there is a matching $f' = \otimes f'_v$ on $\mathbf{H}'(\mathbb{A})$ (in a sense soon to be made precise), and vice versa, such that the geometric sides of the formulae be equal. The resulting equality of spectral sides will then be used to derive the representation theoretic applications.

Choosing a product measure $dh = \otimes dh_v$ on $\mathbf{C}(\mathbb{A})$, and $du = \otimes du_v$ on $\mathbf{U}(\mathbb{A})$, and putting $\psi_v(x) = \boldsymbol{\psi}_v(x + \overline{x})$, the global integrals $\Psi(b, f, \psi)$ become products over all v of the local integrals

$$\Psi(b, f_v, \psi_v) = \int_{U_v} \int_{C_v} f_v(ug_b g_0 h)\psi_v(x) du dh,$$

$$g_b = \begin{pmatrix} b & & 0 \\ & 1 & \\ 0 & & \overline{b}^{-1} \end{pmatrix}, \quad u = \begin{pmatrix} 1 & x & \frac{1}{2}x\overline{x} + iy \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{pmatrix},$$

if $b \in E_v^\times / E_v^\bullet$ and $i \in E_v^\times$ with $i + \overline{i} = 0$ ($g_b =$ diag$(1, 1, b^{-1})$ and $C_v =$ Levi subgroup of type $(2, 1)$ when v is split), and $\Psi(0, f_v, \psi_v) = \int_{U_v/U_{0v}} \int_{C_v} f_v(uh)\psi_v(x) du\, dh$.

Similarly, the integrals of $f' = \otimes f'_v$ on $\mathbf{H}'(\mathbb{A}) = PGL(2, \mathbb{A}_E)$ are the products of

$$\Psi(b, f'_v, \psi_v) = \int_{\mathbf{N}(E_v)} \int_{PGL(2,F_v)} f'_v(n \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \eta h)\psi_v(2x) dn\, dh,$$

if $b \neq 0$ ($\eta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$), and if $b = 0$ of

$$\Psi(0, f'_v, \psi_v) = \int_{\mathbf{N}(E_v)} \int_{PGL(2,F_v)} f'_v(n \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}^{-1} h)\psi_v(2x) dn\, dh.$$

7. Proposition. *Let v be a non-archimedean place of F. For every $f_v \in C_c^\infty(G_v)$ there exists $f'_v \in C_c^\infty(H'_v)$, and for each such f'_v there exists f_v, such that*

$$\Psi(b, f_v, \psi_v) = |2|\ |b|^{1/2}\Psi(b, f'_v, \psi_v) \tag{7.1}$$

for all $b \in E_v^\times / E_v^\bullet$ ($\simeq F_v^\times$ in the split case).

Moreover, if f_v and f'_v satisfy (7.1), then $\Psi(0, f_v, \psi_v) = \Psi(0, f'_v, \psi_v)$.

Remark. (1) The cases of split v and non-split v would require completely different discussion. We do not treat the archimedean cases, although their treatment is not so distant from that of the non-archimedean cases. Consequently our global result would hold only for global fields with no archimedean places, namely function fields.
(2) The proof of Proposition 7 is based on computing the asymptotic behavior of the integrals as $b \to \infty$. Once the proof is completed we shall show that when one of f_v or f'_v is spherical, so can

be chosen the other, in fact these spherical functions would be related by the correspondence of representations, via the theory of the Satake transform. The proof of the global result requires both the transfer result of general functions, as in Proposition 7, and those of the spherical functions, stated once the proof of Proposition 7 is complete, in Propositions 14 and 16.

Definition. Functions $f_v \in C_c^\infty(G_v)$ and $f'_v \in C_c^\infty(H'_v)$ are called *matching* if $\Psi(b, f_v, \psi_v) = |4b|^{1/2}\Psi\ (b, f'_v, \psi_v)$ for all $b \in E_v^\times$ (by Proposition 7, this identity implies that $\Psi(0, f_v, \psi_v) = \Psi(0, f'_v, \psi_v))$.

To prove the local matching theorem, we study the asymptotic behaviour of the local integrals. First we deal with a place which does not split, and use local notations. Let E/F be a quadratic extension of non-archimedean fields of characteristic $\neq 2$,

$$G = \left\{ g \in PGL(3,E); g\mathcal{J}{}^t\overline{g} = \mathcal{J} = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \right\},$$

$$C = \left\{ g \in G; g\mathcal{J}_0 g^{-1} = \mathcal{J}_0 = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix} \right\} = g_0 Z_G \begin{pmatrix} 0 & & \frac{1}{2} \\ & 1 & \\ 2 & & 0 \end{pmatrix} g_0^{-1},$$

$$U = \left\{ u = \begin{pmatrix} 1 & x & \frac{1}{2}x\overline{x} + iy \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{pmatrix};\ x \in E, y \in F \right\},$$

$$g_0 = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \quad g_0^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ \frac{1}{2} & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & & 0 \\ & -1 & \\ 0 & & 2 \end{pmatrix} g_0 \begin{pmatrix} \frac{1}{2} & & 0 \\ & -1 & \\ 0 & & 2 \end{pmatrix},$$

$$g_0^{-1} \begin{pmatrix} b & & 0 \\ & 1 & \\ 0 & & \overline{b}^{-1} \end{pmatrix} g_0 = \begin{pmatrix} (c+1)/2 & d/2 & (c-1)/4 \\ d & c & d/2 \\ c-1 & d & (c+1)/2 \end{pmatrix}$$

with

$$c = (b + \overline{b}^{-1})/2,\ d = (\overline{b}^{-1} - b)/2,\ E = F(i),\ i + \overline{i} = 0,\ \frac{1}{2}(I - \mathcal{J}_0) = {}^t\varepsilon_0\varepsilon_0,\ \varepsilon_0 = (0,1,0).$$

Then $g_0^{-1} \begin{pmatrix} b & & 0 \\ & 1 & \\ 0 & & \overline{b}^{-1} \end{pmatrix} g_0 \in C$ if $b = \overline{b}^{-1} \in E^\bullet = \{z \in E; z\overline{z} = 1\}$, and

$$\Psi(b) = \Psi(b,f) = \Psi(b,f,\psi) = \int_U \int_C f\left(u \begin{pmatrix} b & & 0 \\ & 1 & \\ 0 & & \overline{b}^{-1} \end{pmatrix} g_0 h \right) \psi(x) du\, dh$$

depends only on the projection of b in $E^\times/E^\bullet$.

Replacing f by $F_1(g) = \int_C f(gh)dh$, we obtain that Ψ is the integral over U of the function F_1 on the symmetric space G/C. Via the map $g \mapsto g{}^t\varepsilon_0\varepsilon_0 g^{-1} = (g{}^t\varepsilon_0)\varepsilon_0\mathcal{J}{}^t\overline{g}\mathcal{J}^{-1}$ the space

G/C embeds in the space of 3×3 matrices x over E with trace 1 and rank 1 and with ${}^t\overline{x}=\mathcal{J}^{-1}x\mathcal{J}$. Denote the image by X_0, and put $F_0(x)=F_1(g)$ if $x=-(g^t\varepsilon_0)^t(\overline{g}^t\varepsilon_0)\mathcal{J}$. Then

$$\Psi(b)=\int_F\int_E F_0\left[\begin{pmatrix}\frac{t}{2} & -\frac{xt}{2} & -b\overline{b}t\overline{t}\\ \frac{-\overline{x}}{4b\overline{b}} & \frac{x\overline{x}}{4b\overline{b}} & \frac{\overline{x}\overline{t}}{2}\\ \frac{-1}{4b\overline{b}} & \frac{x}{4b\overline{b}} & \frac{\overline{t}}{2}\end{pmatrix}\right]\psi(x)dx\,dy,$$

where $t=1-\frac{x\overline{x}+2iy}{4b\overline{b}}$.

The function f is compactly supported and locally constant, hence so is F_0. Consequently $\Psi(b)$ is supported on $|b|\geq c_0$ for some $c_0=c_0(f)>0$, and it is locally constant on $b\in E^\times$. We proceed to analyze the asymptotic behavior of $\Psi(b)$ as $b\to\infty$ in E. Extend F_0 to a locally constant compactly supported function on the space of 3×3 matrices over E. In fact we need only the extension of F_0 to a neighborhood of X_0 in this space, and this extension is used only to simplify the exposition.

Since F_0 is compactly supported, the last entry on the top row is bounded, hence $|t|<<|b|^{-1}$. We write $\alpha<<\beta$ if there is a positive constant $c=c(f)$ with $\alpha\leq c\beta$. Hence $\frac{x\overline{x}}{4b\overline{b}}\to 1$ as $b\to\infty$, and for a sufficiently large b we have

$$\Psi(b)=\int_E\int_F F_0\left(\begin{pmatrix}0 & -xt/2 & -b\overline{b}t\overline{t}\\ 0 & 1 & \overline{x}\overline{t}/2\\ 0 & 0 & 0\end{pmatrix}\right)\psi(x)dx\,dy.$$

Changing variables: $x\mapsto 2bx$, $y\mapsto 2b\overline{b}y$, we obtain

$$=|2|^{3/2}|b|^2\int_E\int_F F_0\left(\begin{pmatrix}0 & bxt & -b\overline{b}t\overline{t}\\ 0 & 1 & -\overline{b}\overline{x}\overline{t}\\ 0 & 0 & 0\end{pmatrix}\right)\psi(2bx)dxdy,\qquad t=x\overline{x}-1+iy.$$

Since F_0 is compactly supported there exists some $C\geq 1$ with $|x\overline{x}-1|\leq C/|b|$ and $|iy|\leq|2|C/|b|$. This C can be replaced by any bigger number. We may write $x=\varepsilon(1+z)$, with $|z|\leq C/|b|$ and $\varepsilon\in E^\times/(1+\frac{c}{b}R_E)$ with $\varepsilon\overline{\varepsilon}=1$. Here $c\in E^\times$ with $|c|=C$ (in particular C is chosen in $|E^\times|$). Note that ε is a representative in the class modulo $1+\frac{c}{b}R_E$. Put $z_1=(z+\overline{z})/2$ and $iz_2=(z-\overline{z})/2$ (thus $z=z_1+iz_2$ with $z_1,z_2\in F$), to obtain

$$|2|^{3/2}|b|^2\sum_{\substack{\varepsilon\in E^\times/(1+\frac{c}{b}R_E)\\ \varepsilon\overline{\varepsilon}=1}}\iint_{\substack{|iy|\leq|2|C/|b|\\ |z_1|\leq C/|b|}}dydz_1F_{00}(b\varepsilon(2z_1+iy))\psi(2b\varepsilon)\psi(2b\varepsilon z_1)\int_{|iz_2|\leq C/|b|}\psi(2b\varepsilon iz_2)dz_2.$$

Here $F_{00}(t)=F_0\begin{pmatrix}0 & t & -t\overline{t}\\ 0 & 1 & -\overline{t}\\ 0 & 0 & 0\end{pmatrix}$. Let $c(\psi)$ be the largest number such that $\int_{|a|\leq c}\psi(a)da=0$ ($a\in E$) for all $c>c(\psi)$, $c\in|E^\times|$. Define $b_1,b_2\in F$ by $b\varepsilon=b_1+ib_2$. Then the inner integral over z_2 of $\psi(2i^2b_2z_2)$ would vanish unless $|2i^2b_2|C/|ib|\leq c(\psi)$, namely $|b_2/b|\leq c(\psi)/|2i|C$.

On this domain $|i^2b_2y| \le (C|2|/|b|)(c(\psi)|b|/|2|C) = c(\psi)$, hence $\psi(b\varepsilon iy) = \psi(i^2b_2y) = 1$. This integration over z_2 then yields $(C/|b|)^{1/2}$, if dz_2 is normalized by $\int_{|z_2|\le 1} dz_2 = |i|^{1/2}$. Writing $x = 2z_1 + iy$ ($x \in E; y, z_1 \in F$), $dx = |2|^{1/2}dz_1dy$, the integral becomes $|4b|^{1/2}\Lambda_{\psi_2}(b)\int_E F_{00}(x)\psi(x)dx$, where $\psi_2(b) = \psi(2b)$, $c(\psi_2) = c(\psi)/|2|$, and

$$\Lambda_\psi(b) = C^{1/2} \sum_{\substack{\varepsilon\in E^\times/(1+\frac{c}{b}R_E)\\ \varepsilon\overline{\varepsilon}=1,\ |ib_2/b|\le c(\psi)/C}} \psi(b\varepsilon).$$

Our study of spherical functions below implies that for unramified E/F and ψ, c and C can be taken to be 1 without affecting the values of $\Lambda_\psi(b)$. Moreover it can be shown that it is equal to $\int_{E^\bullet}\psi(b\varepsilon)d\varepsilon$, but we do not prove this since we do not use it.

Clearly

$$\Psi(0,f,\psi) = \int_E\int_C f(uh)\psi(x)dudh = \int_E F_1(u)\psi(x)du \qquad \left(u = \begin{pmatrix} 1 & x & \frac{1}{2}x\overline{x} \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1\end{pmatrix}\right)$$

$$= \int_E F_0(-(u^t\varepsilon_0)^t(\overline{u}^t\varepsilon_0)\mathcal{J})\psi(x)dx = \int_E F_{00}(x)\psi(x)dx.$$

The factor $\Lambda_{\psi_2}(b)$ is independent of f and of $C = C(f)$: C can be replaced by any larger number without changing the value of $\Psi(b)$. Moreover, $\Psi(b)$ depends only on the image of b in $E^\times/E^\bullet$. We obtain the following characterization of the $\Psi(b,f,\psi)$ by means of their asymptotic behavior.

8. Lemma. *The function $\Lambda_\psi(b)$ on $b \in E^\times/E^\bullet$ has the following properties.*
(a) Given $f \in C_c^\infty(G)$ there is $B(f) > 0$ such that for all $|b| \ge B(f)$ we have $\Psi(b,f,\psi) = |4b|^{1/2}\Psi(0,f,\psi)\Lambda_{\psi_2}(b)$. Moreover, if $\Psi(b,f_1,\psi) = \Psi(b,f_2,\psi)$ for all $b \in E^\times/E^\bullet$, then $\Psi(0,f_1,\psi) = \Psi(0,f_2,\psi)$.
(b) Let Ψ be a locally constant function on $E^\times/E^\bullet$, such that $\Psi(b) = 0$ if $|b|$ is sufficiently small, and $\Psi(b) = |4b|^{1/2}\Psi(0)\Lambda_{\psi_2}(b)$ if $|b|$ is sufficiently large; $\Psi(0)$ is a (constant) complex number. Then there exists some $f \in C_c^\infty(G)$ with $\Psi(b,f,\psi) = \Psi(b)$ for all b.

Proof. The asymptotic behavior claimed in (1) is proven above. For (2) note that $f \mapsto \Psi(0,f,\psi)$ is linear in f, hence given Ψ there is some $f_1 \in C_c^\infty(G)$ with $\Psi(b) - \Psi(b,f,\psi)$ compactly supported on $b \in E^\times(/E^\bullet$; thus this difference vanishes for $|b|$ too big or too small). But if $\Psi(b)$ is locally constant and compactly supported on $E^\times/E^\bullet$, then clearly there is some $f_2 \in C_c^\infty(G)$, with $\Psi(b) = \Psi(b,f_2,\psi)$. Finally the last claim in (1) follows at once from the asymptotic behavior of $\Psi(b,f,\psi)$. □

Analogous characterization of the Fourier orbital integrals on H' by means of their asymptotic behavior will be studied next. For this purpose, given $f' \in C_c^\infty(H')$, $H' = PGL(2,E)$ (more precisely, we take f' on $GL(2,E)$ which transforms trivially under the center $Z' \simeq E^\times$), where E is a quadratic extension of F with char $F \ne 2$, and $b \in E^\times$, we put

$$\Psi_1(b,f',\psi) = \int_E\int_H f'\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1\end{pmatrix}\eta h\right)\psi(x/i)dxdh, \qquad \eta = \begin{pmatrix} 1 & i \\ 1 & -i\end{pmatrix},$$

and

$$\Psi_1(0,f',\psi)=\int_{E/F}\int_H f'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right)\psi(x/i)dx\,dh, \qquad H=PGL(2,F).$$

Note that $\Psi_1(b,f',\psi)$ depends only on the image of b in $E^\times/E^\bullet$, hence $\Psi(\overline{b})=\Psi(b)$.

9. Lemma. *For every $f'\in C_c^\infty(H')$ the function $\Psi_1(b,f',\psi)$ is locally constant on $E^\times$, invariant under $E^\bullet$, vanishes near 0, and there is $B(f')>0$ such that*

$$\Psi_1(b,f',\psi)=\Psi_1(0,f',\psi)\Lambda_\psi(b/i), \qquad |b|\geq B(f').$$

Conversely, if $\Psi_1'(b)$ is a function on $E^\times/E^\bullet$ which vanishes in a neighborhood of $0\in E$, and there is some $B'>0$ such that for $|b|\geq B'$ we have

$$\Psi_1'(b)=\Psi_1'(0)\Lambda_\psi(b/i) \qquad (\Psi_1'(0)\in\mathbb{C}),$$

then $\Psi_1'(b)=\Psi_1(b,f',\psi)$ for some $f'\in C_c^\infty(H')$, for all $b\in E^\times/E^\bullet$.

Finally, if $\Psi_1(b,f_1',\psi)=\Psi_1(b,f_2',\psi)$ for all $b\in E^\times/E^\bullet$, then $\Psi_1(0,f_1',\psi)=\Psi_1(0,f_2',\psi)$.

