On Distinguished Representations

Yuval Z. Flicker[1]

Abstract for Zentralblatt. Let E/F be a (separable) quadratic extension of global fields with $charF \neq 2$, and $\mathbb{A}_X, \mathbb{A}_X^\times$ the X-adeles, X-ideles. Put $\underline{G} = GL(2)$. A cuspidal $\underline{G}(\mathbb{A}_E)$-module π' is called *distinguished* if there is a form $\phi \in \pi'$ with $\int_{\underline{G}(F)\underline{Z}(\mathbb{A}_F)\backslash\underline{G}(\mathbb{A}_F)} \phi(h)dh \neq 0$. These π' are characterized in the author's paper "Twisted tensors and Euler products", Bull. Soc. Math. France 116(1988),295-313, by a property of their "twisted-tensor" L-function $L(s,\pi',r)$, that it has a pole at $s = 1$. In this paper the distinguished π' are characterized as the image of the unstable base-change lifting from the unitary group $U = \underline{U}(F) = U(2, E/F)$ of $g \in \underline{G}(E)$ with $\sigma(g) = g$. We put $\sigma(g) = w^t\overline{g}^{-1}w^{-1}, w = \text{antidiag}(1,-1), t =$ transpose, bar = galois action on entries of g. The underlying unstable base-change homomorphism b_κ goes from the dual group $\widehat{U} = \underline{G}(\mathbb{C}) \rtimes W_{E/F}$ of U to that $\widehat{G}' = (\underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})) \rtimes W_{E/F}$ of $G' = \underline{G}(E)$. The Weil group $W_{E/F}$ acts via its quotient $Gal(E/F)$, whose non-trivial element σ acts on the connected component $\widehat{G}'^0$ by $\sigma(g_1, g_2) = (\sigma g_2, \sigma g_1)$, and by $\sigma g = w^t g^{-1} w^{-1}$ on $\widehat{U}^0 = \underline{G}(\mathbb{C})$. The map b_κ is defined using a character κ of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}_F^\times$ has the kernel $F^\times N_{E/F}\mathbb{A}_E^\times$, as follows: $b_\kappa(g) = (g, \sigma g) \quad (g \in \underline{G}(\mathbb{C}))$; $\quad b_\kappa(z) = (\kappa(z), \kappa(\overline{z}))z \quad (z \in W_{E/E} = \mathbb{A}_E^\times/E^\times)$; $\quad b_\kappa(\sigma) = (I, -I)\sigma$. Here I = identity in $\underline{G}(\mathbb{C})$. Let ω be a unitary character of $\mathbb{A}_E^\times/E^\times N_{E/F}\mathbb{A}_E^\times$ which is non-trivial on $\mathbb{A}_F^\times$. Then $\omega'(z) = \omega(z/\overline{z})$ is a unitary character of $\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times$. Put $\kappa'(z) = \kappa(z/\overline{z})$. The main result, Theorem 1, asserts:

(1) Every distinguished cuspidal $\underline{G}(\mathbb{A}_E)$-module π' with central character $\omega'\kappa'^2$ is the unstable base-change lift (via b_κ) of a cuspidal non-degenerate $\underline{U}(\mathbb{A}_F)$-module π with central character ω. (2) Every cuspidal non-degenerate $\underline{U}(\mathbb{A}_F)$-module π with central character ω lifts via the unstable map b_κ either to a distinguished cuspidal $\underline{G}(\mathbb{A}_E)$-module π' with central character $\omega'\kappa'^2$, or to an induced $\underline{G}(\mathbb{A}_E)$-module $I(\mu_1, \mu_2)$, where μ_i are unitary characters of $\mathbb{A}_E^\times/\mathbb{A}_F^\times E^\times$ with $\mu_1 \neq \mu_2$ and $\mu_1\mu_2 = \omega'\kappa'^2$.

In the second half of the work, from Theorem 7 on, a local analogue of this global result, also in the context of $GL(2)$, is studied. The introduction formulates and motivates a conjectural generalization to the context of $GL(n)$ (including the case of $GL(1)$), which the author believes might be provable using the techniques employed here in the preliminary case of $GL(2)$. Indeed this conjecture is reduced in the author's preprint "Distinguished representations and a Fourier trace formula", Bull. Soc. Math. France (1992), to a local technical conjecture.

[1]Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210

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Introduction.

A representation π of a group G is called *distinguished* with respect to (wrt, or by) a subgroup H of G if there exists a non-zero H-invariant linear form on the space of π, namely the representation dual to π contains a copy of the trivial H-module. Although this notion in general, and in the context of finite and real groups, is of much interest, this paper will concentrate on an example, or rather a family thereof, which appears to be crucial and suggestive to further development of the subject, in the case of automorphic and admissible representations of $GL(n)$, $n = 2$. Let then F be a global field and G a reductive F-group, H an F-subgroup of G, Z_H the center of H, $\mathbb{A}$ the ring of adeles of F, and $\mathbb{H} = H(\mathbb{A})$, $H = H(F)$, $\mathbb{Z}_H = Z_H(\mathbb{A})$. An automorphic G-module π is called H-*distinguished* if there is a vector in the space of π (in its automorphic realization) which is integrable over the homogeneous space $\mathbb{Z}_H H\backslash\mathbb{H}$, and whose integral over this space is non-zero. Such π is then distinguished by $\mathbb{H}$ as an abstract representation of $\mathbb{G} = G(\mathbb{A})$, a linear form being given by $L(\phi) = \int_{H\mathbb{Z}_H\backslash\mathbb{H}} \phi(h)dh$. Clearly, if π is H-distinguished, its central character $\omega = \omega_\pi$ has to be trivial on $\mathbb{Z}_H$, but not necessarily on $\mathbb{Z} = Z(\mathbb{A})$, where Z is the center of G. Below when an automorphic π is said to be distinguished, it will be meant in the automorphic sense, unless otherwise specified; the reference to H will be dropped when H is fixed.

The simplest example is of $G = E^\times = GL(1,E)$ and $H = F^\times = GL(1,F)$, where E/F is a quadratic separable extension of local fields. If the character χ' of $E^\times$ is distinguished, then it is trivial on $F^\times$, and so by Hilbert Theorem 90 there exists a character χ on the unitary group $U(1,E/F) = \{z \in E^\times\,;\ z\overline{z} = 1\}$, which is equal to the norm-one subgroup $E^1 = \{z/\overline{z}; z \in E^\times\}$ of E, with $\chi'(z) = \chi(z/\overline{z})$. Namely the base-change lifting $\chi \to \chi'$ establishes a bijection from the set of $U(1,E/F)$-modules to the set of $GL(1,F)$-distinguished $GL(1,E)$-modules, respecting irreducibility and admissibility, and automorphicity in the analogous global case. As we shall see below, this simple example is very suggestive, and fundamental.

Let us consider also the dual example, where $G = GL(1,E) = E^\times$ and $H = U(1,E/F) = E^1$. If the character χ' of $E^\times$ is distinguished by (namely trivial on) E^1, then $\chi' = \overline{\chi}'$, where $\overline{\chi}'(z) = \chi'(\overline{z})$ and $z \to \overline{z}$ is the non-trivial automorphism of E over F. Consequently there exists a character χ of $F^\times = GL(1,F)$ with $\chi'(z) = \chi(z\overline{z})$ $(z \in E^\times)$. The character χ is determined uniquely on the norm subgroup $NE^\times = \{z\overline{z}\,;\ z \in E^\times\}$ of $F^\times$. Since $[F^\times : NE^\times] = 2$, χ is determined uniquely up to multiplication by the unique non-trivial character $\chi_{E/F}$ of $F^\times$ which is trivial on $NE^\times$. We conclude that the base-change-for-$GL(1)$ map $\chi \to \chi'$ yields a bijection from the set of $GL(1,F)$-modules, up to the equivalence $\chi \sim \chi\chi_{E/F}$, to the set of $U(1,E/F)$-distinguished $GL(1,E)$-modules, again respecting irreducibility and admissibility, and automorphicity in the analogous global case. This example is also generalizable; see Remark (4) at the end of this paper.

Another example is known when $G = GL(2,E)$ and $H = GL(2,F)$, and E/F is a separable quadratic extension of global fields. A simple one-page argument (see, e.g., [F1], p. 311) based on the theory of L-functions, shows:

0.1. Proposition. *A cuspidal (automorphic) $GL(2,\mathbb{A}_E)$-module π', with a trivial* *central character ω', is $GL(2,\mathbb{A}_F)$-distinguished if and only if it is the base-change lift (see [F2]) of a cuspidal $GL(2,\mathbb{A}_F)$-module π whose central character ω is the unique non-trivial character $\chi_{E/F}$ of $\mathbb{A}_F^\times$ which is trivial on $F^\times N_{E/F}\mathbb{A}_E^\times$; note that if π base-changes to π' then $\omega'(z) = \omega(z\overline{z})$, hence ω is necessarily trivial on $F^\times N_{E/F}\mathbb{A}_E^\times$ when $\omega' = 1$. Here $\mathbb{A}_E$ denotes the ring of adeles, and $\mathbb{A}_E^\times$ its multiplicative group of ideles, of E, and $\mathbb{A}_F$, $\mathbb{A}_F^\times$ the corresponding objects attached to F.*

This $GL(2)$-example seems, at first glance, to be unlike the ($G = GL(1,E)$, $H = GL(1,F)$)-example described above, where the H-distinguished G-modules are lifts from $U(1,E/F)$. This ambiguity is caused by having put the restrictive assumption that the central character ω' of π' be trivial on $\mathbb{A}_E^\times$. *The purpose of this paper is to show that in fact if a cuspidal $GL(2,\mathbb{A}_E)$-module is $GL(2,\mathbb{A}_F)$-distinguished, then it is a base-change from the unitary group $U(2,\mathbb{A}_E/\mathbb{A}_F)$* (see [F3]; since there are two different such base-change maps, described in [F3], this statement needs to be clarified, and this is done below). *Moreover, our results for $GL(2)$, and the technique of proof, suggest a conjecture in the case of $GL(n)$, and a method for its proof.* To state precisely our results for $GL(2)$, and to motivate the conjecture made below for $GL(n)$, we now recall some results of [F1] about $GL(n,\mathbb{A}_F)$-distinguished representations of $GL(n,\mathbb{A}_E)$, and their relations to L-functions.

Let $\pi' = \otimes_v \pi'_v$ be an (irreducible) automorphic representation of $\mathbb{G}' = G(\mathbb{A}_E)$, $G = GL(n)$. Then there is a finite set V of places of F containing the archimedean places and those which ramify in E, such that: for each place v' of E above a place $v \notin V$ of F, the component $\pi'_{v'}$ of π' at v' is unramified. Thus for each such v' there is an unramified character $(a_{ij}) \mapsto \prod_{1\le i\le n} \mu_{iv'}(a_{ii})$ of the upper triangular subgroup $B(E_{v'})$ of $G(E_{v'})$, and $\pi_{v'}$ is the unique unramified (irreducible) constituent in the composition series of the (unramified) representation $I((\mu_{iv'}))$ unitarily induced from $(\mu_{iv'})$. Let $\underline{\pi} = \underline{\pi}_v$ be a uniformizer of F_v. Denote by $t_{v'} = t(\pi_{v'})$ the semi-simple conjugacy class in $GL(n,\mathbb{C})$ with eigenvalues $(\mu_{iv'}(\underline{\pi}))$. For each v' the map $\pi_{v'} \mapsto t(\pi_{v'})$ is a bijection from the set of equivalence classes of irreducible unramified $G(E_{v'})$-modules to the set of semi-simple conjugacy classes in $G(\mathbb{C})$. Denote by σ the non-trivial element in the galois group $\mathrm{Gal}(E/F)$. Put $\widehat{G}'$ for the semi-direct product $(G(\mathbb{C}) \times G(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)$, where σ acts by $\sigma(x,y) = (y,x)$. If $v \notin V$ splits into v', v'' in E, the component $\pi'_v = \pi'_{v'} \times \pi'_{v''}$ defines a conjugacy class $t_v = t(\pi'_v) = (t_{v'} \times t_{v''}) \times 1$ in $\widehat{G}'_v = G(\mathbb{C}) \times G(\mathbb{C})$. If $v \notin V$ is inert in E, and v' is the place of E above v, then we put π'_v for $\pi'_{v'}$. This π'_v defines a conjugacy class $t_{v'}$ in $G(\mathbb{C})$, and a conjugacy class $t_v = (t_{v'} \times 1) \times \sigma$ in $\widehat{G}'$. Given a finite dimensional complex representation r of $\widehat{G}'$, introduce the partial L-function

$$L(s,r,\pi,V) = \prod_{v\notin V} \det(1 - q_v^{-s} r(t_v))^{-1}$$

in the complex parameter s. Here q_v is the cardinality of the residue field $R_v/\underline{\pi}_v R_v$ of the ring R_v of integers in the completion F_v of F at v.

