# FOURIER ORBITAL INTEGRALS OF SPHERICAL FUNCTIONS ON $U(3)$

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The purpose of this work is to prove the identities of Fourier orbital integrals of spherical functions on the unitary groups in three and two variables, as well as on the general linear groups in three and two variables, which are stated as Propositions 13, 15, 17 and 18 of [F]. These Propositions make the main local technical requirements for the method of [F], which determines the cyclic generic cusp forms on $U(3)$ as the image of the endo-lift from $U(2)$. As the present work is an Appendix to [F], we use its notations and definitions without a change. We hope that a general technique would eventually be found and would provide a more conceptual proof.

**Part 1.** *Proof of Proposition* 13. Note that

$$\begin{pmatrix} 1/2i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},$$

and hence that for $f' \in \mathbb{H}'$ we have

$$\begin{aligned}\Psi(-b/2i, f', \psi) &= \psi(b/i)\int_E dz \int_{F^\times} d^\times a \int_F dn \\ &\cdot f'\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right)\psi(2z) \\ &= \psi(b/i)\iiint f'\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\right)\psi(2z)dzdnd^\times a \quad (d = n + ai).\end{aligned}$$

To study the argument of the spherical $f'$, note that when $|z| \le 1$, it is

$$\begin{aligned}\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{d}{ab} & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}, \qquad \text{if } |d| \le \max(1, |ab|), \\ &\equiv \begin{pmatrix} d & 0 \\ 0 & ab/d \end{pmatrix}\begin{pmatrix} 1 & ab/d \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} d & 0 \\ 0 & ab/d \end{pmatrix}, \qquad \text{if } |d| > \max(1, |ab|).\end{aligned}$$

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Of course we use the Bruhat decomposition of $\begin{pmatrix}1 & 0\\ d & 1\end{pmatrix}$ as recorded in the beginning of the proof of Lemma 12, and note that $f'$ is $K'$-biinvariant. Hence ${}^tg \equiv g$. If $|z|>1$ then

$$\begin{aligned}
&\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & ab\end{pmatrix}\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix} \equiv \begin{pmatrix}z & 0\\ 0 & ab/z\end{pmatrix}\begin{pmatrix}1 & d+ab/z\\ 0 & 1\end{pmatrix}\\
&\quad \equiv \begin{pmatrix}1 & (zd+ab)z/ab\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & ab/z\end{pmatrix} \equiv \begin{pmatrix}z & 0\\ 0 & ab/z\end{pmatrix}, \qquad \text{if } |zd+ab| \le \max(|z|,|ab/z|);\\
&\equiv \begin{pmatrix}1 & 0\\ d+ab/z & 1\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & ab/z\end{pmatrix}\\
&\quad \equiv \begin{pmatrix}d+ab/z & 0\\ 0 & 1/(d+ab/z)\end{pmatrix}\begin{pmatrix}z & 0\\ 0 & ab/z\end{pmatrix}\begin{pmatrix}1 & (ab/z)/(zd+ab)\\ 0 & 1\end{pmatrix}\\
&\quad \equiv \begin{pmatrix}zd+ab & 0\\ 0 & ab/(zd+ab)\end{pmatrix}, \qquad \text{if } |zd+ab| > \max(|z|,|ab/z|).
\end{aligned}$$

It is also useful to recall that $d = n+ai$, $n \in F$, $a \in F^\times$, $-\bar{i} = i \in E^\times$, and so $d + ab/z = n + aR(b/z) + ai(1+I(b/z))$, where for each $x \in E$ the elements $R(x)$ and $I(x)$ in $F$ are uniquely determined by $x = R(x) + iI(x)$. We shall abbreviate below $I$ for $I(b/z)$, and write $\tilde{n}$ for $n + aR(b/z)$. Our method of integration of the triple integral over $a$, $n$, and $z$, which defines $\Psi(-b/2i, f', \psi)$ above, will be to fix $z (\in E)$, and integrate over $a \in F^\times$ and $n \in F$ first.

**13.1 Lemma.** *Suppose that $|z|>1$. If $|1+I| \le |b/z^2|$, then*

$$\int_{F^\times}\int_F f'\left(\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & ab\end{pmatrix}\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\right) dn d^\times a = f'(I) + \sum_{k\ge 1}(1+q^{-1})q^k f'\left(\begin{pmatrix}\boldsymbol{\pi}^{-k} & 0\\ 0 & 1\end{pmatrix}\right).$$

*If $|1+I| > |b/z^2|$ then the integral is equal to*

$$= f'\left(\begin{pmatrix}(1+I)z^2/b & 0\\ 0 & 1\end{pmatrix}\right) + \sum_{k\ge 1}(1+q^{-1})q^k f'\left(\begin{pmatrix}\boldsymbol{\pi}^{-k}(1+I)z^2/b & 0\\ 0 & 1\end{pmatrix}\right).$$

*Proof.* We shall cut the domain of integration into 3 domains (the third will further be cut into two), and integrate over each subdomain.

(1) The first domain consists of the $a$, $n$ with $|d+ab/z| \le 1$ and $|ab/z| \le |z|$. Then $|\tilde{n}| \le 1$ and $|a(1+I)| \le 1$ (we take $i$ with $|i|=1$). Define $k$ by $|\boldsymbol{\pi}|^{-k} = |z^2/ab|$. This $k$ can take any value such that $(|\boldsymbol{\pi}^{-k}| =)\ |z^2/ab| \ge |z^2(1+I)/b|$. The integral is

$$\sum_{\substack{k\ge 0\\ |\boldsymbol{\pi}^{-k}|\ge|z^2(1+I)/b|}} f'\left(\begin{pmatrix}\boldsymbol{\pi}^{-k} & 0\\ 0 & 1\end{pmatrix}\right).$$

(2) The second domain consists of the $a$, $n$ with $|d+ab/z| \le |ab/z^2|$ and $|z| < |ab/z|$. Define $k$ by $|\boldsymbol{\pi}^{-k}| = |ab/z^2| (>1)$. Then $|\tilde{n}| \le |\boldsymbol{\pi}^{-k}|$ and $|a(1+I)| \le |\boldsymbol{\pi}^{-k}| (= |ab/z^2|)$, hence $|1+I| \le |b/z^2|$. The integral over this subdomain is then

$$\sum_{\substack{k>0 \\ |1+I| \le |b/z^2|}} q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(3) The third domain consists of the $a$, $n$ with $|d+ab/z| > \max(1, |ab/z^2|)$. We distinguish between two subcases.
(3a) If $|1+I| \le |b/z^2|$, since $|z| > 1$ we have that $|d+ab/z| = |\tilde{n}|$. Define $k$ by $|\boldsymbol{\pi}^{-k}| = |d+ab/z|^2 |z^2/ab| (>1)$. Then $(|\tilde{n}^2 \boldsymbol{\pi}^k| =) |ab/z^2| < |\tilde{n}|$, namely $|\tilde{n}| < |\boldsymbol{\pi}^{-k}|$. As $\tilde{n}$ ranges over $1 < |\tilde{n}| < |\boldsymbol{\pi}^{-k}|$, the integral over this subdomain is

$$\sum_{\substack{k>0 \\ |1+I| \le |b/z^2|}} (q^{k-1} - 1) f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

At this stage the lemma follows in the case that $|1+I| \le |b/z^2|$.
(3b) If $|1+I| > |b/z^2|$, define $k$ by $|\boldsymbol{\pi}^{-k}| = |d+ab/z|^2/|a(1+I)| (< |(dz+ab)^2/ab|)$. Again we split into two subdomains.
(3b(i)) If $|\tilde{n}| \le |a(1+I)|$ then $|d+ab/z| = |a(1+I)| = |d+ab/z|^2 |\boldsymbol{\pi}|^k$. Hence

$$|\boldsymbol{\pi}^{-k}| = |d+ab/z| = |a(1+I)|,$$

and $|a| = |\boldsymbol{\pi}^{-k}/(1+I)|$, $|\tilde{n}| \le |\boldsymbol{\pi}|^{-k}$.
(3b(ii)) If $|\tilde{n}| > |a(1+I)| = |d+ab/z|^2 |\boldsymbol{\pi}|^k = |\tilde{n}|^2 |\boldsymbol{\pi}|^k$, then $1 < |\tilde{n}| < |\boldsymbol{\pi}^{-k}|$ and $|a| = |\tilde{n}^2 \boldsymbol{\pi}^k/(1+I)|$. Integrating over the subdomains (3b(i)) and (3b(ii)) we obtain

$$\sum_{\substack{k>0 \\ |1+I| > |b/z^2|}} (q^k + q^{k-1} - 1) f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}(1+I)z^2 b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

The lemma now follows also in the case where $|1+I| > |b/z^2|$. □

Next we consider the complementary case, where $|z| \le 1$. Recall that in this case the argument of $f'$ in the integrand is $\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}$ if $|d| \le \max(1, |ab|)$, and $\begin{pmatrix} d & 0 \\ 0 & ab/d \end{pmatrix}$ if $|d| > \max(1, |ab|)$.

**13.2 Lemma.** *Suppose that $|z| \le 1$. If $|b| \ge 1$ then the integral over $a$ and $n$ of Lemma 13.1 is equal to*

$$f'(I) + \sum_{k \ge 1} (1+q^{-1}) q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

*If* $|b| \le 1$ *then it is equal to*

$$f'\left(\begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{k\ge 1}(1+q^{-1})q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}/b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

*Proof.* We split the domain of integration into three.
(1) On $|ab| \le 1$, $|d| \le 1$, we have $|n| \le 1$, $|a| \le 1$, and we define $k$ by $|\boldsymbol{\pi}^k| = |ab|$. The integral over this subdomain is

$$\sum_{\substack{k\ge 0 \\ |\boldsymbol{\pi}^{-k}|\ge|b|^{-1}}} f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(2) On $|ab| > 1$, $|d| \le |ab|$ (hence $|b| \ge 1$) we define $k$ by $|\boldsymbol{\pi}^{-k}| = |ab| (> 1)$. Then $|n| \le |\boldsymbol{\pi}^{-k}|$, and the integral over this subdomain is

$$\sum_{\substack{k>0 \\ |b|\ge 1}} q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(3) On $|d| > \max(1, |ab|)$, define $k$ by $|\boldsymbol{\pi}^{-k}| = |d^2/a| (> |b|)$, if $|b| \le 1$, and then the integrand is $f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}/b & 0 \\ 0 & 1 \end{pmatrix}\right)$, and by $|\boldsymbol{\pi}^{-k}| = |d^2/ab| (> 1)$ if $|b| > 1$, and then the integrand is $f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right)$. To integrate this again we split the domain in two.
(3i) On $|n| \ge |a|$ we have $|n| > 1$ and $|n| > |ab|$. If $|b| \le 1$ then $|\boldsymbol{\pi}^{-k}| = |n^2/a| \ge |n|$. Hence $|a| = |n^2\boldsymbol{\pi}^k|$ and $1 < |n| \le |\boldsymbol{\pi}^{-k}|$. The integral over this subdomain is

$$\sum_{k>0, |b|\le 1} (q^k - 1) f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}/b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

If $|b| > 1$ then $|\boldsymbol{\pi}^{-k}| = |n^2/ab| > |n|$, and so $|a| = |n^2\boldsymbol{\pi}^k/b|$ and $1 < |n| < |\boldsymbol{\pi}^{-k}|$. The integral in this case is

$$\sum_{k>0, |b|>1} (q^{k-1} - 1) f'\left(\begin{pmatrix} \underline{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(3ii) On $|n| < |a|$ we have $|a| > 1$ and $|b| < 1$. Define $k$ by $|\boldsymbol{\pi}^{-k}| = |a|$. The integral is

$$\sum_{\substack{k>0 \\ |b|<1}} q^{k-1} f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}/b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Both claims of the lemma, for $|b| \ge 1$ and $|b| \le 1$, follow at once. $\square$

**13.3 Lemma.** *If $|b| \leq 1$ then $\Psi(b, f', \psi)$ is equal to*

$$f'\left(\begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) + (1+q) f'\left(\begin{pmatrix} b^{-1}\boldsymbol{\pi}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) + (q^2+q-1) f'\left(\begin{pmatrix} b^{-1}\boldsymbol{\pi}^{-2} & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$+ \sum_{k\geq 3} (1+q^{-1})(1-q^{-2}) q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}/b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

*Proof.* Note that $|b/2i| = |b|$ and $\psi(b) = 1$. By Lemma 13.2, the contribution from $|z| \leq 1$ is

$$f'\left(\begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{k\geq 1} (1+q^{-1}) q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}/b & 0 \\ 0 & 1 \end{pmatrix}\right).$$

On $|z| > 1$ we have that $|1 + \operatorname{Im}(b/z)| = 1 > |b/z^2|$. Hence Lemma 13.1 implies that the contribution from the domain of $|z| > 1$ is

$$\int_{|z|>1} [f'\left(\begin{pmatrix} z^2/b & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{k>0} (1+q^{-1}) q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} z^2/b & 0 \\ 0 & 1 \end{pmatrix}\right) \psi(2z) dz.$$

If $|z| = q^m$, $m \geq 1$, then $\int_{|z|=q^m} \psi(2z) dz$ is 0 unless $m = 1$ where $-1$ is obtained. The lemma follows now at once. □

This completes the proof of Proposition 13 in the case where $|b| \leq 1$.

**13.4 Lemma.** *If $|b| \geq 1$ then $\Psi(b, f', \psi)$ is equal to*

$$\left[ f'(I) + q f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) + (1-q^{-2}) \sum_{k>1} q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right) \right] \sum(b).$$

*Proof.* On the domain $|z| \leq 1$, since $\psi(z) = 1$ Lemma 13.2 asserts that the contribution to the integral $\overline{\psi}(b/i)\Psi(-b/2i, f', \psi)$ is

(i) $$f'(I) + \sum_{k\geq 1} (1+q^{-1}) q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

On $|z| > 1$, Lemma 13.1 contributes the sum of the following two integrals:

(ii) $$\int_{\substack{|z|>1 \\ |1+I| \leq |b/z^2|}} \left[ f'(I) + (1+q^{-1}) \sum_{k\geq 1} q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k} & 0 \\ 0 & 1 \end{pmatrix}\right) \right] \psi(2z) dz,$$

and the integral over $z$ with $|z| > 1$ and $|1+I| > |b/z^2|$:

$$\text{(iii)} \quad \int \left[ f'\left(\begin{pmatrix} (1+I)z^2/b & 0 \\ 0 & 1 \end{pmatrix}\right) + (1+q^{-1}) \sum_{k\geq 1} q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k}(1+I)z^2/b & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \psi(2z)dz.$$

On $|z| \leq 1 < |b|$ we have $|b/z| \leq |b/z^2|$ and so $|1+I| \leq |b/z^2|$. Hence the sum of (i) and (ii) is the product of (i) and

$$\text{(iv)} \quad \int_{|1+I|\leq |b/z\overline{z}|} \psi(2z)dz.$$

Put $b' = b/2i$. The inequality

$$|b/z\overline{z}| \geq |1 + b'/z + \overline{b}'/\overline{z}| = |[(z+b')(\overline{z}+\overline{b}') - b'\overline{b}']/b'\overline{b}'| \; |b'\overline{b}/z\overline{z}|$$

is equivalent to

$$|(1+z/b')(1+\overline{z}/\overline{b}') - 1| \leq 1/|b'|.$$

The solutions of this inequality are described by $1 + z/b' = \varepsilon(1+u/b')$, where $u \in R'$, $\varepsilon \in R'^{\times}/(1+b'^{-1}R')$ with $\varepsilon\overline{\varepsilon} = 1$, namely $z = -b' + \varepsilon b' + \varepsilon u$. Hence (iv) is equal to $\overline{\psi}(b/i)\sum(b)$.