Proof. Put $F_1'(g)=\int_H f'(gh)dh$. Then F_1' is a locally constant compactly supported modulo Z' function on $GL(2,E)/GL(2,F)$, with $F_1'(zg)=F_1'(g)(z\in Z')$. This homogeneous space is isomorphic to the space X' of $g\in GL(2,E)$ with $g\overline{g}=1$ via the morphism $h\mapsto h\overline{h}^{-1}\in X'$. Put $F_0'(x)=F_1'(g)$ if $x=g\overline{g}^{-1}$. Then $F_0'(zx)=F_0'(x)(z\in E^\bullet)$. Extend F_0' to a locally constant compactly supported function on the space of 2×2 matrices over E, modulo $E^\bullet$ (in fact, only a neighborhood of $X'/E^\bullet$ is needed). Note that

$$\Psi_1(0,f',\psi)=\int_{E/F}F_0'\left(\begin{pmatrix}1 & x-\overline{x}\\ 0 & 1\end{pmatrix}\right)\psi(x/i)dx.$$

Since $\eta\overline{\eta}^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, we have

$$\begin{aligned}\Psi_1(b,f',\psi)&=\int_E F_0'\left(\begin{pmatrix}x/\overline{b} & b(1-x\overline{x}/b\overline{b})\\ 1/\overline{b} & -\overline{x}/\overline{b}\end{pmatrix}\right)\psi(x/i)dx\\ &=|b|\int_E F_0'\left(\begin{pmatrix}x & b(1-x\overline{x})\\ 0 & -\overline{x}b/\overline{b}\end{pmatrix}\right)\psi(x\overline{b}/i)dx \qquad (|b| \quad \text{large}).\end{aligned}$$

As f_0' is compactly supported, there is $C=C(f')>1$ with $|x\overline{x}-1|\leq C/|b|$ for x in the support of the integrand. Hence such x can be written in the form $x=\varepsilon(1+z)$, $z\in E$, $|z|\leq C/|b|$, $\varepsilon\in E^\times/(1+\frac{c}{b}R_E)$, $\varepsilon\overline{\varepsilon}=1$, where $c\in E^\times$ satisfies $|c|=C$ (and we assume that C is of the form $|c|$ for some $c\in E^\times$). Write $z=z_1+iz_2$ and $\varepsilon\overline{b}=b_1+ib_2$ with $z_1,z_2,b_1,b_2\in F$. Then $\Psi_1(b,f',\psi)$ becomes:

$$|b|\sum_{\substack{\varepsilon\in E^\times/(1+\frac{c}{b}R_E)\\ \varepsilon\overline{\varepsilon}=1}}\psi(\varepsilon\overline{b}/i)\int_{|z_1|\leq C/|b|}F_0'\left(\begin{pmatrix}\varepsilon & -2bz_1\\ 0 & -\overline{\varepsilon}b/\overline{b}\end{pmatrix}\right)\psi(\varepsilon\overline{b}z_1/i)dz_1\int_{|z_2|\leq C/|ib|}\psi(\varepsilon\overline{b}z_2)dz_2.$$

Let $c(\psi) \in |E^\times|$ be the largest real number such that $\int_{|a|\le c} \psi(a)da = 0$ for all $c \in |E^\times|, c > c(\psi)$. Then the inner integral over z_2 above vanishes unless $|b_1|C/|ib| \le c(\psi)$, thus $|b_1/b| \le c(\psi)|i|/C$. For such ε (hence b_1, b_2), we have

$$\psi(\varepsilon\overline{b}z_1/i) = \psi(b_2z_1), \qquad F_0'\left(\begin{pmatrix} \varepsilon & -2bz_1 \\ 0 & -\overline{\varepsilon}b/\overline{b}\end{pmatrix}\right) = F_0'\left(\varepsilon\begin{pmatrix} 1 & 2ib_2z_1 \\ 0 & 1\end{pmatrix}\right).$$

For the last equality note that $-2\overline{\varepsilon}bz_1 = 2ib_2z_1(1 - b_1/ib_2), |b_1/ib_2| \le c(\psi)/C$, and $2ib_2z_1$ is bounded (depending only on the support of f', or F_0').

As $|b_2|_F = |b_2|_E^{1/2} = |b|_E^{1/2}$, changing variables $z_1 \mapsto z_1/b_2$ the integral becomes

$$|b|^{1/2}[(C/|b|)^{1/2} \sum_{\substack{\varepsilon\in E^\times/(1+\frac{C}{b}R_E) \\ \varepsilon\overline{\varepsilon}=1,\ |b_1/b|\le c(\psi)/(C/|i|)}} \psi(\varepsilon\overline{b}/i)] \int_{E/F} F_0'\left(\begin{pmatrix} 1 & x-\overline{x} \\ 0 & 1\end{pmatrix}\right)\psi(x/i)dx,$$

since $F_0'(\varepsilon g) = F_0'(g)$ (F_0' is invariant under multiplication by scalars in $E^\bullet$), and we realize the isomorphism $E/F \xrightarrow{\sim} F$ by $x - \overline{x} = 2z_1 i$. The integral over x is equal to $\Psi(0, f', \psi)$. The sum over ε of $\psi(\varepsilon\overline{b}/i)$ depends only on the projection of b in $E^\times/E^\bullet$, and is independent of C, which can be replaced by any larger number without affecting the value of the sum. The first claim of the lemma follows.

Given a function $\Psi_1'(b)$ on $E^\times/E^\bullet$ which vanishes near 0 and has the asymptotic behavior as $b \to \infty$ as specified in the lemma, by the linearity in f' of $f' \mapsto \Psi_1(f', \psi)$ we have that there exists $f_1' \in C_c^\infty(H')$ such that $\Psi_1'(b) - \Psi_1(b, f_1', \psi)$ is compactly supported on $E^\times/E^\bullet$. But then there exists some $f_2' \in C_c^\infty(H')$ with $\Psi_1'(b) - \Psi_1(b, f_1', \psi) = \Psi(b, f_2', \psi)$, and so $\Psi_1'(b) = \Psi_1(b, f', \psi)$ for $f' = f_1' + f_2'$ by the linearity of $f' \mapsto \Psi(f', \psi)$. The final claim follows at once from the asymptotic behavior. □

Proof of Proposition 7. The integrals Ψ and Ψ_1 on H' are related by

$$\Psi(b, f', \psi) = |i|^{-1}\Psi_1(ib, {}^*f', \psi_2), \qquad {}^*f'(g) = f'\left(\begin{pmatrix} i & 0 \\ 0 & 1\end{pmatrix}^{-1} g\right).$$

Here $\psi_2(x) = \psi(2x)$. Hence the asymptotic behavior of $\Psi(b, f', \psi)$, as $b \to \infty$, is

$$\Psi(b, f', \psi) = |i|^{-1/2}\Psi_1(0, {}^*f', \psi_2)\Lambda_{\psi_2}(b).$$

Since $\Psi_1(0, {}^*f', \psi_2)$ is equal to

$$|i|^{1/2}\int_F\int_H f'\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & 1\end{pmatrix}^{-1} h\right)\psi(2x)dx\,dh = |i|^{1/2}\Psi(0, f', \psi),$$

Lemmas 8 and 9 imply that given $f \in C_c^\infty(G)$ there is $f' \in C_c^\infty(H')$, and given f' there is f, with $|2|\ |b|^{1/2}\Psi(b, f', \psi) = \Psi(b, f, \psi)$ for all $b \in E^\times/E^\bullet$. A pair $f,\ f'$ which satisfies the last equality for all $b \in E^\times/E^\bullet$, satisfies $\Psi(0, f, \psi) = |i|^{-1/2}\Psi_1(0, {}^*f', \psi_2) = \Psi(0, f', \psi)$. The proposition follows, in the case of a place v of F which stays prime in E. □

4. Matching Integrals at a split place.

The next step is that of a non-archimedean place v of F which splits in E. As usual we omit v from the notations. In this case $E = F \oplus F$, $H = PGL(2,F)$, $H' = H \times H$, $\mathbf{f}' = (f_1', f_2')$ is a C_c^∞-function on H' (thus $\mathbf{f}'(g) = f_1'(g_1)f_2'(g_2)$ for $g = (g_1, g_2) \in H'$); by this we mean a smooth function on $GL(2,F) \times GL(2,F)$ which transforms trivially under the center, and is compactly supported modulo the center. Also $\mathbf{b} = (b_1, b_2)$ and $|\mathbf{b}|_E = |b_1 b_2|$ ($|.|$ is the absolute value on F, $|.|_E$ on E); $\mathbf{i} = (i_1, i_2)$ satisfies $\mathbf{i} + \bar{\mathbf{i}} = 0$, thus $i_1 + i_2 = 0$ (as $\bar{\mathbf{i}} = (i_2, i_1)$), so we write $\mathbf{i} = (i, -i)$, and $|\mathbf{i}|_E = |i^2|$; and we put $f_2'^*(h) = f_2'(h^{-1})$, and $f' = f_1' * f_2'^*$, thus $f'(g) = \int_H f_1'(gh^{-1})f_2'^*(h)dh$; H embeds diagonally in H'.

The integral

$$\Psi(b, \mathbf{f}', \psi) = \int_E \int_H \mathbf{f}' \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{b} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{i} \\ 1 & -\mathbf{i} \end{pmatrix} h \right) \psi(2x) dx\, dh$$

of Proposition 7, is equal to

$$\int_F \int_F \int_H f_1' \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} h \right)$$
$$\cdot f_2' \left(\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} h \right) \boldsymbol{\psi}(2x_1 + 2x_2) dx_1 dx_2 dh.$$

Changing variables $h \mapsto \left(\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \right)^{-1} h$, this becomes

$$\int_F \int_F f' \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \right) \boldsymbol{\psi}(2x_1 - 2x_2) dx_1 dx_2.$$

Note that $|\mathbf{b}\mathbf{i}|_E^{1/2} = |b|^{1/2}|i|$, where $b = b_1 b_2$, and $f'\left(* \begin{pmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{pmatrix} * \right) = f'\left(* \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} * \right)$, since f' is defined on $PGL(2,F)$.

The singular integral

$$\Psi(0, \mathbf{f}', \psi) = \int_N \int_H \mathbf{f}' \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{pmatrix}^{-1} h \right) \psi(2x) dx\, dh$$

is equal to $\int_F f' \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \boldsymbol{\psi}(2x) dx$ by a similar change of variables.

The integral $\Psi(b, f, \psi)$ on G, where now $G = PGL(3,F)$ and C is the centralizer of $\mathcal{J}_0 = \operatorname{diag}(1, -1, 1)$, takes the form $\int_U \int_C f(u g_b g_0 h) \psi(u) du\, dh$ for $b \in F^\times$, where $\psi(u) = \boldsymbol{\psi}(p + q)$, and

$$u = \begin{pmatrix} 1 & p & z \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}, \quad g_b = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & b^{-1} \end{pmatrix}, \quad g_0 = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

Further, $\Psi(0,f,\psi)$ is $\int_{U/U_0}\int_C f(uh)\psi(u)du dh$, where $U_0 = U\cap C$.

Definition. The functions $f\in C_c^\infty(PGL(3,F))$ and $f'\in C_c^\infty(PGL(2,F))$ are called *matching* if $\Psi(b,f,\psi)=|4|\ |b|^{1/2}\Psi(b,f',\psi)$ for all $b\in F^\times$, and $\Psi(0,f,\psi)=\Psi(0,f',\psi)$.

In the case of a place v of F which splits in E, Proposition 7 asserts that given $f\in C_c^\infty(G)$ there is a matching $f'\in C_c^\infty(H)$, and given f' there is a matching f, and if f and f' are matching, namely $\Psi(b,f,\psi)=|4|\ |b|^{1/2}\Psi(b,f',\psi)$ for all $b\in F^\times$, then they satisfy also $\Psi(0,f,\psi)=\Psi(0,f',\psi)$. As in the non-split case, we prove this Proposition by characterizing the Fourier orbital integrals via their asymptotic behavior as $b\to\infty$.

10. Lemma. *Put* $\psi_2(x)=\boldsymbol{\psi}(2x)$ *and* $\psi(x)=\boldsymbol{\psi}(x)$. *For every* $f'\in C_c^\infty(H)$ *the function* $\Psi(b,f',\psi)$ *is locally constant on* $F^\times$, *vanishes near* 0, *and there is* $B(f')>0$ *such that*

$$\Psi(-b,f',\psi)=\Psi(0,f',\psi)\int_F\psi_2\left(x-\frac{b}{x}\right)dx\qquad(|b|\ge B(f')).$$

Conversely, if $\Psi'(b)$ *is a locally constant function on* $F^\times$ *which vanishes near* 0 *and there is* $\Psi'(0)\in\mathbb{C}$ *and* $B'>0$ *such that for* $|b|\ge B'$ *we have* $\Psi'(-b)=\Psi'(0)\int_F\psi_2\left(x-\frac{b}{x}\right)dx$, *then there is* $f'\in C_c^\infty(H)$ *with* $\Psi(b,f',\psi)=\Psi'(b)$ *for all* $b\in F$. *Finally, if* $\Psi(b,f_1',\psi)=\Psi(b,f_2',\psi)$ *for all* $b\in F^\times$*,then* $\Psi(0,f_1',\psi)=\Psi(0,f_2',\psi)$.

Remark. The integral $\int_F\psi_2\left(x-\frac{b}{x}\right)dx$ is defined to be the limit of the integrals over $A^{-1}\le|x|\le A$, $A\to\infty$. It does not converge absolutely. An equivalent definition in terms of an absolutely convergent integral is that – when $|b|>1$ – the integral vanishes unless the (normalized) valuation of b is even, and then the integral is equal to $\int_{|x|=|b|^{1/2}}\psi_2\left(x-\frac{b}{x}\right)dx$.

Proof. In the integral

$$\Psi(-b,f',\psi)=\int_F\int_F f'\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}-b&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&y\\0&1\end{pmatrix}\right)\psi_2(x-y)dx\,dy,$$

the function f' is evaluated at $\begin{pmatrix}x&xy-b\\1&y\end{pmatrix}$ in $PGL(2,F)\simeq SO(3)$. The image of this matrix in $SO(3)$ is

$$b^{-1}\begin{pmatrix}x^2&2x(xy-b)&(xy-b)^2\\x&2xy-b&y(xy-b)\\1&2y&y^2\end{pmatrix}.$$

Since f' is compactly supported there is some $C\ge1$ such that $|x|$, $|y|$ and $|xy-b|$ are all bounded by $C|b|^{1/2}$. For a large enough b we conclude that $|b|=|xy|\le C|x|\ |b|^{1/2}$, hence $|x|$, $|y|\ge C^{-1}|b|^{1/2}$. Write $y=\frac{b}{x}+z$. Then

$$C^{-1}|b|^{1/2}|z|\le|xz|=|xy-b|\le C|b|^{1/2}\quad\text{and so }|z|\le C^2.$$

The integral is then equal to

$$\int_F\int_F f'\left(\begin{pmatrix}1&z\\0&b/x^2\end{pmatrix}\right)\psi_2(x-b/x)\psi_2(-z)dzdx=\int_F F(b/x^2)\psi_2(x-b/x)dx,$$

where $F(t)=\int_F f'\left(\begin{pmatrix}1 & z\\ 0 & t\end{pmatrix}\right)\psi_2(-z)dz$ is a locally constant function, compactly supported on $F^\times$. In particular there is a sufficiently small $C_2=C_2(F)>0$ such that $F(t(1-\varepsilon))=F(t)$ for all t and all $|\varepsilon|\le C_2$. Note that

$$x(1-\varepsilon)-\frac{b}{x}(1+\varepsilon+\varepsilon^2+\dots)=x-\frac{b}{x}-\varepsilon(x+\frac{b}{x}+\frac{b}{x}(\varepsilon+\varepsilon^2+\dots)).$$

Replacing x by $x(1-\varepsilon)$, where $|\varepsilon|\le C_2$, we get that $\int_F F(b/x^2)\psi_2(x-b/x)dx$ is equal to the quotient by $\int_{|\varepsilon|\le C_2}d\varepsilon$ of

$$\int_F F(b/x^2)\psi_2(x-b/x)\int_{|\varepsilon|\le C_2}\overline{\psi_2}(\varepsilon(x+b/x))d\varepsilon\, dx.$$

The inner integral over ε vanishes if $|x+b/x|>C_1$ for some $C_1=C_1(k,\psi)$, whence the integral over x can be taken only over those $x\in F^\times$ with $|x+b/x|\le C_1$, for some $C_1>0$, without changing the value of the integral. But for x with $|1+b/x^2|\le C_1/|x|\le CC_1/|b|^{1/2}$, we have that $F(b/x^2)=F(-1)$, since F is locally constant (and $|b|$ is sufficiently large). Obtained is $F(-1)\int\psi_2(x-b/x)dx$, where

$$\begin{aligned}F(-1)&=\int_F f'\left(\begin{pmatrix}1 & z\\ 0 & -1\end{pmatrix}\right)\psi_2(-z)dz\\ &=\int_F f'\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\psi_2(z)dz=\Psi(0,f',\psi),\end{aligned}$$

and the first claim of the lemma follows.

Given a locally constant $\Psi'(b)$ on $b\in F^\times$ which vanishes near 0 and has the indicated asymptotic behavior as $|b|\to\infty$, the linearity of $\Psi(b,f',\psi)$ in f' implies that there is $f_3'\in C_c^\infty(H)$ with $\Psi(b,f_3',\psi)=\Psi'(b)$ on all sufficiently large $b\in F^\times$. The difference $\Psi'(b)-\Psi(b,f_3',\psi)$ is locally constant and compactly supported on $F^\times$, hence it is clear that it is equal to $\Psi(b,f_4',\psi)$ for some $f_4'\in C_c^\infty(H)$, as required. The final claim, that $\Psi(b,f_1',\psi)=\Psi(b,f_2',\psi)$ for all $b\in F^\times$ implies that $\Psi(0,f_1',\psi)=\Psi(0,f_2',\psi)$, follows at once from the asymptotic behavior of the Fourier orbital integrals. □

Remark. Let us clarify the asymptotic expansion of the integral $\int_F F(b/x^2)\psi_2(x-b/x)dx$, using the stationary phase method. This technique asserts that the leading term in the asymptotic expansion is from x in a small neighborhood of the stationary points of the argument $k(x)=x-b/x$ of $\psi_2(x-b/x)$. These are the points where the gradient of k is 0, namely $x^2=-b$. For a sufficiently large b, our integral vanishes unless $-b$ is a square, say β^2, β in F. By Morse Lemma there is a change $x=\beta(1+y+ay^2+\cdots)$ of variables in a small neighborhood of β (i.e., y is small) such that $x-b/x=\beta(2+y^2)$, and $F(b/x^2)=F(-(1-2y))$. Put $\varphi(y)=F(-1+y)$. Our integral is then the sum over β, $\beta^2=-b$, of

$$|1/2|\psi(4\beta)|\beta|\int_F\varphi(y)\psi(\frac{1}{2}\beta y^2)dy=|\beta/2|\psi(4\beta)\gamma_\psi(\beta)|\beta|^{-1/2}\int_F\hat{\varphi}(y)\psi(-\frac{1}{2}\beta^{-1}y^2)dy.$$

The equality follows from the definition of the Weil factor $\gamma_\psi(\beta)$, assuming the Haar measure dy is self-dual, and $\hat{\varphi}$ is the Fourier transform of φ with respect to ψ. This $\hat{\varphi}$ is compactly supported. Hence for a large b the last integral is $\int_F \hat{\varphi}(y)dy = \varphi(0) = F(-1)$. It follows that for a large b, the integral $\int_F F(b/x^2)\psi_2(x - b/x)dx$ vanishes unless $-b$ is a square in F, in which case this integral is equal to $\sum_{i=\pm} \psi(4i\sqrt{-b})\gamma_\psi(i\sqrt{-b})|\beta/4|^{1/2}F(-1)$. This approach is systematically used in [FM2], in the context of the group of similitudes of a symplectic form on a four dimensional space.

11. Lemma. *For every $f \in C_c^\infty(G)$ the function $\Psi(b, f, \psi)$ is locally constant on $F^\times$, vanishes near 0, and there is $B(f) > 0$ such that for all $|b| \geq B(f)$,*

$$\Psi(-b, f, \psi) = |4|\ |b|^{1/2}\Psi(0, f, \psi) \int_F \psi_2(x - b/x)dx.$$

Conversely, if $\Psi(b)$ is a locally constant function on $F^\times$ which vanishes near 0, and there is $\Psi(0) \in \mathbb{C}$ and $B' > 0$ such that for $|b| \geq B'$ we have

$$\Psi(-b) = |b|^{1/2}|4|\Psi(0) \int_F \psi_2(x - b/x)dx,$$

then there is $f \in C_c^\infty(G)$ with $\Psi(b) = \Psi(b, f, \psi)$ for all $b \in F^\times$.