The work of [F1] concerns the n^2-dimensional "twisted-tensor" representation r of $\widehat{G}'$ on $\mathbb{C}^n \otimes \mathbb{C}^n$, defined by $r((a,b))(x\otimes y) = ax \otimes by$ and $r(\sigma)(x\otimes y) = y\otimes x$. Recall that the automorphic $G(\mathbb{A}_E)$-module π' is called distinguished by $G(\mathbb{A}_F)$ if its central character ω' is trivial on $\mathbb{A}_F^\times$ and there is an automorphic form ϕ in the space of π' in $L^2(G(E)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_E))$ whose integral $\int \phi(g)dg$ over the closed subspace $G(F)Z(\mathbb{A})\backslash G(\mathbb{A}_F)$ of $G(E)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_E)$ is non-zero. The main theorem of [F1] asserts that, when every archimedean place of F splits in E, and π' is cuspidal, we have

0.2. Proposition. *The product $L(s,r,\pi',V)$ converges absolutely, uniformly in compact subsets of some right half plane of $s\in\mathbb{C}$, has analytic continuation as a meromorphic function in $Re(s)\geq 1$, and the only possible pole of $L(s,r,\pi',V)$ in $Re(s)\geq 1$ is simple, located at $s=1$. Moreover, $L(s,r,\pi',V)$ has a pole at $s=1$ precisely when π' is distinguished.*

This characterization of distinguished representations by means of L-functions suggests the following procedure. Cuspidal $G(\mathbb{A}_E)$-modules π' are conjecturally parametrized, by the "principle of functoriality" (see [L] or [B′]), by representations $\rho' : W_F \to \widehat{G}'$ of the Weil group (see [T]) of F. Then we need to find those ρ' such that $r\circ\rho'$ contains the trivial representation $\mathbb{1}$, since then

$$L(s,r,\pi',V) = L(s,r\circ\rho',V)$$

is a multiple of $L(s,\mathbb{1},V) = \prod_{v\notin V}(1-q_v^{-s})^{-1}$, which has a pole at $s=1$. This pole – again conjecturally – cannot be canceled by a zero of $L(s,r\circ\rho'/\mathbb{1},V)$.

To find such ρ', we need another basic observation about distinguished π'. The restriction of the non-zero linear form $L(\phi) = \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)}\phi(g)dg$ on π' to its component $\pi'_v = \pi'_{v'}\times\pi'_{v''}$ at a place v which splits into v', v'' in E, is a $G(F_v)$-invariant non-zero form on $\pi'_{v'}\times\pi'_{v''}$. Its existence implies that $\pi'_{v''}$ is the contragredient $\check{\pi}'_{v'}$ of the $G(F_v)$-module $\pi'_{v'}$.

Let now J be the $n\times n$ matrix whose (i,j) entry is $(-1)^{n-i}\delta_{i,n-j+1}$. Write $\sigma(g) = J^t\overline{g}^{-1}J^{-1}$ for g in $G(\mathbb{A}_E)$, where t indicates "transpose", and $\overline{g}=(\overline{a}_{ij})$ if $g=(a_{ij})$, where $a\mapsto\overline{a}$ is the non-trivial automorphism of E over F. Put ${}^\sigma\pi'(g) = \pi'(\sigma g)$. The "principle of functoriality" mentioned above suggests that π' with $\pi'\simeq {}^\sigma\pi'$ are obtained by base-change from $U(n,E/F)$ (this map is described below), and this is indeed known when $n=1$, $n=2$ ([F3]) and $n=3$ ([F4]). In particular, at a place v of F which splits into v', v'' in E, we have $G(E_v) = G(F_v)\times G(F_v)$ ($E_v = E\otimes_F F_v = F_{v'}\oplus F_{v''}$ and $F_{v'} = F_{v''} = F_v$), and σ maps $g=(g',g'')$ to $(\sigma g'',\sigma g')$, where $\sigma h = J^t h^{-1}J^{-1}$ for $h\in G(F_v)$. Thus the component $\pi'_v = \pi'_{v'}\times\pi'_{v''}$ for a σ-invariant π' satisfies $\pi'_{v''}\simeq\check{\pi}'_{v'}$, suggesting that the distinguished π' are base-change lifts from $U(n,E/F)$. Consequently we shall now describe the base-change map from $U(n,E/F)$ to $GL(n,E)$, and examine when does a ρ' obtained by the base-change map have the property that $r\circ\rho'$ contains the trivial representation.

The unitary group $U(n,E/F)$ consists of all $g\in G(E)$ ($G=GL(n)$) with $\sigma(g) =$

g. Its dual group is $\widehat{U} = G(\mathbb{C}) \rtimes W_{E/F}$. The relative Weil group

$$W_{E/F} = \langle z \in W_{E/E}\,,\ \sigma\,;\ \sigma z \sigma^{-1} = \overline{z}\,,\ \sigma^2 \in W_{F/F} - N_{E/F} W_{E/E} \rangle$$

is an extension of $\mathrm{Gal}(E/F) = \langle \sigma \rangle$ by $W_{E/E}(= E^\times$ if E is local, $= \mathbb{A}_E^\times/E^\times$ if E is global; in the global case put $N_{E/E}W_{E/E}$ for $F^\times N_{E/F}\mathbb{A}_E^\times/F^\times)$. We take here the smallest form of the Weil group which suffices for our purposes; as we see below, the form $\widehat{U} = G(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$ is insufficient. Now $W_{E/F}$ acts on $G(\mathbb{C})$ via its projection on $\mathrm{Gal}(E/F)$, and the non-trivial element in this galois group acts by $\sigma(g) = J^t g^{-1} J^{-1}$ on $G(\mathbb{C})$.

Given an unramified irreducible $U_v = U(n, E_v/F_v)$-module π_v there exists an $[n/2]$-tuple of unramified characters (μ_i) of $E_v^\times$ such that π_v is the unique unramified irreducible constituent in the composition series of the U_v-module $I((\mu_i))$ normalizedly induced from the character $(a_{ij}) \mapsto \prod_{1\le i\le n/2} \mu_i(a_{ii})$ of the upper triangular subgroup. Then π_v is parametrized by the conjugacy class $t_v = \mathrm{diag}(\mu_1(\underline{\pi}_v), \dots, \mu_{[n/2]}(\underline{\pi}_v), 1, \dots, 1) \times \sigma$ in $\widehat{U}$; $\underline{\pi}_v$ is a uniformizer in E_v. At a place v which splits in E we have $U_v = G(F_v)$, an unramified irreducible π_v is again associated with an induced $I((\mu_i(1 \le i \le n)))$ and a conjugacy class $t_v = \mathrm{diag}(\mu_i(\underline{\pi}_v)) \times 1$ in $\widehat{U}_v = G(\mathbb{C})$.

The stable base-change homomorphism

$$b : \widehat{U} = G(\mathbb{C}) \rtimes \mathrm{W}_{E/F} \to \widehat{G}' = [G(\mathbb{C}) \times G(\mathbb{C})] \rtimes W_{E/F}$$

restricts to the identity on $W_{E/F}$ and maps $g \in G(\mathbb{C})$ to $(g, \sigma g)$. To determine when does the representation $r \circ b$ of $\widehat{U}$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ contain the trivial representation, namely a $\widehat{U}$-invariant vector, denote by $(x_i;\ 1 \le i \le n)$ the standard basis of $\mathbb{C}^n$. If $g = (a_{ij})$, then $gx_i = \Sigma_j a_{ij} x_j$, and if $g^{-1} = (b_{ij})$ then ${}^t g^{-1} x_i = \Sigma_j b_{ji} x_j$. Since $J^2 = (-1)^{n-1}$ and $Jx_i = (-1)^{n-1} x_{n+1-i}$, we have

$$g x_u \otimes J^t g^{-1} J^{-1} x_{n+1-u} = \sum_i a_{ui} x_i \otimes \sum_j (-1)^{n-j} b_{j,u} x_{n+1-j}\,.$$

Hence the vector

$$w = \sum_i (-1)^i x_i \otimes x_{n+1-i}$$

is invariant under the action of $r(b(g)) = r(g, \sigma g)$ for all $g \in G(\mathbb{C})$. However, we have $r(b(\sigma))w = r(\sigma)w = (-1)^{n-1}w$. We conclude that when n is odd, the representation $r \circ b =$ (twisted tensor) $\circ$ (stable base-change) of $\widehat{G}$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ contains a copy of the trivial $\widehat{U}$-module.

Believing in the conjectural "principle of functoriality" we will conjecture that when n is odd, a cuspidal $G(\mathbb{A}_E)$-module π' is $G(\mathbb{A}_F)$-distinguished when it is the stable base-change lift of a cuspidal $U(n, E/F)(\mathbb{A})$-module π. As usual we say that $\pi = \otimes \pi_v$ lifts to $\pi' = \otimes \pi'_v$ if $t(\pi'_v) = b(t(\pi_v))$ for almost all places v where π_v is unramified, and note that π' is uniquely determined by almost all of its components (if it exists) by rigidity and multiplicity one theorems for $GL(n)$. If π' exists then it is σ-invariant, but this is not a sufficient condition for π' to be a lift. When π' is a lift, it is expected to be the lift of a "packet" $\{\pi\}$ containing π, and a "packet", which in fact can be defined as the preimage of a π', contains precisely one non-degenerate (having a Whittaker model) irreducible π. These conjectures are known to hold for $n = 2$ ([F3]) and $n = 3$ ([F4]).

When n is even the vector w is not $r(b(\widehat{U}))$-invariant. We will find now that w is $\widehat{U}$-invariant, when n is even, as long as the stable base-change map b is replaced by the unstable base-change map of [F3]. This map

$$b_\kappa : \widehat{U} \to \widehat{G}'$$

takes $g \in G(\mathbb{C})$ again to $(g, \sigma g) \in G(\mathbb{C}) \times G(\mathbb{C})$, and $\sigma \in \mathrm{Gal}(E/F)$ to $(I_n, -I_n)\sigma \in \widehat{G}'$. I_n is the identity $n \times n$ matrix. In particular $b_\kappa(\sigma^2) = (-I_n, -I_n)\sigma^2$, hence b_κ does not define a homomorphism from the Galois form $G(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$ of $\widehat{H}$. However, b_κ extends to a homomorphism from the Weil form $G(\mathbb{C}) \rtimes W_{E/F}$ of $\widehat{U}$. Indeed, fix a character κ of $E^\times$ which is trivial on $NE^\times$ but not on $F^\times$ (locally), or of $\mathbb{A}_E^\times$ which is trivial on $F^\times N_{E/F}\mathbb{A}_E^\times$ but not on $\mathbb{A}_F^\times$ (globally). Recalling that $W_{E/F}$ is equal to

$$\langle z \in W_{E/E}(= E^\times \text{ or } \mathbb{A}_E^\times/E^\times)\,,\ \sigma\,;\ \sigma z = \overline{z}\sigma\,,\ \text{and } \sigma^2 \in F{-}NE \text{ or } (\mathbb{A}^\times - N_{E/F}\mathbb{A}_E^\times)/F^\times\rangle$$

it is clear that b_κ extends to a homomorphism $\widehat{U} \to \widehat{G}'$ on putting

$$b_\kappa(z) = (\kappa(z), \kappa(\overline{z}))z \qquad (z \in W_{E/E})\,,$$

since $\kappa(\sigma^2) = -I_n$. Now the twisted tensor map $r : \widehat{G}' \to GL(n^2, \mathbb{C})$ factorizes through $G(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$, namely $r(W_{E/E})$ acts trivially on $\mathbb{C}^n \otimes \mathbb{C}^n$. Since $\kappa(\overline{z})^{-1} = \kappa(z)$ it is clear that w is fixed by the action of $r(b_\kappa(\widehat{U}))$, leading us to conjecture that when n is even, a $G(\mathbb{A}_E)$-module π' is $G(\mathbb{A}_F)$-distinguished when it is the unstable base-change lift (via b_κ) of a $U(n, E/F)(\mathbb{A})$-packet $\{\pi\}$, and of a unique non-degenerate $U(n, E/F)(\mathbb{A})$-module π. More precisely, the discussion above motivates the following.

Conjecture. *The stable (when n is odd) [resp. unstable (when n is even)] base-change map b [resp. b_κ] yields a bijection from the set of cuspidal non-degenerate $U(n, E/F)(\mathbb{A})$-modules π whose image under b [resp. b_κ] is cuspidal, to the set of cuspidal $G(\mathbb{A}_F)$-distinguished $G(\mathbb{A}_E)$-modules π'.*

This answers the question posed in [F1], p. 298, l. 2-3, to characterize the distinguished representations for a general n. Of course, a proof of this conjecture also should contain a proof of the existence of the stable (n odd) and unstable (n even) base-change liftings.

In forming this Conjecture I benefitted from very interesting conversations with Steve Rallis; it should be viewed as a joint conjecture. We hope to study cases thereof later. The purpose of this paper is to verify the Conjecture in the easiest case of $n > 1$, namely when $n = 2$, to ascertain its precise form at least at this initial, illuminating case. The conjecture may simply follow from Proposition 0.1 when $n = 2$. However we prefer to prove it for $n = 2$ on using the Relative Trace Formula, motivated by the recent alternative proof by Ye [Y] of Proposition 0.1, since we believe that it will generalize and will eventually yield a proof of the conjecture for all n. Technically our proof is simply an adaptation of Ye's work from the context of comparison of: (1) π' on $GL(2, \mathbb{A}_E)$ with trivial central character ω' and double cosets $GL(2, F)\backslash GL(2, E)/N(E)$, and (2) π on $GL(2, \mathbb{A}_F)$ with central character $\omega = \chi_{E/F}$ and double cosets $N(F)\backslash GL(2, F)/N(F)$ (where N is the upper triangular unipotent subgroup), to the context of comparison of: (1) π' on $GL(2, \mathbb{A}_E)$ whose central character $\omega'\kappa'$ (where $\kappa'(z) = \kappa(z/\overline{z})$) is trivial (only) on $\mathbb{A}_F^\times$ and double cosets $GL(2, F)\backslash GL(2, E)/N(E)$ as before, and (2) π on $U(2, E/F)(\mathbb{A})$ and double cosets $N(F)\backslash U(2, E/F)/N(F)$. Rather than reproving the required transfer of relative orbital integrals we simply reduce the identities which we need to those stated in [Y]. We start with our global results.

Global Theory.

Let E/F be a separable quadratic extension of global fields with char$F \neq 2$, $\omega = \otimes\omega_v$ a unitary character of $\mathbb{A}_E^\times/E^\times N_{E/F}\mathbb{A}_E^\times$ which is non-trivial on $\mathbb{A}_F^\times$. Then $\omega'(z) = \omega(z/\overline{z})$ is a unitary character on $\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times$.