When $|(1+a/b')(1+\overline{z}/\overline{b}') - 1| > 1$ we have $|z| > |b'| > 1$, and $\int \psi(2z)dz = 0$. When

$$|(1+z/b')(1+\overline{z}/\overline{b}') - 1| = |\boldsymbol{\pi}^j|, \quad |b'|^{-1} < |\boldsymbol{\pi}^j| \leq 1,$$

we have that $1 + z/b' = \varepsilon(1 - \boldsymbol{\pi}^j u)$ with $\varepsilon \in R'^{\times}/(1+\boldsymbol{\pi}^j R')$, $\varepsilon\overline{\varepsilon} = 1$, and $|u| = 1$ if $j \geq 1$, in which case $z = -b' + \varepsilon b' - \varepsilon b' \boldsymbol{\pi}^j u$, but when $j = 0$ we have that $z = -b' + \varepsilon b' - \varepsilon b' u$ with $\varepsilon \in R'^{\times}/(1+\boldsymbol{\pi}R')$, $\varepsilon\overline{\varepsilon} = 1$, and $u \in R' - (1+\boldsymbol{\pi}R')$. Note that the domain in (iii) is of the form $|z| > 1$ and $|b'\boldsymbol{\pi}^j| = |(1+I)z^2/b| > 1$. Integrating over $z$ we obtain, when $j \geq 1$, the product of $\psi(-b/i)\sum_{\varepsilon} \psi(\varepsilon b/i)$ with

$$\int \psi(2z)dz = \begin{cases} 0, & \text{if } |b'\boldsymbol{\pi}^j| > q \\ -1, & \text{if } |b'\boldsymbol{\pi}^j| = q. \end{cases}$$

When $j = 0$, the integral is

$$\int \psi(2z)dz = \psi(-2b') \sum_{\varepsilon} \psi(2\varepsilon b') \int_{R'-(1+\boldsymbol{\pi}R')} \psi(-\varepsilon b' u)du$$

$$= \begin{cases} 0, & |b'| > q, \\ -\psi(-b/i)\sum_{\varepsilon} \psi(\varepsilon b/i), & |b'| = q. \end{cases}$$

In conclusion we must have $|1+I|\ |z^2/b| = q$ to have a non zero integral over $z$, and then $\int \psi(2z)dz$ is $-\psi(-b/i)\sum(b/2i)$. Indeed, for $|b|>1$ we have

$$\sum(b) = \sum_{\varepsilon\in R'^\times/(1+b^{-1}R')} \psi(2\varepsilon b) = \sum_{\varepsilon\in R'^\times/(1+(\boldsymbol{\pi}^{-1}/b)R');\ } \sum_{\varepsilon_1\in(1+(\boldsymbol{\pi}^{-1}/b)R')/(1+b^{-1}R')} \psi(2\varepsilon_1\varepsilon b)$$
$$= -\sum_{\varepsilon\in R'^\times/(1+(\boldsymbol{\pi}^{-1}/b)R')} \psi(2\varepsilon b).$$

It then follows that $\Psi(b,f',\psi)$ is the difference of (i) $\times \sum(b)$ and the term

$$\left[ f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{k\geq 1}(1+q^{-1})q^k f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-k-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \sum(b),$$

obtained from the integral of (iii) with $|(1+I)z^2/b| = |\boldsymbol{\pi}^{-1}|$. This difference is equal to the expression of the lemma. $\square$

*Proof of Proposition* 15. Using the Bruhat decomposition for an element in $U(3)$ recorded in the proof of Lemma 12, we have

$$g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac12 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac12 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \begin{pmatrix} \frac12 & & 0 \\ & 1 & \\ 0 & & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac12 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The change $x\to x-2b$ of variables; the fact that $f\in\mathbb{H}$ is spherical and in particular left invariant under $\mathcal{J} = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$, and using the Iwasawa decomposition $C = ANK_C$ on $C$ and again the invariance of $f$ under $K_C$ on the right, the integral $\Psi(b,f,\psi)$ to be studied becomes

$$\overline{\psi}(2b)\int_E dy \int_F dz \int_{E^\times}\int_F X\psi(-\overline{y})dx d^\times a,$$

where

$$X = f\left(\begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ z & \overline{y} & 1 \end{pmatrix} \begin{pmatrix} 1/2\overline{b} & & 0 \\ & 1 & \\ 0 & & 2b \end{pmatrix} \begin{pmatrix} 1 & 1 & \frac12 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & & 0 \\ & 1 & \\ 0 & & \overline{a} \end{pmatrix} \begin{pmatrix} 1 & & x \\ & 1 & \\ 0 & & 1 \end{pmatrix}\right).$$

Here $z = \frac12 y\overline{y} + it$, $t\in F$, and $dz$ indicates $dt$, so that $\int dz$ ranges over $F$. Similarly $x$ ranges over the $x$ in $E$ with $x+\overline{x}=0$. Hence $\overline{\psi}(b)\Psi(b/2,f,\overline{\psi})$ is equal to

$$\int_E dy \int_F dz \int_{E^\times} dc \int_F dd \left[ f\left(\begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ z & \overline{y} & 1 \end{pmatrix} \begin{pmatrix} c & & 0 \\ & 1 & \\ 0 & & 1/\overline{c} \end{pmatrix} \begin{pmatrix} 1 & a & d \\ 0 & 1 & \overline{a} \\ 0 & 0 & 1 \end{pmatrix}\right) \psi(y)\right].$$

Here $z+\overline{z}=y\overline{y}$, $c=1/a\overline{b}$, $d+\overline{d}=a\overline{a}$, and so once $y$ and $c$ (whence $a$) are chosen, $z$ and $d$ range on a space naturally isomorphic to $F$.

Using the Bruhat decomposition recorded in the proof of Lemma 12, we determine the $K$-double coset of the matrix which appears in the argument of $f$. Since $z+\overline{z}=y\overline{y}$, if $|z|\le 1$ then $|y|\le 1$, and if $|z|\ge 1$ then $|z|\ge|y|$, and similarly for $d+\overline{d}=a\overline{a}$. Suppose first that $|z|>1$. Then

$$\begin{aligned}
&\begin{pmatrix}1&0&0\\ y&1&0\\ z&\overline{y}&1\end{pmatrix}\begin{pmatrix}c&&0\\ &1&\\ 0&&1/\overline{c}\end{pmatrix}\begin{pmatrix}1&a&d\\ 0&1&\overline{a}\\ 0&0&1\end{pmatrix}\\
&\equiv\begin{pmatrix}z&&0\\ &\overline{z}/z&\\ 0&&1/\overline{z}\end{pmatrix}\begin{pmatrix}1&\overline{y}/z&1/z\\ 0&1&y/\overline{z}\\ 0&0&1\end{pmatrix}\begin{pmatrix}c&&0\\ &1&\\ 0&&1/\overline{c}\end{pmatrix}\begin{pmatrix}1&a&d\\ 0&1&\overline{a}\\ 0&0&1\end{pmatrix}\\
&\equiv\begin{pmatrix}zc&&0\\ &1&\\ 0&&1/\overline{zc}\end{pmatrix}\begin{pmatrix}1&a+\overline{y}/cz&d+(1+\overline{acy})/c\overline{c}z\\ 0&1&\overline{a}+y/\overline{cz}\\ 0&0&1\end{pmatrix}\\
&\equiv\begin{pmatrix}1&acz+\overline{y}&dc\overline{c}z\overline{z}+\overline{acyz}+\overline{z}\\ 0&1&\overline{acz}+y\\ 0&0&1\end{pmatrix}\begin{pmatrix}cz&&0\\ &1&\\ 0&&1/\overline{cz}\end{pmatrix}\\
&\equiv\begin{pmatrix}cz&&0\\ &1&\\ 0&&1/\overline{cz}\end{pmatrix}\qquad\text{if } |cdz+\overline{ay}+1/\overline{c}|\le\max\{|cz|,1/|cz|\};
\end{aligned}$$

if, moreover, $|d|\le 1$, then these identities hold with $a$, $d$ replaced by 0, and $c$ by $1/\overline{b}$. In the complementary case we obtain

$$\begin{aligned}
&\equiv\begin{pmatrix}1&0&0\\ a+\overline{y}/cz&1&0\\ d+(\overline{acy}+1)/c\overline{c}z&\overline{a}+y/\overline{cz}&1\end{pmatrix}\begin{pmatrix}cz&&0\\ &1&\\ 0&&1/\overline{cz}\end{pmatrix}\\
&\equiv\begin{pmatrix}cdz+\overline{ay}+1/\overline{c}&&0\\ &1&\\ 0&&1/(\overline{cdz}+ay+1/c)\end{pmatrix}
\end{aligned}$$

if $|cdz+\overline{ay}+1/\overline{c}|>\max\{|cz|,1/|cz|\}$ (or $>\max\{|cd|,1/|cd|\}$). If $|d|>1$, transposing we obtain an analogous situation, hence

$$\equiv\begin{pmatrix}cd&&0\\ &1&\\ 0&&1/\overline{cd}\end{pmatrix}\qquad\text{if } |cdz+\overline{ay}+1/\overline{c}|\le\max\{|cd|,1/|cd|\}.$$

Since the expression $cdz+\overline{ay}+1/\overline{c}$ does not change if $(y,z,c)$ is interchanged with $(a,d,c)$, in the complementary case the same expression as for $|z|>1$ is obtained, with $z$ replaced by $d$ in $\max\{|cz|,1/|cz|\}$. To check the compatibility of the statements, we prove

**15.1 Lemma.** *Suppose that* $|z| > 1$ *and* $|d| > 1$*. Then*

$$|cdz + \overline{ay} + 1/\overline{c}| < \max\{|cz|, 1/|cz|\} \text{ if and only if it is } < \max\{|cd|, 1/|cd|\},$$

*in which case* $|cz|\ |cd| = 1$*.*

*Proof.* Suppose that $|cdz + \overline{ay} + 1/\overline{c}| < |cz|$, $|cz| \geq 1$. Then $|cdz| = |ay + 1/c|$, and $|c^2dz| = |acy + 1|$. If $|acy| > 1$ then $|cdz| = |ay|$, and $|c| = |ay|/|dz| \leq 1/|ay|$ since $|z| > 1$ implies $|z| \geq |y|^2$ and $|d| > 1$ implies $|d| \geq |a|^2$; hence $|acy|$ cannot be $> 1$. On the other hand, if $|c^2dz| < 1$ then $|c^2d| < 1$ and $|c^2z| < 1$, and so $|ac| < 1$ and $|yc| < 1$; also $|acy| = 1$, hence $|a| > 1$ and $|y| > 1$, and so $|d| \geq |a|^2$, $|z| \geq |y|^2$, and $1 > |c^2dz| \geq |c^2a^2y^2| = 1$ is a contradiction. Hence $|c^2dz| = 1$ as required.

In the complementary case we have $|cdz + \overline{ay} + 1/\overline{c}| < 1/|cz|$, with $|cz| < 1$. Then $|c\overline{c}dz\overline{z} + \overline{acyz} + \overline{z}| < 1$, and $|z| = |c\overline{c}dz\overline{z} + \overline{acyz}|$, or $1 = |c\overline{c}dz + \overline{cay}|$. If $|c\overline{c}dz| > 1$ then $|c\overline{c}dz| = |acy|$, and $|c| = |ay|/|dz| \leq 1/|ay|$, a contradiction. If $|c\overline{c}dz| < 1$ then $|acy| = 1$, and so $1 > |c^2dz| \geq |c^2a^2y^2| = 1$. Hence $|cd|\ |cz| = 1$, and the lemma follows. □

To simplify the notations in the course of the proof of the next lemma we make

*Definition.* Write $f(t)$ for $f\begin{pmatrix} t & & 0 \\ & 1 & \\ 0 & & 1/\overline{t} \end{pmatrix}$.

The next lemma computes the integral of Proposition 15.

**15.2 Lemma.** *If* $|b| \leq 1$ *then* $\Psi(b, f, \psi)$ *is equal to*

$$f(b^{-1}) + (q^2+2q-1)f(b^{-1}\boldsymbol{\pi}^{-1}) + (q^4+2q^3-2q)f(b^{-1}\boldsymbol{\pi}^{-1}) + \sum_{\ell\geq 3}(1+q^{-1})^3(1-q^{-1})q^{2\ell}f(b^{-1}\boldsymbol{\pi}^{-\ell}).$$

*When* $|b| > 1$ *it is equal to the product of* $|b|_F \sum(b)$*, where* $|b|_F = |b\overline{b}|_F^{1/2} = |b|^{1/2}$*, and*

$$f(I) + (q^2 + q - 1)f(\boldsymbol{\pi}^{-1}) + (1 + q^{-1})(1 - q^{-2})\sum_{k\geq 2} q^{2k} f(\boldsymbol{\pi}^{-k}).$$

*Proof.* We shall partition the domain over which the integral which represents $\overline{\psi}(b)\Psi(b/2, f, \psi)$ is taken, into six subdomains, which will be further partitioned, and on which the value of $f$ was computed in the lines prior to Lemma 15.1. We will integrate first over $a$ and $d$, and only later over $y$ and $z$. It is useful to observe that there is $\tilde{d}$ with $\tilde{d} + \overline{\tilde{d}} = 0$ which depends on $d - \overline{d}$, such that

$$\left|\frac{dz}{a\overline{b}} + \overline{ay} + \overline{a}b\right| = \left|\frac{z}{ab}\right|\ \left|\tilde{d} + \frac{ab}{z}\cdot\frac{\overline{ab}}{\overline{z}}\left(\frac{\overline{z}}{b} + y\right)\left(\frac{z}{\overline{b}} + \overline{y}\right)\right|.$$

(1) On the domain where $|z|>1$, $|z/ab|\ge 1$ (recall that $c=1/a\overline{b}$) and $|dz/a\overline{b}+\overline{ay}+\overline{a}b|\le |z/ab|$, the function $f$ takes the value $f(z/a\overline{b})$. Define $k$ by $|\boldsymbol{\pi}^{-k}|=|z/ab|(\ge 1)$. Then $k\ge 0$, $|\tilde{d}|\le 1$ and $|\overline{z}/b+y|\le|\boldsymbol{\pi}^{-k}|$. The integral over $a$ and $d$ in this domain yields

$$\sum_{\substack{k\ge 0\\ |z|>1,\ |\overline{z}/b+y|\le|\boldsymbol{\pi}^{-k}|}} f(\boldsymbol{\pi}^{-k}).$$

(2) Suppose $|z|>1$ and $|dz/a\overline{b}+\overline{ay}+\overline{a}b|\le|ab/z|$, where $|ab/z|>1$. Here the function $f$ again takes the value $f(z/a\overline{b})$. Define $k$ by $|\boldsymbol{\pi}^{-k}|=|ab/z|(>1)$, then $k>1$, $|\tilde{d}|\le|\boldsymbol{\pi}^{-2k}|$, and $|\overline{z}/b+y|\le 1$. The integral over $a$ and $d$ in this domain is

$$\sum_{\substack{k>0\\ |z|>1,|\overline{z}/b+y|\le 1}} q^{2k} f(\boldsymbol{\pi}^{-k}).$$

(3) Suppose that $|z|>1$, $|z/ab|\ge 1$, and $|\tilde{d}+\frac{ab}{z}\cdot\frac{\overline{ab}}{\overline{z}}\left(\frac{\overline{z}}{b}+y\right)\left(\frac{z}{\overline{b}}+\overline{y}\right)|>1$. On this domain the value of $f$ is $f(\frac{dz}{a\overline{b}}+\overline{ay}+\overline{a}b)$. We further partition as follows.
(3′) On $|\tilde{d}|<|\frac{ab}{z}(\frac{\overline{z}}{b}+y)|^2$ define $k$ by $|\boldsymbol{\pi}^{-k}|=|\frac{ab}{z}(\frac{\overline{z}}{b}+y)|$. Then $k\ge 1$ and $|\overline{z}/b+y|\ge|\boldsymbol{\pi}^{-k}|$. The integral over $a$ and $d$ is

$$\sum_{\substack{k\ge 1\\ |z|>1,|\boldsymbol{\pi}^{-k}|\le|\overline{z}/b+y|}} q^{2k-1} f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)).$$