Finally, if $\Psi(b, f_1, \psi) = \Psi(b, f_2, \psi)$ for all $b \in F^\times$, then $\Psi(0, f_1, \psi) = \Psi(0, f_2, \psi)$.

Proof. Introduce $F_1(g) = \int_C f(gh)dh$ on G, and $F_0(x) = F_1(g)$ if $x = g^t\varepsilon_0\varepsilon_0 g^{-1}$ on the space X_0 of 3×3 matrices over F with trace and rank equal to one. Extend F_0 to a locally constant compactly supported function in a neighborhood of X_0 in the space of 3×3 matrices over F. Recall that $\varepsilon_0 = (0, 1, 0)$, and that $g_0^{-1} = \mathcal{J}^t g_0 \mathcal{J}^{-1}$; then $\varepsilon_0 g_0^{-1} = (-\frac{1}{2}, 0, 1)$. Carrying out the matrix multiplication we obtain that $\Psi(b, f, \psi)$ is equal to $\int_F \int_F \int_F A \cdot \psi(p+q)dp\,dq\,dz$, where A is

$$F_0\left(\begin{pmatrix} \frac{1}{2}\left(1 - \frac{z}{2b} - \frac{pq}{4b}\right) & -\frac{p}{2}\left(1 - \frac{z}{2b} - \frac{pq}{4b}\right) & -b\left(1 - \frac{z}{2b} - \frac{pq}{4b}\right)\left(1 + \frac{z}{2b} - \frac{pq}{4b}\right) \\ \frac{-q}{4b} & \frac{pq}{4b} & \frac{q}{2}\left(1 + \frac{z}{2b} - \frac{pq}{4b}\right) \\ \frac{-1}{4b} & \frac{p}{4b} & \frac{1}{2}\left(1 + \frac{z}{2b} - \frac{pq}{4b}\right) \end{pmatrix}\right)$$

To study the asymptotic behaviour of this integral we shall regard b as having large $|b|$, and denote by $\alpha << \beta$ the statement: there exists $c > 0$, independent of b, with $\alpha \leq c\beta$. We shall be concerned with the support of the integrand, and attempt to find over which domain can the integration be restricted to, without affecting the value of the integral.

Given any $C_1 > 0$, we may restrict the domain of integration to the set where at least one of $|p|$, $|q|$ is bigger than C_1. Suppose this is not true. Then we consider the integral over the set of $|p|$, $|q| \leq C_1$. Without loss of generality $|1 - \frac{pq}{4b} + \frac{z}{2b}| \leq |1 - \frac{pq}{4b} - \frac{z}{2b}|$. Considering the $(1, 3)$ entry of the matrix at which F_0 is evaluated, since F_0 is compactly supported there is $C_2 > 0$ such that $1 - \frac{pq}{4b} + \frac{z}{2b} = u/b^{1/2}$, $|u| \leq C_2$. But then $1 - \frac{pq}{4b} - \frac{z}{2b} = 2(1 - \frac{pq}{4b}) - u/b^{1/2} \to 2$ as $|b| \to \infty$ (here $u \in \overline{F}$ such that $u/b^{1/2} \in F$, and $|u| \leq C_2$ means: $|u/b^{1/2}| \leq C_2|b|^{-1/2}$). Consequently the absolute value of the $(1, 3)$ entry is $|b| \cdot |u/b^{1/2}| \cdot |2|$, and this is bounded by some $C_3 > 0$, since F_0 is compactly supported. Hence $|u| << |b|^{-1/2}$, and we may write $1 - \frac{pq}{4b} + \frac{z}{2b} = \frac{u}{b}$ with $|u| << 1$. Again $1 - \frac{pq}{4b} - \frac{z}{2b} \to 2$, and over the domain of $|p|$, $|q| \le C_1$ the integral is

$$\iiint F_0\left(\begin{pmatrix} 1 & -p & -2u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \psi(p+q)dp\,dq\,dz.$$

It vanishes since the F_0 part of the integrand is independent of q, and $\int_{|q|\le C_1} \psi(q)dq = 0$ for a sufficiently large C_1.

We may now assume that our integral ranges over a domain where at least one of $|p|$, $|q|$, is larger than any chosen $C_1 > 0$. Since F_0 is locally constant we have that $F_0(\boldsymbol{\varepsilon} g \boldsymbol{\varepsilon}^{-1}) = F_0(g)$ for $\boldsymbol{\varepsilon} = \operatorname{diag}(1, \varepsilon, 1)$, ε in some neighborhood of 1. Changing variables on p and q, we may replace the factor $\psi(p+q)$ in the integrand by $\psi(\varepsilon p + q/\varepsilon)$. Integrating $\psi(\varepsilon p + q/\varepsilon)$ over ε in this neighborhood of 1 (which depends on F_0) it is easy to see that 0 is obtained unless $|p| = |q|$. Considering the entry $(2,2)$, since F_0 is compactly supported we have that the integral can be taken only over $|p| = |q| \le C_4|b|^{1/2}$ for some $C_4 > 0$ depending only on F_0 and ψ. Taking a sufficiently large C_1, the entries $(1,2), (2,3)$ can be used to show that $|pq| = |4b|$. Hence $|b|^{1/2}C_4^{-1} \le |p| = |q| \le C_4|b|^{1/2}$. The entries $(1,2)$ and $(2,3)$ imply that the integral ranges only over

$$\left|1 - \frac{pq}{4b} - \frac{z}{2b}\right| << |p|^{-1} << |b|^{-1/2}, \quad \left|1 - \frac{pq}{4b} + \frac{z}{2b}\right| << |q|^{-1} << |b|^{-1/2}.$$

Hence $|1 - \frac{pq}{4b}| << |b|^{-1/2}$, and $q = 4b/p + u$, $|u| << 1$, and $|z| << |b|^{1/2}$. Over the domain of integration the integral takes the form

$$\iiint F_0\left(\begin{pmatrix} 0 & \frac{p}{2}\left(\frac{up}{4b} + \frac{z}{2b}\right) & -b\left(\frac{up}{4b} + \frac{z}{2b}\right)\left(\frac{up}{4b} - \frac{z}{2b}\right) \\ 0 & 1 & -\frac{2b}{p}\left(\frac{up}{4b} - \frac{z}{2b}\right) \\ 0 & 0 & 0 \end{pmatrix}\right) \psi\left(p + \frac{4b}{p} + u\right) dp\,dz\,du.$$

Writing $x = \frac{p}{2}\left(\frac{up}{4b} + \frac{z}{2b}\right)$, $y = -\frac{2b}{p}\left(\frac{up}{4b} - \frac{z}{2b}\right)$, the Jacobian $\partial(u,z)/\partial(x,y)$ is $|4b/p|$. We obtain

$$\iiint F_0\left(\begin{pmatrix} 0 & x & xy \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}\right) \psi\left(p + \frac{4b}{p}\right) \psi\left(\frac{4b}{pp}x - y\right) \left|\frac{4b}{p}\right| dx\,dy\,dp.$$

Sublemma. *The value of the integral does not change if the integration is restricted to p with $|p - \frac{4b}{p}| < |b|^{2/5}$.*

Proof. It suffices to show that the integral over the p with $|p - \frac{4b}{p}| \ge |b|^{2/5}$ of $|p|^{-1}\psi(p + \frac{4b}{p} + \frac{4bx}{pp})$ is zero. Take ε with $|\varepsilon| \le |b|^{-1/3}$. Then $|\varepsilon|^2|b|^{1/2} \le |b|^{-1/6}$, and $\varepsilon(p - 4b/p)$ ranges over the set of a with $|a| \le |b|^{1/15}$ ($\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$). The domain $|p - 4b/p| \ge |b|^{2/5}$ is stable under $p \mapsto p(1-\varepsilon)$, since

$$p(1-\varepsilon) - \frac{4b}{p}(1 + \varepsilon + \varepsilon^2 + \dots) = p - \frac{4b}{p} - \varepsilon\left(p + \frac{4b}{p}\right) - \varepsilon^2\frac{4b}{p}(1 + \varepsilon + \varepsilon^2 + \dots)$$

has absolute value $= |p - 4b/p|$ (indeed, $\frac{2}{5} > \frac{1}{2} - \frac{1}{3}$). On the other hand,

$$p(1-\varepsilon) + \frac{4b}{p}(1+\varepsilon+\varepsilon^2+\dots) + \frac{4bx}{pp}(1+2\varepsilon+3\varepsilon^2+\dots) = p + \frac{4b}{p} + \frac{4bx}{pp} - \varepsilon\left(p - \frac{4b}{p}\right) + o(1),$$

where $o(1)$ means a quantity with an absolute value as small as desired, provided $|b|$ is large enough. It follows that the integral specified at the first sentence of our proof contains a factor of the form

$$\int_{|\varepsilon|\le|b|^{-1/3}} \psi(\varepsilon(p - 4b/p))d\varepsilon,$$

which is zero if $|p - 4b/p| \ge |b|^{2/5}$, for a sufficiently large b. The sublemma follows. □

The Sublemma implies that $|1 - 4b/p^2| < |b|^{2/5}/|p| << |b|^{-1/10}$, and so $\psi(4bx/p^2)$ can be replaced by $\psi(x)$ in the integrand. The integral over p will vanish unless p is taken over the domain $|p|^2 = |4b|$. We are left with

$$|2|\ |b|^{1/2} \int_F \int_F F_0\left(\begin{pmatrix} 0 & x & xy \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}\right) \psi(x-y)dxdy \int_F \psi(p + 4b/p)dp.$$

This is equal to

$$|4|\ |b|^{1/2} \cdot \Psi(0, f, \psi) \cdot \int_F \psi(2(p + b/p))dp.$$

Replacing b by $-b$, the first claim of the lemma follows. The other claims of the lemma follow as usual. Then the proof of Proposition 7 is complete also in the case of a place v of F which splits in E, by Lemmas 10 and 11, and since $|4|_F = |2|_E$. □

Remark. At a place of F which splits in E, the kernel $E^\bullet$ of the norm map $E = F \oplus F \to F$, $(b_1, b_2) \mapsto b_1 b_2$, is $E^\bullet = \{(b, b^{-1})\}$. For $\mathbf{b} = (-b, 1) \in E^\times$, with $|b| > 1$, the integral is

$$\int_{\varepsilon=(z,z^{-1})\in E^\bullet} \psi(2\mathbf{b}\varepsilon)d\varepsilon = \int_{F^\times} \psi(2(z - b/z))d^\times z = |b|^{-1/2}\left(1 - \frac{1}{q}\right)^{-1} \int_{|z|=|b|^{1/2}} \psi(2(z - b/z))dz,$$

where $d^\times z = (1 - 1/q)^{-1} dz/|z|$ is the relation between the normalized measures $d^\times z$ and dz on $F^\times$ and F.

5. Matching Integrals of Spherical Functions.

The equality of the geometric sides of the Fourier summation formulae is proven for global test functions whose local components are almost all equal to the unit element in the Hecke algebra of spherical functions on the local group. We need to show that when f_v and f'_v are taken to be these unit elements, they are matching. More generally, the correspondence of unramified local representations introduced in the Introduction defines a homomorphism of the convolution Hecke algebras on the groups. The isolation argument used below to determine the cyclic cuspidal representations is based on the fact that such corresponding spherical functions are matching. The treatment of the two cases of split and non-split cases is entirely different. We start with the easier case of a place which stays prime in E.

We shall use local notations. Let E/F be an unramified quadratic non-archimedean field extension of residual characteristic $\neq 2$, R' the ring of integers in E (R in F),

$$G = \{g \in PGL(3,E); g\mathcal{J}{}^t\overline{g} = \mathcal{J}\} \quad \text{and} \quad K = \{g \in PGL(3,R'); g\mathcal{J}{}^t\overline{g} = \mathcal{J}\}.$$

Denote by $\mathbb{H}$ the convolution algebra (for simplicity we choose the Haar measure on G which assigns the maximal compact subgroup K the volume 1) of compactly supported K-biinvariant functions on G. By the theory of the Satake transform, the function $f \in \mathbb{H}$ is determined by the values of the traces $\operatorname{tr} \pi(f)$ of the convolution operators $\pi(f)$, where π runs through the variety of unramified irreducible representations on G. Any such π is the unique unramified constituent in the composition series (of length one or two) of a representation $I(\mu)$ of G which is normalizedly induced from an unramified character $\begin{pmatrix} a & & * \\ & 1 & \\ 0 & & \overline{a}^{-1} \end{pmatrix} \mapsto \mu(a)$ of the upper triangular subgroup of G. The character $\mu : E^\times \to \mathbb{C}^\times$ is not uniquely determined by π, but $\{\mu, \mu^{-1}\}$ is, and so is $\operatorname{tr} I(\mu, f) = \operatorname{tr} \pi(f)$, for every $f \in \mathbb{H}$. In fact

$$\operatorname{tr} \pi(f) = \operatorname{tr} I(\mu, f) = \sum_{n\in\mathbb{Z}} F_f(n)\mu(\boldsymbol{\pi})^n,$$

where $\boldsymbol{\pi}$ is a generator of the maximal ideal in the local ring R, and

$$F_f(n) = F_f(\mathbf{a}) = \frac{|a-1|_E |a\overline{a}-1|_F}{|a|_E} \int f(\mathbf{u}^{-1}\mathbf{a}\mathbf{u})d\mathbf{u} = |a|_E \int f_U(\mathbf{a}\mathbf{u})d\mathbf{u}.$$

Here $\mathbf{a} = \operatorname{diag}(a, 1, \overline{a}^{-1})$, $|a| = |\boldsymbol{\pi}|^n$, $\mathbf{u}$ is the upper triangular unipotent matrix in G with top row $(1, u, \frac{1}{2}u\overline{u} + v)$ (the $(2,3)$ entry is then $\overline{u}$), where u ranges over E, and v over the elements in E with $v + \overline{v} = 0$ (this subspace is isomorphic to F by $v \mapsto (v - \overline{v})/(v_0 - \overline{v}_0)$ for a fixed $v_0 \neq 0$). Clearly $F_f(n) = F_f(-n)$, so we may always take $|a| \geq 1$.

Analogously, let $\mathbb{H}'$ be the Hecke convolution algebra of $H' = PGL(2,E)$ with respect to $K' = PGL(2,R')$. Again the $f' \in \mathbb{H}'$ are determined by the values of the traces $\operatorname{tr} \pi'(f')$, where π' runs through the unramified irreducible representations of H'. Each such π' is the unique unramified irreducible constituent in the H'-module $I'(\mu)$ normalizedly induced from the character $\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto \mu(a/b)$ of the upper triangular subgroup, and

$$\operatorname{tr} \pi'(f') = \operatorname{tr} I'(\mu, f') = \sum_{n\in\mathbb{Z}} F_{f'}(n)\mu(\boldsymbol{\pi})^n.$$

It will be more convenient to use the isomorphism $PGL(2,E) \xrightarrow{\sim} SO(3,E)$, which maps

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mapsto \mathbf{u} = \begin{pmatrix} 1 & 2u & u^2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mapsto \mathbf{a} = \begin{pmatrix} a & & 0 \\ & 1 & \\ 0 & & a^{-1} \end{pmatrix}.$$

Then

$$F_{f'}(n) = F_{f'}(\mathbf{a}) = \frac{|a-1|}{|a|^{1/2}} \int f'(\mathbf{u}^{-1}\mathbf{a}\mathbf{u})d\mathbf{u} = |a|^{1/2} \int f(\mathbf{a}\mathbf{u})d\mathbf{u}.$$

Here $a \in E^\times$ with $|a| = |\boldsymbol{\pi}|^n$ ($|.|$ is $|.|_E$), and $F_{f'}(-n) = F_{f'}(n)$, hence we may deal only with $|a| \geq 1$.

Recall that the correspondence of unramified H'-modules to such G-modules was defined by $I'(\mu) \to I(\mu)$. The dual map $D : \mathbb{H} \to \mathbb{H}'$ of Hecke algebras is defined by $D : f \mapsto f'$ if $\operatorname{tr} I'(\mu, f') = \operatorname{tr} I(\mu, f)$, for all unramified characters μ of $E^\times$. Namely $f \in \mathbb{H}$ *corresponds* to $f' \in \mathbb{H}'$ precisely when $F_f(n) = F_{f'}(n)$ for all $n \in \mathbb{Z}$.

The Hecke algebra $\mathbb{H}$ of G is spanned – as a vector space – by the characteristic functions $f^k = \operatorname{ch} K\mathbf{a}_k K$, where $\mathbf{a}_k = \operatorname{diag}(\boldsymbol{\pi}^{-k}, 1, \boldsymbol{\pi}^k)$, where $k \geq 0$. Similarly, the Hecke algebra $\mathbb{H}'$ of $H' = SO(3, E)$ is spanned by the characteristic functions f'^k of the double cosets $K'\mathbf{a}_k K'$ in H'. To describe explicitly the Hecke algebra homomorphism, we prove:

12. Lemma. *The image $D(f^k)$ of $f^k \in \mathbb{H}$ in $\mathbb{H}'$ is*

$$q^k f'^k + (1 - q^{-1})q^k \sum_{0 \leq m < k} f'^m \qquad (q = \operatorname{card}(R/(\boldsymbol{\pi})),\ k \geq 0).$$

Proof. Our first step is to compute $F(\mathbf{a}_j, f'^k)$. We use the Bruhat decomposition

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix} \begin{pmatrix} 1 & 1/u \\ 0 & 1 \end{pmatrix},$$

and the corresponding decomposition in $SO(3)$. As usual, $j,\ k \geq 0$. We write $g_1 \equiv g_2$ for $g_i \in H'$ if $K'g_1K' = K'g_2K'$. Note that ${}^t g \equiv g$. If $|u| > 1$, then

$$\begin{pmatrix} \boldsymbol{\pi}^{-j} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}^{-j} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1/u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}^{-j} & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} u/\boldsymbol{\pi}^j & 0 \\ 0 & 1/u \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{\pi}^j/u \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} u/\boldsymbol{\pi}^j & 0 \\ 0 & 1/u \end{pmatrix}.$$

It follows that

$$F(\mathbf{a}_j, f'^k) = \begin{cases} |\boldsymbol{\pi}|^{-j/2} = q^j, & j = k \\ (1 - q^{-2})|\boldsymbol{\pi}|^{-k/2} = (1 - q^{-2})q^k, & k > j \geq 0, \quad k \equiv j \pmod 2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that $|.| = |.|_E$, hence $|\boldsymbol{\pi}| = q^{-2}$.