1. THEOREM. *Every cuspidal $GL(2, \mathbb{A}_F)$-distinguished $GL(2, \mathbb{A}_E)$-module π' with central character $\omega'\kappa'$ is the unstable base-change lift (via b_κ) of a cuspidal non-degenerate $U(2, E/F)(\mathbb{A})$-module π with central character ω. In particular such π' is σ-invariant. Conversely, every cuspidal non-degenerate $U(2, E/F)(\mathbb{A})$-module π with central character ω lifts via the unstable map b_κ either to a cuspidal $\mathbb{G}$-distinguished $\mathbb{G}'$-module π' with central character $\omega'\kappa'$, or to an induced $\mathbb{G}'$-module $I(\mu_1, \mu_2)$, where μ_i are unitary characters of $\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times$ with $\mu_1\mu_2 = \omega'\kappa'$ and $\mu_1 \neq \mu_2$.*

By (global) lifting we mean here quasi-lifting, defined in terms of almost all places. We do not deduce here local lifting, and do not prove that the set of cuspidal non-degenerate $\mathbb{U}$-modules admits multiplicity one and rigidity ("strong multiplicity one") theorem. The existence of the (stable and) unstable base-change lifting from $U(2, E/F)$ to $GL(2, E)$ was proven in [F3] on using a different technique (comparison of the trace formula of $U(2, E/F)$ with the stabilized trace formula of $GL(2, E)$ twisted by σ). The results of [F3] concern global lifting in the strong sense of all places, rather than only quasi-lifting. Complete local results, based on character relations, are obtained, and multiplicity one and rigidity theorem for the cuspidal spectrum of $U(2, E/F)$ are proven. The virtue of our technique here, suggested by [Y], is in the determination of the set of the cuspidal $\mathbb{G}$-distinguished $\mathbb{G}'$-modules as the set of those cuspidal π' which are obtained as the image of the unstable base-change lifting from $\mathbb{U}$. According to [F3], this set makes one "half" of the set of cuspidal σ-invariant $\mathbb{U}$-modules, the other "half" being the image of the stable base-change lifting. Further, in Theorem 7 below, stated after the proof of Theorem 1 is completed, we deduce from our global Theorem 1, and the description in [F3] of the local lifting (via b_κ), a complete description of the set of $GL(2,F_v)$-distinguished admissible $GL(2,E_v)$-modules π'_v. This set is precisely the image under the unstable local base-change map b_κ of the set of infinite-dimensional $U(2,E_v/F_v)$-modules (and one-dimensional G'_v-modules of the form $\chi'_v(z)=\chi_v(z/\overline{z})$). The proof of the local Theorem 7 uses in particular the observation that a local square-integrable distinguished G'_v-module can be embedded as a component of a global cuspidal distinguished $\mathbb{G}'$-module. This, and several related results, are proven in the context of $GL(n)$. In any case we begin with the proof of Theorem 1.

Proof. This is based on a comparison of two relative trace formulae. Put $\underline{G}=GL(2)$; $\underline{G}'=\mathrm{Res}_{E/F}\underline{G}$ the reductive group obtained by restriction of scalars from E to F, thus $\underline{G}'(F)=\underline{G}(E)$; $\underline{U}=U(2,E/F)$ the quasi-split unitary subgroup of $\underline{G}'$ defined as the fixed point set of the involution $\sigma(g)=J{}^t\overline{g}^{-1}J^{-1}$ where $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$; $\underline{N}$ the upper triangular unipotent subgroup $\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}\ ;\ x\in\underline{G}_{\mathrm{add}/F}\right\}$ of $\underline{G}$ or $\underline{U}$ (depending on the context), and $\underline{N}'=\mathrm{Res}_{E/F}\underline{N}$ the corresponding subgroup of $\underline{G}'$; $\underline{Z}=$ center of $\underline{G}$ or $\underline{U}$, $\underline{Z}'=$ center of $\underline{G}'$, $\underline{A}=$ diagonal subgroup of $\underline{G}$ or $\underline{U}$ (depending on context), and $\underline{A}'$ of $\underline{G}'$. For each place v of F the index v indicates the group of F_v-valued points (e.g. $G_v=\underline{G}(F_v)$, $G'_v=\underline{G}'(F_v)=\underline{G}(E_v)$ $(=G_v\times G_v$ if v splits in $E)$, etc.), and a special roman letter $\mathbb{G},\mathbb{G}'(=\mathbb{G}(\mathbb{A}_E))$, $\mathbb{N},\mathbb{Z},\mathbb{U},\mathbb{A}$, etc., indicates the group of F-adele (i.e. $\mathbb{A}_F$-) valued points. Choose product measures $dx=\otimes dx_v$, $d^\times x=\otimes d^\times x_v$ on $\mathbb{A}=\mathbb{A}_F$ and $\mathbb{A}^\times$, normalized say by assigning the volume 1 to the ring R_v of integers in F_v and to its group $R_v^\times$ of units, for all non-archimedean v. This yield measures on $\mathbb{N},\mathbb{Z},\mathbb{A}$. Fix also Haar measures on G_v, G'_v, U_v such that the product of the volumes of the maximal compact subgroups $K_v=\underline{G}(R_v)$, $K'_v=\underline{G}'(R_v)$, $K_v=\underline{U}(R_v)$ converges. The trace formulae apply with test functions $f=\otimes f_v$ on $\mathbb{U}$ and $f'=\otimes f'_v$ on $\mathbb{G}'$, where for all v the component is smooth and compactly supported modulo the center, satisfying $f_v(zg)=\omega_v(z)^{-1}f_v(g)$ $(z\in Z_v)$ and $f'_v(zg)=\kappa'_v\omega'_v(z^{-1})f'_v(g)$ $(z\in Z'_v)$. For almost all v the component f_v is the unit element f_v^0 in the convolution algebra $\mathbb{H}_v$ of K_v-biinvariant, compactly-supported modulo-center, functions on U_v, and f'_v is the unit element ${f'_v}^0$ in the convolution algebra $\mathbb{H}'_v$ of K'_v-biinvariant compactly-supported modulo-center, functions on G'_v. Fix a non-trivial additive character $\psi=\otimes\psi_v$ on $\mathbb{A}\mathrm{mod}F$, write $\mathrm{tr}z$ for $z+\overline{z}$ (z in E or $\mathbb{A}_E$) and $Nz=z\overline{z}$, and define $\psi'=\psi\circ\mathrm{tr}$ on $\mathbb{A}_E\mathrm{mod}E$. Then ψ,ψ' define characters on $\mathbb{N}$ and $\mathbb{N}'$. Of course the functions f,f',ψ,ψ' are all complex-valued.

Let $L^2_\omega(U)$ be the space of functions $\varphi:\mathbb{U}\to\mathbb{C}$ such that $\varphi(z\gamma g)=\omega(z)\varphi(g)$ $(g\in\mathbb{U},\ \gamma\in U=\underline{U}(F),\ z\in\mathbb{Z})$ and $\int_{\mathbb{Z}U\backslash\mathbb{U}}|\varphi(g)|^2dg<\infty$. The convolution operator

$$(\rho(f)\varphi)(g)=\int_{\mathbb{Z}\backslash\mathbb{U}}f(h)\varphi(gh)dh=\int_{\mathbb{Z}U\backslash\mathbb{U}}K_f(g,h)\varphi(h)dh$$

is an integral operator with kernel

$$K_f(g,h) = \sum_{\gamma \in Z\backslash U} f(g^{-1}\gamma h) \qquad\qquad (Z = \underline{Z}(F))\,.$$

The theory of Eisenstein series decomposes $L^2_\omega(U)$ as the direct sum of the three mutually orthogonal invariant subspaces: the space $L^2_{\omega,0}(U)$ of cusp forms, the space $L^2_{\omega,1}(U)$ of functions $\varphi(g) = \chi(\det g)$ with $\chi^2 = \omega$, and the continuous spectrum $L^2_{\omega,c}(U)$. Correspondingly,

$$K_f(g,h) = K_{f,0}(g,h) + K_{f,1}(g,h) + K_{f,c}(g,h)\,,$$

where

$$K_{f,1}(g,h) = \frac{1}{2}\sum_{\chi^2=\omega} \chi(\det g)\overline{\chi}(\det h)\int_{\mathbb{Z}\backslash\mathbb{U}} f(u)\chi(\det u)du\,.$$

On the analogous space $L^2_{\omega'\kappa'}(G')$ we have the analogous decomposition

$$K_{f'}(g,h) = \sum_{\delta\in Z'\backslash G'} f'(g^{-1}\delta h) = K_{f',0}(g,h) + K_{f',1}(g,h) + K_{f',c}(g,h)$$

with

$$K_{f',1}(g,h) = \frac{1}{2}\sum_{\chi^2=\omega'\kappa'} \chi(\det g)\overline{\chi}(\det h)\int_{\mathbb{Z}'\backslash\mathbb{G}'} f'(u)\chi(\det u)du\,.$$

Put $G = \underline{G}(F)$, $G' = \underline{G}'(F)$ $(= GL(2,E))$, $N = \underline{N}(F)$, $N' = \underline{N}'(F)$, etc. The first step in the proof is

2. Proposition. *For every $f' = \otimes f'_v$ there exists $f = \otimes f_v$, and for every f there exists f', such that*

$$\int_{\mathbb{G}/\mathbb{Z}G}\int_{\mathbb{N}'/N'}\sum_{\delta\in Z'\backslash G'} f'(g\delta n)\psi'(n)dgdn = \int_{\mathbb{N}/N}\int_{\mathbb{N}/N}\sum_{\gamma\in Z\backslash U} f(n_1\gamma n_2)\psi(n_1n_2)dn_1dn_2\,.$$

Moreover, if $E_v/F_v, \psi_v$ and κ_v are unramified, and f'_v is spherical ($\in \mathbb{H}'_v$), then the component f_v can be chosen to be the image of f'_v in $\mathbb{H}_v$ under the homomorphism $\mathbb{H}'_v \to \mathbb{H}_v$ dual to the unstable base-change map b_κ.

Since ψ and ψ' are non-trivial, we have $\int_{\mathbb{A}/F}\psi(x)dx = 0$ and $\int_{\mathbb{A}_E/E}\psi'(x)dx = 0$. Hence

$$\int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{\mathbb{N}'/N'} K_{f',1}(g,n)\psi'(n)dgdn = 0\,,$$

and

$$\int_{\mathbb{N}/N}\int_{\mathbb{N}/N} K_{f,1}(n_1,n_2)\psi(n_2n_1^{-1})dn_1dn_2 = 0\,,$$

and Proposition 2 has the

Corollary. *For f, f' as above we have*

$$\int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{\mathbb{N}'/N'} K_{f',0}(g,n)\psi'(n)dgdn - \int_{\mathbb{N}/N}\int_{\mathbb{N}/N} K_{f,0}(n_1,n_2)\psi(n_2n_1^{-1})dn_1dn_2$$
$$= \int_{\mathbb{N}/N}\int_{\mathbb{N}/N} K_{f,c}(n_1,n_2)\psi(n_2n_1^{-1})dn_1dn_2 - \int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{\mathbb{N}'/N'} K_{f',c}(g,n)\psi'(n)dgdn\,.$$

Writing out $K_{f,c}$ and $K_{f',c}$ after Proposition 2 is proven, we will conclude essentially (but not precisely!) that both sides in the Corollary vanish, and deduce Theorem 1.

By the Bruhat decomposition, the expression on the right of the identity of Proposition 2 is the sum of

$$\sum_{a\in Z\backslash A}\int_{\mathbb{N}}\int_{\mathbb{N}} f(nJan')\psi(nn')dndn' \qquad \left(J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

and

$$\sum_{a\in Z\backslash A}\int_{\mathbb{N}} dn \int_{\mathbb{N}/N} f(nan')\psi(nn')dn'$$
$$= \sum_{\alpha\in E^\times/E^1}\int\int f\left(\begin{pmatrix} 1 & x+x'\alpha\overline{\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix}\right)\psi(x+x')dxdx'$$
$$= \mathrm{vol}(\mathbb{A}_F/F)\int_{\mathbb{N}} f(n)\psi(n)dn \quad \left(a = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix},\ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ n' = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}\right).$$

The expression on the left is

$$\int_{\mathbb{G}/\mathbb{Z}}\int_{\mathbb{N}'/N'} \sum_{\delta\in Z'G\backslash G'} f'(g\delta n)\psi'(n)dgdn\,.$$

If B' is the upper triangular subgroup of G', then G'/B' is the projective line over E, and G, acting on the left, has two orbits, represented by $\binom{0}{1}$ and $\binom{\sqrt{\theta}}{1}$, where $E = F(\sqrt{\theta})$. Hence G' decomposes as a disjoint union $G' = G\eta B' \cup GB'$, where $\eta = \begin{pmatrix} -\sqrt{\theta} & \sqrt{\theta} \\ 1 & 1 \end{pmatrix}$. Note that $\widetilde{G} = \eta^{-1}G\eta = \left\{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix};\ a\in E,\ b\in E,\ a\overline{a}\neq b\overline{b}\right\}$. To describe the quotient $Z'G\backslash G\eta B'$, consider the stabilizer

$$g\eta\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} n = z\eta\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} n' \qquad (n,\ n'\in N';\ a,b\in E^\times;\ g\in G;\ z\in Z'\cong E^\times)$$

or

$$\widetilde{g} = \eta^{-1}g\eta = z\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} n'n^{-1}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}^{-1}.$$