(3″) On $|\tilde{d}|\ge|\frac{ab}{z}(\frac{\overline{z}}{b}+y)|^2$ we have $|\tilde{d}|>1$, and the value of $f$ is $f(z\tilde{d}/ab)=f(\boldsymbol{\pi}^{-\ell})$ if we define $\ell$ by $|z\tilde{d}/ab|=|\boldsymbol{\pi}^{-\ell}|$. Then $|\tilde{d}|\ge|\frac{\tilde{d}}{\boldsymbol{\pi}^{-\ell}}(\frac{\overline{z}}{b}+y)|^2$, and $|\tilde{d}|\le|\boldsymbol{\pi}^{-2\ell}|/|\overline{z}/b+y|^2$, as well as $|\tilde{d}|\le|\boldsymbol{\pi}^{-\ell}|$ (as $|z/ab|\ge 1$). We split again:
($3''_1$) On $|\boldsymbol{\pi}^{-\ell}|\le|\boldsymbol{\pi}^{-2\ell}|/|\overline{z}/b+y|^2$ we have $|\boldsymbol{\pi}^{-\ell}|\ge|\overline{z}/b+y|^2$, and $\tilde{d}$ ranges over $1<|\tilde{d}|\le|\boldsymbol{\pi}^{-\ell}|$. Integrating over this subdomain we obtain

$$\sum_{\substack{\ell\ge 1\\ |z|>1,|\boldsymbol{\pi}^{-\ell}|\ge|\overline{z}/b+y|^2}} (q^{\ell}-1) f(\boldsymbol{\pi}^{-\ell}).$$

($3''_2$) On $|\boldsymbol{\pi}^{-\ell}|>|\boldsymbol{\pi}^{-2\ell}|/|\overline{z}/b+y|^2$ define $k$ by $|\boldsymbol{\pi}^{-k}|=|\boldsymbol{\pi}^{-\ell}|/|y+\overline{z}/b|$. There $|\boldsymbol{\pi}^{-k}|<|y+\overline{z}/b|$, and $\tilde{d}$ ranges over $1<|\tilde{d}|\le|\boldsymbol{\pi}^{-2k}|$. Integrating over $a$, $d$ in this subdomain we obtain

$$\sum_{\substack{k\ge 1\\ |z|>1,|\boldsymbol{\pi}^{-k}|<|\overline{z}/b+y|}} (q^{2k}-1) f(\boldsymbol{\pi}^{-k}(y+\overline{z}/b)).$$

(4) The next domain is where $|z|>1$ and

$$1<\left|\frac{ab}{z}\right|<\left|\frac{dz}{a\overline{b}}+\overline{ay}+\overline{a}b\right|=\left|\frac{z}{ab}\right|\ \left|\tilde{d}+\frac{ab}{z}\cdot\frac{\overline{ab}}{\overline{z}}\left(\frac{\overline{z}}{b}+y\right)\left(\frac{z}{\overline{b}}+\overline{y}\right)\right|.$$

It is cut into two. The value of $f$ here is $f(\frac{dz}{a\overline{b}}+\overline{ay}+\overline{a}b)$.
(4′) On $|\tilde{d}|\le(|\frac{ab}{z}|\;|\frac{\overline{z}}{b}+y|)^2$ we have $|\overline{z}/b+y|>1$. Define $k$ by $|\boldsymbol{\pi}^{-k}|=|ab/z|\;|\overline{z}/b+y|$. Then $|\boldsymbol{\pi}^{-k}|>|\overline{z}/b+y|>1$, and $|\tilde{d}|\le|\boldsymbol{\pi}|^{-2k}$. Integrating over $a$ and $d$ we obtain

$$\sum_{\substack{k\ge1\\ |z|>1,\;|\boldsymbol{\pi}^{-k}|>|\overline{z}/b+y|>1}} q^{2k}f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)).$$

(4″) On $|\tilde{d}|>(|\frac{ab}{z}|\;|\frac{\overline{z}}{b}+y|)^2$ we have $|\tilde{d}|>|ab/z|^2>1$. Then $|\tilde{d}|>|ab/z|^2A$, where we put $A=\|(1,\overline{z}/b+y)\|^2$ ($\|(u,v)\|=\max\{|u|,|v|\}$). Define $\ell$ by $|\boldsymbol{\pi}^{-\ell}|=|z\tilde{d}/ab|(>|ab/z|A=|\tilde{d}/\underline{\pi}^{-\ell}|A)$. Hence $|\tilde{d}|<|\boldsymbol{\pi}^{-2\ell}|A^{-1}$. Note that $|a|=|z\tilde{d}|/|b\boldsymbol{\pi}^{-\ell}|$ by definition of $\ell$. Since $|\boldsymbol{\pi}^{-\ell}|<|\tilde{d}|$, the integral over $a$ and $d$ is

$$\sum_{q^{\ell-1}>\|(1,\overline{z}/b+y)\|^2\ge1,|z|>1} (q^{2\ell-1}\|(1,\overline{z}/b+y)\|^{-2}-q^{\ell})f(\boldsymbol{\pi}^{-\ell}).$$

(5) On the domain where $|z|\le1$ and $|d|\le\|(1,(ab)^2)\|$, the value of $f$ is $f(\overline{a}b)=f((a\overline{b})^{-1})$. Here $\tilde{d}=d-\overline{d}$.
(5′) If $|z|\le1$ and $|d|\le1$, then $|a|\le1$, $|\tilde{d}|\le1$, and we define $k$ by $|\boldsymbol{\pi}^{-k}|=|a|^{-1}\ge1$.
($5_1'$) On the subdomain where $|ab|\le1$ we have $|\boldsymbol{\pi}^{-k}|\ge|b|$, and obtained is

$$\sum_{\substack{k\ge0\\ |z|\le1}} f(\boldsymbol{\pi}^{-k}/\overline{b})\quad\text{if } |b|\le1,\text{ and }\sum_{\substack{k\ge0\\ |z|\le1}} f(\boldsymbol{\pi}^{-k})\text{ if } |b|\ge1.$$

($5_2'$) On the subdomain where $|ab|>1$ we have $1\le|\boldsymbol{\pi}^{-k}|<|b|$, hence $|b|>1$, and writing $|\boldsymbol{\pi}^{-\ell}|=|\boldsymbol{\pi}^{k}b|$ obtained is

$$\sum_{1<|\boldsymbol{\pi}^{-\ell}|\le|b|} f(\boldsymbol{\pi}^{-\ell}).$$

(5″) On $|z|\le1$ and $|d|>1$ we have $|d|\le|ab|^2$.
($5_1''$) If $|\tilde{d}|\le|a|^2$ then $1<|a|=|\boldsymbol{\pi}^{-k}|$ and $|b|\ge1$. Then $k\ge1$, $|\tilde{d}|\le|\boldsymbol{\pi}^{-2k}|$, and we obtain

$$\sum_{\substack{k\ge1\\ |z|\le1,|b|\ge1}} q^{2k}f(\boldsymbol{\pi}^{-k}b).$$

($5_2''$) If $|\tilde{d}|>|a|^2$ and $|a|\ge1$, define $k$ by $|\boldsymbol{\pi}^{-k}|=|a|\ge1$; then $k\ge0$, and $|\boldsymbol{\pi}^{-2k}|<|\tilde{d}|\le|\boldsymbol{\pi}^{-2k}|\;|b|^2$, and $|b|>1$. Obtained is

$$\sum_{\substack{k\ge0\\ |z|\le1,|b|>1}} q^{2k}(|b|^2-1)f(\boldsymbol{\pi}^{-k}b).$$

($5_3''$) If $|\tilde{d}| > |a|^2$ and $|a| < 1$, then $1 < |\tilde{d}| \leq |ab|^2 = |\boldsymbol{\pi}^{-k}|^2$ defines $k(\geq 1)$, and we obtain

$$\sum_{\substack{k\geq 1\\ |z|\leq 1, |\boldsymbol{\pi}^{-k}|<|b|}} (q^{2k}-1)f(\boldsymbol{\pi}^{-k}).$$

(6) On $|z| \leq 1$, $|d| > 1$ and $|d| > |ab|^2$,the value of $f$ is $f(d/a\overline{b})$.
($6'$) If $|\tilde{d}| \leq |a|^2$ then $|a| > 1$ and $|b| < 1$. Define $k$ by $|\boldsymbol{\pi}^{-k}| = |a| = |d/a| > 1$, then $k \geq 1$, and obtained is

$$\sum_{\substack{k\geq 1\\ |z|\leq 1, |b|<1}} q^{2k} f(\boldsymbol{\pi}^{-k}/\overline{b}).$$

($6''$) If $|\tilde{d}| > |a|^2$ then $k$ is defined by $1 < |\tilde{d}| = |a\boldsymbol{\pi}^{-k}| < |\tilde{d}/a|\ |\boldsymbol{\pi}^{-k}| = |\boldsymbol{\pi}^{-2k}|$. Since $|\tilde{d}| > |ab|^2 = |\tilde{d}/\boldsymbol{\pi}^{-k}|^2|b|^2$, we also have that $1 < |\tilde{d}| < |\boldsymbol{\pi}^{-2k}/b^2|$. We obtain

$$\sum_{\substack{k\geq 1\\ |z|\leq 1, |b|\leq 1}} (q^{2k-1}-1)f(\boldsymbol{\pi}^{-k}/\overline{b}) + \sum_{\substack{k\geq 1\\ |z|\leq 1, |b|>1}} (q^{2k-1}-1)f(\boldsymbol{\pi}^{-k}).$$

This complete the first stage of the proof of our lemma. The next step will be to add up to various contributions depending on the value of $z$, and $b$. There are four such sums.
(A) On $|z| > 1$ and $|\overline{z}/b + y| \leq 1$ there is a contribution from (1), (2), ($3_1''$) and ($4''$), and the sum is

$$\begin{aligned} &\sum_{k\geq 0} f(\boldsymbol{\pi}^{-k}) + \sum_{k\geq 1} q^{2k} f(\boldsymbol{\pi}^{-k}) + \sum_{k\geq 1}(q^k-1)f(\boldsymbol{\pi}^{-k}) + \sum_{k\geq 2}(q^{2k-1}-q^k)f(\boldsymbol{\pi}^{-k}) \\ &= f(I) + (q^2+q)f(\boldsymbol{\pi}^{-1}) + \sum_{k\geq 2}(q^{2k}+q^{2k-1})f(\boldsymbol{\pi}^{-k}). \end{aligned}$$

(B) On $|z| > 1$ and $|\overline{z}/b + y| > 1$ there are contributions from (1), ($3'$), ($3_1''$), ($3_2''$), ($4'$), ($4''$), as follows:

$$\begin{aligned} &\sum_{k\geq 0} f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) + \sum_{1\leq|\pi^{-k}|\leq|\overline{z}/b+y|} q^{2k-1} f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) \\ &\quad + \sum_{|\boldsymbol{\pi}^{-k}|\geq|\overline{z}/b+y|} (q^k|\overline{z}/b+y|-1)f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) \\ &\quad + \sum_{1<|\boldsymbol{\pi}^{-k}|<|\overline{z}/b+y|} (q^{2k}-1)f(\boldsymbol{\pi}^{-k}(y+\overline{z}/b)) + \sum_{|\boldsymbol{\pi}^{-k}|>|\overline{z}/b+y|} q^{2k} f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) \\ &\quad + \sum_{|\boldsymbol{\pi}^{-k}|>q|\overline{z}/b+y|} (q^{2k-1}-q^k|\overline{z}/b+y|)f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) \\ &= f(\overline{z}/b+y) + \sum_{1<|\boldsymbol{\pi}^{-k}|\leq|\overline{z}/b+y|} (q^{2k}+q^{2k-1})f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) \\ &\quad + (q^2+q)|\overline{z}/b+y|^2 f(\boldsymbol{\pi}^{-1}(\overline{z}/b+y)^2) + \sum_{|\boldsymbol{\pi}^{-k}|>q|\overline{z}/b+y|} (q^{2k}+q^{2k-1})f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)) \\ &= f(\overline{z}/b+y) + \sum_{k\geq 1}(q^{2k}+q^{2k-1})f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y)). \end{aligned}$$

(C) On $|z| \le 1$ and $|b| \le 1$ there are contributions from $(5_1')$, $(5_1'')$, $(6')$, and $(6'')$:

$$\sum_{k\ge 0} f(\boldsymbol{\pi}^{-k}/\overline{b}) + \sum_{\substack{k\ge 1\\ |b|=1}} q^{2k} f(\boldsymbol{\pi}^{-k}/\overline{b}) + \sum_{\substack{k\ge 1\\ |b|<1}} q^{2k} f(\boldsymbol{\pi}^{-k}/\overline{b}) + \sum_{k\ge 1}(q^{2k-1}-1) f(\boldsymbol{\pi}^{-k}/\overline{b})$$
$$= f(\overline{b}^{-1}) + (q^2+q) f(\overline{b}^{-1}\pi^{-1}) + \sum_{k\ge 2} q^{2k}(1+q^{-1}) f(\boldsymbol{\pi}^{-k}/\overline{b}).$$

(D) On $|z| \le 1$ and $|b| > 1$ there are contributions from $(5_1')$, $(5_2')$, $(5_1'')$, $(5_2'')$, $(5_3'')$, $(6'')$:

$$\sum_{k\ge 0} f(\boldsymbol{\pi}^{-k}) + \sum_{1<|\boldsymbol{\pi}^{-k}|\le|b|} f(\boldsymbol{\pi}^{-k}) + \sum_{k\ge 1} q^{2k} f(\boldsymbol{\pi}^{-k} b) + \sum_{k\ge 0} q^{2k}(|b|^2-1) f(\boldsymbol{\pi}^{-k} b)$$
$$+ \sum_{1<|\boldsymbol{\pi}^{-k}|<|b|} (q^{2k}-1) f(\boldsymbol{\pi}^{-k}) + \sum_{k\ge 1}(q^{2k-1}-1) f(\boldsymbol{\pi}^{-k})$$
$$= f(I) + (q^2+q) f(\boldsymbol{\pi}^{-1}) + \sum_{q^2\le|\boldsymbol{\pi}^{-k}|\le|b|} (q^{2k}+q^{2k-1}) f(\boldsymbol{\pi}^{-k}) + \sum_{|\boldsymbol{\pi}^{-k}|>|b|} (q^{2k}+q^{2k-1}) f(\boldsymbol{\pi}^{-k})$$
$$= f(I) + \sum_{k\ge 1} q^{2k}(1+q^{-1}) f(\boldsymbol{\pi}^{-k}).$$

We are now in a position to compute $\Psi(b,f,\psi)$ and we start with the case of $|b| \le 1$. There are two contributions to the integral over $z$ (and $y$), the first being the expression of (C), which is the integral over $|z| \le 1$ (where $\psi(y)=1$). On the domain of $|z|>1$ there is a contribution only from (B), since $|z|>1$ and $|\overline{z}/b+y| \le 1$ imply $|y| = |z/b| > 1$, hence $|by| = |z| \ge |y\overline{y}|$ and $1 < |y| \le |b| \le 1$. When $|z|>1$ and $|\overline{z}/b+y|>1$, then $|\overline{z}/b| > |y|$ and so $|\overline{z}/b+y| = |z/b| > 1$ (clearly if $|y| \le 1$; if $|y|>1$ then $|z/b| \ge |z| > |y|$). If $|y|>1$ then we may take $|y| = q'(=q_E)$, since $\int_{|y|=q'^m} \psi(y)dy = 0$ for $m \ge 2$. Note that $\int_{|y|=q'} \psi(y)dy = -1$. We get three contributions from (B), corresponding to: $|y| \le 1$, $|y| = q'$ and $|\tilde{z}| \le q'^2$, and $|y| = q'$ and $|\tilde{z}| > q'^2$ (here $\tilde{z} = z - \overline{z}$). They are:

$$\int_{|\tilde{z}|>1} [f(\tilde{z}/b) + \sum_{\ell\ge 1}(1+q^{-1}) q^{2\ell} f(\boldsymbol{\pi}^{-\ell}\tilde{z}/b)] d\tilde{z}$$
$$- \int_{|\tilde{z}|\le q'^2} [f(b^{-1}\boldsymbol{\pi}^{-2}) + \sum_{\ell\ge 1}(1+q^{-1}) q^{2\ell} f(\boldsymbol{\pi}^{-2-\ell} b^{-1})] d\tilde{z}$$
$$- \int_{|\tilde{z}|>q'^2} [f(z'/b) + \sum_{\ell\ge 1}(1+q^{-1}) q^{2\ell} f(\boldsymbol{\pi}^{-\ell}\tilde{z}/b)] d\tilde{z}$$
$$= (q-1)[f(\boldsymbol{\pi}^{-1}/b) + (1+q^{-1}) \sum_{\ell\ge 1} q^{2\ell} f(b^{-1}\boldsymbol{\pi}^{-\ell-1})]$$
$$+ ((q^2-q)-q^2)[f(\boldsymbol{\pi}^{-2}/b) + \sum_{\ell\ge 1}(1+q^{-1}) q^{2\ell} f(b^{-1}\boldsymbol{\pi}^{-\ell-2})].$$

The sum of this and the expression for (C) is the expression given in the lemma for the value of $\Psi(b,f,\psi)$ at $|b|\le 1$.