Next we compute the orbital integrals $F(\mathbf{a}_j, f^k)$ on G. We use

$$\begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & \overline{u} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \overline{u}/\overline{v} & 1/v \\ 0 & 1 & u/v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v & & 0 \\ & \overline{v}/v & \\ 0 & & 1/\overline{v} \end{pmatrix} \begin{pmatrix} 1 & \overline{u}/v & 1/v \\ 0 & 1 & u/\overline{v} \\ 0 & 0 & 1 \end{pmatrix}$$

where $v+\overline{v}=u\overline{u}$, and note that if $|v|\le 1$ then $|u|\le 1$, and if $|v|\ge 1$ then $|v|\ge |u|$. Note that if $|v|>1$ then $|v|>|u|$, $|\boldsymbol{\pi}^j u/v|<1$, $|\boldsymbol{\pi}^{2j}/v|<1$, and

$$\begin{pmatrix}1&0&0\\u&1&0\\v&\overline{u}&1\end{pmatrix}\begin{pmatrix}\boldsymbol{\pi}^{-j}&&0\\&1&\\0&&\boldsymbol{\pi}^j\end{pmatrix}\equiv\begin{pmatrix}v&&0\\&\overline{v}/v&\\0&&1/\overline{v}\end{pmatrix}\begin{pmatrix}1&\overline{u}/v&1/v\\&1&u/\overline{v}\\0&&1\end{pmatrix}\begin{pmatrix}\boldsymbol{\pi}^{-j}&&0\\&1&\\0&&\boldsymbol{\pi}^j\end{pmatrix}$$
$$\equiv\begin{pmatrix}v\boldsymbol{\pi}^{-j}&&0\\&\overline{v}/v&\\0&&\boldsymbol{\pi}^j/\overline{v}\end{pmatrix}\begin{pmatrix}1&\boldsymbol{\pi}^j\overline{u}/v&\boldsymbol{\pi}^{2j}/v\\&1&\boldsymbol{\pi}^j u/\overline{v}\\0&&1\end{pmatrix}\equiv\begin{pmatrix}v\boldsymbol{\pi}^{-j}&&0\\&\overline{v}/v&\\0&&\boldsymbol{\pi}^j/\overline{v}\end{pmatrix}.$$

Hence

$$F(\mathbf{a}_j,f)=|\boldsymbol{\pi}^{-j}|\iint f\left(\mathbf{a}_j\begin{pmatrix}1&u&v\\0&1&\overline{u}\\0&0&1\end{pmatrix}\right)du\,dv$$
$$=|\boldsymbol{\pi}|^{-j}\int_{|v|\le 1} f\left(\left(\begin{matrix}\boldsymbol{\pi}^{-j}&&0\\&1&\\0&&\boldsymbol{\pi}^j\end{matrix}\right)\right)dv+|\boldsymbol{\pi}|^{-j}\int_{|v|>1} f\left(\left(\begin{matrix}v/\boldsymbol{\pi}^j&&0\\&\overline{v}/v&\\0&&\boldsymbol{\pi}^j/\overline{v}\end{matrix}\right)\right)dv$$
$$=|\boldsymbol{\pi}|^{-j}f\left(\left(\begin{matrix}\boldsymbol{\pi}^{-j}&&0\\&1&\\0&&\boldsymbol{\pi}^j\end{matrix}\right)\right)+\sum_{\ell\ge 1}|\boldsymbol{\pi}|^{-j}\cdot\int_{|v|=|\boldsymbol{\pi}|^{-\ell}}dv\cdot f\left(\left(\begin{matrix}\boldsymbol{\pi}^{-j-\ell}&&0\\&1&\\0&&\boldsymbol{\pi}^{j+\ell}\end{matrix}\right)\right)$$
$$=q^{2j}f(\mathbf{a}_j)+(1-q^{-3})\sum_{\ell\ge 1}q^{2j+4\ell}f(\mathbf{a}_{j+2\ell})+(q-1)\sum_{\ell\ge 0}q^{2j+4\ell}f(\mathbf{a}_{j+2\ell+1}).$$

Indeed, if $\ell(>1)$ is even, $\operatorname{vol}\{v\in E;|v|=|\boldsymbol{\pi}^{-\ell}|\}$ is the sum of $\operatorname{vol}\{u\in E,t\in F;|u|=|\boldsymbol{\pi}|^{-\ell/2},$ $|t|\le|\boldsymbol{\pi}|^{-\ell}\}$ and $\operatorname{vol}\{u\in E,t\in F;|u|<|\boldsymbol{\pi}|^{-\ell/2},|t|=|\boldsymbol{\pi}|^{-\ell}\}$ (since $v=\frac{1}{2}u\overline{u}+it$, and $|2|=|i|=1$) and these two volumes are equal to $(1-q^{-2})q^{2\ell}$ and $(1-q^{-1})q^{-2}q^{2\ell}$. If $\ell(>1)$ is odd then $\operatorname{vol}\{v\in E;|v|=|\boldsymbol{\pi}|^{-\ell}\}$ is $\operatorname{vol}\{u\in E,t\in F;|u|\le|\boldsymbol{\pi}|^{-(\ell+1)/2},\ |t|_F=q^\ell\}$, and this is $q^{-1}(1-q^{-1})q^{2\ell}$. In other words,

$$F(\mathbf{a}_j,f^k)=q^{2k}\begin{cases}1, & \text{j=k},\\ 1-q^{-3}, & k-j=2\ell,\ell\ge 1,\\ q^{-1}(1-q^{-1}), & k-j=2\ell+1,\ell\ge 0,\\ 0, & \text{otherwise}.\end{cases}$$

It is clear that the image in $\mathbb{H}'$ of $q^{-2k}f^k\in\mathbb{H}$ is $f'=\sum_{0\le m\le k}c_m q^{-m}f'^m$, $c_m\in\mathbb{C}$. Using $F(\mathbf{a}_j,f^k)=F(\mathbf{a}_j,f')$ with $j=k$, we conclude that $c_k=1$. When $k-j=2\ell>1$, we obtain

$$1-q^{-3}=c_j+(1-q^{-2})[c_{j+2}+c_{j+4}+\cdots+c_k].$$

In particular, when $j=k-2$, thus $\ell=1$, $1-q^{-3}=c_{k-2}+1-q^{-2}$, and $c_{k-2}=q^{-2}-q^{-3}$. For $k-(j-2)=2(\ell+1)>1$ we have

$$1-q^{-3}=c_{j-2}+(1-q^{-2})[c_j+c_{j+2}+\cdots+c_k].$$

Hence $c_{j-2} = q^{-2}c_j$, and $c_{k-2\ell} = q^{2-2\ell}c_{k-2} = q^{-2\ell}(1-q^{-1})$, for $\ell (1 \le \ell \le k/2)$.

On the other hand, when $k - j = 2\ell + 1 \ge 1$, we get

$$q^{-1}(1-q^{-1}) = c_j + (1-q^{-2})[c_{j+2} + c_{j+4} + \cdots + c_{k-1}],$$

and $c_{k-1} = q^{-1}(1-q^{-1})$ when $\ell = 0$. For $k-(j-2) = 2(\ell+1)-1 > 1$ we obtain

$$q^{-1}(1-q^{-1}) = c_{j-2} + (1-q^{-2})[c_j + c_{j+2} + \cdots + c_{k-1}].$$

Hence $c_{j-2} = q^{-2}c_j$, and $c_{k-1-2\ell} = q^{-2\ell}c_{k-1} = q^{-1-2\ell}(1-q^{-1})$. Namely $c_{k-m} = q^{-m}(1-q^{-1})$ if $1 \le m \le k$, or $c_m = q^{m-k}(1-q^{-1})$ for $0 \le m < k$. Since $q^{-2k}f^k$ then corresponds to $q^{-k}(f'^k + (1-q^{-1})\sum_{0\le m<k} f'^m)$, the image of f^k is as claimed in the lemma. □

Denote by f^{-1} and f'^{-1} the zero functions.

Corollary. *The image in* $\mathbb{H}'$ *of* $q^{-k-1}f^{k+1} - q^{-k}f^k \in \mathbb{H}$ *is*

$$f'^{k+1} + (1-q^{-1})f'^k - f'^k = f'^{k+1} - q^{-1}f'^k, \quad k \ge -1. \qquad \square$$

We shall compute the Fourier orbital integrals on H' and G for our spherical functions $f' \in \mathbb{H}'$ and $f \in \mathbb{H}$, where E/F is an unramified quadratic field extension (of residual characteristic $\neq 2$), and $\psi : E \to \mathbb{C}^\times$ is defined by $\psi(x) = \boldsymbol{\psi}(x + \overline{x})$, where $\boldsymbol{\psi} : F \to \mathbb{C}^\times$ is a fixed additive character which is trivial on R but not on $\boldsymbol{\pi}^{-1}R$. In [F8] we prove:

13. Proposition. *If* $|b| > 1$ *and* $m \ge 0$*, then*

$$\Psi(b, f'^m, \psi) = \sum(b) \times \begin{cases} 1, & m = 0, \\ q, & m = 1, \\ (1-q^{-2})q^m, & m \ge 2, \end{cases}$$

where

$$\sum(b) = \sum_\varepsilon \psi(2b\varepsilon) \quad (\varepsilon \in R'^\times/(1+b^{-1}R'),\ \varepsilon\overline{\varepsilon} = 1).$$

If $|b| = |\boldsymbol{\pi}|^j \le 1$*,* $j \ge 0$*, and* $m \ge 0$*, then*

$$\Psi(b, f'^m, \psi) = \begin{cases} 0, & 0 \le m < j, \\ 1, & m = j, \\ 1+q, & m = j+1, \\ q^2+q-1, & m = j+2, \\ (1+q^{-1})(1-q^{-2})q^{m-j}, & m-j \ge 3. \end{cases}$$

In particular, $\sum(b)$ is $\Lambda_{\psi_2}(b)$.

Our aim is to prove the following specification of Proposition 7.

14. Proposition. *For every $b \in E^\times$ we have $|b|^{-1/2}\Psi(b, f, \psi) = \Psi(b, D(f), \psi)$.*

Here f ranges over $\mathbb{H}$, and $D : \mathbb{H} \to \mathbb{H}'$ is the homomorphism of Hecke algebras specified above by the correspondence of unramified representations. If Proposition 14 is true, then the values of $\Psi(b, f, \psi)$ are given as follows. Since $\mathbb{H}$ is spanned by the $f^k (k \geq 0)$, we consider only f of the form f^k. Suppose first that $|b| > 1$. Then Propositions 13 and 14 imply:

$$
\begin{aligned}
&|b|^{-1/2}\Psi(b, f^0, \psi) = \Psi(b, f'^0, \psi) = \sum(b);\\
&|b|^{-1/2}\Psi(b, f^0, \psi) = q\Psi(b, f'^1 + (1 - q^{-1})f'^0, \psi)\\
&= q[q + (1 - q^{-1})]\sum(b) = (q^2 + q - 1)\sum(b);\\
&|b|^{-1/2}\Psi(b, f^k, \psi) = q^k[\Psi(b, f'^k, \psi) + (1 - q^{-1}) \sum_{0 \leq m < k} \Psi(b, f'^m, \psi)]\\
&= q^k[(1 - q^{-2})q^k + (1 - q^{-1})(1 + q + (q^2 - 1) + (q^3 - q) + \dots\\
&\quad + (q^{k-2} - q^{k-4}) + (q^{k-1} - q^{k-3}))]\sum(b)\\
&= q^{2k}(1 + q^{-1})(1 - q^{-2})\sum(b).
\end{aligned}
$$

On the other hand, when $|b| = |\boldsymbol{\pi}|^j \leq 1$, Propositions 13 and 14, as well as Corollary 12, would imply that

$$|b|^{-1/2}[q^{-k-1}\Psi(b, f^{k+1}, \psi) - q^{-k}\Psi(b, f^k, \psi)] = \Psi(b, f'^{k+1}, \psi) - \frac{1}{q}\Psi(b, f'^k, \psi)$$

is equal to 0 if $j > k + 1$, and to

$$
\begin{cases}
1, & j = k + 1,\\
1 + q - q^{-1}, & j = k,\\
q^2 + q - 1 - \frac{1}{q}(1 + q), & j = k - 1,\\
\left(1 + \frac{1}{q}\right)\left(1 - \frac{1}{q^2}\right) - \frac{1}{q}(q^2 + q - 1), & j = k - 2,\\
\left(1 + \frac{1}{q}\right)\left(1 - \frac{1}{q^2}\right)\left(q^{k+1-j} - q^{k-1-j}\right), & j \leq k - 3.
\end{cases}
$$

15. Proposition. *The function $\Psi(b, f^k, \psi)$ takes the values just specified for all $b \in E^\times$.*

Propositions 13 and 15 are proven in [F8], Part 1, by direct computation. Then Proposition 14 follows at once by virtue of Lemma 12 and its Corollary.

6. Spherical Matching in the split case.

Our next aim is to show that corresponding spherical functions are matching in the split case. Here the two groups are $G = PGL(3, F)$ and $H = PGL(2, F)$, where F is a p-adic field. Denote by R the ring of integers of F, and put $K = PGL(3, R)$, $K' = PGL(2, R)$. Fixing Haar measures on G and H, say the ones which assign K and K' the volumes 1, we define $\mathbb{H}$ (and $\mathbb{H}'$) to be the convolution algebra(s) of K- (and K'-) biinvariant compactly supported complex valued functions f (and f') on G (and H, respectively). A function $f \in \mathbb{H}$ (and $f' \in \mathbb{H}'$) is uniquely determined by the values of the traces $\operatorname{tr} I(\mu, f)$ and $\operatorname{tr} I'(\mu, f')$ where μ runs through the variety of unramified characters of the diagonal subgroup. Recall that the correspondence of representations maps $I'(\mu)$, the representation of H normalizedly induced from the character $\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto \mu(a/b)$, where μ is an unramified character of $F^\times$, to the representation $I(\mu, \mu^{-1}, 1)$ of G induced from the character $\operatorname{diag}(a, b, c) \mapsto \mu(a/b)$ of the upper triangular subgroup (which is 1 on the unipotent subgroup).

Definition. The function $f \in \mathbb{H}$ *corresponds* to the function $f' \in \mathbb{H}'$ if

$$\operatorname{tr} I'(\mu, f') = \operatorname{tr} I((\mu, \mu^{-1}, 1); f)$$

for all unramified characters $\mu : F^\times \to \mathbb{C}^\times$.

Note that f' is uniquely determined by f, but f is not uniquely determined by f'. Put $t = \mu(\boldsymbol{\pi}^{-1})$. Then $\operatorname{tr} I'(\mu, f') = \sum_n F(n, f')t^n$, where

$$\begin{aligned} F(n, f') = F_{f'}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) &= \left|\frac{(a-b)^2}{ab}\right|^{1/2} \int_F f'\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx \\ &= \left|\frac{a}{b}\right|^{1/2} \left|1 - \frac{b}{a}\right| \int_F f'\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & x(1-b/a) \\ 0 & 1 \end{pmatrix}\right) dx \\ &= q^{n/2} \int_F f'\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx \end{aligned}$$

for any $a \neq b$ with $|a/b| = q^n$ (note that $q = q_F$ and $|.|$ is the absolute value on F). Since $F(-n, f') = F(n, f')$ we may assume that $n \geq 0$, choose $b = 1$ and $|a| \geq 1$.

Using the Bruhat decomposition for $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ recorded at the beginning of the proof of Lemma 12, it is easy to see that

$$K'\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} K' = K' \begin{pmatrix} ax & 0 \\ 0 & b/x \end{pmatrix} K' \quad \text{if } |x| > 1 \text{ and } |a| \geq |b|,$$

hence that

$$F(n, f') = q^{n/2}\left[f'\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) + (1 - q^{-1}) \sum_{m>0} q^m f'\left(\begin{pmatrix} a\boldsymbol{\pi}^{-m} & 0 \\ 0 & b\boldsymbol{\pi}^m \end{pmatrix}\right)\right].$$

for $|a/b| = |\boldsymbol{\pi}^{-n}|$. In particular, for $|a/b| = |\boldsymbol{\pi}^{-n}|$ we have

$$|b/a|^{1/2}(F(n, f') - F(n+2, f')) = f'\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) - f'\left(\begin{pmatrix} a\boldsymbol{\pi}^{-1} & 0 \\ 0 & b\boldsymbol{\pi} \end{pmatrix}\right).$$

On the side of G we have

$$\mathrm{tr}\ I((\mu,\mu^{-1},1);f)=\sum_{n,m}F((n+m,m,0),f)t^n,$$

where

$$F((n,m,k),f)=F_f(g)=\left|\frac{(a-b)(b-c)(a-c)}{abc}\right|\int_U f(u^{-1}gu)du=|a/c|\int_U f(gu)du.$$

Here $|a|=q^n$, $|b|=q^m$, $|c|=q^k$, $g=\mathrm{diag}\,(a,b,c)$, and U is the upper triangular unipotent subgroup of G. The equality $\mathrm{tr}\ I'(\mu,f')=\mathrm{tr}\ I((\mu,\mu^{-1},1),f)$ for all μ implies that

$$F(n,f')=\sum_m F((n+m,m,0),f).$$

Hence

$$f'(I)-f'\left(\begin{pmatrix}\boldsymbol{\pi}^{-1} & 0\\ 0 & \boldsymbol{\pi}\end{pmatrix}\right)=F(0,f')-F(2,f')$$
$$=\sum_m[F((m+1,m+1,0),f)-F((m+2,m,0),f)],$$

and

$$f'\left(\begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix}\right)-f'\left(\begin{pmatrix}1 & 0\\ 0 & b\boldsymbol{\pi}^2\end{pmatrix}\right)=|b|^{1/2}\left[F_{f'}\left(\begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix}\right)-F_{f'}\left(\begin{pmatrix}1 & 0\\ 0 & b\boldsymbol{\pi}^2\end{pmatrix}\right)\right]$$
$$=|b|^{1/2}\sum_m\left[F_f\left(\begin{pmatrix}b^{-1}\boldsymbol{\pi}^{-m-1} & & 0\\ & \boldsymbol{\pi}^{-m-1} & \\ 0 & & 1\end{pmatrix}\right)-F_f\left(\begin{pmatrix}b^{-1}\boldsymbol{\pi}^{-2-m} & & 0\\ & \boldsymbol{\pi}^{-m} & \\ 0 & & 1\end{pmatrix}\right)\right].$$

These last identities are used in the proof in [F8] of

16. Proposition. *If $f\in\mathbb{H}$ and $f'\in\mathbb{H}'$ are corresponding, and*

$$f(\,\mathrm{diag}\,(\boldsymbol{\pi}^{-1},1,1))=f(\,\mathrm{diag}\,(\boldsymbol{\pi}^{-2},\boldsymbol{\pi}^{-1},1))=0,$$

then they are matching, namely $\Psi(b,f,\psi)=|b|^{1/2}\Psi(b,f',\psi)$ for all $b\in F^\times$.

The first step in the proof is the computation of the Fourier orbital integral on H. We shall deal with a slightly more general situation of a K'-biinvariant function f' on $GL(2,F)$ which transforms under the center via an unramified character η of order at most two, and is compactly supported modulo the center. We need only the case of $\eta=1$, but dealing with $\eta\neq 1$ too does not complicate the proof: it clarifies it a little.