It is clear that the last equation holds only when $n' = n$, $a/b \in E^1$, and $g = z$. Hence the part of the integral corresponding to the big (open) cell $G\eta B'$ in G' is

$$\sum_{a \in E^\times/E^1} \int_{\mathbb{G}/\mathbb{Z}} \int_{\mathbb{N}'} f'\left(g\eta \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} n\right) \psi'(n) dg dn\,.$$

The small (closed) cell GB' contributes

$$\sum_{\delta \in Z'B\backslash B'} \int_{\mathbb{G}/\mathbb{Z}} \int_{\mathbb{N}'/N'} f'(g\delta n)\psi'(n) dg dn\,.$$

A set of representatives for $Z'\backslash B'$ is given by $\begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix}$, $c \in E^\times$, $b \in E$. A set of representatives for the quotient of $\left\{\begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \; ; \; c \in E - F\,,\; b \in E\right\}$ by left multiplication under $\left\{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} \; ; \; \gamma \in F^\times\,,\; \beta \in F\right\}$, is given by $\left\{\begin{pmatrix} 1 & f \\ 0 & c \end{pmatrix} \; ; \; c \in (E-F)/F^\times\,,\; f \in F\right\}$. By the Iwasawa decomposition $G = KAN$ we have $g = k\begin{pmatrix} 1 & y \\ 0 & a \end{pmatrix}$, and the corresponding part of the integral becomes

$$\sum_{c \in (E-F)/F^\times} \sum_{f \in F} \int_{\mathbb{K}} dk \int_{\mathbb{A}_F^\times} (d^\times a/\|a\|) \int_{\mathbb{A}_F} dy \int_{\mathbb{A}_E/E} f'\left(k\begin{pmatrix} 1 & y \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & f+x \\ 0 & c \end{pmatrix}\right)\psi'(x)dx$$
$$= \sum_{c \in (E-F)/F^\times} \int_{\mathbb{K}} dk \int_{\mathbb{A}_F^\times} (d^\times a/\|a\|) \int_{\mathbb{A}_E} f'\left(k\begin{pmatrix} 1 & x \\ 0 & ac \end{pmatrix}\right)\psi'(x)dx \int_{\mathbb{A}_F/F} \psi'(-yc)dy\,.$$

Since $\int_{\mathbb{A}_F/F} \psi(-y(c+\overline{c}))dy = 0$ unless $c + \overline{c} = 0$, the sum over c reduces to a single term, represented by $c = \sqrt{\theta}$. Writing $x = x_1 + x_2\sqrt{\theta}$ $(x_1, x_2 \in \mathbb{A}_F)$, and $g = k\begin{pmatrix} 1 & x_2 \\ 0 & a \end{pmatrix}$, we obtain

$$\mathrm{vol}(\mathbb{A}_F/F) \int_{\mathbb{G}/\mathbb{Z}} \int_{\mathbb{N}} f'\left(g\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\theta} \end{pmatrix} n\right)\psi'(n) dg dn\,.$$

The remaining part of $Z'B\backslash B'$, represented by $\left\{\begin{pmatrix} 1 & b\sqrt{\theta} \\ 0 & 1 \end{pmatrix} \; ; \; b \in F\right\}$, contributes

$$\sum_{b \in F} \int_{\mathbb{G}/\mathbb{Z}} \int_{\mathbb{N}'/N'} f'\left(g\begin{pmatrix} 1 & b\sqrt{\theta} \\ 0 & 1 \end{pmatrix} n\right)\psi'(n) dg dn\,.$$

Writing $g = k\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, we obtain

$$\int_{\mathbb{K}} dk \int_{\mathbb{A}_F^\times/F^\times} (d^\times a/\|a\|) \int_{\mathbb{A}_E} dx \int_{\mathbb{A}_F/F} dy \cdot f'\left(k\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}\right)\psi'(x)\,.$$

But this is zero since $\int_{\mathbb{A}_F/F} \psi(2y)dy = 0$.

Proposition 2 then follows at once from the local

3. Proposition. *For every place v in F, given f_v there is f'_v, and given f'_v there is f_v, such that for every a in $E_v^\times$ we have*

$$\int_{N_v}\int_{N_v} f_v\left(nJ\begin{pmatrix}\overline{a}^{-1} & 0\\ 0 & a\end{pmatrix}n'\right)\psi_v(nn')dndn'$$
$$= \lambda_v(\psi_v)\kappa_v(a)\int_{G_v/Z_v}\int_{N'_v} f'_v\left(g\eta\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}n\right)\psi'_v(n)dgdn\,,$$

and

$$\int_{N_v} f_v(n)\psi_v(n)dn = \kappa_v(-1)|2|_v\int_{G_v/Z_v}\int_{N_v} f'_v\left(g\begin{pmatrix}1 & 0\\ 0 & \sqrt{\theta}\end{pmatrix}n\right)\psi'_v(n)dndg\,.$$

Here $\lambda_v(\psi_v)$ is the constant $\lambda_{F_v/E_v}(\psi_v)$ of [Y], end of p. 65; it is 1 if v splits or unramified in E, and $\prod_v \lambda_v(\psi_v) = 1$. Moreover, these identities are held when f_v is the image of f'_v under the map $\mathbb{H}'_v \to \mathbb{H}_v$ dual to the unstable base-change map b_κ.

Definition. Functions f_v, f'_v which satisfy the equalities of Proposition 3 are called *matching.*

To prove this, we need to distinguish between two cases, when v splits or does not split in E. Since the situation is local, we omit the index v. When v splits, $G' = G\times G$, $f'(g') = f_1(g_1)f_2(g_2)$, etc., and the right side of the first identity in Proposition 3 is

$$\kappa_1(a/\overline{a})\int_{G/Z}\int_N\int_N f_1\left(g\eta\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}n_1\right)f_2\left(g\overline{\eta}\begin{pmatrix}1 & 0\\ 0 & \overline{a}\end{pmatrix}n_2\right)\psi(n_1n_2)dgdn_1dn_2\,.$$

Note that $\kappa(a) = \kappa_1(a)\kappa_2(\overline{a}) = \kappa_1(a/\overline{a})$ since $\kappa_1 = \kappa_2$ (κ is trivial on $NE^\times$, and this is $F^\times$ when v splits). Since $\overline{\eta}^{-1}\eta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, a change of variables on g yields

$$\kappa_1(a/\overline{a})\int_{G/Z}\int_N\int_N f_1\left(g\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}n_2J\begin{pmatrix}\overline{a}^{-1} & 0\\ 0 & a\end{pmatrix}n_1\right)f_2(g)\psi(n_1n_2)dgdn_1dn_2\,.$$

Define $\widetilde{f}_2$ by $\widetilde{f}_2(x) = f_2\left(x^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$, and f by $f = \kappa_1\cdot f_1 * \widetilde{f}_2$, namely

$$f(x) = \kappa_1(\det x)\int_{G/Z} f_1(gx)f_2\left(g\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)dg\,.$$

We obtain

$$\int_N\int_N f\left(nJ\begin{pmatrix}\overline{a}^{-1} & 0\\ 0 & a\end{pmatrix}n'\right)\psi(nn')dndn'\,,$$

which is the left side of the first identity in Proposition 3.

The right side of the second identity is

$$|2|\int_{G/Z}\int_N f_1\left(g\begin{pmatrix}1&0\\0&\sqrt{\theta}\end{pmatrix}n\right)f_2\left(g\begin{pmatrix}1&0\\0&-\sqrt{\theta}\end{pmatrix}n\right)\psi'(n)dgdn$$
$$=\int_{G/Z}\int_F f_1\left(g\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&2x\\0&1\end{pmatrix}\right)f_2(g)\psi(2x)dgd(2x)=\int_N f(n)\psi(n)dn\,,$$

as required.

This proves the existence of f, once $f'=(f_1,f_2)$ is given. Given f on G there is $f'=(f_1,f_2)$ on G'/Z with $f=f_1*\widetilde{f_2}$, and then f,f' are matching.

When v stays prime in E, writing $f'(x)=\omega(-\det\overline{x\eta}^{-1})\kappa(\det\eta x^{-1})f'_w(\overline{x})$ where f'_w is the function on G'_v $(=GL(2,E_v))$ of (1), [Y], p. 82 or p. 68, and noting that

$$\begin{pmatrix}-\sqrt{\theta}&-2\sqrt{\theta}\,\overline{\beta}\\1&0\end{pmatrix}=\begin{pmatrix}-\sqrt{\theta}&\sqrt{\theta}\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&-\overline{\beta}\end{pmatrix}=\eta\begin{pmatrix}1&0\\0&-\overline{\beta}\end{pmatrix}\begin{pmatrix}1&\overline{\beta}\\0&1\end{pmatrix},$$

we rewrite (1) of [Y], p. 82 or p. 68 in the form

$$\lambda(\psi)\int_{G/Z}\int_{N'}f'\left(g\eta\begin{pmatrix}1&0\\0&-\overline{\beta}\end{pmatrix}n\right)\psi'(n)dgdn$$
$$=\omega(\beta)\kappa(-\overline{\beta})^{-1}\int_N\int_N f_v\left(-n\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\0&\beta\overline{\beta}\end{pmatrix}n'\right)\psi(nn')dndn'\,.$$

The function f_v on the right is, as in [Y], a function on the subgroup $G_0=\{g\in G;\ \det g\in NE^\times\}$ of G which satisfies $f_v(zg)=\omega(z)^{-1}f_v(g)$ $(z\in F^\times,\ g\in G_0)$. Extend f_v of [Y] to a function on $Z'G_0$ by $f_v(zg)=\omega(z)^{-1}f_v(g)$ $(z\in E^\times,\ g\in G_0)$. Since each g in $U=U(2,E/F)$ can be written in the form

$$g=\begin{pmatrix}z&0\\0&\overline{z}^{-1}\end{pmatrix}s=\overline{z}^{-1}\begin{pmatrix}z\overline{z}&0\\0&1\end{pmatrix}s\qquad(z\in E^\times\,,\ s\in SL(2,F))\,,$$

we have $Z'U=Z'G_0$, and the identity (1) of [Y], p. 82 or p. 68 can be written as

$$\lambda(\psi)\int_{G/Z}\int_{N'}f'\left(g\eta\begin{pmatrix}1&0\\0&-\overline{\beta}\end{pmatrix}n\right)\psi'(n)dgdn$$
$$=\kappa(-\overline{\beta})^{-1}\int_N\int_N f\left(nJ\begin{pmatrix}-\beta^{-1}&0\\0&-\overline{\beta}\end{pmatrix}n'\right)\psi(nn')dndn'\,,$$

where $f=f_v$. Taking $a=-\overline{\beta}$ we obtain the first identity of Proposition 3 when v stays prime (archimedean or not) in E.

Since

$$k\begin{pmatrix}1 & x_1+x_2\sqrt{\theta}\\ 0 & a\sqrt{\theta}\end{pmatrix} = k\begin{pmatrix}1 & x_2\\ 0 & a\end{pmatrix}\begin{pmatrix}1 & x_1\\ 0 & \sqrt{\theta}\end{pmatrix} = g\begin{pmatrix}1 & 0\\ 0 & \sqrt{\theta}\end{pmatrix}n$$

($x_1, x_2 \in F$; $a \in F^\times$; $n \in N$), the left side of [Y], (2) on p. 82 or p. 69, can be written as

$$\kappa(-1)|2|\int_N f'\left(g\begin{pmatrix}1 & 0\\ 0 & -\sqrt{\theta}\end{pmatrix}n\right)\psi'(n)dn\,,$$

while the right side of [Y], (2) on p. 82 or p. 69 is already $\int_N f(n)\psi(n)dn$. Consequently the second identity of Proposition 3 follows.

To complete the proof we need to show that spherical f_v, f'_v are matching if f_v is the image of f'_v under the Hecke algebra morphism $b^*_\kappa : \mathbb{H}'_v \to \mathbb{H}_v$. To establish this we first recall the definition of b^*_κ. Again we use local notations, namely omit v, and we assume that the extension E/F is unramified. Recall that $\widehat{U} = G(\mathbb{C}) \rtimes \langle\sigma\rangle$, where $\sigma g = J {}^t g^{-1} J^{-1}$, and $\widehat{G}' = (G(\mathbb{C}) \times G(\mathbb{C})) \rtimes \langle\sigma\rangle$, where $\sigma(g, g') = (g', g)$, and $b_\kappa : \widehat{U} \to \widehat{G}'$ maps g to $(g, \sigma g)$, σ to $(I, -I)\sigma$ (b_κ is a homomorphism only after an extension to the Weil form of $\widehat{U}$ and $\widehat{G}'$, but it suffices to work with the galois form in the present context of the Satake transform). The Satake transform (see [C]) defines an isomorphism from the convolution algebra $\mathbb{H}$ of K-biinvariant compactly supported functions f on U, to the algebra of W (= Weyl group) -invariant Laurent polynomials, $\mathbb{C}[\widehat{T}\times\sigma]^W$, where $\widehat{T}$ is the diagonal subgroup of $G(\mathbb{C})$. For any rational integer n, let

$$F_f(n) = \Delta(\gamma)\int_{U/A} f(g\gamma g^{-1})dg\,, \qquad \gamma = \begin{pmatrix}a & 0\\ 0 & \overline{a}^{-1}\end{pmatrix}, \quad |a| = |\underline{\pi}|^n\,,$$

denote the normalized (by $\Delta(\gamma) = |a - \overline{a}^{-1}|_F = |a\overline{a} - 1|_F/|a\overline{a}|_F^{1/2}$) orbital integral of the spherical f at a regular γ; since f is K-biinvariant, $F_f(n)$ is independent of the choice of the representative $\gamma = \gamma(n)$. Up to conjugacy by $G(\mathbb{C})$, any element of $\widehat{T}\times\sigma$ is representable by $\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\times\sigma$ for some $t \in \mathbb{C}^\times$, and the Satake transform is defined by

$$f^\vee\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\times\sigma\right) = \sum_n F_f(n)t^n\,.$$

Analogously, the Satake transform defines an isomorphism from the convolution algebra $\mathbb{H}'$ of K'-biinvariant compactly supported functions f' on G'/Z', to the algebra $\mathbb{C}[\widehat{T}'\times\sigma]^W$ of W'-invariant Laurent polynomials on the torus $\widehat{T}' = \widehat{T}_0\times\widehat{T}_0$, where $\widehat{T}_0$ = diagonal subgroup of $S = SL(2,\mathbb{C})$. Note that $(t, t')\times\sigma$ is conjugate to $(tt', 1)\times\sigma$ under $S\times S$. The Satake transform is defined by

$$f'^\vee\left(\left(\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}, I\right)\times\sigma\right) = \sum_n F_{f'}(n)t^n\,,$$

where

$$F_{f'}(n) = \Delta(\delta)\int_{G'/A'} f'(g\delta g^{-1})dg\,, \qquad \delta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad |a/b| = |\underline{\pi}|^n, \quad a \neq b,$$

and

$$\Delta(\delta) = |(a-b)^2/ab|_E^{1/2}.$$

The normalized orbital integral $F_{f'}(n)$ is independent of the choice of the representative δ, since f' is K-biinvariant and Z'-invariant.