It remains to compute the value of $\overline{\psi}(b)\Psi(b/2,f,\overline{\psi})$ for $|b|>1$. This is the sum of three expressions. The first is the integral over $|z|\le 1$, and this is given by (D). The contribution over $|z|>1$ and $|\overline{z}/b+y|\le 1$ is obtained from (A); it is

$$[f(I)+(1+q^{-1})\sum_{k\ge 1}q^{2k}f(\boldsymbol{\pi}^{-k})]\int_{|z|>1,|\overline{z}/b+y|\le 1}\psi(y)dz.$$

The contribution over $|z|>1$ and $|\overline{z}/b+y|>1$ is obtained from (B):

$$\int_{|z|>1,|\overline{z}/b+y|>1}[f(\overline{z}/b+y)+\sum_{k\ge 1}(1+q^{-1})q^{2k}f(\boldsymbol{\pi}^{-k}(\overline{z}/b+y))]\psi(y)dz.$$

We shall name these three terms: the first, second and third.

Concerning the second term, where $|z|>1$, $|\overline{z}/b+y|\le 1$, or $|\overline{z}+by|\le|b|$, or $|\tilde{z}+y\overline{y}+by+\overline{by}|\le|b|$, we have $|\tilde{z}|\le|b|$ and $|(1+y/\overline{b})(1+\overline{y}/b)-1|\le 1/|b|$, hence $y/\overline{b}+1=\varepsilon(1+u/\overline{b})$, $\varepsilon\in R'^{\times}/(1+b^{-1}R')$, $u\in R'$, $\varepsilon\overline{\varepsilon}=1$. Then $y=-\overline{b}+\varepsilon\overline{b}+\varepsilon u$. When $\varepsilon\in 1+b^{-1}R'$ we have $|y|\le 1$, and then $|\tilde{z}|=|z|>1$, so $1<|\tilde{z}|\le|b|$. Hence the integral in the second term is

$$|b|_F\sum_{\varepsilon\in(R'^{\times}/(1+b^{-1}R'))^{\times},\varepsilon\overline{\varepsilon}=1}\psi(\varepsilon\overline{b})\overline{\psi}(\overline{b})+|b|_F-1=|b|_F\sum(b/2)-1.$$

Consequently the sum of the first and second terms is (D) times $\sum(b/2)$, which is the second term with the integral replaced by $\sum(b/2)$.

Concerning the third term where $|z|>1$ and $|\overline{z}/b+y|>1$, we claim that only the $z$ with $|\overline{z}/b+y|=|\boldsymbol{\pi}^{-1}|$ contribute to the integral. To see this, write $|\overline{z}/b+y|=|\boldsymbol{\pi}|^{-m}$, where $m\le 1$. Then $|\tilde{z}+(y+\overline{b})(\overline{y}+b)-b\overline{b}|=|\overline{z}+by|=|b\boldsymbol{\pi}^{-m}|$, and there are two cases to deal with: (1) $|\tilde{z}|=|b\boldsymbol{\pi}^{-m}|$, then $|(1+y/\overline{b})(1+\overline{y}/b)-1|\le|\boldsymbol{\pi}^{-m}/b|$; (2) $|\tilde{z}|<|b\boldsymbol{\pi}^{-m}|$, then $|(1+y/\overline{b})(1+\overline{y}/b)-1|=|\boldsymbol{\pi}^{-m}/b|$. We shall integrate $\psi(y)$ over the $y$ subject to $|\overline{z}/b+y|=|\boldsymbol{\pi}^{-m}|$. There are three possibilities.

(i) If $|\boldsymbol{\pi}^{-m}|>|b|$, in case (1) we have $|y|^2\le|b\boldsymbol{\pi}^{-m}|$, and so $\int\psi(y)dy=0$. In case (2) we have $|y|^2=|b\boldsymbol{\pi}^{-m}|\ge q^3$, and $\int_{|y|=q^\ell}\psi(y)dy=0$ if $\ell\ge 2$.

(ii) If $|\boldsymbol{\pi}^{-m}|=|b|$, in case (1) we have $|y|\le|b|$, and $\int\psi(y)dy=0$. In case (2) $y$ ranges over $\{|y|\le|b|\}-\{y/\overline{b}+1=\varepsilon(1+\boldsymbol{\pi}u),\ u\in R',\ \varepsilon\in R'/(1+\boldsymbol{\pi}R'),\ \varepsilon\overline{\varepsilon}=1\}$, hence $\int\psi(y)dy$ is

$$-\psi((\varepsilon-1)\overline{b})\int_{|u|\le 1}\psi(\boldsymbol{\pi}\overline{b}u)du=\begin{cases}0, & |b|>|\boldsymbol{\pi}^{-1}|,\\ -\psi((\varepsilon-1)\overline{b}), & |b|=|\boldsymbol{\pi}^{-1}|,\ \text{i.e. } m=1\end{cases}$$

(iii) If $|\boldsymbol{\pi}^{-m}|<|b|$ then $y/\overline{b}+1=\varepsilon(1+\boldsymbol{\pi}^{-m}\overline{b}^{-1}u)$, $\varepsilon\in R'^{\times}/(1+\boldsymbol{\pi}^{-m}\overline{b}^{-1}R')$, $\varepsilon\overline{\varepsilon}=1$, thus $y=(\varepsilon-1)\overline{b}+\varepsilon\boldsymbol{\pi}^{-m}u$, and $|u|\le 1$. In case (1) this $u$ ranges all over $|u|\le 1$, then

$\int_{|u|\le 1}\psi(\varepsilon u\boldsymbol{\pi}^{-m})du=0$. In case (2) this $u$ ranges over the complement in $|u|\le 1$ of $\{y/\overline{b}+1=\varepsilon(1+\boldsymbol{\pi}^{-m+1}\overline{b}^{-1}u);\ \varepsilon\in R'^{\times}/(1+\boldsymbol{\pi}^{-m+1}\overline{b}^{-1}R'),\ \varepsilon\overline{\varepsilon}=1,\ |u|\le 1\}$, and then $\int\psi(y)dy$ is

$$-\psi((\varepsilon-1)\overline{b})\int_{|u|\le 1}\psi(\varepsilon u\boldsymbol{\pi}^{1-m})du=\begin{cases}0, & m>1\\ -\psi((\varepsilon-1)\overline{b}), & m=1(\text{i.e. } |b|>|\boldsymbol{\pi}^{-1}|).\end{cases}$$

In summary, we may take only $z$ with $|\overline{z}/b+y|=|\boldsymbol{\pi}^{-1}|$ in the integral of the third term, only case (2) applies, namely $\tilde{z}$ ranges over $|\tilde{z}|<|b\boldsymbol{\pi}^{-m}|$, $m=1$, thus $|\tilde{z}|\le|b|$, hence obtained is the product of $-|b|_F\overline{\psi}(\overline{b})\sum(b/2)$ and

$$f(\boldsymbol{\pi}^{-1})+\sum_{k\ge 1}q^{2k}(1+q^{-1})f(\boldsymbol{\pi}^{-k-1})=f(\boldsymbol{\pi}^{-1})+(1+q^{-1})\sum_{k\ge 2}q^{2k-2}f(\boldsymbol{\pi}^{-k}).$$

The sum of all terms is then

$$[f(I)+(q^2+q-1)f(\boldsymbol{\pi}^{-1})+(1+q^{-1})(1-q^{-2})\sum_{k\ge 2}q^{2k}f(\boldsymbol{\pi}^{-k})]|b|_F\overline{\psi}(b)\sum(b/2).$$

But this is the assertion of Lemma 15.2, completing the proof of this lemma, and so also of Proposition 15. □

*Remark.* The result of the last computation can also be expressed in the form

$$|b|^{-1/2}q^{-k-1}\Psi(b,f^{k+1},\psi)=\sum(b)\begin{cases}q+1-1/q, & k=0,\\ (1+q^{-1})(1-q^{-2})q^{k+1}, & k\ge 1,\end{cases}$$

and

$$|b|^{1/2}q^{-k}\Psi(b,f^{k},\psi)=\sum(b)\begin{cases}1, & k=0,\\ q+1-1/q, & k=1,\\ (1+q^{-1})(1-q^{-2})q^{k}, & k\ge 2,\end{cases}$$

when $|b|>1$. The difference is easily compared with the corresponding difference of the values of $\Psi(b,f'^{k+1},\psi)-q^{-1}\Psi(b,f'^{k},\psi)$ described in Proposition 13. Similar comment applies in the case of $|b|\le 1$.

As noted after Proposition 15, any two of Propositions 13, 14, 15 imply the third. We proved 13 and 15, and conclude 14, which asserts that corresponding spherical functions are matching, at a place of the global field $F$ which is non-split and unramified in the quadratic extension $E$ (we also took the conductor of $\psi$ to be $R'$, but this is an easily removable condition, and only the case of conductor $(\psi)=R'$ will be used anyway).

**Part 2.** *Proof of Proposition* 17. In the course of this proof (only) we write $K'=GL(2,R)$. We first consider the argument of $f'$ in the integral. We ask, for $n\ge 0$, for which $x,y\in F$ and $c\in F^{\times}$, we have

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}-b/c & 0\\ 0 & 1/c\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\in K'\begin{pmatrix}\boldsymbol{\pi}^{-n} & 0\\ 0 & 1\end{pmatrix}K'.$$

(1) If $|y| \le 1$ and $|x| \le \|(1,b)\| (= \max\{1,|b|\})$, or $|x| \le 1$ and $|y| \le \|(1,b)\|$, our matrix is in the $K'$-double coset of

$$\begin{pmatrix} b/c & 0 \\ 0 & 1/c \end{pmatrix}\begin{pmatrix} 1 & x/b \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} b/c & 0 \\ 0 & 1/c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ y/b & 1 \end{pmatrix}\begin{pmatrix} b/c & 0 \\ 0 & 1/c \end{pmatrix} \equiv \begin{pmatrix} b/c & 0 \\ 0 & 1/c \end{pmatrix},$$

and then $|c| = 1$ and $|b| = q^n$, or $|b| = |c| = q^n$.
(2) If $|x| \le 1$ and $|y| > \|(1,b)\|$, using the Bruhat decomposition recorded in the proof of Lemma 12, our matrix lies in the $K'$-double coset of

$$\begin{pmatrix} b/c & 0 \\ 0 & 1/c \end{pmatrix}\begin{pmatrix} 1 & 1/y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/y & 0 \\ 0 & y \end{pmatrix} \equiv \begin{pmatrix} b/cy & 0 \\ 0 & y/c \end{pmatrix},$$

Thus $|y| = |c|$ and $|b/c^2| = q^n$, or $|b| = |cy|$ and $|y/c| = q^n$.
(3) If $|y| \le 1$ and $|x| > \|(1,b)\|$, our matrix is in the $K'$-doble coset of

$$\begin{pmatrix} b/c & 0 \\ 0 & 1/c \end{pmatrix}\begin{pmatrix} 1 & 1/x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/x & 0 \\ 0 & x \end{pmatrix} \equiv \begin{pmatrix} 1 & b/x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b/cx & 0 \\ 0 & x/c \end{pmatrix},$$

thus $|x| = |c|$ and $|b/c^2| = q^n$, or $|b| = |cx|$ and $|x/c| = q^n$.
(4) If $|x| > 1$ and $|y| > 1$ then

$$\begin{aligned}
&\equiv \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -b/c & 0 \\ 0 & 1/c \end{pmatrix}\begin{pmatrix} 1 & 1/y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/y & 0 \\ 0 & y \end{pmatrix} \equiv \begin{pmatrix} -b/cy & 0 \\ 0 & y/c \end{pmatrix}\begin{pmatrix} 1 & -(xy-b)y/b \\ 0 & 1 \end{pmatrix} \\
&\equiv \begin{pmatrix} 1 & (xy-b)/y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b/cy & 0 \\ 0 & y/c \end{pmatrix} \equiv \begin{pmatrix} b/cy & 0 \\ 0 & y/c \end{pmatrix} \qquad \text{if } |xy-b| \le \|(y,b/y)\|; \\
&\equiv \begin{pmatrix} b/cy & 0 \\ 0 & y/c \end{pmatrix}\begin{pmatrix} 1 & y/(xy-b) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y/(xy-b) & 0 \\ 0 & (xy-b)/y \end{pmatrix} \\
&\equiv \begin{pmatrix} 1 & -b/y(xy-b) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b/c(xy-b) & 0 \\ 0 & (xy-b)/c \end{pmatrix} \\
&\equiv \begin{pmatrix} b/c(xy-b) & 0 \\ 0 & (xy-b)/c \end{pmatrix} \qquad \text{if } |xy-b| > \|(y,b/y)\|;
\end{aligned}$$

Hence – by symmetry of $x$ and $y$ – we also have

$$\begin{aligned}
&\equiv \begin{pmatrix} b/cx & 0 \\ 0 & x/c \end{pmatrix} \qquad \text{if } |xy-b| \le \|(x,b/x)\|; \\
&\equiv \begin{pmatrix} b/c(xy-b) & 0 \\ 0 & (xy-b)/c \end{pmatrix} \qquad \text{if } |xy-b| > \|(x,b/x)\|.
\end{aligned}$$

To check compatibility in the case where $|x| > 1$ and $|y| > 1$, we now claim that $|xy-b| \le \|(x,b/x)\|$ if and only if $|xy-b| \le \|(y,b/y)\|$, and then $|xy| = |b|$. Suppose that $|xy-b| \le \|(y,b/y)\|$. We deal separately with two cases. (i) If $1 < |y| \le |b/y|$ then $|xy-b| \le |b/y|$, and since $|y|>1$ we have $|xy|=|b|$. If $|xy-b|>\|(x,b/x)\|$ then $|b/y|\geq|xy-b|>|x|=|b/y|$, a contradiction. (ii) If $|b/y|<|y|>1$ then $|xy-b|\leq|y|$, and since $|x|>1$ we have $|xy|=|b|$. Then $|xy-b|>\|(x,b/x)\|$ leads to $|y|\geq|xy-b|>|b/x|=|y|$, again a contradiction, and the claim follows.