17. Proposition. *The value of* $\Psi(-b, f', \psi) =$

$$\int_F \int_F f'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)\psi_2(x-y)dx\,dy$$

is

$$\eta(b^{1/2})^{-1}\left[f'(I) - f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-1} & 0 \\ 0 & \boldsymbol{\pi} \end{pmatrix}\right)\right]\int_F \psi_2\left(y - \frac{b}{y}\right)dy \quad \textit{if } |b| > 1$$

(the last integral ranges only over y *with* $|y| = |b|^{1/2}$*, it is zero unless the valuation of* b *is even, and then the value of the unramified* η *at* $b^{1/2}$ *is defined), and*

$$f'\left(\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\right) - 2f'\left(\begin{pmatrix} 1 & 0 \\ 0 & b\boldsymbol{\pi}^2 \end{pmatrix}\right)\eta(\boldsymbol{\pi}) + \eta(\boldsymbol{\pi}^2)f'\left(\begin{pmatrix} 1 & 0 \\ 0 & b\boldsymbol{\pi}^4 \end{pmatrix}\right) \quad \textit{if } |b| \le 1.$$

This is proven in [F8], Part 2.

To prove the matching Proposition 16 it is not necessary to compute $\Psi(b, f, \psi)$ explicitly, but we need to show that it is related to the result of Proposition 17 via the identities in the few lines prior to Proposition 16. Since $g_0\mathcal{J}_0 g_0^{-1} = g_1^{-1}\mathcal{J}_1 g_1$, for $\mathcal{J}_0 = \operatorname{diag}(1,-1,1), \mathcal{J}_1 = (1,1,-1)$ and

$$g_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}\begin{pmatrix} 1 & & -1/2 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 4 \end{pmatrix},$$

we have that $\Psi(b, f, \psi) = \int X\psi(x+y)dx\,dy\,dz\,dy\,d^\times\alpha d^\times\beta$, where $\psi = \boldsymbol{\psi}$ and

$$X = f\left(\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}\begin{pmatrix} 1/b & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & -1/4 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1/\alpha & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right).$$

Here we used the Iwasawa decomposition on $H^+ = \left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}\right\} \subset G$, and noted that f is K-invariant on the right, and that f is K-invariant on the left (we only used the fact that $f(\mathcal{J}g) = f(g)$ with $\mathcal{J} = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}$, and changed variables on z. The argument of f in X can also be expressed as

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}\begin{pmatrix} 1/\alpha b & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & u & -\alpha/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ -\alpha/4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\alpha b & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It suffices then to show the following

18. Proposition. *The value of* $\Psi(-b,f,\psi)|b|^{-1/2}$ *for* $f\in\mathbb{H}$ *with* $f(\operatorname{diag}(\boldsymbol{\pi}^{-1},1,1))=f(\operatorname{diag}(\boldsymbol{\pi}^{-2},\boldsymbol{\pi}^{-1},1))=0$ *is the product of* $\int_y\psi_2(y-b/y)dy$ *and*

$$\sum_m[F((m+1,m+1,0),f)-F((m+2,m,0),f)]$$

when $|b|>1$*. But when* $|b|=|\boldsymbol{\pi}^j|\leq 1$*, it is*

$$\begin{aligned}&\sum_m[F((m+1+j,m+1,0),f)-F((m+2+j,m,0),f)]\\&-q^{-1}\sum_m[F((m+3+j,m+1,0),f)-F((m+4+j,m,0),f)]\end{aligned}.$$

This is proven in [F8], Part 2, by a direct computation.

7. Applications to Cyclic Representations.

Now that we have all of the required results about matching local Fourier orbital integrals, we can return to the global Fourier summation formulae of Proposition 1 on $\mathbf{H}'=\operatorname{Res}_{E/F}PGL(2)$ and of Corollary 6 on $\mathbf{G}=PU(3,E/F)$. Let E/F be a quadratic extension of global fields, $f=\otimes f_v$, $f_v\in C_c^\infty(G_v)$ for all v, $f_v=f_v^0$ for almost all v, and $f'=\otimes f'_v$, $f'_v\in C_c^\infty(H'_v)$ for all v, $f'_v=f'^0_v$ for almost all v. Assuming the validity of Proposition 7 at the archimedean places, we may take E/F to be number fields. Since Proposition 7 is proven above only for finite places, our applications are fully proven only for function fields E/F.

19. Proposition. *Suppose that* f_v *and* f'_v *are matching for all places* v *of* F*. Then*

$$\sum_{\pi'}(W_\psi\overline{P}_H)_{\pi'}(f')+(1.2)+(1.3)=\sum_\pi n(\pi)(W_\psi\overline{P})_\pi(f)+(6.1)+(6.2).$$

The sum on the left ranges over all cuspidal distinguished (there is $\phi\in\pi'$ *with* $P_H(\phi)\neq 0$*) representations of* $PGL(2,\mathbb{A}_E)$*. On the right* π *ranges over all generic (*$W_\psi(\phi)\neq 0$*) cyclic (*$P(\phi)\neq 0$ *for some* $\phi\in\pi$*) cuspidal representations of* $PU(3,E/F)_{\mathbb{A}}$*.*

Proof. Propositions 14 and 16 imply – in the non-split and split cases respectively – that f'^0_v and f_v^0 are matching, and Proposition 7 asserts that for each $f_v\in C_c^\infty(G_v)$ there is a matching $f'_v\in C_c^\infty(H'_v)$, and for each such f'_v there is a matching f_v. Hence the assumption of the proposition is not vacuous. Since f_v and f'_v are matching for all v, by definition we have that $\Psi(b,f,\psi)=\Psi(b,f',\psi)$ for all $b\in E^\times/E^\bullet$, and for $b=0$. Hence the "geometric" sides of the summation formulae of Proposition 1 and Corollary 6 are equal. Our proposition simply concludes that for such matching f and f' the spectral sides are equal. □

To use Proposition 19 to derive representation theoretic consequences, we need to isolate a single contribution π' or π on the left or right of the identity of Proposition 19.

20. Proposition. *Let V be a finite set of places of F including all the places v where ψ_v is not unramified or v is ramified (including archimedean). For each $v \notin V$ fix an irreducible unramified unitarizable generic representation $I'(\mu_v)^0$ of $H'_v = PGL(2, E_v)$ (it is of the form $I'(\mu_v)^0 \times I'(\mu_v^{-1})^0$ if v splits in E), and denote by $I(\mu_v)^0$ the corresponding representation of G_v. For each $v \in V$, let $f'_v \in C_c^\infty(H'_v)$ and $f_v \in C_c^\infty(G_v)$ be matching functions, and put $f' = \otimes f'_v$ with $f'_v = f'^0_v$ for all $v \notin V$, and $f = \otimes f_v$ with $f_v = f_v^0$ for all $v \notin V$. Then*

$$\sum_{\pi'} (W_\psi \overline{P}_H)_{\pi'}(f') + (1.3) = \sum_\pi n(\pi)(W_\psi \overline{P})_\pi(f), \tag{20.1}$$

where π' ranges over the cuspidal distinguished representations of $\mathbf{H}'(\mathbb{A}) = PGL(2, \mathbb{A}_E)$ with $\pi'_v = I'(\mu_v)^0$ for all $v \notin V$, and π over the cuspidal generic cyclic representations of $\mathbf{G}(\mathbb{A}) = PU(3, E/F)_\mathbb{A}$ with $\pi_v = I(\mu_v)^0$ for $v \notin V$.

Proof. Let $v_1 \notin V$ be a place which stays prime in E, and use the identity of Proposition 19 with f, f' whose components are corresponding spherical f_{v_1} and f'_{v_1}. The operators $\pi'_{v_1}(f'_{v_1})$ and $\pi_{v_1}(f_{v_1})$ act as zero on the spaces of π'_{v_1} and π_{v_1}, except on the K'_{v_1}- and K_{v_1}- fixed vectors which exist only when π'_{v_1} and π_{v_1} are unramified. Hence the identity of Proposition 19 can be expressed in the form

$$\begin{aligned}&(1.3) + \sum_{\mu_{v_1}} \operatorname{tr} I'(\mu_{v_1}, f'_{v_1}) \sum_{\pi'} (W_\psi \overline{P}_H)_{\pi'}(f'_1)\\ &+ \sum_{\mu_{v_1}} \int \operatorname{tr} I'(\mu_{v_1}\nu_{v_1}^s, f'_{v_1}) d(\mu_{v_1}, f'_1, s)ds = \sum_{\mu_{v_1}} \operatorname{tr} I(\mu_{v_1}, f_{v_1}) \sum_\pi n(\pi)(W_\psi \overline{P})_\pi(f_1) + (6.1)\end{aligned}$$

of a standard kind (f'_1 is f' with f'_{v_1} replaced by $f'^0_{v_1}$, f_1 is f with f_{v_1} replaced by $f^0_{v_1}$; the sum over π' ranges over those π' whose component at v_1 is $I'(\mu_{v_1})$, and similarly for the sum over π; and s ranges over $i(\mathbb{R}/(\ell n q_{v_1})^{-1}\mathbb{Z})$).

Since $\operatorname{tr} I(\mu_{v_1}, f_{v_1}) = \operatorname{tr} I'(\mu_{v_1}, f'_{v_1})$, a standard argument based on the unitarizability of the π'_{v_1} and the π_{v_1}, the absolute convergence of the sums and integrals in the Fourier summation formula, and the Stone-Weierstrass Theorem (see, e.g., [F4], Proposition 5), imply the identity of our Proposition 20, but with arbitrary matching f, f' whose component at v_1 is $f^0_{v_1}$, $f'^0_{v_1}$, and where π, π' range only over those representations whose component at v_1 is the fixed $\pi_{v_1} = I(\mu_{v_1})^0$ and $\pi'_{v_1} = I'(\mu_{v_1})^0$. Note that the continuous sum, coming from (1.2) and (6.2) , vanishes in the comparison with the discrete sums of the last displayed identity.

Such argument of "generalized linear independence" of characters $\operatorname{tr} I'(\mu_{v_1}, f'_{v_1})$ ($= \operatorname{tr} I(\mu_{v_1}, f_{v_1})$) can be carried out at any finite number of places $v_1 \notin V$ which stay prime in E. A similar argument can be employed at any finite number of places $v_2 \notin V$ which split in E, but with a little difference. At a place v_2 which splits in E, an unramified generic unitarizable component π_{v_2} of π which appears in the sum of Proposition 19 is *a-priori* any representation $I(\mu_1, \mu_2, \mu_3)$ of $G_{v_2} = PGL(3, F_{v_2})$ induced from the Borel subgroup. But only π_{v_2} of the form $I(\mu, \mu^{-1}, 1)$ correspond to representations (generic unramified unitarizable) π'_{v_2}, necessarily of the form $I'(\mu) \times I'(\mu^{-1})$, of $H'_{v_2} = PGL(2, F_{v_2}) \times PGL(2, F_{v_2})$.

We may then apply "linear independence of characters" on G_{v_2}, and conclude that on fixing a local representation $I(\mu_{1v_2},\mu_{2v_2},\mu_{3v_2})^0$ in the statement of the proposition, the sum (1.3) and the one over π' are empty unless $\mu_{1v_2}\mu_{2v_2}=1$ and $\mu_{3v_2}=1$. This can be used at a later stage to show that any unramified component at v_2 of a π which occurs in the sum of Proposition 19 is necessary of the form $I(\mu_{v_2},\mu_{v_2}^{-1},1)$. However, Proposition 0 of [F7] – which is proven by purely local means in the context of $GL(n,F_v)$ – implies when $n=3$ the following.

20.2 Lemma. *The non-trivial unitarizable irreducible representations of $PGL(3,F_v)$ which have a non-zero $GL(2,F_v)$-invariant form are of the form $I(1\times\rho_2)$ – where ρ_2 is a representation of $PGL(2,F_v)$ – induced from the representation $1\times\rho_2$ of a maximal parabolic subgroup of $PGL(3,F_v)$.* □

In particular the unramified representations of the Lemma are of the form $I(1,\mu,\mu^{-1})$, and they correspond to representations $I'(\mu)\times I'(\mu^{-1})$ of $H'_{v_2}=PGL(2,F_{v_2})\times PGL(2,F_{v_2})$. Consequently at v_2 we may apply "linear independence of characters" on $PGL(2,F_{v_2})$. We may repeat this argument at any finite number of places $v_2\notin V$ which split in E. Proposition 20 now follows from a simple induction-type argument, as in [F4], Lemmas on p. 758. Note that (6.1) does not appear on the right side of (20.1) since $I'(\mu)$ with $\mu=1$ does not appear in (1.3). □

Remark. (1) By rigidity theorem for $GL(2)$, since we fix in Proposition 20 a representation ${\pi'_v}^0$ of H'_v for almost all v, there is at most one cuspidal π' with these local components, and at most one representation $\pi'=I'(\mu)$, with $\mu^2\neq 1$, $\mu|\mathbb{A}_F^\times=1$, with these local components, and there is at most one non-zero contribution to the left of (20.1) (namely the sum of (1.3) or the sum over π' is empty).
(2) The uniqueness of the Whittaker functional on π'_v, and of the $PGL(2,F_v)$-invariant functional on the $PGL(2,E_v)$-module π'_v, imply that each of the global functionals W_ψ and P_H split as a product of local ones, W_{ψ_v} and P_{H_v}, on the local representations. Consequently, we can make the following.

Definition. Let $\{\phi(\pi'_v)\}$ indicate a K_v-finite orthonormal basis of the space of π'_v. Then

$$(W_{\psi_v}\overline{P}_{H_v})_{\pi'_v}(f'_v)=\sum_{\phi(\pi'_v)}W_{\psi_v}(\pi'_v(f'_v)\phi(\pi'_v))\overline{P}_{H_v}(\phi(\pi'_v))$$

defines a functional on $C_c^\infty(H'_v)$.

This functional depends on π'_v, is independent of the choice of the basis $\{\phi(\pi'_v)\}$, it is zero unless π'_v is generic and distinguished, and for inequivalent π'_{vi} $(1\le i\le k)$, the $(W_{\psi_v}\overline{P}_{H_v})_{\pi'_{vi}}$ are linearly independent. It is invariant under right translations by $H_v=PGL(2,F_v)$ and transforms under left translations of f'_v by $\begin{pmatrix}1&x\\0&1\end{pmatrix}\in N'_v$ $(x\in E_v)$ according to ψ_v.

For every cuspidal π' and non-trivial character ψ there is a constant $c(\pi',\psi)$ such that

$$(W_\psi\overline{P}_H)_{\pi'}(f')=c(\pi',\psi)\prod_v(W_{\psi_v}\overline{P}_{H_v})_{\pi'_v}(f'_v)$$

if $f' = \otimes f'_v$. Both sides are zero if some π'_v is not generic and distinguished. The global cuspidal π' is (automorphically) distinguished precisely when all $\pi_{v'}$ are distinguished, and $c(\pi',\psi) \neq 0$. The constant $c(\pi',\psi)$ depends on the various normalizations involved. For example we may choose the orthonormal basis $\{\phi(\pi'_v)\}$ of an unramified π'_v to contain a K'_v-invariant vector ξ^0_v, and require that $P_{H_v}(\xi^0_v) = 1 = W_{\psi_v}(\xi^0_v)$ when ψ_v is unramified. Then $(W_{\psi_v}\overline{P}_{H_v})_{\pi'_v}(f'_v) = \mathrm{tr}\ \pi'_v(f'_v)$ at a spherical (K'_v-biinvariant) f'_v.

We shall use these remarks to deduce from Proposition 20 the following.

21. Proposition. *Let π' be a distinguished cuspidal representation of $PGL(2,\mathbb{A}_E)$, or an induced representation of the form $I'(\mu)$, with $\mu : \mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times \to \mathbb{C}^\times$ and $\mu^2 \neq 1$. Then π' corresponds to a generic cyclic cuspidal representation π of $PU(3,E/F)_{\mathbb{A}}$.*

Proof. Consider the case where π' is cuspidal distinguished. The case where π' is $I'(\mu)$ can be handled similarly. We shall use Proposition 20 with a finite set V which contains all places of F which ramify in E (including the archimedean primes), and all places where ψ_v or π'_v are ramified. The fixed local representations $I'(\mu_v)^0$ are chosen to be the components π'_v of π' for all $v \notin V$. The left side of (20.1) consists of a single summand, that indexed by our π', by the Remark (1) above. The sum of (1.3) is empty.

To complete the proof we simply need to produce $f'_v \in C_c^\infty(H'_v)$ with $(W_{\psi_v}\overline{P}_{H_v})_{\pi'_v}(f'_v) \neq 0$ for each v. This f'_v needs to be matching some $f_v \in C_c^\infty(G_v)$. By the normalization specified at the end of Remark (2) above, we have $(W_{\psi_v}\overline{P}_{H_v})_{\pi'_v}(f'^0_v) = 1$ for all $v \notin V$ (and f'^0_v matches f^0_v by Propositions 14 and 16). At each v in V, let $\phi_1(\pi'_v)$ be a smooth vector in the space of π'_v with $P_{H_v}(\phi_1(\pi'_v)) \neq 0$, and $\phi_2(\pi'_v)$ a smooth vector with $W_{\psi_v}(\phi_2(\pi'_v)) \neq 0$. We may assume that either $\phi_2(\pi'_v) = \phi_1(\pi'_v)$, or that $\phi_2(\pi'_v)$ is orthogonal to $\phi_1(\pi'_v)$. Each of $\phi_i(\pi'_v)$ can be multiplied by a scalar to have length one, and we extend $\{\phi_i(\pi'_v); i = 1,2\}$ to an orthonormal basis of π'_v. Since $\{\pi'_v(f'_v); f'_v \in C_c^\infty(H'_v)\}$ span the algebra of endomorphisms of π'_v, we may choose f'_v such that $\pi'_v(f'_v)\phi_i(\pi'_v)$ is 0 unless $i = 1$, and it is $\phi_2(\pi'_v)$ if $i = 1$. Then

$$(W_{\psi_v}\overline{P}_{H_v})_{\pi'_v}(f'_v) = W_{\psi_v}(\pi'_v(f_v)\phi_1(\pi'_v))\overline{P}_{H_v}(\phi_1(\pi'_v)) = W_{\psi_v}(\phi_2(\pi'_v))\overline{P}_{H_v}(\phi_1(\pi'_v)) \neq 0.$$

By Proposition 7 any f'_v matches some $f_v \in C_c^\infty(G_v)$. Applying Proposition 20 we conclude that the left side of (20.1) is non-zero, hence so is the right side, namely there is a cuspidal generic cyclic π which corresponds to our π', as required. □

In the opposite direction we prove the following.

22. Proposition. *Let π^0 be a generic cyclic cuspidal (irreducible) representation of $PU(3,E/F)_{\mathbb{A}}$. Then there is either a unique cuspidal distinguished (irreducible) representation π' of $PGL(2,\mathbb{A}_E)$, or a unique induced representation $I'(\mu)$ of $PGL(2,\mathbb{A}_E)$, where μ is a character of $\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times$ with $\mu^2 \neq 1$, which corresponds to π^0.*

Proof. This will be proven along the lines of the proof of Proposition 21, except that the rigidity theorem for $PU(3,E/F)$ is deeper than that for $PGL(2)$. In any case we choose a finite set V consisting of all places of F which ramify in E, and where ψ or π is ramified; this includes the archimedean places. We apply Proposition 20 with a fixed local representation

$I'(\mu_v)^0$ which corresponds to the components π_v^0 of π_v at each $v \notin V$. We need to show that the right side of (20.1) is non-zero.