The map $b_\kappa^* : \mathbb{H}' \to \mathbb{H}$ dual to the unstable base-change map $b_\kappa : \widehat{U} \to \widehat{G}'$ is defined by $f^\vee(t\times\sigma) = f'^\vee(b_\kappa(t\times\sigma))$. Since

$$b_\kappa\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\times\sigma\right) = \left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -t & 0 \\ 0 & -1 \end{pmatrix}\right)\times\sigma$$

is conjugate to

$$\left(\begin{pmatrix} -t & 0 \\ 0 & -t^{-1} \end{pmatrix}, I\right)\times\sigma\,,$$

we obtain

$$\sum_n F_f(n)t^n = \sum_n F_{f'}(n)(-t)^n \qquad \text{if} \qquad f' = b_\kappa^*(f)\,,$$

hence $F_f(n) = (-1)^n F_{f'}(n)$ for every rational integer n.

Definition. Spherical functions f and f' are called *corresponding* if $f = b_\kappa^*(f')$, namely $F_f(n) = (-1)^n F_{f'}(n)$ for every rational integer n.

A standard change of variables shows that

$$F_{f'}(n) = q_F^n \int_E f'\begin{pmatrix} \underline{\pi}^n & x \\ 0 & 1 \end{pmatrix} dx$$

and

$$F_f(n) = |\alpha\overline{\alpha}|_F^{1/2}\int_F f\left(\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx \qquad (|\alpha|_E = |\alpha\underline{\alpha}|_F = q_E^{-n} = q_F^{-2n})\,.$$

Extending f to $Z'U$ by $f(zg) = \omega(z)^{-1}f(g)$ ($z \in Z'$, $g \in U$; note that $\omega(z) = (-1)^n$ if $|z| = |\underline{\pi}|^n$), we conclude that

$$F_f(n) = (-1)^n q_F^n \int_F f\begin{pmatrix} \underline{\pi}^{2n} & x \\ 0 & 1 \end{pmatrix} dx\,,$$

and so

$$\int_F f\begin{pmatrix} \underline{\pi}^{2n} & x \\ 0 & 1 \end{pmatrix} dx = \int_E f'\begin{pmatrix} \underline{\pi}^n & x \\ 0 & 1 \end{pmatrix} dx \qquad \text{if} \quad f = b_\kappa^*(f')\,.$$

Now the restriction of f from $Z'G_0 = Z'U$ to G_0 is a $GL(2,R)$-biinvariant function on $G = GL(2,F)$ which is supported on G_0 and transforms under $Z(G) = F^\times$ via $\omega \mid F^\times = \chi_{E/F}$. The orbital integral of f on G, which is

$$F_f^G(n) = \Delta(\gamma) \int_{G/A} f(g\gamma g^{-1})dg = q_F^{n/2} \int_F f\begin{pmatrix} \pi^n & x \\ 0 & 1 \end{pmatrix} dx\,,$$

is then zero when n is odd, and satisfies $F_f^G(2n) = F_{f'}(n)$ for all n. It follows that our f, f' are the functions f_v, f'_w of [Y], p. 99, l. 2, and p. 98, l. 1, on G, G' with central characters $\chi_{E/F}$, 1; $f = f_v$ is the image of $f' = f'_w$ under the Hecke algebra homomorphism $\mathbb{H}' \to \mathbb{H}_G$ dual to the base-change-for-$GL(2)$ map $G(\mathbb{C}) \times \mathrm{Gal}(E/F) \to \widehat{G}'$, which maps g to (g,g) and reduced to the identity on the second factor. Now the corresponding f, f' are shown in [Y], §V, to satisfy the identities (1), (2) on [Y], p. 97, which are the same as (1), (2) on [Y], p. 82, which, as noted above, are the same as the identities of our Proposition 3 since f transforms under Z' according to ω. This completes the proof of Proposition 3 for spherical f, f' when E/F is unramified. The case where f, f' are spherical and $E = F \oplus F$ easily follows from the definitions. Proposition 3 follows. As noted above so does Proposition 2, and its Corollary, which we proceed to study.

We now return to global notations. Thus E/F is a separable quadratic extension of global fields, and ω is a unitary character of the center $\mathbb{Z} \cong \mathbb{A}_E^1/E^1$ $(= \{z/\overline{z}\,;\ z \in \mathbb{A}_E^\times/E^\times\})$ of $\mathbb{U}$. For any unitary character μ of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}_E^1$ is ω, and a complex number s, consider the Hilbert space $H(\mu,s)$ of functions $\phi : \mathbb{U} \to \mathbb{C}$ which satisfy

$$\phi\left(\begin{pmatrix} a & * \\ 0 & \overline{a}^{-1} \end{pmatrix} g\right) = |a|_E^{s+1/2}\mu(a)\phi(g) \qquad (a \in \mathbb{A}_E^\times\,,\ g \in \mathbb{U})$$

and $\int_{\mathbb{K}} |\phi(k)|^2 dk < \infty$. The restriction-to-$\mathbb{K}$ map $\phi \to \phi \mid \mathbb{K}$ defines an isomorphism from $H(\mu,s)$ to $H(\mu) = H(\mu,0)$; we identify the spaces $H(\mu,s)$ with the fixed space $H(\mu)$ via this map, and denote by $\phi(\mu,s)$ the element in $H(\mu,s)$ corresponding to $\phi(\mu)$ in $H(\mu)$. Let $I(\mu,s)$ be the representation of $\mathbb{U}$ on $H(\mu,s)$ by right translation. The Eisenstein series

$$E(g,\phi,\mu,s) = \sum_{\gamma \in B\backslash U} \phi(\gamma g,\mu,s) \qquad (\phi = \phi(\mu) \in H(\mu))$$

converges absolutely on $\mathrm{Re}(s) > \frac{1}{2}$, and has analytic continuation to $\mathbb{C}$. The kernel on the continuous spectrum is given by

$$K_{f,c}(x,y) = \frac{1}{4\pi} \sum_\mu \sum_\phi \int_{-\infty}^{\infty} E(x, I(\mu,it;f)\phi,\mu,it)\overline{E}(y,\phi,\mu,it)dt\,,$$

where ϕ ranges over an orthonormal basis $\{\phi_\alpha\}$ of $H(\mu)$. Here μ ranges over a set of representative under the equivalence relation $\mu' \sim \mu$ if $\mu'(a) = \mu(a)|a|^s$ $(s \in \mathbb{C})$ for all $a \in \mathbb{A}_E^\times$.

On $\mathrm{Re}(s) > \frac{1}{2}$ we have

$$(*) \qquad E_\psi(\phi,\mu,s) = \int_{\mathbb{N}/N} E(n,\phi,\mu,s)\overline{\psi}(n)dn = \int_{\mathbb{N}} \phi(Jn,\mu,s)\overline{\psi}(n)dn$$
$$= L(\mu,2s+1,V)^{-1} \prod_{v\in V} \int_{N_v} \phi_v(Jn,\mu_v,s)\overline{\psi}_v(n)dn\,,$$

assuming $\phi = \otimes\phi_v$, $\phi_v \in H(\mu_v,s)$ (= local analogue of $H(\mu,s)$), as we do. Here

$$L(\mu,t,V) = \prod_{v\notin V}(1-q_v^{-t}\mu_v(\underline{\pi}_v))^{-1}\,,$$

and V is a finite set of places of F containing the archimedean ones and those where E_v/F_v, μ_v or ψ_v are ramified, and $\phi_v \neq \phi_v^0$. Every local integral on the right of $(*)$ is holomorphic on $i\mathbb{R}$, $L(\mu,2t+1,V)\neq 0$ on $t\in i\mathbb{R}$ and its inverse is $O(t^n)$ for some $n>0$; hence the left side of $(*)$ is $O(t^n)$ on $s=it\in i\mathbb{R}$, namely slowly increasing. Also

$$(I(\mu,s;f)\phi,\phi') = \int_{\mathbb{K}}\int_{\mathbb{U}/\mathbb{Z}} f(g)\phi(kg,\mu,s)\overline{\phi}'(k,\mu,s)dkdg$$
$$= \int_{\mathbb{A}_E^\times/\mathbb{A}_E^1} |a|_E^{s+\frac{1}{2}}\mu(a)\int_{\mathbb{K}}\int_{\mathbb{K}}\int_{\mathbb{N}} f\left(k^{-1}\begin{pmatrix} a & 0\\ 0 & \overline{a}^{-1}\end{pmatrix} nk'\right)\phi'(k)\overline{\phi}'(k)dndkdk'd^\times a$$
$$= \prod_v \int_{E_v^\times/E_v^1} |a|_{E_v}^{s+\frac{1}{2}}\mu_v(a)\int_{K_v}\int_{K_v}\int_{N_v} f_v\left(k^{-1}\begin{pmatrix} a & 0\\ 0 & \overline{a}^{-1}\end{pmatrix} nk'\right)\phi_v(k')\overline{\phi}'_v(k)dndkdk'd^\times a\,,$$

if $f=\otimes f_v$, $\phi=\otimes\phi_v$, $\phi'=\otimes\phi'_v$ (v ranges over the places of F). At a place v where f_v is K_v-biinvariant, the local integral vanishes unless $\phi_v=\phi_v^0$ and $\phi'_v=\phi_v^0$ (= the unique right K_v-invariant element of $H(\mu_v)$ whose value on K_v is $1/\mathrm{vol}(K_v)$), and μ_v is unramified. In this case the local integral is the value

$$f_v^\vee(\tau(\mu_v,s))\,,\qquad \tau(\mu_v,s)=\tau(I(\mu_v,s)) = \begin{pmatrix}\mu_v(\underline{\pi}_v)q_v^{-2s-1} & 0\\ 0 & 1\end{pmatrix}\times\sigma\in\widehat{U}$$

of the Satake transform $f_v^\vee$ of $f_v\in\mathbb{H}_v$ at the class in $\widehat{U}$ which parametrizes $I(\mu_v,s)$. In any case, $(I(\mu,it;f)\phi,\phi')$ is rapidly decreasing in t, being the Fourier transform of a compactly supported (modulo $\mathbb{Z}$) smooth function. We conclude:

4. Proposition. *Let V be a finite set of F-places, containing the archimedean and ramified places. Suppose $f_v\in\mathbb{H}_v$ for all $v\notin V$, $f_v=f_v^0$ for almost all v, and $f=\otimes f_v$. Then*

$$\int_{\mathbb{N}/N}\int_{\mathbb{N}/N} K_{f,c}(n,n')\psi(n'n^{-1})dndn'$$
$$= \frac{1}{4\pi}\sum_\mu\int_{-\infty}^{\infty}\left[\prod_{v\notin V} f_v^\vee(\tau(\mu_v,it))\right]\cdot\left[\sum_{\alpha,\beta} E_\psi(\phi_\alpha,\mu,it)\cdot\overline{E}_\psi(\phi_\beta,\mu,it)\cdot\int_{\mathbb{K}}dk\int_{\mathbb{K}}dk'\right.$$
$$\left.\phi_\beta(k')\overline{\phi}_\alpha(k)\cdot\prod_{v\in V}\int_{E_v^\times/E_v^1}|a|_{E_v}^{it+1/2}\mu_v(a)\int_{N_v} f_v\left(k_v^{-1}\begin{pmatrix} a & 0\\ 0 & \overline{a}^{-1}\end{pmatrix}nk'_v\right)dnd^\times a\right]dt\,.$$

For any pair μ_1, μ_2 of unitary characters on $\mathbb{A}_E^\times/E^\times$ with $\mu_1\mu_2 = \omega'\kappa'$, where $\omega'(x) = \omega(x/\overline{x})$, and a complex number s, consider the Hilbert space

$$H(\mu_1,\mu_2,s) = \{\Phi : \mathbb{G} \to \mathbb{C};\ \Phi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = |a/b|_E^{s+1/2}\mu_1(a)\mu_2(b)\phi(g)\,,$$
$$\int_{\mathbb{K}'} |\Phi(k)|^2 dk < \infty\}\,,$$

the $\mathbb{G}$-module structure $I(\mu_1,\mu_2,s)$, the identification (of spaces) of $H(\mu_1,\mu_2,s)$ with $H(\mu_1,\mu_2) = H(\mu_1,\mu_2,0)$, an orthonormal basis $\{\Phi_\alpha\}$, and the Eisenstein series
$E(g,\Phi,\mu_1,\mu_2,s)$, all defined analogously to the case of $\mathbb{U}$. The kernel on the continuous spectrum is

$$K_{f',c}(x,y) = \frac{1}{4\pi}\sum_{\mu_1,\mu_2}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(I(\mu_1,\mu_2,it;f')\Phi_\beta,\Phi_\alpha)E(x,\Phi_\alpha,\mu_1,\mu_2,it)$$
$$\overline{E}(y,\Phi_\beta,\mu_1,\mu_2,it)dt\,.$$