We are now ready to perform the integration of the proposition. If $|b|\leq 1$, the contributions from the domains (1), (2), (3), (4) yield

$$\begin{aligned} f'\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) &+ 2\int_{|y|>1} f'\left(\begin{pmatrix} b/y & 0 \\ 0 & y \end{pmatrix}\right)\psi_2(y)dy \\ &+ \iint_{|x|>1,|y|>1} f'\left(\begin{pmatrix} b/(xy-b) & 0 \\ 0 & xy-b \end{pmatrix}\right)\psi_2(x-y)dxdy \\ &= f'\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) - 2f'\left(\begin{pmatrix} b\boldsymbol{\pi}^2 & 0 \\ 0 & 1 \end{pmatrix}\right)\eta(\boldsymbol{\pi}) + \eta(\boldsymbol{\pi})^2 f'\left(\begin{pmatrix} b\boldsymbol{\pi}^4 & 0 \\ 0 & 1 \end{pmatrix}\right), \end{aligned}$$

as asserted in the proposition.

When $|b|>1$ the contribution from the domains (1), (2), (3) to the integral is zero since $\int_{|y|\leq|b|}\psi_2(y)dy$ and $\int_{|y|>|b|}\psi_2(y)dy$ are zero. The contribution from the domain (4) is the sum of

$$\iint_{\substack{|x|>1,|y|>1 \\ |xy-b|\leq\|(y,b/y)\|=\|(x,b/x)\|}} f'\left(\begin{pmatrix} b/y & 0 \\ 0 & y \end{pmatrix}\right)\psi_2(x-y)dx\,dy$$

and

$$\iint_{\substack{|x|>1,|y|>1 \\ |xy-b|>\|(y,b/y)\|}} f'\left(\begin{pmatrix} b/(xy-b) & 0 \\ 0 & xy-b \end{pmatrix}\right)\psi_2(x-y)dx\,dy.$$

To compute the first integral, we split the domain of integration into two.
(i) On $|x|>|b/x|=|y|>1$ we have $|xy-b|\leq|x|$ and so $|y-b/x|\leq 1$, and we obtain

$$\int_{|b|>|x|>|b|^{1/2}} f'\left(\begin{pmatrix} x & 0 \\ 0 & b/x \end{pmatrix}\right)\psi_2(x-b/x)dx.$$

This is zero since $\int_{|x|=q^m}\psi_2(x-b/x)dx\neq 0$ only when $|b|=q^{2m}$ (i.e. $|x|=|b|^{1/2}$).
(ii) On $|y|\geq|b/y|$ we have $|xy-b|\leq|y|$ and $|x-b/y|\leq 1$, so the integral is

$$\int_{1<|y|\geq|b|^{1/2}} f'\left(\begin{pmatrix} b/y & 0 \\ 0 & y \end{pmatrix}\right)\psi_2(y-b/y)dy.$$

The only non-zero contribution to this integral is obtained from $y$ with $|y|=|b|^{1/2}$, and so in particular the valuation of $b$ is even. We then get

$$\eta(b^{1/2})f'(I)\int_{|y|=|b|^{1/2}}\psi_2(y-b/y)dy.$$

To compute the second integral, note that on its domain $|x-b/y| > \|(1,b/y^2)\|$. If $|x-b/y| > q$ then $|x+u-b/y| = |x-b/y| > q$ for all $u \in F$ with $|u| = q$, and the integral over the corresponding subdomain clearly vanishes. It remains to consider $|x-b/y| = q$; there $x = b/y + u$, $|u| = q$. The integral takes the form

$$\int f'\left(\begin{pmatrix} b/y\boldsymbol{\pi}^{-1} & 0 \\ 0 & y\boldsymbol{\pi}^{-1} \end{pmatrix}\right)\psi_2(y-b/y)dy \int_{|u|=q}\psi_2(u)du.$$

The integral over $u$, $|u| = q$, is equal to $-1$. Non zero contribution to the integral is obtained only from $y$ with $|y| = |b|^{1/2}$. Hence we obtain

$$\eta(b^{1/2}\boldsymbol{\pi}^{-1})^{-1} f'\left(\begin{pmatrix} \boldsymbol{\pi}^{-1} & 0 \\ 0 & \boldsymbol{\pi} \end{pmatrix}\right)\int \psi_2(y-b/y)dy,$$

and the proposition follows. □

**Proof of Proposition** 18. Our approach will be to compute both sides in a direct way, and then compare them. We start with $\Psi(b,f,\psi)$.

**18.1 Lemma.** *It suffices to take the integration in* $\Psi(b,f,\psi)$

$$= \int f\left(\begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ 1/4\alpha & 0 & 1 \end{pmatrix}\begin{pmatrix} -\alpha/b & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right)\psi(x+y)dx\,dy\,dz\,du\,d^\times\alpha d^\times\beta$$

*over* $|x| \le q$ *and* $|y| \le q$ *or* $|x| = |y| \ge q^2$*, without changing the value of the integral.*

*Proof.* Since $f$ is spherical we may conjugate its argument by $\operatorname{diag}(1,\varepsilon,\varepsilon^{-1})$, $|\varepsilon| = 1$, then change variables on $x(\mapsto \varepsilon x)$, $y(\mapsto y/\varepsilon)$, and $u$, to obtain that our integral – for fixed $x, y, z, u, \alpha, \beta$ – factorizes through

$$\int_{|\varepsilon|=1}\psi(x\varepsilon + y/\varepsilon)d\varepsilon.$$

If $|x| \le 1$ and $|y| > q$ this is clearly zero. If $|x| \ge |y| \ge q$, and $|y| \le q^{2m}$, $m \ge 1$, the integral factorizes through the $\varepsilon = 1 - \eta\boldsymbol{\pi}^m$, $|\eta| \le 1$, and $\varepsilon^{-1} = 1 + \eta\boldsymbol{\pi}^m + \eta^2\boldsymbol{\pi}^{2m} + \dots$. Hence

$$\psi(x\varepsilon + y/\varepsilon) = \psi(x + y + (y-x)\eta\boldsymbol{\pi}^m),$$

and its integral over $|\eta| \le 1$ is zero unless $|x-y| \le q^m$, thus $|x| = |y|$ if $|y| > q$, and $|x| \le q$ if $|y| = q$ (we take minimal $m$, namely $m = 1$ if $|y| = q$), as required. □

Another simple observation is the following

**18.2 Lemma.** *The integral*

$$\iiint f\left(\begin{pmatrix}1&0&0\\x&1&0\\z&y&1\end{pmatrix}\begin{pmatrix}u&&0\\&v&\\0&&w\end{pmatrix}\right)\psi(x+y)dx\,dy\,dz$$

*vanishes unless* $|u|\ge|v|\ge|w|$.

*Proof.* Indeed the integral is equal to

$$\iiint f\left(\begin{pmatrix}u&&0\\&v&\\0&&w\end{pmatrix}\begin{pmatrix}1&0&0\\xu/v&1&0\\zu/w&yv/w&1\end{pmatrix}\right)\psi(x+y)dx\,dy\,dz$$

$$=\left|\frac{v}{u}\frac{w}{u}\frac{w}{v}\right|\iiint f\left(\begin{pmatrix}u&&0\\&v&\\0&&w\end{pmatrix}\begin{pmatrix}1&0&0\\x&1&0\\z&y&1\end{pmatrix}\right)\psi(xv/u+yw/v)dx\,dy\,dz.$$

If $|v/u|>1$, change $x\mapsto x+\eta$, and note that $\int_{|\eta|\le1}\psi(\eta v/u)d\eta=0$, as required. □

Our next step is to split the domain of integration for $\Psi(-b,f,\psi)$ into five. We take the argument of $f$, as in Lemma 18.1, with $|x|\le q$, $|y|\le q$.

(I) $$\equiv\begin{pmatrix}1&0&0\\x&1&0\\z&y&1\end{pmatrix}\begin{pmatrix}\alpha/b&&0\\&\beta&\\0&&1\end{pmatrix},\qquad\text{if } |u|\le1\le|\alpha|;$$

(II) $$\equiv\begin{pmatrix}1/\alpha&&0\\&1&\\0&&\alpha\end{pmatrix}\begin{pmatrix}1&&1/4\alpha\\&1&\\0&&1\end{pmatrix}\begin{pmatrix}\alpha/b&&0\\&\beta&\\0&&1\end{pmatrix}\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}$$

$$\equiv\begin{pmatrix}1/b&&0\\&\beta&\\0&&\alpha\end{pmatrix}\begin{pmatrix}1&x&z+b/4\\0&1&y\\0&0&1\end{pmatrix}$$

$$\equiv\begin{pmatrix}1&0&0\\x&1&0\\z+b/4&y&1\end{pmatrix}\begin{pmatrix}1/b&&0\\&\beta&\\0&&\alpha\end{pmatrix},\qquad\text{if } |u|\le1,\ |\alpha|<1;$$

(III) $$\equiv\begin{pmatrix}u&&0\\&1/u&\\0&&1\end{pmatrix}\begin{pmatrix}1&1/u&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\alpha/b&&0\\&\beta&\\0&&1\end{pmatrix}\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}$$

$$\equiv\begin{pmatrix}\alpha u/b&&0\\&\beta/u&\\0&&1\end{pmatrix}\begin{pmatrix}1&x+\beta b/\alpha u&z+y\beta b/\alpha u\\0&1&y\\0&0&1\end{pmatrix}$$

$$\equiv\begin{pmatrix}1&0&0\\x+\beta b/\alpha u&1&0\\z+y\beta b/\alpha u&y&1\end{pmatrix}\begin{pmatrix}\alpha u/b&&0\\&\beta/u&\\0&&1\end{pmatrix},\qquad\text{if } |u|>1,\ |\alpha|\ge1;$$

$$
\text{(IV)}\quad \equiv \begin{pmatrix}1&0&0\\u&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&4\alpha\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&&-1\\&1&\\1&&0\end{pmatrix}
$$

$$
\cdot\begin{pmatrix}1/4\alpha&&0\\&1&\\0&&4\alpha\end{pmatrix}\begin{pmatrix}\alpha/b&&0\\&\beta&\\0&&1\end{pmatrix}\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}
$$

$$
\equiv \begin{pmatrix}1&0&0\\0&1&-u\\0&0&1\end{pmatrix}\begin{pmatrix}1/4b&&0\\&\beta&\\0&&4\alpha\end{pmatrix}\begin{pmatrix}1&x&z+4b\\0&1&y\\0&0&1\end{pmatrix}
$$

$$
\equiv \begin{pmatrix}1/b&&0\\&\beta&\\0&&\alpha\end{pmatrix}\begin{pmatrix}1&x&a+4b\\0&1&y-4\alpha u/\beta\\0&0&1\end{pmatrix}
$$

$$
\equiv \begin{pmatrix}1&0&0\\x&1&0\\z+4b&y-4\alpha u\beta&1\end{pmatrix}\begin{pmatrix}1/b&&0\\&\beta&\\0&&\alpha\end{pmatrix},\qquad \text{if } 1<|u|\le 1/|\alpha|;
$$

$$
\text{(V)}\quad \equiv \begin{pmatrix}1&&0\\&1&\\1/4\alpha&&1\end{pmatrix}\begin{pmatrix}1&1/u&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}u&&0\\&1/u&\\0&&1\end{pmatrix}
$$

$$
\cdot\begin{pmatrix}1&1/u&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\alpha/b&&0\\&\beta&\\0&&1\end{pmatrix}\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}
$$

$$
\equiv \begin{pmatrix}1&0&0\\0&1&0\\0&-1/4\alpha&1\end{pmatrix}\begin{pmatrix}\alpha u/b&&0\\&\beta/u&\\0&&1\end{pmatrix}\begin{pmatrix}1&x+\beta b/\alpha u&z+y\beta b/\alpha u\\0&1&y\\0&0&1\end{pmatrix}
$$

$$
\equiv \begin{pmatrix}\alpha u/b&&0\\&\beta/u\alpha&\\0&&\alpha\end{pmatrix}\begin{pmatrix}1&x+\beta b/\alpha u&z+y\beta b/\alpha u\\0&1&y-4\alpha u/\beta\\0&0&1\end{pmatrix}
$$

$$
\equiv \begin{pmatrix}1&0&0\\x+\beta b/\alpha u&1&0\\z+y\beta b/\alpha u&y-4\alpha u/\beta&1\end{pmatrix}\begin{pmatrix}\alpha u/b&&0\\&\beta/u\alpha&\\0&&\alpha\end{pmatrix},\qquad \text{if } 1<|\alpha|^{-1}\le|u|.
$$

We shall now integrate the integrand of Lemma 18.1 on each of these five domains. Put

$$
U=\begin{pmatrix}1&0&0\\x&1&0\\z&y&1\end{pmatrix},\qquad t=\begin{pmatrix}\alpha\beta&&0\\&\beta&\\0&&1\end{pmatrix},
$$

and note that the following integrals are taken over $|\alpha|\ge 1$, $|\beta|\ge 1$, and against $dx\,dy\,dz\,du\,d^\times\alpha\,d^\times\beta$. Here they are:
$(\int_I)$. Change $\alpha\mapsto\alpha\beta b$ to obtain $(*)=\int_{|\alpha|\ge 1/|\beta b|}f(Ut)\psi(x+y)$.

($\int_{II}$). Change: $\alpha \mapsto 1/\alpha$, $\beta \mapsto \beta/\alpha$, $\alpha \mapsto \alpha\beta b$, $z \mapsto z - b/4$ (in this order) to obtain

$$\int_{|\alpha|>1/|\beta b|} f(Ut)\psi(x+y).$$

($\int_{III}$). Change: $z \mapsto y\beta b/\alpha u$, $x \mapsto x - \beta b/\alpha u$, $\beta \mapsto \beta u$, $\alpha \mapsto \alpha\beta b/u$ (in this order) to obtain

$$\int f(Ut)\psi(x+y)\cdot|\alpha| \int\limits_{|\alpha|^{-1}<|u|\leq|\beta b|} \psi(u)du.$$

($\int_{IV}$). Change $z \mapsto z - 4b$, $y \mapsto y + 4u\alpha/\beta$, $\alpha \mapsto \alpha^{-1}$, $\beta \mapsto \beta/\alpha$, $\alpha \mapsto \alpha\beta b$, to obtain

$$\int f(Ut)\psi(x+y)\cdot|\beta| \int_{|\beta|^{-1}<|u|\leq|\alpha b|} \psi(u)du.$$

($\int_{V}$). Change: $z \mapsto z - y\beta b/\alpha u$, $y \mapsto y + 4\alpha u/\beta$, $x \mapsto x - \beta b/\alpha u$, $\beta \mapsto \beta\alpha^2 u$, $u \mapsto u\beta b$, $\alpha \mapsto \alpha^{-1}$, $\alpha \leftrightarrow u$, to obtain

$$\int f(Ut)\psi(x+y) \int\limits_{1<|u|\leq|\alpha\beta b|} \psi(-b\beta/u + 4u/\beta)d^\times u\cdot|\beta b|d\alpha$$

$$= |b| \int f(Ut)\psi(x+y)|\alpha\beta|\cdot \int\limits_{|\beta|^{-1}<|u|\leq|\alpha b|} |u|^{-1}\psi(4u - b/u)du.$$

**18.3 Lemma.** *When $|b|>1$, the integral $\Psi(-b,f,\psi)$ is equal to*

$$|b|^{1/2}\int_{|u|=|b|^{1/2}} \psi_2(u-b/u)du\cdot\int_{|\alpha|\geq1,|\beta|\geq1} |\alpha\beta| f(Ut)\psi(x+y)dU\,d^\times\alpha\,d^\times\beta.$$

*When $|b|\leq 1$ it is*

$$\int\limits_{1\leq|\beta|\leq|b|^{-1}} f\left(U\begin{pmatrix}1/b & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\right)\psi(x+y)d^\times\beta dU$$

$$+ \int\limits_{|\alpha|>1,1\leq|\beta|\leq1/|b|} f\left(U\begin{pmatrix}\alpha/b & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\right)\psi(x+y)|\alpha|d^\times\alpha\,d^\times\beta\,dU$$

$$+ \int\limits_{|\alpha|\geq|\beta b|,1<|\alpha|\leq|\beta|} f\left(U\begin{pmatrix}\alpha/b & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\right)\psi(x+y)|\alpha|d^\times\alpha\,d^\times\beta\,dU$$

$$+ \int\limits_{\substack{|\alpha|\geq1,|\beta|\geq1\\|\alpha\beta b|>1}} f(Ut)\psi(x+y)|\alpha|d^\times\alpha d^\times\beta \int\limits_{1/|\beta|<|u|\leq|\alpha b|} \psi_2(u-b/u)\frac{du}{|u|}.$$

*Proof.* When $|b|>1$, $(\int_I)$ is $(*)$, $(\int_{II})$ is $(*)$, $(\int_{III})$ is $(*)$ (times $|\alpha|(1-1/|\alpha|-1)=1) = -(*)$, and $(\int_{IV})$ is $(*)$ (times $|\beta|(1-1/|\beta|-1)) = -(*)$, so they cancel each other, and we are left with $(\int_V)$, which is (after changing $u\mapsto u/2$)

$$\int_{\substack{|\beta|\geq 1\\ |\alpha|\geq 1}} f(Ut)\psi(x+y)|\alpha|d^\times\alpha|\beta b|d^\times\beta\cdot \int_{|\beta|^{-1}<|u|\leq|\alpha b|} |u|^{-1}\psi_2(u-b/u)du.$$

The inner integral can be taken only over $|u|=|b|^{1/2}$, and the lemma follows when $|b|>1$.