Using Bernstein's "multiplicity one" theorem recorded in the Appendix (asserting the uniqueness of P_v below), we have the product formula $(W_\psi \overline{P})_\pi(f) = c(\psi,\pi)\prod_v (W_{\psi_v}\overline{P}_v)_{\pi_v}(f_v)$ for each cuspidal generic cyclic π. Here $(W_{\psi_v}\overline{P}_v)_{\pi_v}(f_v)$ is $\sum W_{\psi_v}(\pi_v(f_v)\phi(\pi_v))\overline{P}_v(\phi(\pi_v))$, as in the case of $PGL(2,\mathbb{A}_E)$. Also P_v is a $U(2,E_v/F_v)$-invariant form on $PU(3,E_v/F_v)$, $\{\phi(\pi_v)\}$ is an orthonormal basis of smooth vectors in π_v, and W_{ψ_v} is a ψ_v-Whittaker functional on π_v. We assume that $\{\phi(\pi_v)\}$ contains a K_v-fixed vector ξ_v^0, and that $P_v(\xi_v^0) = 1 = W_{\psi_v}(\xi_v^0)$ for all $v \notin V$. Then $(W_{\psi_v}\overline{P}_v)_{\pi_v^0}(f_v^0) = 1$ for all $v \notin V$.

Consider a place $v \in V$ such that π_v^0 is supercuspidal. Such v is necessarily finite and it stays prime in E. Then there are smooth vectors $\phi_i(\pi_v^0)$ $(i = 1, 2)$ of length one with $P_v(\phi_1(\pi_v^0)) \neq 0$ and $W_{\psi_v}(\phi_2(\pi_v^0)) \neq 0$. We may assume that $\phi_2 = \phi_1$ or that ϕ_2 is orthogonal to ϕ_1. Extend $\{\phi_1, \phi_2\}$ to an orthonormal basis of π_v^0. The matrix coefficient $f_v^0(x) = (\pi_v^0(x)\phi_2, \phi_1)$ is a supercusp form which satisfies $\pi_v^0(f_v^0)\phi = 0$ for all ϕ orthogonal to ϕ_1, and $\pi_v^0(f_v^0)(\phi_1) = \phi_2$ (up to a non-zero multiple). Consequently $(W_{\psi_v}\overline{P}_v)_{\pi_v}(f_v^0) = 0$ for all π_v inequivalent to π_v^0, and $(W_{\psi_v}\overline{P}_v)_{\pi_v^0}(f_v^0) \neq 0$. Using f_v^0 at each place $v \in V$ where π_v^0 is supercuspidal, we conclude that the sum on the right of (20.1) extends only over π whose components at these v are the supercuspidal π_v^0.

Next we consider a place $v \in V$ such that π_v^0 is not supercuspidal. Then π_v^0 is the unique generic constituent in the composition series of an induced representation $I(\mu_v)$ (or $I(1 \times \rho_{2v})$, where ρ_{2v} is a generic unitarizable irreducible representation of $PGL(2,F_v)$ if v splits in E). As usual we may choose a basis $\phi_1, \phi_2, \dots$ for π_v^0, and $f_v \in C_c^\infty(G_v)$ with $\pi_v^0(f_v)\phi_i = 0$ $(i \neq 1)$, and $\pi_v^0(f_v)\phi_1 = \phi_2$, so that $(W_{\psi_v}\overline{P}_v)_{\pi_v^0}(f_v) \neq 0$.

Applying Bernstein's decomposition theorem (which is based on Bernstein's center, see [BD], a forthcoming work by Bernstein, and a summary in [F5], pp. 165/6), we may assume that f_v is of the form $f_v * f_v^0$, where $f_v^0 \in C_c^\infty(G_v)$ is a function with the property that $\pi_v(f_v^0)$ acts as 0 on any π_v whose infinitesimal character lies in a Bernstein component different than that of π_v^0, and $\pi_v^0(f_v^0)\phi_1 = \phi_1$. Namely the sum over π on the right of (20.1) will range over all π whose component at the $v \notin V$ or at the v where π_v^0 is supercuspidal, is the same as that of π^0, but at the remaining finite set of places, where π_v^0 is the generic constituent of a full induced $I(\mu_v)$, or $I(\mu_v, \mu_v^{-1}, 1)$ in the split case, we only know that π_v is a constituent of $I(\mu_v\nu_v^s)$ or $I(\mu_v\nu_v^s, \mu_v^{-1}\nu_v^{-s}, 1)$ for some $s \in \mathbb{C}$ (here $\nu_v(x) = |x|_v$). It appears the sum over π may range over a set larger than π^0 alone, and cancellations may cause this sum on the right of (20.1) to vanish.

At this stage we need to invoke the rigidity theorem for automorphic representations of $U(3,E/F)$ from [F3] and [F3′] which asserts, in particular, that: there is at most one cuspidal representation of $U(3,E/F)_{\mathbb{A}}$ almost all of whose components are specified, and whose remaining finite set of components consists of generic constituents of induced $U(3,E_v/F_v)$-modules. Note that this is a weak form of the rigidity theorem of [F3]. It does not use the structure of $U(3)$-packets as described in [F3], nor multiplicity one theorem for $U(3)$, which is proven in [F3] using some of the work of [GP]. The part of the rigidity theorem used here relies on no results stated in [GP]. In any case the part of the rigidity theorem of [F3] just quoted implies that there is only one non-zero term in the sum on the right of (20.1), it is indexed by our π^0, and the right side of (20.1) is non-zero for our choice of f_v for $v \notin V$.

It now follows that the left side of (20.1) is non-zero, and we obtain either a cuspidal distinguished π' or an induced $I'(\mu)$ which corresponds to our π^0, as required. □

8. Local Cyclic Representations.

We shall now turn to the local theory of cyclic representations, namely those representations of $PU(3, E_v/F_v)$ which admit a $U(2, E_v/F_v)$-invariant form, and those representations of $PGL(3, F_v)$ which admit a $GL(2, F_v)$-invariant form. Here F_v is a non-archimedean local field, and E_v is a separable quadratic extension, of characteristic $\neq 2$. All representations which occur in our summation formulae are generic, hence all results derived here from the summation formulae will concern only generic local representations.

A complete characterization of the distinguished representations π'_v of $GL(2, E_v)$, namely those representations which admit a non-zero $GL(2, F_v)$-invariant form, is given in [F5]. We shall return to this parametrization shortly, but note that if v splits in E/F, then $E_v = F_v \oplus F_v$ and $\pi'_v = \pi_{1v} \times \pi_{2v}$ where π_{iv} is a representation of $GL(2, F_v)$, and π'_v is distinguished precisely when $\pi_{2v} \simeq \check{\pi}_{1v}$ (= contragredient of π_{1v}). The local application of Propositions 20/21/22, is the following.

23. Proposition. *For each generic distinguished representation π'^0_v of $PGL(2, E_v)$ which is a component of a cuspidal distinguished representation π'^0 of $PGL(2, \mathbb{A}_E)$, or of an induced $\pi'^0 = I'(\mu)$, $\mu : \mathbb{A}_E^\times / E^\times \mathbb{A}_F^\times \to \mathbb{C}^\times$, $\mu^2 \neq 1$, there exists a unique finite set $\{\pi_v\}$ of representation of $PU(3, E_v/F_v)$, and constants $c(\psi_v, \pi_v)$ such that the π_v are components of cyclic generic cuspidal representations π^0 of $PU(3, E/F)_{\mathbb{A}}$, and all of the π_v lie in one packet (a notion introduced and studied in [F3]) which is uniquely determined by π'^0_v; such that for all matching $f'_v \in C_c^\infty(H'_v)$ and $f_v \in C_c^\infty(G_v)$ we have*

$$(W_{\psi_v} \overline{P}_{H_v})_{\pi'^0_v}(f'_v) = \sum_{\pi_v \in \{\pi_v\}} c(\psi_v, \pi_v)(W_{\psi_v} \overline{P}_v)_{\pi_v}(f_v). \tag{23.1}$$

Conversely, for every component π_v^0 of a generic cyclic cuspidal representation π^0 of $PU(3, E/F)_{\mathbb{A}}$, there exists a unique generic distinguished representation π'^0_v of $PGL(2, E_v)$ which is a component of a cuspidal distinguished representation π'^0 of $PGL(2, \mathbb{A}_E)$, or of an induced $\pi'^0 = I'(\mu)$, $\mu : \mathbb{A}_E^\times / E^\times \mathbb{A}_F^\times \to \mathbb{C}^\times$, $\mu^2 \neq 1$, such that (23.1) holds for all matching $f'_v \in C_c^\infty(H'_v)$ and $f_v \in C_c^\infty(G_v)$, with $\{\pi_v\} = \{\pi_v\}(\pi'^0_v)$.

If v splits in E then $\{\pi_v\}$ consists of a single representation.

Proof. Given a global π^0 or the corresponding global π'^0, we set up the identity (20.1) such that π'^0 indexes the only term on the left, and π^0 occurs on the right. At each place $v_1 \in V$, $v_1 \neq v$, we choose f_{v_1} as in the proof of Proposition 22, to guarantee that $(W_{\psi_{v_1}} \overline{P}_{v_1})_{\pi^0_{v_1}}(f_{v_1}) \neq 0$ and that the π which index non-zero terms on the right will have the component $\pi^0_{v_1}$ at each place $v_1 \neq v$ in V.

As in the proof of Proposition 22, we used here the rigidity theorem for $U(3, E/F)$ of [F3]. We derive the identity (23.1) for all matching f'_v and f_v, where the sum on the right ranges over a subset of the packet of π_v^0, by virtue of a special case of the rigidity theorem for $U(3,E/F)$ of [F3]: If $\pi=\otimes\pi_v$ is a cuspidal representation of $U(3,\mathbb{A}_E/\mathbb{A}_F)$ lifted (in terms of almost all places) from a generic representation of $U(2,\mathbb{A}_E/\mathbb{A}_F)$, then the local components at v of any cuspidal representation of $U(3)$ which is almost everywhere equivalent to π must lie in the packet of π_v. Note that for each π_v which contributes a non-zero entry to the right side of (23.1), the global representation $\pi_v \otimes (\bigotimes_{v_1\neq v} \pi_{v_1}^0)$ needs to be cuspidal, cyclic and generic, in order to appear in (20.1).

The $\{\pi_v\}$ is uniquely determined by π'_v, and π'_v is uniquely determined by π_v, since the distributions $(W_{\psi_v}\overline{P}_v)_{\pi_v}$ are linearly independent, and the subset $\{\pi_v\}$ of the packet of π_v is uniquely determined by the rigidity theorem and the conditions which we put by fixing all other components of π^0. □

Definition. A distinguished generic representation π'_v of $PGL(2,E_v)$ and a cyclic generic representation π_v of $PU(3,E_v/F_v)$ are said to *correspond* if they satisfy the relation (23.1) for all matching f'_v and f_v.

Remark. (1) Clearly, each component of a generic representation is generic.
(2) It is shown in [F3] that once a twisted analogue of the theorem of Rodier [Ro], relating the asymptotic behavior of a character with the number of Whittaker models of the representation, is made available, then the packet $\{\pi_v^0\}$ of the π_v^0 of Proposition 23 will contain precisely one generic member, namely π_v^0 itself, and the sum on the right of (23.1) consists of a single non-zero summand, indexed by this π_v^0.

Proposition 14 of [F5] asserts (see also the Remark following B17 in [FH]) that: *each distinguished infinite dimensional supercuspidal representation π_v of $GL(2,E_v)$ can be viewed as a component of a cuspidal distinguished representation π of $GL(2,\mathbb{A}_E)$*, in fact with supercuspidal distinguished components at any prescribed finite set of places. If π_v has trivial central character, π can be chosen to have trivial central character.

The proof of Proposition 14 of [F5] is based on a simple compactness argument which produces a global test function f with the preassigned local components, such that the geometric side of the summation formula reduces to a single non-zero term, hence the spectral side is also non-zero. Analogous proof applies in the case of cyclic generic representations, and we record only the result.

24. Proposition. *Let E/F be a separable quadratic extension of global fields, and π_v a generic supercuspidal cyclic representation of $PU(3,E_v/F_v)$ where E_v is a field. Then there exists a generic cuspidal cyclic representations π of $PU(3,E/F)_{\mathbb{A}}$ whose component at v is the give π_v.*

Again, the proof of this follows closely that of Proposition 14 in [F5], and will not be given here. □

To obtain explicit description of the local cyclic representation, we recall some of the results of [F5] and [F3]. Our recollection here will be very brief; for a full description of the definitions and results see the original papers. According to [F5], the distinguished representations of $GL(2,\mathbb{A}_E)$ (and $GL(2,E_v)$) are precisely those cuspidal (and admissible (irreducible)) representations of these groups which are obtained as the image of the unstable base change lifting from the unitary group $U(2,E/F)_{\mathbb{A}}$ (or $U(2,E_v/F_v)$) in two variables associated with the quadratic field extension E/F (or E_v/F_v) which is implicit in the definition of a distinguished representation. Note that the unstable base-change lifting depends on a choice of a character $\kappa : \mathbb{A}_E^\times / E^\times N_{E/F}\mathbb{A}_E^\times \to \mathbb{C}^\times$ whose restriction to $\mathbb{A}_F^\times$ is not trivial (or locally on a choice of a $\kappa_v : E_v^\times / N_{E/F}E_v^\times \to \mathbb{C}^\times$ with restriction $\kappa_v|F_v^\times \neq 1$). The image of the lifting is independent of κ (or κ_v).

Combining the unstable base change lifting from $U(2)$ to $GL(2,E)$, with the global correspondence – defined in terms of almost all components – from the set of distinguished cusp forms on $PGL(2,\mathbb{A}_E)$ to the set of cyclic generic cusp forms on $PU(3,E/F)_{\mathbb{A}}$, which is studied in this paper, we obtain the endoscopic lifting from $U(2,E/F)_{\mathbb{A}}^\kappa$ (the superscript κ indicates representations with central character κ^{-1}) to $PU(3,E/F)_{\mathbb{A}}$, which depends on a choice of κ as above. This endoscopic lifting is studied in [F3].

In the local case of a place v of F which splits in E, we obtain:

25. Proposition. *Let ρ_v be a square-integrable representation of $PGL(2,F_v)$. Then the representation $I(1 \times \rho_v)$ of $PGL(3,F_v)$, which is normalizedly induced from the representation of a maximal parabolic subgroup defined by ρ_v on its Levi subgroup, is cyclic, namely admits a non-zero $GL(2,F_v)$-invariant form.*

Remark. This statement involves the local non-archimedean field F_v, but no quadratic extension E_v thereof. A purely local proof of this is given in [F7], Propositions 0 and 0.1. Proposition 23 asserts also that $\pi_v = I(1 \times \rho_v)$ corresponds to $\pi_v' = \rho_v \times \rho_v$, namely that $(W_{\psi_v}\overline{P}_{H_v})_{\pi_v'}(f_v') = c(W_{\psi_v}\overline{P}_v)_{\pi_v}(f_v)$ for all matching f_v' and f_v, where c is a constant depending on ψ_v and ρ_v. This relation of Whittaker - period distributions does not follow from the methods of [F7].

Proof. Given ρ_v we construct – along standard lines, using the usual trace formula on $U(2,E/F)_{\mathbb{A}}$ and a pseudo-coefficient of ρ_v whose existence (for non supercuspidals) is proven in Kazhdan [K] – a cuspidal representation ρ of $U(2,E/F)_{\mathbb{A}}$ (with the required central character) whose component at v is ρ_v. The lift of ρ to $PU(3,E/F)_{\mathbb{A}}$ is cyclic, and its component at v is $I(1\times\rho_v)$ (by following the diagram in the introduction and using [F3]), which is cyclic as a component of a global cyclic representation. □

In the case where E_v/F_v is a separable quadratic extension of local non-archimedean fields, we conclude the following. Recall that $G_v = PU(3,E_v/F_v)$.

26. Proposition. *Every G_v-packet which is the image via the endoscopic lifting of a square-integrable representation of $U(2,E_v/F_v)^{\kappa_v}$ contains a generic cyclic square-integrable representation π_v. Conversely, the packet of any generic cyclic square-integrable representation π_v of G_v is the endoscopic lift of a square-integrable $U(2,E_v/F_v)$-module (with central character κ_v^{-1}). Equivalently, the correspondence establishes a bijection between the set of G_v-packets containing a generic cyclic square-integrable representation of G_v, and the set of square-integrable distinguished $PGL(2,E_v)$-modules or the induced $I'(\mu_v)$ where $\mu_v : E_v^\times/F_v^\times \to \mathbb{C}^\times$ satisfies $\mu_v^2 \neq 1$.*

Proof. For the first claim we need to repeat the proof of Proposition 25, namely that any square-integrable representation ρ_v of $U(2,E_v/F_v)^{\kappa_v}$ can be viewed as a component of a cuspidal representation of $U(2,E/F)^{\kappa}_{\mathbb{A}}$. For the "conversely", the case of a supercuspidal π_v is handled by Proposition 24. A list of the reducible principal series representations of $U(3,E_v/F_v)$ is given in [F2], (3.2), p. 558. There are two cases where the full induced representation has a square integrable constituent. The case of the square integrable subrepresentation of the induced $I(\mu_v\kappa_v\nu_v^{1/2})$, where $\mu_v : E_v^\times/F_v^\times \to \mathbb{C}^\times$, is dealt with by the first assertion of our proposition: it is the generic cyclic lift of the "special" representation of $U(2,E_v/F_v)$. The other square integrable non supercuspidal $U(3,E_v/F_v)$-module is the ("special") subrepresentation $sp(\mu_v)$ of $I(\mu_v\nu_v)$, $\mu_v : E_v^\times/F_v^\times \to \mathbb{C}^\times$. It is generic, but not an endo-lift from $U(2,E_v/F_v)$ (see [F3]). Hence we need to show that $sp(\mu_v)$ is not cyclic. This is done in Proposition 29(b) by means of a purely local proof. This completes the proof of Proposition 26. □

Remark. (1) There is one more case of a reducible principal series $U(3,E_v/F_v)$-module ([F2], (3.2(1)), p. 558), namely $I(\mu_v)$, for some $\mu_v : E_v^\times/F_v^\times \to \mathbb{C}^\times$. It is the lift of the reducible tempered $U(2,E_v/F_v)$-module $I_0(\mu_v/\kappa_v)$. Since the two constituents of $I_0(\mu_v/\kappa_v)$ are elliptic (their characters are not identically zero on the elliptic regular set of $U(2,E_v/F_v)$), they have pseudo-coefficients as in [K], and so an irreducible constituent of $I_0(\mu_v/\kappa_v)$ makes a local component of a cuspidal $U(2,E/F)_{\mathbb{A}}$-module, whose endo-lift will then be a cyclic generic cuspidal $U(3,E/F)_{\mathbb{A}}$-module whose component at v is cyclic, generic and a constituent of $I(\mu_v)$.
(2) According to Proposition 26, each G_v-packet which is the endoscopic lift of a square-integrable $U(2,F_v/F_v)$-module contains a generic member, which is also cyclic. All of the elements in this packet are square-integrable. Assuming the validity of the twisted analogue of [Ro] we conclude in [F3] that this packet contains only one generic element. Then Proposition 26 will assert that the generic member of the packet is cyclic, all generic square-integrable cyclic G_v-modules are so obtained, and they correspond to the $PGL(2,E_v)$-modules specified in the Proposition. Without using the twisted analogue of [Ro] we may have several generic cyclic modules in a packet lifted from $U(2,E_v/F_v)$, and these contribute to the right side of the correspondence relation (23.1).