The first sum ranges over the pairs μ_1,μ_2 with $\mu_1\mu_2 = \omega'\kappa'$ up to the equivalence $(\mu_1,\mu_2) \sim (\mu_1\nu^s, \mu_2\nu^{-s})$ ($s \in \mathbb{C}$, $\nu(x) = |x|_E$ on $x \in \mathbb{A}_E^\times$). We have to compute

$$\int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{\mathbb{N}'/N'} K_{f',\mathrm{cont}}(g,n)\psi'(n)dgdn = \lim_{c\to\infty}\int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{\mathbb{N}'/N'} T^c K_{f',\mathrm{cont}}(g,n)\psi'(n)dgdn\,,$$

where T^c ($c > 1$) is the usual truncation operator, whose definition is recalled in [JL], (1), p. 264. As in the case of U, the integral

$$\int_{\mathbb{N}'/N'} E(n,\Phi,\mu_1,\mu_2,s)\overline{\psi}'(n)dn = E_{\psi'}(\Phi,\mu_1,\mu_2,s)$$

is slowly increasing (i.e. $O(t^n)$ for some $n > 0$) on $s = it$, real t. Further we have that

$$(I(\mu_1,\mu_2,it;f')\Phi_\beta,\Phi_\alpha) = \prod_{v\notin V} f_v'^{\vee}(\tau(\mu_{1v},\mu_{2v};it)) \cdot \int_{\mathbb{K}'} dk \int_{\mathbb{K}'} dk' \cdot \Phi_\beta(k')\overline{\Phi}_\alpha(k)\cdot$$
$$\prod_{v\in V}\int_{A_v'/Z_v'} \mu_{1v}(a)\mu_{2v}(b)|a/b|_{E_v}^{it+1/2}\int_{N_v'} f_v'\left(k^{-1}\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} nk'\right)(d^\times a d^\times b)dn$$

is a Schwartz function, rapidly decreasing, in $t \in \mathbb{R}$. Thus we need to consider

$$\int_{\mathbb{Z}G\backslash\mathbb{G}} T^c E(g,\Phi,\mu_1,\mu_2,it)dg\,.$$

We introduce some notations. Write μ for (μ_1,μ_2). Let $\delta(\mu)$ be 0 unless μ_1 is trivial on $\mathbb{A}_F^\times$, in which case $\delta(\mu) = \mathrm{vol}(\mathbb{A}_F^u/F^\times)$, where $\mathbb{A}_F^u = \{a \in \mathbb{A}_F^\times;\ \|a\| =$

1}. Put $\varepsilon(\mu) = 0$ unless $\mu_1\overline{\mu}_2 = 1$, in which case $\varepsilon(\mu) = \mathrm{vol}(\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times)$. The intertwining operator $M(\mu, s) : H(\mu) \to H(\widetilde{\mu})$, where $\widetilde{\mu} = (\mu_2, \mu_1)$, is defined by

$$(M(\mu,s)\Phi)(g,\widetilde{\mu},-s) = \int_{\mathbb{N}'} \Phi(Jng,\mu,s)dn$$

for $\mathrm{Re}(s) > 1/2$, and has analytic continuation to the entire complex plane. Recall that

$$G' = GB' \cup G\eta B' = B'G \cup B'\eta^{-1}G\,, \qquad\qquad \eta = \begin{pmatrix} -\sqrt{\theta} & \sqrt{\theta} \\ 1 & 1 \end{pmatrix},$$

and put

$$T = G \cap \eta B'\eta^{-1} = \left\{ \eta \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \eta^{-1}\,;\ a \in E^\times \right\} = \left\{ \begin{pmatrix} \alpha & \beta\theta \\ \beta & \alpha \end{pmatrix}\,;\ \alpha,\beta \in F \right\} \cap G\,.$$

If $\mu_1\overline{\mu}_2 = 1$ define

$$J(\mu,s)\Phi = \int_{T(\mathbb{A}_F)\backslash\mathbb{G}} \Phi(\eta^{-1}g,\mu,s)dg$$

for $\mathrm{Re}(s) > 1/2$ and by analytic continuation on $\mathbb{C}$.

Lemma. *The integral of $T^cE(g,\Phi,\mu,s)$ over $\mathbb{Z}G\backslash\mathbb{G}$ is equal to*

$$\frac{\delta(\mu)}{2s}\left[c^s\int_{\mathbb{K}}\Phi(k)dk - c^{-s}\int_{\mathbb{K}}(M(\mu,s)\Phi)(k)dk\right] + \varepsilon(\mu)J(\mu,s)\Phi\,.$$

If Φ is $\mathbb{K}$-finite, then

$$J_1(\mu,s)\Phi = J(\mu,s)\Phi \cdot L(1+2s,\chi_{E/F}\cdot\mu_1 \mid F)/L(2s,\mu_1 \mid F)$$

is an elementary function of s (i.e. a linear combination of products of rational and exponential functions of s) which is holomorphic on $Re(s) = 0$. Here $\mu_1 \mid F$ denotes the restriction of μ_1 to $\mathbb{A}_F^\times$. Moreover, $\int T^cE(g,\Phi,\mu,s)dg$ is analytic and of polynomial growth on $i\mathbb{R}$.

Proof. A proof of this can be found in [JL], §8, in the special case when $\mu_1\mu_2 = 1$. The general case follows along the same lines, and we merely indicate the changes to be made in [JL]. The $\chi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ of [JL], p. 287, has to be replaced by a pair $\chi = (\chi_1,\chi_2)$ ($= \mu = (\mu_1,\mu_2)$ in our notations). Thus $\mathbb{H}(s,\chi)$ (of [JL]) should be read: $\mathbb{H}(s,\chi_1,\chi_2)$, and $\chi(a/b)$ of (1), p. 287, should be $\chi_1(a)\chi_2(b)$. Also the γ on p. 287, lines $-10/-9/-8$ should be χ. $\mathbb{H}(\overline{\chi})$ on p. 288, l. 1, should be $\mathbb{H}(\widetilde{\chi})$, $\widetilde{\chi} = (\chi_2,\chi_1)$. The assumption in "(6) Lemma", p. 288, and the surrounding lines, should be $\chi_1 = \chi_2$ (or $\chi_1/\chi_2 = 1$), instead of $\chi^2 = 1$; χ^2 should be read as χ_1/χ_2. In "(7) Lemma" the assumption should be replaced by: $\chi_1/\chi_2 = \chi_1\overline{\chi}_2 \neq 1$, and $\chi_1 \mid \mathbb{A}_F^\times = 1$. In the proof, $\overline{\chi}^2$ (p. 288, l. -1) should be $\overline{\chi}_1/\overline{\chi}_2$, and e.g. on

p. 289, l. 7, χ_w^2 should be χ_{1w}/χ_{2w}, $\overline{\chi}_w^2$ should be $\overline{\chi}_{1w}/\overline{\chi}_{2w}$, on l. 17: $\chi \to \chi_1$, $\chi_{v1} \to \chi_{1v1}$, l. 18: $\chi_{v2} \to \chi_{2v2}$. On p. 290, l. 13, 17, 18, 19, 24, replace χ^2 by χ_1/χ_2, and $\chi(\det g)$ by $\chi_1(\det g)$; also note that $|\det g|$ on l. 13, 24, is $|\det g|_E$ (see p. 285, l. -1). On p. 292, l. 5 and below, replace χ by χ_1, on l. 10 put a comma between $E_{\mathbb{A}}^\times$ and F_∞^+, on l. -3 replace $\lambda^{-1}1\lambda$ by $\lambda\ell\lambda^{-1}$ (1 on l. -2 should also be ℓ). On p. 293, l. 2, replace $\chi(t^{\sigma-1})$ by $\chi_1(t)\chi_2(\overline{t})$ (and note that $|t^{\tau-1}| = 1$, and $t \in F_{\mathbb{A}}^\times E^\times \backslash E_{\mathbb{A}}^\times$). On l. 3, 4, replace $\chi^{\sigma-1}$ by $\chi_1\overline{\chi}_2$, and erase all mention of μ. In the expression for $h(g,s)$ on l. 11/12 (and l. -3), replace χ^2 by χ_1/χ_2, $\chi(\det g)$ by $\chi_1(\det g)$, $|\det g|$ by $|\det g|_E$, and add $d^\times t$. On l. -8, $F \to F_{\mathbb{A}}$. On l. -9, "$1 =$" should be "$\ell =$", with the same ℓ which appears previously on this line; $1 \to \ell$ also on l. -6. Now μ^2 on l. -6, -2, should be replaced by χ_1. The same applies to p. 294, l. 3, 7, 8, -12, -2; $\chi^2 \to \chi_1/\chi_2$ on l. 2, 7. On l. 13, χ^{s-1} (intended to be $\chi^{\sigma-1}$) should be $\chi_1\overline{\chi}_2$, and on l. -8, $E \to T^cE$ and $F \to F_{\mathbb{A}}$. This completes the proof of the first two assertions in our lemma. For the last assertion, on p. 295, l. -9, -8, -4, -1, replace χ by χ_1, $\chi^{\sigma-1}$ by $\chi_1\overline{\chi}_2$, $\chi^2 \neq 1$ by $\chi_1 \neq \chi_2$, erase the sentences on l. $-3/-1$, and replace μ^2 by $\chi_1 \mid \mathbb{A}_F^\times$ in the following lines (l. -1, and p. 296, l. 1, 2, 4, 9). Again, on p. 296, $\chi^2 \to \chi_1/\chi_2$ (l. 1, 4), $\chi \to \chi_1$ (l. 3), $\chi^{\sigma-1} \to \chi_1\overline{\chi}_2$ (l. 3), and note that our $\chi_{E/F}$ is denoted in [JL] by η. The lemma follows.

By virtue of the lemma, $\int\int K_{f',\mathrm{cont}}(g,n)\psi'(n)dgdn$ can be computed as in [Y], pp. 112/3, where the case of $(\mu_1,\mu_2) = (\mu,\mu^{-1})$ is considered. We merely have to replace (μ,μ^{-1}) of [Y] by (μ_1,μ_2) on p. 112, l. 2, 3, -7, -6, -5, -3, -1, and p. 113, l. 1, 3, 5, 6, 7, 8, 10, 11, 12, 13, -3, -2, -1. Moreover, on p. 112, l. -9, and p. 113, l. 4, 9, -4, replace $\mu \mid F_{\mathbb{A}}^\times$ by $\mu_1 \mid \mathbb{A}_F^\times$ and $\mu \mid E_{\mathbb{A}}^0$ by $\mu_1\mu_2$. In summary:

5. Proposition. *Give $f' = \otimes f'_v$ with $f'_v \in \mathbb{H}'_v$ for all $v \notin V$, and $f'_v = {f'_v}^0$ for* *almost all v, we have*

$$\int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{\mathbb{N}'/N'} K_{f',c}(g,n)\psi'(n)dgdn$$

$$= \frac{vol(\mathbb{A}_E^\times/\mathbb{A}_F^\times E^\times)}{4\pi}\sum_\mu \int_{-\infty}^{\infty}\left\{\left[\prod_{v\notin V} f_v'^{\vee}(\tau(\mu_v,\overline{\mu}_v^{-1};it))\right]\cdot\sum_{\alpha,\beta}\overline{E}_{\psi'}(\Phi_\beta,\mu,\overline{\mu}^{-1};it)\right.$$

$$\cdot\, J(\mu,\overline{\mu}^{-1};it)\Phi_\alpha\cdot\int_{\mathbb{K}'}dk\int_{\mathbb{K}'}dk'\cdot\left[\Phi_\beta(k')\overline{\Phi}_\alpha(k)\left\{\prod_{v\in V}\int_{A_v'/Z_v'}\mu_v(a/\overline{b})|a/b|_{E_v}^{it+1/2}\right.\right.$$

$$\left.\left.\left.\cdot\int_{N_v'} f_v'\left(k_v^{-1}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} nk_v'\right)(d^\times a d^\times b)dn\right\}\right]\right\} dt$$

$$+\frac{vol(\mathbb{A}_F^u/F^\times)}{4\pi}\sum_{\mu_1\neq\mu_2\ ,\ \mu_i|\mathbb{A}_F^\times=1}\left[\prod_{v\notin V} f_v'^{\vee}(\tau(\mu_{1v},\mu_{2v};0))\right]$$

$$\cdot\sum_{\alpha,\beta}\int_{\mathbb{K}}\Phi_\alpha(k)dk\cdot\overline{E}_{\psi'}(\Phi_\beta;\mu_1,\mu_2;0)\cdot\int_{\mathbb{K}'}dk'\int_{\mathbb{K}'}dk\cdot\left[\Phi_\beta(k')\overline{\Phi}_\alpha(k)\right.$$

$$\left.\cdot\prod_{v\in V}\int_{A_v'/Z_v'}\mu_{1v}(a)\mu_{2v}(b)|a/b|_{E_v}^{1/2}\int_{N_v'} f_v'\left(k_v^{-1}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix} nk_v'\right)(d^\times a d^\times b)dn\right].$$

Next we describe spectrally the kernels on the cuspidal spectra, and their contributions to the identity of Corollary (to Proposition) 2. As always, $f=\otimes f_v$ and $f'=\otimes f_v'$ are matching, and $f_v'\in\mathbb{H}_v'$, $f_v=b_\kappa^*(f_v')\in\mathbb{H}_v$ for v outside some fixed finite set V. Put $\mathbb{K}'(V)=\prod_{v\notin V}K_v'$ and $\mathbb{K}(V)=\prod_{v\notin V}K_v$. Given a cuspidal $\mathbb{G}'$-module $\pi'=\otimes\pi_v'$ with central character $\omega'\kappa'$, let $\pi'^{\mathbb{K}'(V)}$ denote its space of $\mathbb{K}'(V)$-fixed vectors, and denote by $\{\Lambda\}$ an orthonormal basis of $\pi'^{\mathbb{K}'(V)}$ (if $\neq\{0\}$). Then

$$K_{f',0}(x,y)=\sum_{\pi'}\left[\prod_{v\notin V} f_v'^{\vee}(t(\pi_v'))\right]\sum_\Lambda\left(\prod_{v\in V}\pi_v'(f_v')\right)\Lambda(x)\overline{\Lambda}(y)\,.$$

Similarly

$$K_{f,0}(x,y)=\sum_{\pi}\left[\prod_{v\notin V} f_v^{\vee}(t(\pi_v))\right]\sum_\lambda\left(\prod_{v\in V}\pi_v(f_v)\right)\lambda(x)\overline{\lambda}(y)\,,$$

where π ranges over all cuspidal $\mathbb{U}$-modules with central character ω and $\pi^{\mathbb{K}(V)}\neq\{0\}$, and $\{\lambda\}$ is an orthonormal basis of $\pi^{\mathbb{K}(V)}$.