When $|b|\leq 1$, $(\int_I)+(\int_{II})$ is equal to the sum of the first integral in the statement of the lemma (for $|b|\leq 1$), and

$$2\int_{|\alpha|\geq 1,|\beta|\geq 1,|\alpha\beta|>1/|b|} f(Ut)\psi(x+y).$$

Also

$$(\int_{III}) = \int_{|\alpha|\geq 1,|\beta|\geq 1,|\alpha\beta|>1/|b|} f(Ut)\psi(x+y)|\alpha| \left\{\begin{array}{ll} |\beta b|-1/|\alpha|, & \text{if } |\beta b|\leq 1 \\ 1-1/|\alpha|-1, & \text{if } |\beta b|>1 \end{array}\right\},$$

and

$$(\int_{IV}) = \int_{|\alpha|\geq 1,|\beta|\geq 1,|\alpha\beta|>1/|b|} f(Ut)\psi(x+y)|\beta| \left\{\begin{array}{ll} |\alpha b|-1/|\beta|, & \text{if } |\alpha b|\leq 1 \\ 1-1/|\beta|-1, & \text{if } |\alpha b|>1 \end{array}\right\}.$$

The sum of the last three displayed lines is equal to the sum of the second and third terms in the lemma. The fourth is directly obtained from $(\int_V)$, and the lemma follows. □

Our next aim is to compute the integral

$$\int_{|\alpha|\geq 1,|\beta|\geq 1} f(Ut)\psi(x+y)dx\,dy\,dz|\alpha\beta|d^\times\alpha d^\times\beta \tag{18.4.1}$$

which occurs in the expression of Lemma 18.3 for the integral $\Psi(-b,f,\psi)$ when $|b|>1$. We shall compute it, and the result of the computation is a sum of many terms which do not fit together in an easily expressed formula. In retrospect it is then natural and non-surprising that our method of proof will be to cut the domain of integration into as many subdomains as necessary (they will be denoted by (1), (2), ...), and then perform the integration.

(1) If $|y|\leq 1$, $|z|\leq 1$ and $|x|\leq 1$, then the argument of $f$ is $\begin{pmatrix}\alpha\beta & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}$, and we obtain

$$\int_{|\alpha|,|\beta|\geq 1} f\left(\begin{pmatrix}\alpha\beta & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\right)|\alpha\beta|d^\times\alpha d^\times\beta.$$

(2) If $|y|\le 1$, $|z|\le 1$, $|x|=q$, we obtain

$$\begin{aligned}&\int_{|\beta|\ge q,|\alpha|\ge 1} f\left(\begin{pmatrix}\alpha\beta\mathbf{q} & & 0\\ & \beta/\mathbf{q} & \\ 0 & & 1\end{pmatrix}\right)\psi(x)|\alpha\beta|dx\,d^\times\alpha d^\times\beta\\ &+\int_{|\alpha|\ge 1} f\left(\begin{pmatrix}\alpha\mathbf{q}^2 & & 0\\ & \mathbf{q} & \\ 0 & & 1\end{pmatrix}\right)\psi(x)|\alpha|dxd^\times\alpha\\ &=-q^{-1}\int_{|\alpha|\ge q^2,|\beta|\ge 1} f\left(\begin{pmatrix}\alpha\beta & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\right)|\alpha\beta|d^\times\alpha d^\times\beta\\ &-q^{-2}\int_{|\alpha|\ge q^2} f\left(\begin{pmatrix}\alpha & & 0\\ & \mathbf{q} & \\ 0 & & 1\end{pmatrix}\right)|\alpha|d^\times\alpha.\end{aligned}$$

We write $\mathbf{q}$ for $\boldsymbol{\pi}^{-1}$, thus $|\mathbf{q}|=q$.

(3) If $|y|\le 1$ and $|z|\ge q\ge |x|$, the argument of $f$ is $\begin{pmatrix}\alpha\beta z & & 0\\ & \beta & \\ 0 & & 1/z\end{pmatrix}\equiv\begin{pmatrix}\alpha\beta z^2 & & 0\\ & \beta z & \\ 0 & & 1\end{pmatrix}$, and the integral factorizes through $\int\psi(x)dx$, $|x|\le q$, which is 0.

*Remark.* When $|y|>1$ the integral takes the form

$$\begin{aligned}&\int f\left(\begin{pmatrix}1 & & 0\\ & \varepsilon & \\ 0 & & \eta\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ z & 1 & 0\\ z/y-x & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\beta & & 0\\ & \beta y & \\ 0 & & 1/y\end{pmatrix}\begin{pmatrix}1 & & 0\\ & 1/\varepsilon & \\ 0 & & 1/\eta\end{pmatrix}\right)\psi(x+y)|\alpha\beta|\\ &=\int f\left(\begin{pmatrix}1 & 0 & 0\\ \varepsilon z & 1 & 0\\ \eta z/y-\eta x & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\beta & & 0\\ & \beta y & \\ 0 & & 1/y\end{pmatrix}\right)\psi(x+y)|\alpha\beta|,\end{aligned}$$

for any $|\varepsilon|=1=|\eta|$. Integrating over $\varepsilon=\eta$ in this domain we conclude that $|x|\le q$ (the integral over $|x|>q$ is zero). Integrating over $\eta$ and fixing $\varepsilon=1$ we conclude that it suffices to integrate over $|y|\le q$. We use this remark to list the next domains.

As we have dealt in (1), (2), (3) with $|y|\le 1$, we assume in all the following cases that $|y|=q$.

(4) If ($|y| = q$ and) $|z| > q^2$, $|x| \leq q$, the argument of $f$ can be taken to be

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ x - z/y & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta & & 0 \\ & \beta y & \\ 0 & & 1/y \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z/y - x & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta z & & 0 \\ & \beta y/z & \\ 0 & & 1/y \end{pmatrix}$$

$$\equiv \begin{pmatrix} \alpha\beta z & & 0 \\ & \beta & \\ 0 & & 1/z \end{pmatrix} \equiv \begin{pmatrix} \alpha\beta z^2 & & 0 \\ & \beta z & \\ 0 & & 1 \end{pmatrix} \quad \text{since} \quad z/y - x = (z/y)(1 - xy/z).$$

But then the integral factorizes through $\int_{|x| \leq q} \psi(x) dx = 0$.

(5) If $|z| = q^2$ and $|z/y - x| \leq 1$ then ($|x| = q$ and) the argument of $f$ is $\operatorname{diag}(\alpha\beta yz, \beta y^2/z, 1)$. Also $\psi(x+y) = \psi(y + z/y)$, and the integral is easily computed, after a change $z \mapsto zy$, to be

$$q \int_{|\alpha| \geq 1, |\beta| \geq 1} f\left(\begin{pmatrix} \alpha\beta\mathbf{q}^2 & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right) |\alpha\beta| d^\times\alpha d^\times\beta$$

$$= q^{-2} \int_{|\alpha| \geq q^3, |\beta| \geq 1} f\left(\begin{pmatrix} \alpha\beta & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right) |\alpha\beta| d^\times\alpha d^\times\beta.$$

(6) If $|z| \leq q = |x|$, the value of $f$ in the integral is

$$f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta x & & 0 \\ & \beta y & \\ 0 & & 1/xy \end{pmatrix}\right) = f\left(\begin{pmatrix} \alpha\beta x^2 y & & 0 \\ & \beta x y^2 & \\ 0 & & 1 \end{pmatrix}\right).$$

The integrand contains also a factor $\psi(x+y)$. Integrating over $|x| = q = |y|$ and $|z| \leq q$ we obtain

$$q^{-2} \int_{|\alpha| \geq 1, |\beta| \geq q^3} f\left(\begin{pmatrix} \alpha\beta & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right) |\alpha\beta| d^\times\alpha \, d^\times\beta.$$

(7) If $|z| = q^2$ and $|z/y - x| = q$ write $u = x - z/y$, and note that the integral is

$$\int_{\substack{|u|=q, |y|=q, |z|=q^2 \\ |\alpha| \geq 1, |\beta| \geq 1}} f\left(\begin{pmatrix} \alpha\beta z & & 0 \\ & \beta y u/z & \\ 0 & & 1/uy \end{pmatrix}\right) \psi(y + \frac{z}{y} + u) |\alpha\beta| d^\times\alpha d^\times\beta \, du \, dy \, dz$$

$$= -q \int_{|\alpha| \geq 1, |\beta| \geq 1} f\left(\begin{pmatrix} \alpha\beta\mathbf{q}^4 & & 0 \\ & \beta\mathbf{q}^2 & \\ 0 & & 1 \end{pmatrix}\right) |\alpha\beta| d^\times\alpha d^\times\beta$$

$$= -q^{-3} \int_{|\beta| \geq q^2, |\alpha| \geq q^2} f\left(\begin{pmatrix} \alpha\beta & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right) |\alpha\beta| d^\times\alpha d^\times\beta.$$

(8) If $|z| = q$ and $|x| \leq 1$, the integrand is $f\left(\begin{pmatrix} \alpha\beta z & & 0 \\ & \beta y/z & \\ 0 & & 1/y \end{pmatrix}\right)\psi(y)$, and the integral is

$$- (q-1)\int_{|\alpha|,|\beta|\geq 1} f\left(\begin{pmatrix} \alpha\beta\mathbf{q}^2 & & 0 \\ & \beta\mathbf{q} & \\ 0 & & 1 \end{pmatrix}\right)|\alpha\beta|d^\times\alpha d^\times\beta$$
$$= -\frac{1}{q}(1-\frac{1}{q})\int_{|\alpha|,|\beta|\geq q} f\left(\begin{pmatrix} \alpha\beta & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right)|\alpha\beta|d^\times\alpha d^\times\beta.$$

(9) If $|z| \leq 1$ and $|x| \leq 1$ then the integrand is $f\left(\begin{pmatrix} \alpha\beta y & & 0 \\ & \beta y^2 & \\ 0 & & 1 \end{pmatrix}\right)\psi(y)$, and the integral is then

$$-\int_{|\alpha|\geq q,|\beta|\geq 1} f\left(\begin{pmatrix} \alpha\beta\boldsymbol{\pi} & & 0 \\ & \beta\mathbf{q}^2 & \\ 0 & & 1 \end{pmatrix}\right)|\alpha\beta|d^\times\alpha d^\times\beta - \int_{|\beta|\geq 1} f\left(\begin{pmatrix} \beta\mathbf{q}^2 & & 0 \\ & \beta\mathbf{q} & \\ 0 & & 1 \end{pmatrix}\right)|\beta|d^\times\beta$$
$$= -\frac{1}{q}\int_{|\alpha|\geq 1,|\beta|\geq q^2} f\left(\begin{pmatrix} \alpha\beta & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right)|\alpha\beta|d^\times\alpha d^\times\beta - \frac{1}{q}\int_{|\beta|\geq q} f\left(\begin{pmatrix} \beta\mathbf{q} & & 0 \\ & \beta & \\ 0 & & 1 \end{pmatrix}\right)|\beta|d^\times\beta.$$

Next we add up (1) – (9). We write $f(a,b)$ for $f\left(\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & 1 \end{pmatrix}\right)$. The main term is the product of

(18.4.2) $$\int_{|\alpha|\geq 1}\int_{|\beta|\geq 1} f(\alpha\beta,\beta)|\alpha\beta|d^\times\alpha d^\times\beta$$

and

$$1 - \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^2} - \frac{1}{q^3} - \frac{1}{q} + \frac{1}{q^2} - \frac{1}{q} = (1-\frac{1}{q})^3.$$

The other terms are

$$\frac{1}{q}\int_{|\beta|\geq 1} f(\beta,\beta)|\beta|d^\times\beta + \int_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta|d^\times\beta - \frac{1}{q^2}\int_{|\alpha|\geq q^2} f(\alpha,\mathbf{q})|\alpha|d^\times\alpha$$

from (2),

$$-\frac{1}{q^2}\int_{|\beta|\geq 1} f(\beta,\beta)|\beta|d^\times\beta - \frac{1}{q}\int_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta|d^\times\beta - \int_{|\beta|\geq 1} f(\beta\mathbf{q}^2,\beta)|\beta|d^\times\beta$$

from (5),

$$-\frac{1}{q^2}\int_{|\alpha|\geq 1} f(\alpha,1)|\alpha|d^\times\alpha - \frac{1}{q^2}\int_{|\alpha|\geq q} f(\alpha,\mathbf{q})|\alpha|d^\times\alpha - \frac{1}{q^2}\int_{|\alpha|\geq q^2} f(\alpha,\mathbf{q}^2)|\alpha|d^\times\alpha$$

from (6),

$$\begin{aligned} &q^{-3}\int_{|\beta|\geq q^2} f(\beta,\beta)|\beta|d^\times\beta + q^{-2}\int_{|\beta|\geq q^2} f(\beta\mathbf{q},\beta)|\beta|d^\times\beta \\ &+ q^{-3}\int_{|\alpha|\geq 1} f(\alpha,1)|\alpha|d^\times\alpha + q^{-3}\int_{|\alpha|\geq q} f(\alpha,\mathbf{q})|\alpha|d^\times\alpha \end{aligned}$$

from (7),

$$q^{-1}(1-q^{-1})\int_{|\beta|\geq q} f(\beta\beta)|\beta|d^\times\beta + q^{-1}(1-q^{-1})\int_{|\alpha|\geq 1} f(\alpha,1)|\alpha|d^\times\alpha$$

from (8), and

$$q^{-1}\int_{|\alpha|\geq 1} f(\alpha,1)|\alpha|d^\times\alpha + q^{-1}\int_{|\alpha|\geq q} f(\alpha,\mathbf{q})|\alpha|d^\times\alpha + q^{-1}f(\mathbf{q},1) - q^{-1}\int_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta|d^\times\beta$$

from (9).