The global result is very much the same. Again $\mathbb{G} = PU(3,E/F)_{\mathbb{A}}$.

27. Proposition. *Every $\mathbb{G}$-packet which is an endoscopic lift of a cuspidal $U(2,E/F)_{\mathbb{A}}$-packet contains a generic cyclic cuspidal representation π. Conversely, the $\mathbb{G}$-packet of any generic cyclic cuspidal π is the endoscopic lift of a cuspidal $U(2,E/F)_{\mathbb{A}}$-module with central character κ^{-1}. Equivalently, the correspondence establishes a bijection between the set of $\mathbb{G}$-packets containing a generic cyclic cuspidal representation of $\mathbb{G}$, and the set of cuspidal distinguished $PGL(2,\mathbb{A}_E)$-modules or the induced $I'(\mu)$, where $\mu : \mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times \to \mathbb{C}^\times$ satisfies $\mu^2 \neq 1$.* □

The summation formula is used above to show that certain non fully induced generic representations of $PU(3,E_v/F_v)$ are cyclic. Our next aim is to study the remaining generic cyclic representations. In the split case we have Proposition 0 of [F7], whose special case of $n=3$ is quoted in Lemma 20.2 above.

Before turning to the determination of the induced $PU(3, E_v/F_v)$-modules $I(\mu_v)$ which are cyclic, we record here (a more detailed version of) Proposition 9 of [F5], which determines the distinguished $GL(2, E_v)$-modules $I(\mu_v, \overline{\mu}_v^{-1})$. The proof is relegated to B17, [FH], to save space here. Since the problem is local, we use local notations, as follows. Let E/F be a quadratic separable extension of local non-archimedean fields, and μ a unitary character of $E^\times$.

Note that by [F5], Proposition 12, the non supercuspidal infinite dimensional distinguished representations of $GL(2, E)$ are of the form $I(\mu, \overline{\mu}^{-1})$, where $\overline{\mu}(x) = \mu(\overline{x})$, or of the form $I(\mu_1, \mu_2)$, with $\mu_i|NE^\times = 1$, and $\mu_1 \neq \mu_2$, or they are the "special" square-integrable subrepresentation $sp(\mu)$ of $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$, where μ is a character of $E^\times/NE^\times$.

28. Proposition. *(a) The representation* $I_s = I(\mu\nu^s, \overline{\mu}^{-1}\nu^{-s})$ *of* $GL(2, E)$ *is distinguished* $(s \in \mathbb{C})$. *(b) The representation* $I(\mu_1, \mu_2)$, $\mu_1 \neq \mu_2$, *is distinguished precisely when* $\mu_i|F^\times = 1$. *(c) The representation* $sp(\mu)$ *is distinguished precisely when* $\mu|F^\times \neq 1$, *but* $\mu|NE^\times = 1$.

Proof. This is Proposition B17 of [FH], which expands the proof of [F5], Proposition 9. □

Remark. (1) In the split case $E = F \oplus F$, and a representation $\pi_1 \times \pi_2$ of $GL(2, E) = GL(2, F) \times GL(2, F)$ is distinguished precisely when π_2 is the contragredient $\check{\pi}_1$ of π_1. When $\pi_1 = I(\mu_1, \mu_2)$ is induced, then $\check{\pi}_1 = I(\mu_2^{-1}, \mu_1^{-1})$. Let $K = GL(2, R)$ be the standard maximal compact subgroup of $G = GL(2, F)$, and A the diagonal subgroup of G. Define a $G = GL(2, F)$-invariant form L_s on $(\phi_1, \phi_2) \in I(\mu\nu^s, \mu^{-1}\nu^{-s}) \times I(\mu\nu^s, \mu^{-1}\nu^{-s})$ (G acts via (g, wgw), $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) by

$$L_s(\phi_1, \phi_2) = \int_{A\backslash G} \phi_1\left(\begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix}\right) \phi_2\left(\begin{pmatrix} a_2 & b_2 \\ b_1 & a_1 \end{pmatrix}\right) dg.$$

If $A\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} K \neq AK$ then $|x| > 1$. Put $x = (1-b)^{-1}$, note that

$$\begin{pmatrix} 1 & 1 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-b)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

and that $dx = |1-b|^{-2} db$. If $\mu = 1$, for K -invariant ϕ_1, ϕ_2 we have

$$L_s(\phi_1, \phi_2) = \int\limits_{A\cap K\backslash K} dk + \int\limits_{\substack{|b|=1 \\ |1-b|<1}} \phi_1\left(\begin{pmatrix} 1 & 1 \\ b & 1 \end{pmatrix}\right) \phi_2\left(\begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix}\right) \frac{db}{|1-b|^2}.$$

Normalize the measure to assign $A \cap K\backslash K$ the volume 1. Since $\begin{pmatrix} 1 & 1 \\ b & 1 \end{pmatrix} \in N\begin{pmatrix} 1-b & 0 \\ 0 & 1 \end{pmatrix} K$ and $\begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix} \in N\begin{pmatrix} 1/(1-b) & 0 \\ 0 & 1 \end{pmatrix} K$, when ϕ_1 and ϕ_2 are the unit vector ϕ_s^0 in $I(\nu^s, \nu^{-s})$, we have

$$L_s(\phi_s^0, \phi_s^0) = 1 + \int_{|1-b|<1} |1-b|^{2s+1} |1-b|^{-2} db$$

$$= 1 + \sum_1^\infty q^{-m(2s-1)} \int_{|b-1|=q^{-m}} db = 1 + (1-q^{-1}) \frac{q^{-2s}}{1-q^{-2s}} = \frac{L(2s)}{L(2s+1)}.$$

The cases of other ϕ_1 and ϕ_2, and of ramified μ, is similarly handled.
(2) The explicit form of the functional L_s discussed in (1) in the split case, and in the proof of [FH], B17(a), in the non-split case, is the local component of the global integral $\mathcal{J}(\mu,s)\Phi$ of [F5], p. 156, ℓ. 6. Our analysis here yields an alternative – and more direct and natural – proof of the Lemma of [F5], p. 156.

Next we discuss the analogue of Proposition 28 for $PU(3,E/F)$, where – as there – E/F is a separable quadratic extension of local non-archimedean fields, and μ is a unitary character of $E^\times$. Note that the $PU(3,E/F)$-module $I(\mu\kappa)$ is the endo-lift of the $U(2,E/F)$-module $I_0(\mu)$, whose central character is κ^{-1}, namely $\mu\kappa=1$ on $E^\bullet=\{a\in E^\times; a\overline{a}=1\}$. Hence we need to show that $I(\mu)$ is cyclic precisely when $\mu|E^\bullet=1$.

29. Proposition. *(a) The representation $I_s=I(\mu\nu^s)$ of $G=PU(3,E/F)$ is cyclic precisely when $\mu|E^\bullet=1$. (b) The ("special") subrepresentation $sp(\mu)$ of $I(\mu\nu)$, $\mu:E^\times/F^\times\to\mathbb{C}^\times$, is not cyclic.*

Proof. (a) Recall that C is the centralizer of $\mathcal{J}_0=\operatorname{diag}(1,-1,1)$. The centralizer C_1 of $\mathcal{J}_1=g_0^{-1}\mathcal{J}_0g_0$ is $g_0^{-1}Cg_0$. The space of I_s consists of the smooth $\varphi:G\to\mathbb{C}$ with

$$\varphi\left(\begin{pmatrix} a & & * \\ & 1 & \\ 0 & & \overline{a}^{-1}\end{pmatrix} g\right)=\mu(a)|a|_E^{s+1}\varphi(g)\quad (a\in E^\times, g\in G).$$

Define a C_1-invariant form L_s on I_s by $L_s(\varphi)=\int_{B\cap C_1\backslash C_1}\varphi(h)dh$. Here $B=AN$ is the upper triangular subgroup of G, A the diagonal subgroup, N the unipotent upper triangular, and $B\cap C_1$ consists of $\operatorname{diag}(a,b,a)$ with a,b in $E^\bullet$. Hence L_s is zero unless $\mu|E^\bullet=1$. Using the Bruhat decomposition on G it is easy to see that

$$C_1=(C_1\cap B)\cup(C_1\cap NA\mathcal{J}N)=(C_1\cap K)\cup(C_1\cap NA\mathcal{J}N-K\cap C_1),$$

where $K=G\cap PGL(3,R_E)$ and $\mathcal{J}=\begin{pmatrix}0 & & 1\\ & -1 & \\ 1 & & 0\end{pmatrix}$. Moreover $C_1\cap NA\mathcal{J}N$ consists of

$$\begin{pmatrix}\varepsilon & & 0\\ & 1 & \\ 0 & & \varepsilon\end{pmatrix}\begin{pmatrix}1 & x & y/2\\ 0 & 1 & \overline{x}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}(\overline{y}-1)/2 & & 0\\ & 1 & \\ 0 & & \frac{2}{y-1}\end{pmatrix}\begin{pmatrix}0 & & 1\\ & -1 & \\ 1 & & 0\end{pmatrix}\begin{pmatrix}1 & x & y/2\\ 0 & 1 & \overline{x}\\ 0 & 0 & 1\end{pmatrix}$$
$$=\frac{1}{y-1}\begin{pmatrix}\varepsilon & & 0\\ & 1 & \\ 0 & & \varepsilon\end{pmatrix}\begin{pmatrix}y & x & 1/2\\ 2\overline{x} & 1+\overline{y} & \overline{x}\\ 2 & 2x & y\end{pmatrix}$$

where $y=x\overline{x}+t\in E$; $\varepsilon\in E^\bullet, x\in E$; $t+\overline{t}=0$. We use these two forms to conclude that the indicated matrix lies in K if $|y-1|=1$ or $|y|>1$ (and then $|y|>|x|$) in the normalization of the right side. The matrix does not lie in K if $|y-1|<1$.

Note that it suffices to work with an unramified E/F and μ, since only the tail of the sum matters for convergence. Take $\mu=1$ – adjusting the value of s if necessary – and let φ be the unit vector φ_0 in L_s. We normalize the measure on $B\cap C_1\backslash C_1$ to assign $B\cap C_1\backslash K\cap C_1$ the volume 1. Then

$$L_s(\varphi_0)=\int_{C_1\cap B\backslash C_1\cap K}\varphi_0(k)dk+(1-q^{-2})^{-1}\sum_1^\infty\int_{|y-1|_E=q_E^{-m}}|y-1|_E^{s+1}|y-1|_E^{-2}dxdt$$
$$=1+\sum_1^\infty q^{-2m(s-1)}(1+q^{-1})q^{-2m}.$$

The last equality follows from the Lemma in the proof of [FH], B17(a), asserting that

$$\int_{|x\overline{x}+t-1|_E\le q_E^{-m}}dx\,dt=q^{-m}\cdot q^{-m}(1+q^{-1}).$$

Hence $L_s(\varphi_0)=L(\mu\circ N_{E/F},2s)/L(\mu\circ N_{E/F}\chi,2s+1)$ where χ is again the quadratic character of $F^\times/NE^\times$. We conclude that L_s converges on $\operatorname{Re}(s)>0$, and that $L(\mu\circ N_{E/F},2s)^{-1}L_s$ has analytic continuation to $\operatorname{Re}(s)\ge 0$, with neither zeroes nor poles. Hence $I_s=I(\mu\nu^s)$ is cyclic when $\mu|E^\bullet=1$, as asserted in (a) (For $\operatorname{Re}(s)<0$ note that I_s is cyclic iff its contragredient is). We show that it is not cyclic when $\mu|E^\bullet\ne 1$, together with (b).
(b) We proceed as in the proof of (b) and (c) of B17 in [FH]. By virtue of Proposition 2 we have the disjoint decomposition $G=BC\cup Bg_0C$. Hence any C-invariant form on any subspace of I_s is a linear combination of the following two forms, ℓ_0 and ℓ_1. Put $B_C=B\cap C$. Then ℓ_0 is the average on $B_C\backslash C$ of φ, namely the integral of $\varphi(h)dh$ on $B_C\backslash C$, while ℓ_1 is the integral of φ over $B\cap C_1\backslash C_1$, namely it is L_s of the proof of (a) above. Denote by p an upper triangular matrix with diagonal $\operatorname{diag}(a,1,1/\overline{a})$. Then on C we have the measure decomposition $dh=|a|_E^{-1}dpdk$, and $\varphi(pk)=\mu(a)|a|_E^{s+1}\varphi(k)$. Hence $\varphi(h)dh=\mu(a)|a|_E^s\varphi(k)dpdk$ is left B-invariant precisely when $\mu=1$ and $s=0$. On the other hand as noted in (a) above, L_s vanishes unless $\mu=1$ on $E^\bullet$. This completes the proof of (the opposite direction of) (a).

To prove (b), we take $s=1$, then ℓ_0 is 0, and we need to show that L_s is also zero on the sub $sp(\mu)$ of our $I(\mu\nu)$, whose length is two. Note that $sp(\mu)$ consists of the φ in $I(\mu\nu)$ which are orthogonal to the unique, one-dimensional, quotient μ of $I(\mu\nu)$. Namely $sp(\mu)$ consists of the φ with $\int_{B\backslash G}\mu^{-1}(g)\varphi(g)dg=0$. Since the volume of $B_C\backslash C$ with respect to dg is zero, and μ is trivial on $E^\bullet$, and hence on G, the last integral is no other than the integral which defines L_s. We conclude that L_s is zero on $sp(\mu)$, and so $sp(\mu)$ is not cyclic, as required. □

Remark. It is easy to see that π is cyclic if and only if its contragredient $\check{\pi}$ is. Hence it suffices to show that I_s is cyclic only for $\operatorname{Re}(s)\ge 0$.

In the split case we complement Proposition 25 with the following.

30. Proposition. *For any principal series $PGL(2,F)$-module ρ, the normalizedly induced $PGL(3,F)$-module $I(1\times\rho)$ is cyclic if it is irreducible.*

Remark. This is proven – using [BZ] – in Proposition 0.1 of [F7]. The explicit proof below shows that the invariant linear form is non-zero at the K-fixed vector.

Proof. We need to show that the representation $I_s = I(\mu\nu^s, 1, \mu^{-1}\nu^{-s})$ of $PGL(3,F)$ is cyclic. As remarked prior to the statement of the proposition, we may assume that $s \geq 0$, and that μ is a unitary character of $F^\times$. The shape of a non-zero $GL(2,F)$-invariant form on I_s is suggested by the double coset decomposition $B\backslash G/H^0$ of [F7], Proposition 13. Here B is the standard Borel subgroup of $G = PGL(3,F)$, and $H^0(= GL(2,F))$ is the group of (a_{ij}) in G with $a_{ij} = 0$ when $i+j$ is odd. Put

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The required linear form on I_s will be given by integration on $B\backslash B\varepsilon r H^0$, namely on $\varepsilon r \cdot \{\operatorname{diag}(a,a,b)\}\backslash H^0$. We may as well use at once the Iwasawa decomposition $H^0 = N^0A^0K^0$, and the measure decomposition $dh = |a/b|^{-1} d^\times a\, dx\, dk$ if $h = na'k$, k in the standard maximal compact subgroup K^0 of H^0, n is the upper triangular unipotent matrix in H^0 whose $(1,3)$ entry is x, and a' denotes the diagonal matrix with entries $(a,1,b)$.

Recall that I_s consists of the smooth $\varphi : G \to \mathbb{C}$ with

$$\varphi\left(\begin{pmatrix} a & & * \\ & 1 & \\ 0 & & b \end{pmatrix} g\right) = \mu(a/b)|a/b|^{s+1}\varphi(g).$$

Write $\varphi'(g) = \int_{K^0} \varphi(gk)dk$. Then an H^0-invariant linear form L_s on I_s is given by mapping φ to its integral (against dh) on $\varepsilon r \cdot \{\operatorname{diag}(a,a,b)\}\backslash H^0$, thus

$$L_s(\varphi) = \iint \varphi'\left(\varepsilon r \begin{pmatrix} a & & x \\ & 1 & \\ 0 & & 1 \end{pmatrix}\right) |a|^{-1} d^\times a dx = \iint \mu(a)|a|^s \varphi'\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & x & 1 \end{pmatrix}\right) d^\times a\, dx.$$

We shall show that $L_s(\varphi)$ converges for $s > 0$, where it is not identically zero. We cut the domain of integration of $a \in F^\times$, $x \in F$, into 4 subdomains. In each case it will be clear that the integral is absolutely convergent, and that the question of convergence is equivalent to that where μ is unramified (when μ is ramified, the first few terms in our sums will vanish, but the convergence of course depends only on the end of the sum). So we assume that $\mu = 1$, and that $s \in \mathbb{C}$. For the same reasons we assume that $\varphi \mid K = 1$. In any case,the subdomains are as follows.

(1) $|a| \leq 1$, $|x| \leq 1$, where the integral is $= \int_{|a|\leq 1} |a|^s d^\times a = \sum_{n=0}^{\infty} (q^{-s})^n$, and this is convergent to $(1-q^{-s})^{-1} = L(s)$ when $\operatorname{Re}(s) > 0$.