Corollary 2 can be restated as follows: put $f_V = \bigotimes_{v\in V} f_v$, $f'_V = \bigotimes_{v\in V} f'_v$. Then

$$\sum_{\pi'} a(\pi', f'_V)\left[\prod_{v\notin V} f_v'^{\vee}(t(\pi'_v))\right] - \sum_{\pi} b(\pi, f_V)\left[\prod_{v\notin V} f_v'^{\vee}(b_\kappa(t(\pi_v)))\right]$$

$$+ \sum_{\mu_1\neq\mu_2, \mu_i|\mathbb{A}_F^\times=1} c(\mu_1,\mu_2; f'_V)\left[\prod_{v\notin V} f_v'^{\vee}(t(I(\mu_{1v},\mu_{2v})))\right]$$

$$= \sum_{\mu}\int_{-\infty}^{\infty} d(\mu,t,f_V)\left[\prod_{v\notin V} f_v'^{\vee}(t[I(\mu_v\kappa_v\nu_v^{it}, \overline{\mu}_v^{-1}\kappa_v\nu_v^{-it})])\right] dt$$

$$- \sum_{\mu}\int_{-\infty}^{\infty} e(\mu,t,f'_V)\left[\prod_{v\notin V} f_v'^{\vee}(t[I(\mu_v\nu_v^{it}, \overline{\mu}_v^{-1}\nu_v^{-it})])\right] dt\,.$$

Since all sums and integrals in the trace formula are absolutely convergent, we have that

$$\sum_{\pi'} |a(\pi')|\,,\quad \sum_{\pi} |b(\pi)|\,,\quad \sum_{\mu_1\neq\mu_2} |c(\mu_1,\mu_2)|\,,\quad \sum_{\mu}\int_{-\infty}^{\infty} (|d(\mu,t)| + |e(\mu,t)|)\,dt$$

are finite. Note that $\nu_v(x) = |x|_{E_v}$. Consequently, a standard argument of "generalized linear independence of characters", due to Langlands [L], based on the Stone-Weierstrass theorem and elementary unitarity estimates, implies that both sides in the equality above, the discrete and the continuous measures, are both zero, and moreover we have:

6. Proposition. *Fix V and classes $t'_v \in \widehat{G}'_v$ for every v in V. Then*

$$\sum_{\pi'} a(\pi') + \sum_{\mu_1\neq\mu_2\,,\,\mu_i|\mathbb{A}_F^\times=1} c(\mu_1,\mu_2) = \sum_{\pi} n(\pi) b(\pi)\,.$$

The first sum ranges over all cuspidal $\mathbb{G}'$-modules π' with central character $\omega'\kappa'$, $\pi'^{\mathbb{K}'(V)} \neq 0$, and $t(\pi'_v) = t'_v$. The second ranges over all unordered pairs μ_1, μ_2 of distinct characters of $\mathbb{A}_E^\times/E^\times\mathbb{A}_F^\times$ with $\mu_1\mu_2 = \omega'\kappa'$ and which are unramified outside V, with $t(I(\mu_{1v},\mu_{2v})) = t'_v$. The third sum ranges over all cuspidal $\mathbb{U}$-modules with central character ω, unramified outside V and $b_\kappa(t(\pi_v)) = t'_v$, up to equivalence; $n(\pi)$ is the multiplicity of π in $L^2_{\omega,0}(U)$.

The rigidity theorem for $GL(2)$ implies that the sum over π' contains at most one term, so does the sum over (μ_1,μ_2), and at most one of the two sums is non-zero.

To prove Theorem 1, write

$$W_{\Lambda,\psi'}(g) = \int_{\mathbb{N}'/N'} \Lambda(ng)\overline{\psi}'(n)dn$$

for the Whittaker function associated with the cusp form Λ in $\pi' \subset L^2_{\omega'\kappa',0}(G')$, and

$$W_{\lambda,\psi}(g) = \int_{\mathbb{N}/N} \lambda(ng)\overline{\psi}(n)dn \qquad (\lambda \in \pi \subset L^2_{\omega,0}(U))\,.$$

The map $\alpha' : \Lambda \mapsto W_{\Lambda,\psi'}$ is a $\mathbb{G}'$-module embedding of π' in $\mathrm{Ind}(\psi'; \mathbb{G}', \mathbb{N}')$ for every cuspidal π', and $\alpha'(\pi')$, called the "Whittaker model" of π', has multiplicity one in $\mathrm{Ind}(\psi'; \mathbb{G}', \mathbb{N}')$. The map $\alpha : \lambda \mapsto W_{\lambda,\psi}$ $(\lambda \in \pi \subset L^2_{\omega,0}(U))$ may be the zero map, in which case we say that π has no Whittaker model. Otherwise it defines an embedding of π in $\mathrm{Ind}(\psi, \mathbb{U}, \mathbb{N})$, and the "Whittaker model" $\alpha(\pi)$ of π has multiplicity (at most) one in $\mathrm{Ind}(\psi; \mathbb{U}, \mathbb{N})$. If π (or π') has a Whittaker model, we say that it is non-degenerate. Also put

$$D(\Lambda) = \int_{\mathbb{Z}G\backslash\mathbb{G}} \Lambda(g)dg\,.$$

Then π' is (G-)distinguished if and only if D does not vanish on $\pi' \subset L^2_{\omega'\kappa',0}(G')$. Now the coefficients in Proposition 6 are as follows:

$$\begin{aligned} a(\pi') &= a(\pi', f'_V) = \sum_{\Lambda\in\pi'} D(\pi'_V(f'_V)\Lambda)\overline{W}_{\Lambda,\psi'}(e)\,, \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \\ b(\pi) &= b(\pi, f_V) = \sum_{\lambda\in\pi} W_{\pi_V(f_V)\lambda,\psi}(e)\overline{W}_{\lambda,\psi}(e)\,, \\ c(\mu_1,\mu_2) &= c(\mu_1,\mu_2,f'_V) \\ &= (2\pi)^{-1}\mathrm{vol}(\mathbb{A}^u_F/F^\times)\sum_{\Phi}\int_{\mathbb{K}}(I(\mu_1,\mu_2;f'_V)\Phi)(k)dk\cdot\overline{E}_{\psi'}(\Phi,\mu_1,\mu_2,0)\,. \end{aligned}$$

Note that $a(\pi')$, $b(\pi)$, $c(\mu_1,\mu_2)$ are independent of the choice of bases for π', π and $H(\mu_1,\mu_2)$, and they depend on $\pi', \pi, I(\mu_1,\mu_2)$ only up to equivalence.

To prove Theorem 1, suppose that π' is distinguished. Then there is $\Lambda' \in \pi'$ with $D(\Lambda') \neq 0$. Choose a finite set V of places of F such that Λ' is $\mathbb{K}'(V)$-invariant. Choose $\Lambda \in \pi'^{\,\mathbb{K}'(V)}$ with $W_{\Lambda,\psi'}(e) \neq 0$. Since the set $\{\pi'_V(f'_V);$ all $f'_V\}$ acts transitively on $\pi'^{\,\mathbb{K}'(V)}$, we may and do choose f'_V with $\pi'_V(f'_V)\Lambda = \Lambda'$, and $\pi'_V(f'_V)\Lambda_1 = 0$ for every Λ_1 orthogonal to Λ. Apply Proposition 6 with $\{t'_v = t(\pi'_v)\,;\ v \notin V\}$. Then the left side of the identity of Proposition 6 is

$$a(\pi') = D(\Lambda')\overline{W}_{\Lambda,\psi'}(e) \neq 0\,.$$

Hence the sum on the right is non-empty, and there is some cuspidal $\mathbb{U}$-module π with $b(\pi) \neq 0$. Hence π' is the unstable (via b_κ) base-change lift of the non-degenerate π.

Given $\mu_1, \mu_2 : \mathbb{A}^\times_E/\mathbb{A}^\times_F E^\times \to \mathbb{C}^\times$, since $I = I(\mu_1,\mu_2)$ is non-degenerate there is $\Phi \in H(\mu_1,\mu_2)$ with $E_{\psi'}(\Phi,\mu_1,\mu_2,0) \neq 0$. Clearly there is some $\mathbb{K}$-invariant

Φ' in $H(\mu_1,\mu_2)$. Choose a sufficiently large set V so that both Φ and Φ' be $\mathbb{K}'(V)$-invariant. Then there is f'_V with $I(f'_V)\Phi=\Phi'$, and $I(f'_V)\Phi_1=0$ for every $\mathbb{K}'(V)$-invariant $\Phi_1\in H(\mu_1,\mu_2)$ orthogonal to Φ. With these choices $c(\mu_1,\mu_2)\neq 0$. Choosing $\{t_v : v\notin V\}$ so that the only term on the left of the identity of Proposition 6 is that corresponding to (μ_1,μ_2), we conclude that there exists a non-degenerate (with $b(\pi)\neq 0$) cuspidal $\mathbb{U}$-module π whose unstable base-change lift is $I(\mu_1,\mu_2)$.

The remaining claim of Theorem 1 asserts that each cuspidal non-degenerate $\mathbb{U}$-module π_0 lifts via the unstable base-change map to an automorphic $\mathbb{G}'$-module π' which is either cuspidal or of the form $I(\mu_1,\mu_2)$, $\mu_i:\mathbb{A}_E^\times/\mathbb{A}_F^\times E^\times\to\mathbb{C}^\times$, $\mu_1\neq\mu_2$. This, and in fact stronger results, are proven in [F3]; but our proof is independent of [F3]. Fix such π_0. Since π_0 is non-degenerate, there is some vector λ_0 in π_0 with $W_{\lambda_0,\psi}(e)\neq 0$. Choose a sufficiently large finite set V such that λ_0 is $\mathbb{K}(V)$-invariant. For each v in V choose a compact open subgroup K_{1v} in K_v such that λ_0 is K_{1v}-invariant. Put $\mathbb{K}_1=\mathbb{K}(V)\prod_{v\in V}K_{1v}$. Choose an orthonormal basis to the space of $\mathbb{K}_1$-fixed vectors in $L^2_{\omega,0}(U)$, and extend it to an orthonormal basis of the space of $\mathbb{K}(V)$-invariant vectors in $L^2_{\omega,0}(U)$. Let f_v ($v\in V$) be the unit element in the convolution algebra of K_{1v}-biinvariant complex-valued functions on U_v which transform under Z_v by ω_v^{-1} and are compactly supported modulo Z_v. Put $f_V=\bigotimes_{v\in V}f_v$. If λ is a $\mathbb{K}(V)$-fixed vector in $\pi\subset L^2_{\omega,0}(U)$, then $\pi_V(f_V)$ acts trivially on λ if λ is $\mathbb{K}_1$-invariant, and it maps λ to 0 if λ is in the orthogonal complement of this subspace. Apply Proposition 6 with the set V and sequence $\{t'_v=b_\kappa(t(\pi_{0v}))\,;\ v\notin V\}$. The right side of the identity in Proposition 6 is

$$\sum_\pi n(\pi)\sum_{\lambda\in\pi}W_{\pi_V(f_V)\lambda,\psi}(e)\overline{W}_{\lambda,\psi}(e)=\sum_\pi n(\pi)\sum_{\lambda\in\pi^{\mathbb{K}_1}}|W_{\lambda,\psi}(e)|^2\,.$$

Since $\lambda_0\in\pi_0^{\mathbb{K}_1}$ this is positive. Consequently either there is a pair (μ_1,μ_2) with $c(\mu_1,\mu_2)\neq 0$, and π lifts to $I(\mu_1,\mu_2)$ via the unstable base-change map b_κ, or there is a cuspidal π' with $a(\pi')\neq 0$, namely distinguished, which is the unstable base-change lift of π. This completes the proof of Theorem 1.

Local Theory.

Our next aim is to establish the following local analogue of Theorem 1. We adopt local notations, thus E/F is a quadratic separable extension of local fields, say with $\mathrm{char}F\neq 2$, $G'=GL(2,E)$, $G=GL(2,F)$, $U=U(2,E/F)$, etc. Recall that a G'-module π' is called (G-) distinguished if there exists a non-zero G-invariant form D on (the space of) π', and π' is irreducible. Recall that κ is a character on $E^\times$, trivial on $NF^\times$ and non-trivial on $F^\times$.