These "other terms" add up to:

$$\begin{aligned} &\left(\frac{2}{q}-\frac{2}{q^2}+\frac{1}{q^3}\right)\int_{|\beta|\geq 1} f(\beta,\beta)|\beta|d^\times\beta + (1-q^{-1})^2\int_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta|d^\times\beta \\ &+ q^{-1}(1-q^{-1})^2\int_{|\alpha|\geq 1} f(\alpha,\mathbf{q})|\alpha|d^\times\alpha \\ &+ \left(\frac{2}{q}-\frac{2}{q^2}+\frac{1}{q^3}\right)\int_{|\alpha|\geq 1} f(\alpha,1)|\alpha|d^\times\alpha \\ &- \int_{|\beta|\geq 1} f(\beta\mathbf{q}^2,\beta)|\beta|d^\times\beta - q^{-2}\int_{|\alpha|\geq 1} f(\alpha,\mathbf{q}^2)|\alpha|d^\times\alpha \\ &+ (-q^{-1}+q^{-2}-q^{-3})f(1,1) + (q^{-1}-q^{-2})f(\mathbf{q},\mathbf{q}) \\ &+ q^{-2}f(\mathbf{q}^2,1) + (q^{-2}-q^{-3})f(\mathbf{q},1). \end{aligned} \tag{18.4.3}$$

The result of this computation is the following

**18.4 Lemma.** *The integral (18.4.1) is equal to the sum of (18.4.3) and the product of* $(1-q^{-1})^3$ *and (18.4.2).* □

We shall now continue to study the assertion of Proposition 18 for $|b|>1$, by considering the sum over $m$. A standard change of variables implies that

$$F_f\left(\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}\right)$$

$$= \left|\frac{(a-b)(b-c)(a-c)}{abc}\right| \int f\left(\begin{pmatrix} 1 & -x & xz-y \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\right) dx\,dy\,dz$$

is equal to

$$\left|\frac{a}{c}\right| \int f\left(\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\right) dx\,dy\,dz$$

$$= \left|\frac{a}{c}\right| \int f\left(\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}\right) dx\,dy\,dz.$$

We shall then write the sum over $m$ of Proposition 18 (for $|b| > 1$) in the form

$$\sum_{m\geq 0} (q^{m+1} I_m - q^{m+2} J_m).$$

Here

$$I_m = \iiint f\left(\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^{m+1} & & 0 \\ & \mathbf{q}^{m+1} & \\ 0 & & 1 \end{pmatrix}\right) dx\,dy\,dz$$

is the sum of

$$\iint f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^{m+1} & & 0 \\ & \mathbf{q}^{m+1} & \\ 0 & & 1 \end{pmatrix}\right) dy\,dz \qquad (|x| \leq 1)$$

and

$$\int_{|x|>1} \iint f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} x\mathbf{q}^{m+1} & & 0 \\ & \mathbf{q}^{m+1}/x & \\ 0 & & 1 \end{pmatrix}\right)$$

$$= q \int_{|x|\geq 1} f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} x\mathbf{q}^{m+2} & & 0 \\ & \mathbf{q}^{m}/x & \\ 0 & & 1 \end{pmatrix}\right).$$

Also

$$J_m = \iiint f\left(\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^{m+2} & & 0 \\ & \mathbf{q}^{m} & \\ 0 & & 1 \end{pmatrix}\right)$$

$$= \iint f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^{m+2} & & 0 \\ & \mathbf{q}^{m} & \\ 0 & & 1 \end{pmatrix}\right)$$

$$+ \int_{|x|>1} \iint f\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\right)\left(\begin{pmatrix} x\mathbf{q}^{m+2} & & 0 \\ & \mathbf{q}^{m}/x & \\ 0 & & 1 \end{pmatrix}\right).$$

Hence

$$q^{m+1}I_m - q^{m+2}J_m = q^{m+1}\int f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\mathbf{q}^{m+1}&&0\\&\mathbf{q}^{m+1}&\\0&&1\end{pmatrix}\right)$$

$$+ q^{m+2}(1-q^{-1})\int f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\mathbf{q}^{m+2}&&0\\&q^{m}&\\0&&1\end{pmatrix}\right)$$

$$- q^{m+2}\int f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\mathbf{q}^{m+2}&&0\\&\mathbf{q}^{m}&\\0&&1\end{pmatrix}\right)$$

$$= q^{m+1}\int f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\mathbf{q}^{m+1}&&0\\&\mathbf{q}^{m+1}&\\0&&1\end{pmatrix}\right)$$

$$- q^{m+1}\int f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\mathbf{q}^{m+2}&&0\\&\mathbf{q}^{m}&\\0&&1\end{pmatrix}\right).$$

We conclude

**18.5 Lemma.** *The sum* $\sum_m [F((m+1,m+1,0),f) - F(m+2,m,0),f)]$ *of Proposition 18 is equal to* $A-B$*, where*

$$A = \iiint f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\beta&&0\\&\beta&\\0&&1\end{pmatrix}\right)|\beta|d^{\times}\beta\,dy\,dz$$

*and*

$$B = \iiint f\left(\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\beta\mathbf{q}&&0\\&\beta/\mathbf{q}&\\0&&1\end{pmatrix}\right)|\beta|d^{\times}\beta\,dy\,dz. \qquad \square$$

We shall now compute directly $A-B$. The result is:

**18.6 Lemma.** *The difference* $A-B$ *is the sum of*

$$\iint\limits_{|\alpha||\beta|\geq 1} f(\alpha\beta,\beta)|\alpha\beta|d^{\times}\alpha d^{\times}\beta \text{ times } [1-q^{-2}-q^{-1}(1-q^{-1})^2-2q^{-1}(1-q^{-1})] = (1-q^{-1})^3,$$

$$\int\limits_{|\beta|\geq 1} f(\beta,\beta)|\beta|d^{\times}\beta \text{ times } [q^{-2}+q^{-1}-2q^{-2}+q^{-3}+q^{-1}-q^{-2}] = 2q^{-1}-2q^{-2}+q^{-3},$$

$$\int\limits_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta|d^{\times}\beta \text{ times } [-q^{-1}(1-q^{-1})+(1-q^{-1})^2+(1-q^{-1})] = 2(1-q^{-1})^2,$$

$$\int\limits_{|\beta|\ge 1} f(\beta\mathbf{q}^2,\beta)|\beta|d^\times\beta \ \text{ times } [-q+q-1]=-1,$$

$$\int\limits_{|\alpha|\ge 1} f(\alpha,1)|\alpha|d^\times\alpha \ \text{ times } [q^{-2}+q^{-1}(1-q^{-1})^2+q^{-1}(1-q^{-1})]=q^{-1}[1+(1-q^{-1})^2],$$

$$\int\limits_{|\alpha|\ge 1} f(\alpha,\mathbf{q})|\alpha|d^\times\alpha \ \text{ times } \ q^{-1}(1-q^{-1})^2,$$

$$\int\limits_{|\alpha|\ge 1} f(\alpha,\mathbf{q}^2)|\alpha|d^\times\alpha \ \text{ times } [-q^{-1}+q^{-1}(1-q^{-1})]=-q^{-2},$$

$$f(1,1) \ \text{ times } [-q^{-2}+q^{-1}(1-q^{-1})^2-2q^{-1}(1-q^{-1})^2]=-q^{-1}+q^{-2}-q^{-3},$$
$$f(\mathbf{q},1) \ \text{ times } [q^{-1}-q^{-2}-(1-q^{-1})^2-q^{-1}(1-q^{-1})^2]=-(1-q^{-1})(1-q^{-1}-q^{-2})$$
$$f(\mathbf{q}^2,\mathbf{q}) \ \text{ times } [1-q^{-1}+1-q+q(1-q^{-1})^2-q(1-q^{-1})^2]=2-q-q^{-1},$$
$$f(\mathbf{q}^2,1) \ \text{ times } [q^{-1}-q^{-1}(1-q^{-1})]=q^{-2},$$

*and*

$$f(\mathbf{q},\mathbf{q}) \ \text{ times } \ q^{-1}(1-q^{-1}).$$

*Proof.* Let us compute $A$. First we evaluate the integrand on various subdomains of integration, denoted by (1(i)), ....

(1) Suppose that $|\beta|\ge 1$. (i) If $|y|$, $|z|\le 1$, then the integrand is $f(\beta,\beta)$ (as usual we write $f(a,b)$ for $f\left(\begin{pmatrix} a & & 0\\ & b & \\ 0 & & 1\end{pmatrix}\right)$.

(ii) If $|y|\ge\|(z,\mathbf{q})\|$ ($\|(a,b)\|=\max(|a|,|b|)$) then the integrand is

$$f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&z&1\end{pmatrix}\begin{pmatrix}1&0&1/y\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&&-1\\&1&\\1&&0\end{pmatrix}\begin{pmatrix}y&&0\\&1&\\0&&1/y\end{pmatrix}\right.$$
$$\left.\cdot\begin{pmatrix}1&&1/y\\&1&\\0&&1\end{pmatrix}\begin{pmatrix}\beta&&0\\&\beta&\\0&&1\end{pmatrix}\right)=f\left(\begin{pmatrix}1&-z&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\beta y&&0\\&\beta&\\0&&1/y\end{pmatrix}\right)=f(\beta y^2,\beta y).$$

(iii) If $|z|\ge\|(y,\mathbf{q})\|$, denoting by $r(12)$ a matrix representing the reflection (12), the integrand is

$$f\left(r(12)\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}\begin{pmatrix}\beta&&0\\&\beta&\\0&&1\end{pmatrix}r(12)\right)=f\left(\begin{pmatrix}1&0&0\\0&1&0\\z&y&1\end{pmatrix}\begin{pmatrix}\beta&&0\\&\beta&\\0&&1\end{pmatrix}\right)=f(\beta z^2,\beta z).$$

(2) Suppose that $|\beta|<1$. (i) If $|y|$, $|z|\le|\beta|^{-1}$, then the integrand is

$$f\left(\begin{pmatrix}\beta&&0\\&\beta&\\0&&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\\beta y&\beta z&1\end{pmatrix}\right)=f(\beta,\beta).$$

(ii) If $|y| \geq \|(z, \mathbf{q}/\beta)\|$ then the integrand is

$$f\left(\begin{pmatrix}\beta & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\begin{pmatrix}1 & & 1/\beta y\\ & 1 & \\ 0 & & 1\end{pmatrix}\begin{pmatrix}0 & & -1\\ & 1 & \\ 1 & & 0\end{pmatrix}\begin{pmatrix}\beta y & & 0\\ & 1 & \\ 0 & & 1/\beta y\end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix}1 & & 1/\beta y\\ & 1 & \\ 0 & & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & \beta y & 1\end{pmatrix}\right)$$

$$= f\left(\begin{pmatrix}1/y & & 0\\ & \beta & \\ 0 & & \beta y\end{pmatrix}\begin{pmatrix}1 & -\beta z & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right) = f\left(\begin{pmatrix}y & & 0\\ & 1 & \\ 0 & & 1/\beta y\end{pmatrix}\right) = f(\beta y^2, \beta y).$$

(iii) If $|z| \geq \|(y, \mathbf{q}/\beta)\|$, applying the reflection (12) to the pevious case we obtain that the integrand is $f(\beta z^2, \beta z)$.

We conclude that $A$ is:

$$A = \int_{|\beta|\geq 1} f(\beta,\beta)|\beta| d^\times\beta + (1+q^{-1}) \iint_{\substack{|z|>1\\ |\beta|\geq 1}} f(\beta z^2, \beta z)|\beta z| d^\times\beta\, dz$$

$$+ \int_{|\beta|<1} f(\beta\beta)|\beta|^{-1} d^\times\beta + (1+q^{-1}) \iint_{|z|>|\beta|^{-1}>1} f(\beta z^2, \beta z)|\beta z| d^\times\beta\, dz,$$

and this is the sum of

$$f(1,1) + \int_{|\beta|>1} [f(\beta,\beta) + f(\beta,1)]|\beta| d^\times\beta$$

and

$$(1-q^{-2}) \iint_{1<|\alpha|\leq|\beta|} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta + (1-q^{-2}) \iint_{|\alpha|>|\beta|>1} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta$$

$$= (1-q^{-2}) \iint_{|\alpha|,|\beta|>1} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta.$$

We shall write $B$ in the form I + II + III, where

$$\mathrm{I} = q \iiint_{|\beta|\geq 1} f\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ y & z & 1\end{pmatrix}\begin{pmatrix}\beta\mathbf{q}^2 & & 0\\ & \beta & \\ 0 & & 1\end{pmatrix}\right)|\beta| d^\times\beta\, dy\, dz;$$

$$\mathrm{II} = \iint f\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ y & z & 1\end{pmatrix}\begin{pmatrix}\mathbf{q}^2 & & 0\\ & 1 & \\ 0 & & \mathbf{q}\end{pmatrix}\right) dy\, dz;$$

$$\mathrm{III} = q \iiint_{|\beta|>q} f\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ y & z & 1\end{pmatrix}\begin{pmatrix}\mathbf{q}^2 & & 0\\ & 1 & \\ 0 & & \beta\end{pmatrix}\right)|\beta|^{-1} d^\times\beta.$$

Each of I, II, III will be expressed now as a sum of integrals on subdomains.
(I) We first deal with I, and denote the contributions from the various subdomains by (1), (2), ....

(1) If $|y|$, $|z| \leq 1$, the contribution to I is $q\int_{|\beta|\geq 1} f(\beta\mathbf{q}^2, \beta)|\beta| d^\times\beta$.

(2) On $|y| \geq \|(z, \mathbf{q})\|$, the integrand is $f(\operatorname{diag}(\beta\mathbf{q}^2 y, \beta, 1/y))$, and the contribution is

$$q(1-q^{-1}) \int_{|\beta|\geq 1} \int_{|\alpha|>1} f(\alpha^2\beta q^2, \alpha\beta)|\alpha^2\beta| d^\times\alpha d^\times\beta$$
$$= q^{-1}(1-q^{-1}) \int_{|\alpha|>q^2} \int_{|\beta|\geq|\alpha|q^{-2}} f(\alpha\beta, \beta)|\alpha\beta| d^\times\alpha d^\times\beta.$$

(3) On $|z| > q|y|$, $|z| > q$, the integrand is $f(\operatorname{diag}(\beta\mathbf{q}^2, \beta z, 1/z))$, and we obtain

$$q(1-q^{-1}) \int_{|z|>q} \int_{|\beta|\geq 1} (|z|^2/q^2)|\beta| f(\beta z^2, \beta z q^2) d^\times\beta d^\times z$$
$$= q^{-1}(1-q^{-1}) \int_{|\alpha|\geq 1} \int_{|\beta|\geq q^4|\alpha|} f(\alpha\beta, \beta)|\alpha\beta| d^\times\alpha d^\times\beta.$$

(4) On $|z| = q > |y|$, the integrand is $f(\operatorname{diag}(\beta\mathbf{q}^2, \beta\mathbf{q}, 1/\mathbf{q}))$, and we obtain

$$q(q-1) \int_{|\beta|\geq 1} f(\beta\mathbf{q}^3, \beta\mathbf{q}^2)|\beta| d^\times\beta = q^{-1}(1-q^{-1}) \int_{|\beta|>q} f(\beta\mathbf{q}, \beta)|\beta| d^\times\beta.$$

(5) On $|z| = q|y| > q$ the integrand contains

$$f\left(\begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta\mathbf{q}^2 & & 0 \\ & \beta\mathbf{q}y & \\ 0 & & 1/\mathbf{q}y \end{pmatrix}\right).$$

The intgrand over this subdomain is equal to

$$q(q-1)(1-q^{-1}) \int_{|\beta|\geq 1} \int_{|y|>1} f(\beta y^2\mathbf{q}^3, \beta y\mathbf{q}^2)|\beta y^2| d^\times\beta d^\times y$$
$$= q^{-1}(1-q^{-1})^2 \int_{|\alpha|>q} \int_{|\beta|\geq|\alpha|q} f(\alpha\beta, \beta)|\alpha\beta| d^\times\alpha d^\times\beta.$$

(II) Next we compute II, again by splitting into subdomains.
(1) If $|y| \leq 1$ and $|z| \leq q$, the integral is $qf(\mathbf{q}^2, \mathbf{q})$.