(2) $|a| > 1$, $|a| \geq |x|$, where

$$N \begin{pmatrix} 1 & & 0 \\ & 1 & \\ a & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} K$$

$$= N \begin{pmatrix} 1/a & & 0 \\ & 1 & \\ 0 & & a \end{pmatrix} \begin{pmatrix} 0 & & -1 \\ & 1 & \\ 1 & & 0 \end{pmatrix} \begin{pmatrix} 1 & & 1/a \\ & 1 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} K = N \begin{pmatrix} 1/a & & 0 \\ & 1 & \\ 0 & & a \end{pmatrix} K,$$

the integral is

$$\int_{|a|>1}\left(\int_{|x|\le|a|}dx\right)|a|^s|a|^{-2s-2}d^\times a=\int_{|a|>1}|a|^{-s-1}d^\times a,$$

which converges to $q^{-1-s}(1-q^{-1-s})^{-1}=q^{-1-s}L(s+1)$ when $\mathrm{Re}\,(s)>-1$.
(3) $|x|>1\ge|a|$, where the integral is the product of $\int_{|a|\le 1}|a|^s d^\times a=L(s)$ and $\int_{|x|>1}|x|^{-s-1}dx$ $=(1-q^{-1})q^{-s}L(s)$, both integrals converge on $\mathrm{Re}\,(s)>0$.
(4) $|x|>|a|>1$, where

$$N\begin{pmatrix}1&0&0\\0&1&0\\0&x&1\end{pmatrix}\begin{pmatrix}1&&0\\&1&\\a&&1\end{pmatrix}K$$
$$=N\begin{pmatrix}1&&0\\&1/x&\\0&&x\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1/x\\0&0&1\end{pmatrix}\begin{pmatrix}1&&0\\&1&\\a&&1\end{pmatrix}K$$
$$=N\begin{pmatrix}1&&0\\&1/x&\\0&&x\end{pmatrix}\begin{pmatrix}1&0&0\\a&1&0\\0&0&1\end{pmatrix}K=N\begin{pmatrix}1/a&&0\\&a/x&\\0&&x\end{pmatrix}K.$$

Hence the integral is

$$\iint\limits_{1<|a|\le|x|}|a|^s|ax|^{-s-1}dx\,d^\times a=\int\limits_{|a|>1}|a|^{-s-1}d^\times a\cdot\int\limits_{|x|\ge1}|x|^{-s-1}dx,$$

the last two factors converge on $\mathrm{Re}\,(s)>0$, as noted in (2) and (3). It then follows that $L_s(\varphi)$ converges absolutely to a rational function in q^{-s} on $\mathrm{Re}\,(s)>0$, it is not identically zero, and its denominator is of the form $L(s)^2L(2s)$. The H^0-invariant functional can be defined then on $\mathrm{Re}\,(s)=0$ as the value of $L(2s)^{-1}L(s)^{-2}L_s$, and the definition on $\mathcal{R}(s)<0$ can be given by analytic continuation, or by noting that π is cyclic precisely when its contragredient is, and that the contragredient of $I(\mu\nu^s,1,\mu^{-1}\nu^{-s})$, where it is irreducible, is $I(\mu^{-1}\nu^{-s},1,\mu\nu^s)$.

Remark. The $PGL(n,F)$-module $I_s=I(\mu\nu^s,\mathbb{1},\mu^{-1}\nu^{-s})$ normalizedly induced from the indicated character of the parabolic subgroup of type $(1,n-2,1)$, consists of the smooth $\varphi:PGL(n,F)\to\mathbb{C}$ with

$$\varphi\left(\begin{pmatrix}a&*&*\\0&h&*\\0&0&b\end{pmatrix}g\right)=\left|\frac{a}{b}\right|^{(n-1)(s+1)/2}\mu\left(\frac{a}{b}\right)\varphi(g)\quad(a,b\in F^\times;\ h\in GL(n-2,F)).$$

Put $H=\left\{\begin{pmatrix}h&0\\0&1\end{pmatrix}\in PGL(n,F);h\in GL(n-1,F)\right\}$. Then I_s can be shown to be H-cyclic by studying the linear form L_s on I_s, defined by integrating φdh over

$$\varepsilon\cdot\left\{\begin{pmatrix}a&&0\\&h&\\0&&a\end{pmatrix};a\in F^\times,h\in GL(n-2,F)\right\}\backslash H,$$

where $\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Put $\varphi'(g) = \int_{K_H} \varphi(gk)dk$. Then

$$L_s(\varphi) = \iint \varphi'\left(\varepsilon \begin{pmatrix} a & x & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) d^\times a\, dx$$
$$= \iint \mu(a)|a|^{(n-1)(s+1)/2}\varphi'\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ a & x & 1 \end{pmatrix}\right) d^\times a\, dx,$$

where $a \in F^\times$ and x ranges over the $(n-2)$ rows over F. It can be shown that this integral is absolutely convergent to a non-zero rational function in q^{-s} on $\mathrm{Re}\,(s) > 0$ (when μ is unitary).

9. Appendix. Multiplicity one theorems.

The following are special cases of unpublished Theorems of J. Bernstein. Let F be a local field, $G = G_n = GL(n, F)$, $C = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right); a \in G_{n-1}\}$.

Theorem (J. Bernstein). *Let* (π, V) *be any admissible irreducible representation of* G. *Then* $\dim_{\mathbb{C}}[\mathrm{Hom}_C(V, \mathbb{C})] \leq 1$.

Proof. Given an ℓ-space ([BZ]) X, put $S^*(X)$ for the space of distributions on X, namely the space of linear maps from $S(X) = C_c^\infty(X)$ to $\mathbb{C}$. If a group J acts on X, denote by $S^*(X)^J$ the space of $E \in S^*(X)$ fixed by J. Let T be a finite group which acts on J and on X, let ε be a character of T, extended to the semi-direct product $J \times T$ trivially on J. Put $S^*(X)^{J\times T,\varepsilon}$ for the $E \in S^*(X)^J$ on which T acts via ε.

We shall repeatedly use below three tools from Bernstein [B]. The first asserts that if Z is a closed subset of X, and U is its (open) complement, then the sequence $0 \to S^*(Z) \to S^*(X) \to S^*(U) \to 0$ is exact ([BZ], §1; [B], p. 57), and so is $0 \to S^*(Z)^J \to S^*(X)^J \to S^*(U)^J$, when J maps Z to Z and U to U (same conclusion when J is replaced by $\{J\times T, \varepsilon\}$).

The second is the Localization Principle ([B], p. 58). Let $q : X \to Y$ be a continuous map of ℓ-spaces. Then $S(X)$ and hence $S^*(X)$ are naturally $S(Y)$-modules. Put $X_y = q^{-1}(y)$ for the fiber of $y \in Y$. Identify $S^*(X_y)$ with the subspace $S^*_{X_y}(X)$ of $S^*(X)$ of distributions supported on X_y. Let W be a closed $S(Y)$-submodule of $S^*(X)$. Then the closure of the span of the union of the subspaces $W_y = W \cap S^*(X_y)$ $(y \in Y)$, is equal to W.

The third is Frobenius Reciprocity ([B], p. 60). Suppose an ℓ-group J acts on an ℓ-space X, and $p : X \to Z$ is a continuous J-equivariant map, where Z is a homogeneous J-space. Fix $z_0 \in Z$. Put $X_0 = p^{-1}(z_0)$ in X, and $H = \mathrm{Stab}_J(z_0)$ in J. Then for any $\mu \in S^*(Z)^J$ there is a canonical isomorphism from $S^*(X_0)^H$ to $S^*(X)^J$, explicitly given in [B], p. 60.

We now return to the notations of the theorem. By a criterion of Gelfand-Kazhdan [GK] (cf. [P1], Lemma 4.2; [F5], p. 163), it suffices to show that $S^*(G)^{C\times C} = S^*(G)^{C\times C\times T}$, where $C \times C$ acts on G by $(g, h)x = gxh^{-1}$, and T is the group generated by an involution

t of G which preserves C (thus $t^2 = 1, t(xy) = t(y)t(x), t(C) = C$). Alternatively, it suffices to show that $S^*(G)^{C\times C\times T,\varepsilon} = \{0\}$, where $\varepsilon(t) = -1$. The support supp$E$ of $E \in S^*(G)$ is closed in $G \subset X = X_n$ = space of $n \times n$ matrices over F. Replacing E by its product with the characteristic function of supp$E$ in X, it can be viewed as an element of $S^*(X)$. We will show that $S^*(X)^{C\times C\times T,\varepsilon} = \{0\}$, where $t(x)$ = transpose of $x \in X$.

Put $k = n-1$. For $0 \le r \le k$, put $X'^r = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X; a \in X_k, \mathrm{rk}(a) = r, b \in M_{k\times 1}, c \in M_{1\times k}, d \in F\}$. Here $M_{u\times v}$ signifies the space of $u \times v$ matrices over F. The theorem would follow (using the first tool), once we show, for each r, the vanishing of $S^*(X'^r)^{C\times C\times T,\varepsilon}$. Fix $d_0 \in F^\times$. By the Localization Principle, it suffices to show this with X'^r replaced by its subset X^r defined by $d = d_0$. This we proceed to show.

Consider the $C\times C$-equivariant map $p : X^r \to Z^r = \{a \in X_k; \mathrm{rk}(a) = r\}$. The group $C \times C$ acts transitively on Z^r by $(g,h)a = gah^{-1}$. Denote by I_r the identity $r \times r$ matrix. The stabilizer of $\begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} \in Z^r$ in $C \times C$ contains $C' = G_{k-r} \times G_r$, embedded diagonally as $(\mathrm{diag}(A,D), \mathrm{diag}(A,D))$ $(A \in G_{k-r}, D \in G_r)$. By Frobenius Reciprocity, it suffices to show that $S^*(\{(b,c)\})^{C'\times T,\varepsilon}$ is zero. Here C' acts by $g(b,c) = (gb, cg^{-1})$, and $t(b,c) = (t(c), t(b))$. Write $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $c = (c_1, c_2)$, where b_1, b_2 are columns over F of lengths $k-r, r$, while c_1, c_2 are rows of lengths $k-r, r$. For any u, v in F, put $Y_{u,v} = \{(b,c); c_1b_1 = u, c_2b_2 = v\}$. By the Localization Principle, it suffices to show that $S^*(Y_{u,v})^{C'\times T,\varepsilon}$ is zero for all u, v. This we do next.

The group C' acts transitively on $Y_{u,v}$ when $uv \neq 0$. Put $h = \mathrm{diag}(1,\cdots,1,u,1,\cdots,1,v)$, with u at the $(k-r)$th place, and $t'(b,c) = (ht(c), t(b)h^{-1})$. Choose $c_0 = (c_{01}, c_{02})$, $c_{0i} = (0,\cdots,0,1)$, and $b_0 = ht(c_0)$. By Frobenius reciprocity, the space $S^*(Y_{u,v})^{C'\times T,\varepsilon}$ is contained in $S^*(\{(b_0,c_0)\})^{T',\varepsilon}$; T' is the group generated by t'. This last space is zero since $t'(b_0,c_0) = (b_0,c_0)$. Hence $S^*(Y_{u,v})^{C'\times T,\varepsilon} = 0$ when $uv \neq 0$. To study the case where $u = 0, v \neq 0$, put $Y_v = \{(b,c); c_2b_2 = v\}$. Then $S^*(Y_v)^{C'\times T,\varepsilon} = S^*(Y_{0,v})^{C'\times T,\varepsilon}$. Introduce an action of $\lambda \in F^\times$ on $Y_{0,v}$ by $\lambda(b,c) = (\begin{pmatrix} \lambda b_1 \\ b_2 \end{pmatrix}, (\lambda c_1, c_2))$. There are four C'-orbits in $Y_{0,v}$, defined by $\{b_1 \neq 0, c_1 \neq 0\}$, $\{b_1 \neq 0, c_1 = 0\}$, $\{b_1 = 0, c_1 \neq 0\}$, $\{b_1 = 0, c_1 = 0\}$. Each is preserved under the action of $\lambda \in F^\times$. Hence $\lambda \in F^\times$ acts on $S^*(Y_{0,v})^{C'\times T,\varepsilon}$ with eigenvalues 1.

Fix an additive non trivial complex valued character ψ of F, and define the partial Fourier transform $\mathfrak{F}_1$ on $S^*(Y_v)$ by $\mathfrak{F}_1 E = E\mathfrak{F}_1$, where

$$\mathfrak{F}_1 f(b,c) = \int\int f(\begin{pmatrix} \beta_1 \\ b_2 \end{pmatrix}, (\gamma_1, c_2))\psi(\gamma_1 b_1 + c_1\beta_1)d\beta_1 d\gamma_1.$$

It is an automorphism of $S^*(Y_v)^{C'\times T,\varepsilon}$, and $\lambda \in F^\times$ acts on $\mathfrak{F}_1 E$ with eigenvalues $|\lambda|^{-2(k-r)}$. Indeed,

$$\lambda \cdot \mathfrak{F}_1 E(f) = \mathfrak{F}_1 E(\lambda^{-1} f) = E(\mathfrak{F}_1(\lambda^{-1} f)) = |\lambda|^{-2(k-r)} E(\lambda(\mathfrak{F}_1 f)),$$

since

$$\mathfrak{F}_1(\lambda^{-1}f)(b,c) = \int\int f(\left(\begin{smallmatrix}\lambda\beta_1\\ b_2\end{smallmatrix}\right), (\lambda\gamma_1, c_2))\psi(\gamma_1 b_1 + c_1\beta_1)d\beta_1 d\gamma_1$$
$$= |\lambda|^{-2(k-r)}\mathfrak{F}_1 f(\left(\begin{smallmatrix}\lambda^{-1}b_1\\ b_2\end{smallmatrix}\right), (\lambda^{-1}c_1, c_2)) = |\lambda|^{-2(k-r)}\lambda\cdot\mathfrak{F}_1 f(b,c).$$

In summary, $\lambda \in F^\times$ acts on $E \in S^*(Y_{0,v})^{C'\times T,\varepsilon}$ with eigenvalues 1. On the other hand, E can be viewed as an element of $S^*(Y_v)^{C'\times T,\varepsilon}$ of the form $\mathfrak{F}_1 E_1$, where $E_1 \in S^*(Y_v)^{C'\times T,\varepsilon} = S^*(Y_{0,v})^{C'\times T,\varepsilon}$. Hence $\lambda \in F^\times$ acts on $E = \mathfrak{F}_1 E_1$ with eigenvalues $|\lambda|^{-2(k-r)}$. Taking λ with $|\lambda| \neq 1$, we conclude that $E = 0$, hence that $S^*(Y_{0,v})^{C'\times T,\varepsilon}$ is zero. The proof of $S^*(Y_{u,0})^{C'\times T,\varepsilon} = 0$ for $u \neq 0$ is carried out analogously, on introducing the partial Fourier transform $\mathfrak{F}_2$ on $\{(b_2, c_2)\}$. The proof that $S^*(Y_{0,0})^{C'\times T,\varepsilon} = 0$ can be carried out now in the same way, on using the Fourier transform $\mathfrak{F} = \mathfrak{F}_1\mathfrak{F}_2$. This completes the proof of the theorem. □

From now on, E/F will signify a quadratic extension of local fields, and $x \mapsto \overline{x}$ the action of $\mathrm{Gal}(E/F)$. Put $\mathcal{J} = \left(\begin{smallmatrix}0&0&1\\0&-1&0\\1&0&0\end{smallmatrix}\right)$.

Proposition. *The statement of the Theorem holds with* $G = U(2,1;E/F) = \{g \in GL(3,E);\ \mathcal{J}\,{}^t\overline{g}^{-1}\mathcal{J} = g\}$*, the quasi-split unitary group in* 3 *variables over* F *which splits over* E*, and with* C *the* $U(1,1;E/F)$*-factor (defined by* $a_{2,2} = 1, a_{i,j} = 0$ *for odd* $i+j$*) in the centralizer of* $\mathrm{diag}(1,-1,1)$ *in* G*.*

Proof. By the criterion of [GK] mentioned above, it suffices to show that each double C coset in G is fixed by the transpose t. We use the Bruhat decomposition, $G = B \cup B\mathcal{J}N$, where B is the upper triangular subgroup of G, and N is its unipotent subgroup. Write $N' = \{n(x) = \left(\begin{smallmatrix}1&x&x\overline{x}/2\\0&1&\overline{x}\\0&0&1\end{smallmatrix}\right); x \in E\}$, and note that $G = CN'\mathcal{J}N'C$. As $\mathcal{J} \in C$, G is the union over x, y in E of $X(x,y) = C\,{}^t n(y)n(x)C$. For any $a \in E^\times$, we have $X(x,y) = X(ax, y/a)$. Also ${}^tX(x,y) = X(y,x)$, and $X(x,0) = X(0,-\overline{x})$. If $xy \neq 0$, then for $a = x/y$ we have ${}^tX(x,y) = X(y,x) = X(ay, x/a) = X(x,y)$. Of course ${}^tX(0,0) = X(0,0)$, and ${}^tX(x,0) = X(0,x) = X(-\overline{x},0) = X((-\overline{x}/x)x,0) = X(x,0)$, as required. □

Proposition 11 of [F5], p. 163, shows that the assertion of the Theorem holds for $G = GL(n,E)$, $C = GL(n,F)$, E/F is as usual a quadratic extension of local fields. For such E/F, fix $\theta \in F - N_{E/F}E$. The quaternion algebra D over F can be presented as the algebra of the matrices $\left(\begin{smallmatrix}a&b\theta\\ \overline{b}&\overline{a}\end{smallmatrix}\right); a, b \in E$.

Remark 1. *The assertion of the Theorem holds with $G = GL(2, E)$ and C = multiplicative group of D.*

Indeed, $G = BC = CBC$, where B is the upper triangular subgroup of G. The involution $t(g) = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}^{-1} {}^t g \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$ preserves C. By the criterion of [GK] it suffices to show, for any $u, v, w \in E^\times$ (the case of $v = 0$ is trivial) the existence of $a, b \in E$ with $a\overline{a} \neq b\overline{b}\theta$ such that

$$\begin{pmatrix} a & b\theta \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} u & 0 \\ v/\theta & w \end{pmatrix} \begin{pmatrix} a & \overline{b}\theta \\ b & \overline{a} \end{pmatrix}.$$

The solution is given by $a = (u\overline{b} - wb)\theta/v$.

Remark 2. *The assertion of the Theorem holds for the groups $G = U(2, 1; E/F)$ and the anisotropic $C = C_\theta = U(1) \times U(2)$ of Proposition 1 of [F9] below ($\theta \in F - N_{E/F}E$, E/F = quadratic extension of local fields).*

Indeed, Proposition 1 of [F9] asserts that $G = BC_\theta (= C_\theta B C_\theta)$. Consider the set $X = C'_\theta B C'_\theta$, where C'_θ is the group of the matrices h which are displayed in Corollary 1 of [F9] (but we no longer require (1) and (2) there). It suffices to check that the criterion of [GK] applies with the involution $t(x) = d^{-1}{}^t x d$, where $d = \mathrm{diag}(2\theta, 1, 1/2\theta)$. Given $x, z \in E^\times$, $u \in E$ with $u + \overline{u} = x\overline{x}$, it can be checked that $a = (b - \overline{b}\overline{z})/(2\theta\overline{x}\overline{z})$ and $c = (bu + \overline{b}\overline{u}\overline{z})z/(\overline{x}(1 - z\overline{z}))$ (or $b = (c\overline{x} - \overline{c}xz)/uz$ when $z\overline{z} = 1$) satisfy

$$\begin{pmatrix} z & zx & zu \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1/\overline{z} \end{pmatrix} \begin{pmatrix} a & b/2\theta & c/2\theta \\ \overline{b} & \overline{a}+\overline{c} & \overline{b}/2\theta \\ 2\theta c & b & a \end{pmatrix} = \begin{pmatrix} a & \overline{b}/2\theta & c/2\theta \\ b & \overline{a}+\overline{c} & b/2\theta \\ 2\theta c & \overline{b} & a \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ 2\theta xz & 1 & 0 \\ 4\theta^2 uz & 2\theta\overline{x} & 1/\overline{z} \end{pmatrix}.$$

If $x = 0$, take $b = 0$ and $a = c(1 - z\overline{z})/(2\theta uz\overline{z})$.

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