(2) On $|y| \geq \|(z, \mathbf{q})\|$ the value of $f$ in the integrand is

$$f\left(\begin{pmatrix} 1 & -z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^2 y & & 0 \\ & 1 & \\ 0 & & \mathbf{q}/y \end{pmatrix}\right) = f(y^2\mathbf{q}, y/\mathbf{q}),$$

and the integral is

$$(1-q^{-1}) \int\limits_{|\beta|>1} f(\beta^2\mathbf{q}, \beta/\mathbf{q})|\beta|^2 d^\times\beta = q^2(1-q^{-1}) \int\limits_{|\beta|\geq 1} f(\beta^2\mathbf{q}^3, \beta)|\beta|^2 d^\times\beta.$$

(3) On $|z| > \|(\mathbf{q}y, \mathbf{q})\|$ the integrand contains

$$f\left(\begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^2 & & 0 \\ & z & \\ 0 & & \mathbf{q}/z \end{pmatrix}\right) = f\left(\begin{pmatrix} z^2/\mathbf{q} & & 0 \\ & z\mathbf{q} & \\ 0 & & 1 \end{pmatrix}\right),$$

and the integral is

$$(1-q^{-1}) \int\limits_{|z|\geq q^2} (|z|^2/q^2) f(z^2/\mathbf{q}, z\mathbf{q}) d^\times z = q^{-4}(1-q^{-1}) \int\limits_{|\beta|\geq q^3} f(\beta^2 q^{-3}, \beta)|\beta|^2 d^\times\beta.$$

(4) On $|z| = q|y| > q$ the integrand contains $f(\operatorname{diag}(\mathbf{q}^2 y, \mathbf{q}, 1/y))$, and we obtain

$$(1-q^{-1})(q-1) \int\limits_{|y|>1} f(y^2\mathbf{q}^2, y\mathbf{q})|y|^2 d^\times y = q^{-1}(1-q^{-1})^2 \int\limits_{|\beta|>q} f(\beta^2, \beta)|\beta|^2 d^\times\beta.$$

(III) Finally we compute the integral III, again by partition into subdomains. Recall that in III $\beta$ ranges over $|\beta| \geq q^2$.
(1) On $|y| \leq |\beta|/q^2$, $|z| \leq |\beta|$, the integral is $q^{-1} \int\limits_{|\beta|\geq q^2} f(\beta, \mathbf{q}^2)|\beta| d^\times\beta$.
(2) On $|y| \geq \|(z, \beta/\mathbf{q})\|$, the value of $f$ is

$$f\left(\begin{pmatrix} 1 & -z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^2 y & & 0 \\ & 1 & \\ 0 & & \beta/y \end{pmatrix}\right) = f(y^2\mathbf{q}^2/\beta, y/\beta),$$

and the integral is

$$\begin{aligned} q(1-q^{-1}) &\int\limits_{|\beta|\geq q^2} \int\limits_{|y|\geq 1/q} f(y^2\beta\mathbf{q}^2, y)|y^2\beta| d^\times\beta d^\times y \\ = q^{-1}(1-q^{-1}) &\int\limits_{|\beta|\geq 1/q} \int\limits_{|\alpha|\geq q^4|\beta|} f(\alpha\beta, \beta)|\alpha\beta| d^\times\alpha d^\times\beta. \end{aligned}$$

(3) On $|z| > \|(\beta, \mathbf{q}y)\|$ the value of $f$ is

$$f\left(\begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{q}^2 & & 0 \\ & z & \\ 0 & & \beta/z \end{pmatrix}\right) = f(z^2/\beta, z\mathbf{q}^2/\beta),$$

and the integral is

$$q(1-q^{-1}) \int\limits_{|z|>1} \int\limits_{|\beta|\geq q^2|z|} f(z\beta, z\mathbf{q}^2)(|z\beta|/q^2) d^\times\beta d^\times z$$
$$= q^{-1}(1-q^{-1}) \int\limits_{|\beta|>q^2} \int\limits_{|\alpha|\geq q^{-2}|\beta|} f(\alpha\beta, \beta)|\alpha\beta| d^\times\alpha d^\times\beta$$

(4) On $|z| = q|y| \geq |\beta|$ the value of $f$ is

$$f\left(\begin{pmatrix} \mathbf{q}^2 y & & 0 \\ & \mathbf{q} & \\ 0 & & \beta/\mathbf{q}y \end{pmatrix}\right) = f(y^2\mathbf{q}^3/\beta, y\mathbf{q}^2/\beta) \ (= f(\mathbf{q}^2 y, \mathbf{q}) \text{ if } |z| = q|y| = |\beta|),$$

and the integral is

$$q(1-q^{-1})(q-1) \int\limits_{|y|\geq 1/q} \int\limits_{|\beta|\geq q^2|\beta|} f(y\mathbf{q}^3\beta, y\mathbf{q}^2)|y\beta| d^\times\beta d^\times y$$
$$= q^{-1}(1-q^{-1})^2 \int\limits_{|\beta|>1} \int\limits_{|\alpha|\geq q|\beta|} f(\alpha\beta, \beta)|\alpha\beta| d^\times\alpha d^\times\beta.$$

We now come to putting together the expression for $A - B$. This is merely a matter of careful bookkeeping. Then $A - B$ is the sum of

$$\int\limits_{|\beta|\geq 1} f(\beta,\beta)|\beta| d^\times\beta + \int\limits_{|\alpha|\geq 1} f(\alpha,1)|\alpha| d^\times\alpha - q \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q}^2,\beta)|\beta| d^\times\beta$$

$$- q^{-1}(1-q^{-1}) \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta| d^\times\beta + q^{-1}(1-q^{-1}) f(\mathbf{q},1) + (1-q^{-1}) f(\mathbf{q}^2,\mathbf{q})$$

$$- q^{-1} \int\limits_{|\alpha|\geq 1} f(\alpha,\mathbf{q}^2)|\alpha| d^\times\alpha + q^{-1} f(\mathbf{q}^2,1) + f(\mathbf{q}^2,\mathbf{q}) - q f(\mathbf{q}^2,\mathbf{q})$$

$$- q^2(1-q^{-1}) \int\limits_{|\beta|\geq 1} f(\beta^2\mathbf{q}^3,\beta)|\beta|^2 d^\times\beta - q^2(1-q^{-1}) \int\limits_{|\beta|\geq 1} f(\beta^2\mathbf{q}^3,\beta\mathbf{q}^3)|\beta|^2 d^\times\beta$$

$$- q^{-1}(1-q^{-1})^2 \int\limits_{|\beta|\geq 1} f(\beta^2,\beta)|\beta|^2 d^\times\beta + q^{-1}(1-q^{-1})^2 f(1,1) + q(1-q^{-1})^2 f(\mathbf{q}^2,\mathbf{q})$$

$$+ (1-q^{-2}) \iint\limits_{\substack{|\alpha|\geq 1\\|\beta|\geq 1}} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta - (1-q^{-2}) \int\limits_{|\beta|\geq 1} f(\beta,\beta)|\beta| d^\times\beta$$

$$- (1-q^{-2}) \int\limits_{|\alpha|\geq 1} f(\alpha,1)|\alpha| d^\times\alpha - q^{-2} f(1,1),$$

the product of $-q^{-1}(1-q^{-1})^2$ and

$$\left[\int\limits_{|\alpha|>q} \int\limits_{|\beta|>|\alpha|} + \int\limits_{|\beta|>1} \int\limits_{|\alpha|>|\beta|}\right] f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta, \tag{18.6.1}$$

and the product of $-q^{-1}(1-q^{-1})$ and the integral, which we name (18.6.2), of $f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta$ over

$$\left[\int\limits_{|\alpha|>q^2} \int\limits_{|\beta|\geq|\alpha|q^{-2}} + \int\limits_{|\beta|>q^2} \int\limits_{|\alpha|\geq|\beta|q^{-2}} + \int\limits_{|\alpha|\geq 1} \int\limits_{|\beta|\geq q^4|\alpha|} + \int\limits_{|\beta|\geq 1/q} \int\limits_{|\alpha|\geq q^4|\beta|}\right].$$

The integral (18.6.1) is equal to

$$\iint\limits_{|\alpha|,|\beta|\geq 1} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta - \int\limits_{|\alpha|=|\beta|\geq 1} f(\beta^2,\beta)|\beta|^2 d^\times\beta - \int\limits_{|\alpha|\geq 1} f(\alpha,1)|\alpha| d^\times\alpha$$

$$- \int\limits_{|\beta|\geq 1} f(\beta,\beta)|\beta| d^\times\beta - q \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta| d^\times\beta + q^2 f(\mathbf{q}^2,\mathbf{q}) + q f(\mathbf{q},1) + 2f(1,1).$$

The domains of the four integrals of (18.6.2) can be split as follows.
(1) $|\beta| \geq |\alpha| > q^2$, union with (i) $q|\beta| = |\alpha| > q^2$, (ii) $q^2|\beta| = |\alpha| > q^2$.
(2) $|\alpha| > |\beta| > q^2$, union with (iii) $|\alpha| = |\beta| > q^2$, (iv) $q|\alpha| = |\beta| > q^2$, and (v) $q^2|\alpha| = |\beta| > q^2$.
(3) $|\beta| \geq |\alpha| \geq 1$, minus (iii) $|\beta| = |\alpha| \geq 1$, (iv) $|\beta| = q|\alpha| \geq q$, (v) $|\beta| = q^2|\alpha| \geq q^2$, and $|\beta| = q^3|\alpha| \geq q^3$.
(4) $|\alpha| > |\beta| \geq 1$, union $|\alpha| \geq q^3$, $|\beta| = 1/q$, minus (i) $|\alpha| = q|\beta| \geq q$, (ii) $|\alpha| = q^2|\beta| \geq q^2$, and $|\alpha| = q^3|\beta| \geq q^3$.
We use the labels (i) – (v) to relate subdomains of the various integrals.

The union of the main subdomains in (1) and (2) is $|\alpha| > q^2$, $|\beta| > q^2$, which is $|\alpha| \geq 1$, $|\beta| \geq 1$, minus (5): $[|\alpha| = 1, |\beta| \geq 1]$, $[|\alpha| = q, |\beta| \geq 1]$, $[|\alpha| = q^2, |\beta| \geq 1]$, $[|\alpha| > q^2, |\beta| = 1]$, $[|\alpha| > q^2, |\beta| = q]$, $[|\alpha| > q^2, |\beta| = q^2]$. The union of the main subdomains in (3), (4) is $|\alpha| \geq 1$, $|\beta| \geq 1$. We conclude that (18.6.2) is equal to:

$$
\begin{aligned}
&2 \iint\limits_{|\alpha|,|\beta|\geq 1} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta \\
&- [\int\limits_{|\beta|\geq 1} f(\beta,\beta)|\beta| d^\times\beta + q \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta| d^\times\beta + q^2 \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q}^2,\beta)|\beta| d^\times\beta \\
&+ \int\limits_{|\alpha|>q^2} f(\alpha,1)|\alpha| d^\times\alpha + \int\limits_{|\alpha|>q^3} f(\alpha,\mathbf{q})|\alpha| d^\times\alpha + \int\limits_{|\alpha|>q^4} f(\alpha,\mathbf{q}^2)|\alpha| d^\times\alpha]_{(5)} \\
&- [q^3 f(\mathbf{q}^3,\mathbf{q}) + q f(\mathbf{q},1)]_{(i)} - q^2 f(\mathbf{q}^2,1)_{(ii)} - [q^4 f(\mathbf{q}^4,\mathbf{q}^2) + q^2 f(\mathbf{q}^2,\mathbf{q}) + f(1,1)]_{(iii)} \\
&- [q^3 f(\mathbf{q}^3,\mathbf{q}^2) + q f(\mathbf{q},\mathbf{q})]_{(iv)} - q^2 f(\mathbf{q}^2,\mathbf{q}^2)_{(v)} \\
&- q^3 \int\limits_{|\beta|\geq 1} f(\beta^2\mathbf{q}^3,\beta\mathbf{q}^3)|\beta|^2 d^\times\beta_{(3)} - q^3 \int\limits_{|\beta|\geq 1} f(\beta^2\mathbf{q}^3,\beta)|\beta|^2 d^\times\beta_{(4)} \\
&+ q^{-1} \int\limits_{|\alpha|\geq q^3} f(\alpha,\mathbf{q})|\alpha| d^\times\alpha_{(4)}.
\end{aligned}
$$

The indices (i)–(v), (3)–(5) in the expressions above are meant to indicate the origin of each expression. Let us rewrite (18.6.2) as follows.

$$
\begin{aligned}
&2 \iint\limits_{|\alpha|,|\beta|\geq 1} f(\alpha\beta,\beta)|\alpha\beta| d^\times\alpha d^\times\beta - q^3 \int\limits_{|\beta|\geq 1} f(\beta^2\mathbf{q}^3,\beta\mathbf{q}^3)|\beta|^2 d^\times\beta - q^3 \int\limits_{|\beta|\geq 1} f(\beta^2 q^3,\beta)|\beta|^2 d^\times\beta \\
&- \int\limits_{|\beta|\geq 1} f(\beta,\beta)|\beta| d^\times\beta - q \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q},\beta)|\beta| d^\times\beta - q^2 \int\limits_{|\beta|\geq 1} f(\beta\mathbf{q}^2,\beta)|\beta| d^\times\beta \\
&- \int\limits_{|\alpha|\geq 1} f(\alpha,1)|\alpha| d^\times\alpha - (1-q^{-1}) \int\limits_{|\alpha|\geq 1} f(\alpha,\mathbf{q})|\alpha| d^\times\alpha - \int\limits_{|\alpha|\geq 1} f(\alpha,\mathbf{q}^2)|\alpha| d^\times\alpha \\
&+ (1-q^{-1}) f(\mathbf{q},1) - f(\mathbf{q},\mathbf{q}) + f(\mathbf{q}^2,1).
\end{aligned}
$$

Inserting this expression for (18.6.2), and the one for (18.6.1), in the expression for $A - B$ where (18.6.1) and (18.6.2) were first introduced, we obtain Lemma 18.6. $\square$

Since (18.4.2) times $(1-q^{-1})^3$ is the first displayed expression in Lemma 18.6, and the sum of the other terms in Lemma 18.6 is equal to (18.4.3) for a function $f$ with $f(\mathbf{q},1) = 0 = f(\mathbf{q}^2,\mathbf{q})$, Proposition 18 follows for such $f \in \mathbb{H}$ for $|b| > 1$. Similar computations, which we do not record here, establish the required identity of Proposition 18 also for $|b| \leq 1$.

As explained in the lines leading to Proposition 18, we have that Proposition 16 follows too, at least for all $f \in \mathbb{H}$ which lie in the span of the characteristic functions of the $K$-double cosets $K$ diag $(\mathbf{q}^a, \mathbf{q}^b, 1)K$, $a \geq b$, with $(a,b) \neq (1,0)$ or $(2,1)$. $\square$

*Remark.* It is tempting to guess that Propositions 16 and 18 hold for all $f \in \mathbb{H}$, and that our computation of the coefficients of the terms $f(\mathbf{q},1)$ and $f(\mathbf{q}^2,\mathbf{q})$ are erroneous in (18.4.3) or in Lemma 18.6, but we have detected no error in these computations, and the validity of Proposition(s) 16 (and 18) for all $f \in \mathbb{H}$ with $f(\mathbf{q},1) = f(\mathbf{q}^2,\mathbf{q}) = 0$ suffices to establish all of our qualitative results.

## References

[F] Y. Flicker, Cyclic automorphic forms on a unitary group, *J. Math. Kyoto Univ.*, preprint.