7. Theorem. *An infinite dimensional G'-module π' is G-distinguished if and only if it is the unstable (via b_κ) base-change lift of a U-module π. A one-dimensional G'-module $g\mapsto\chi'(g)$ is distinguished if and only if χ' is trivial on $F^\times$.*

Corollary. *A cuspidal $\mathbb{G}'$-module π' is (automorphically) $\mathbb{G}$-distinguished if all of its components π'_v are G_v-distinguished and at least one of the components is square-integrable or of the form $I(\mu_1, \mu_2)$, $\mu_1 \neq \mu_2$, $\mu_i : E_v^\times / F_v^\times \to \mathbb{C}^\times$.*

This corollary follows at once from the characterizations in Theorems 1 and 7 of the global and local G-distinguished G'-modules as the image of the unstable base-change map, and the results of [F3] which assert that a cuspidal $\mathbb{G}'$-module is an unstable base-change lift if each of its components is an unstable lift, and one of the components is as specified, and consequently not in the image of the stable base-change lifting. Note that each component of a distinguished global π' is clearly distinguished (see Lemma in proof of Proposition 8 below). By definition, a cuspidal π' is abstractly distinguished if all its components are, but it is not true that such π' is automorphically distinguished. In fact there are cuspidal π' whose components are all of the form $I(\mu_v, \overline{\mu}_v^{-1})$, for example π' might be everywhere unramified, and in the image of the stable lifting. Such a π' is not distinguished, but all of its local components are. The two notions of abstract and automorphic distinguishability differ also in the triple product situation considered in Harris-Kudla [HK] and D. Prasad [P], depending on whether the L-function $L(s, \pi_1 \times \pi_2 \times \pi_3)$ vanishes at $s = \frac{1}{2}$, or not.

The local unstable base-change lifting b_κ has been defined in [F3] in terms of character identities. The existence of this lifting is proven in [F3], and its basic properties established. Thus the central character of any (irreducible) σ-invariant (${}^\sigma\pi' \cong \pi'$) G'-module π' is trivial on $F^\times$, and is obtained as the base-change of a unique U-packet (a notion introduced in [F3]; note that a U-packet consists of one or two irreducible U-modules, square-integrable if so is π', and precisely one of which is non-degenerate if so is π') via either the stable b or the unstable b_κ (but not both!) maps. If the U-module π lifts to π' via b_κ and the central character of π is ω, then that of π' is $\omega'\kappa'$ (the central character of $b(\pi)$ is ω'). For induced U and U'-modules we have $b(I(\mu)) = I(\mu, \overline{\mu}^{-1})$, $b_\kappa(I(\mu)) = I(\mu\kappa, \overline{\mu}^{-1}\overline{\kappa}^{-1})$, where $I(\mu)$ is the U-module normalizedly induced from the character $\begin{pmatrix} a & * \\ 0 & \overline{a}^{-1} \end{pmatrix} \mapsto \mu(a)$ of the upper triangular subgroup of U. A character χ of E^1 defines a one-dimensional representation $\pi(\chi)$ of U which is a constituent in the composition series (of length two) of $I(\chi'\nu^{1/2})$, $\chi'(z) = \chi(z/\overline{z})$, $\nu(z) = |z|_E$. The one-dimensional constituent of $I(\chi'\nu^{1/2}, \chi'\nu^{-1/2})$ is denoted by $\pi(\chi', \chi')$ (note that $\overline{\chi}'^{\,-1} = \chi'$). The complement of $\pi(\chi)$ in $I(\chi'\nu^{1/2})$ is the square-integrable special U-module $sp(\chi)$, and that of $\pi(\chi', \chi')$ in $I(\chi'\nu^{1/2}, \chi'\nu^{-1/2})$ is the special G'-module $sp(\chi', \chi')$. We have $b(\pi(\chi)) = \pi(\chi', \chi')$, $b_\kappa(\pi(\chi)) = \pi(\chi'\kappa, \chi'\kappa)$, $b(sp(\chi)) = sp(\chi', \chi')$, $b_\kappa(sp(\chi)) = sp(\chi'\kappa, \chi'\kappa)$. The G'-module $I(\mu_1, \mu_2)$, where $\mu_1 \neq \mu_2$ are characters of $E^\times/F^\times$, is the unstable base-change lift of a U-packet consisting of two supercuspidals; it is not obtained by the stable lifting. The G'-module $I(\mu_1\kappa, \mu_2\kappa)$, $\mu_1 \neq \mu_2$ as above, is the stable lift of a supercuspidal U-packet of cardinality two, but it is not in the image of the unstable lifting. The remaining σ-invariant induced G'-modules $I(\mu_1, \mu_2\kappa)$, $\mu_i : E^\times/F^\times \to \mathbb{C}^\times$, are not obtained as base-change lifts, and their central characters are non-trivial on $F^\times/NE^\times$. This summarizes the relevant local results of [F3]. Our purpose here is to determine the set of distinguished G'-modules, as in Theorem 7. The statement concerning one-dimensional G'-modules being obvious, we consider infinite dimensional π' below.

8. Proposition. *Let π' be either a square-integrable G'-module or one of the form $I(\mu_1, \mu_2)$, $\mu_i : E^\times/F^\times \to \mathbb{C}^\times$, $\mu_1 \neq \mu_2$, which is an unstable base-change lift of a U-module π. Then π' is distinguished.*

Proof. Since our proof is global, we fix a quadratic separable extension E/F of global fields such that at some place u the completion E_u/F_u is the local extension of the proposition, and denote the π' of the proposition by π'_u. This π'_u is the unstable base-change lift of a square-integrable (supercuspidal unless π'_u is special) U_u-packet $\{\pi_u\}$. Using a standard argument, based on the simple trace formula for U, we construct a cuspidal U-module π whose component at u is $\pi_u \in \{\pi_u\}$, and its component at some place v which splits E/F is supercuspidal. Such π lifts via the unstable base-change map to a cuspidal G'-module π', which is distinguished by virtue of Theorem 1. We have:

Lemma. *A local component π'_v of a global distinguished $\mathbb{G}'$-module π' is distinguished.*

Proof of Lemma. The restriction to the component π'_v of the $\mathbb{G}$-invariant form $D(\phi) = \int_{\mathbb{Z}G\backslash\mathbb{G}} \phi(g)dg$ is non-zero.

By construction, the π'_u of the proposition is the component at u of the distinguished π', hence π'_u is distinguished and the proposition follows.

9. Proposition. *For any unitary character $\mu : E^\times \to \mathbb{C}^\times$ and complex s, the G'-module $I_s = I(\mu\nu^s, \overline{\mu}^{-1}\nu^{-s})$ is distinguished.*

Proof. Recall that I_s consists of all $\varphi : G' \to \mathbb{C}$ which satisfy

$$\varphi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = \mu(a/\overline{b})|a/b|^{1/2+s}\varphi(g) \qquad (a, b \in E^\times;\ g \in G'),$$

and $G' = GB' \cup G\eta B' = B'G \cup B'\eta^{-1}G$. Put

$$T = G \cap \eta B'\eta^{-1} = \eta\left\{\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix};\ a \in E^\times\right\}\eta^{-1} = \left\{\begin{pmatrix} a & b\theta \\ b & a \end{pmatrix};\ a, b \in F,\ a^2 - \theta b^2 \neq 0\right\},$$

and consider the G-invariant linear form

$$L_s(\varphi) = \int_{T\backslash G} \varphi(\eta^{-1}g)dg \text{ on } I_s.$$

Since for any $a, b \in E^\times$ with $c = a\overline{a} - b\overline{b} \neq 0$ we have $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = n\begin{pmatrix} c/d & 0 \\ 0 & d \end{pmatrix}k$ with $n \in N', k \in K'$, and $d \in E^\times$ with $|d| = \max(|a|, |b|)$, the integral which defines

L_s converges for s with $\mathrm{Re}(s) \geq 1/2$. Defining L_s by analytic continuation for all s, the proposition follows.

This completes the proof of half of Theorem 7, claiming that each infinite dimensional G'-module which is the unstable lift of a U-module, is distinguished. To show that there are no other distinguished G'-modules, we shall prove that each local (infinite dimensional) distinguished G'-module not of the form $I(\mu, \overline{\mu}^{-1})$ is a component of a global cuspidal distinguished G'-module. We begin with

10. Proposition. *Put $G' = GL(n,E)$ and $G = GL(n,F)$. Then given $g \in G'$ there exist $x, y \in G$ with $g^{-1} = x\overline{g}y$.*

Proof. (1) The map $G'/G \to S = \{g \in G';\ g\overline{g} = 1\}$, by $g \mapsto g\overline{g}^{-1}$, is a bijection. Indeed, it is clearly well defined and injective, and the surjectivity follows at once from the triviality of $H^1(\mathrm{Gal}(E/F), GL(n,E))$ (if $g\overline{g} = 1$, $a_\sigma = g$ defines a cocycle, which is then a coboundary, namely there is $x \in GL(n,E)$ with $g = a_\sigma = x\overline{x}^{-1}$).

Now $g^{-1}\overline{g}$ and $\overline{g}g^{-1}$ lie in S, and as $g^{-1}\overline{g} = g^{-1} \cdot \overline{g}g^{-1} \cdot g$, they are conjugate in G'. Since the map $G'/G \to S$ bijects the double coset GgG with the orbit $Ad(G)(g\overline{g}^{-1})$ under the adjoint action of G, it suffices to show:

(2) *If $g, h \in S$ are conjugate by an element of G', then they are conjugate by an element of G.* To prove this, suppose $x \in G'$ satisfies $gx = xh$. Since $g\overline{g} = 1$ and $h\overline{h} = 1$, we have $g\overline{x} = \overline{x}h$. Put $a = \frac{1}{2}(x + \overline{x})$, $b = (x - \overline{x})/2\sqrt{\theta}$. Then $x = a + b\sqrt{\theta}$, $ga = ah$ and $gh = bh$. Since a, b are $n \times n$ matrices with entries in F, we are done if $\det a$ or $\det b$ are non-zero. The polynomial $p(t) = \det(a + tb)$ has degree $\leq n$ and coefficients in F. It is non-zero since $p(\sqrt{\theta}) = \det x \neq 0$. As long as F has more than n elements, there exists $t \in F$ with $p(t) \neq 0$. With this t, the matrix $a + tb$ lies in G, and it conjugates g to h.

This elementary result is used to establish

11. Proposition. *Any irreducible $G' = GL(n,E)$-module π' admits at most one – up to a scalar multiple – $G = GL(n,F)$-invariant linear form on its space.*

Proof. We shall use the following result of Gelfand-Kazhdan [GK], for whose (one-page) proof we refer to Lemma 4.2 of (the Harvard 1989 thesis of) D. Prasad [P]:

Let G' be an ℓ-group (in the sense of Bernstein-Zelevinski [BZ]), and G a closed subgroup such that G'/G carries a G-invariant measure. Suppose that $x \mapsto x^*$ is an involution ($(xy)^* = y^*x^*$, $x^{**} = x$) which leaves G invariant and fixes any G-biinvariant distribution on G'. Then for any irreducible admissible G'-module (π', V), we have $\dim(V^{*G}) \cdot \det(\widetilde{V}^{*G}) \leq 1$. Here V^* is the space of linear forms on V, $\widetilde{V}$ is the contragredient of V, and V^{*G} is the space of G-invariant forms on V. The dimension is over $\mathbb{C}$.

This applies with our $G' = GL(n,E)$ and $G = GL(n,F)$, and by virtue of Proposition 10, with the involution $x^* = \overline{x}^{-1}$. It suffices to show that when $\dim(V^{*G}) \neq 0$, then $\dim(\widetilde{V}^{*G}) \neq 0$, for then both dimensions will be one. Gelfand-Kazhdan have shown that the contragredient $(\widetilde{\pi}', \widetilde{V})$ of the irreducible (π', V) can be realized on the same space $\widetilde{V} = V$, with $\widetilde{\pi}'(g) = \pi'({}^t g^{-1})$ (t for transpose). If $D_{\pi'}$ is a non-zero G-invariant form on (π', V), define $D_{\widetilde{\pi}'}$ by $D_{\widetilde{\pi}'}(\phi) = D_{\pi'}(\phi)$ ($\phi \in V$). Then $D_{\widetilde{\pi}'}(\widetilde{\pi}'(g)\phi) = D_{\pi'}(\widetilde{\pi}'(g)\phi) = D_{\pi'}(\pi'({}^t g^{-1})\phi)$ is a non-zero G-invariant linear form on $(\widetilde{\pi}', V)$, and the proposition follows.

Remark. If π' is distinguished, then so is $\overline{\pi}'$ ($\overline{\pi}'(g) = \pi'(\overline{g})$): put $D_{\overline{\pi}'}(\phi) = D_{\pi'}(\phi)$, $\phi \in V$.

12. Proposition. *If π' is a $G = GL(n,F)$-distinguished $G' = GL(n,E)$-module, then it is σ-invariant: ${}^\sigma\pi \cong \pi$, where $\sigma(g) = J{}^t\overline{g}^{-1}J^{-1}$.*

Proof. Denote by $D_{\pi'}$ the unique (up to scalar) G-invariant form on (the space of) π'. Then $D_{\widetilde{\pi}'}$ lies in the space $\widetilde{V}^*$ dual to $\widetilde{V}$. Since $\pi'(f')$ is an operator of finite rank, $\pi'(f')D_{\widetilde{\pi}'}$ lies in the space $\widetilde{\widetilde{V}}$ contragredient to $\widetilde{V}$. But $\widetilde{\widetilde{V}} = V$, and so we can define the linear form $\mathbb{D}_{\pi'}(f') = D_{\pi'}(\pi'(f')D_{\widetilde{\pi}'})$ on the Hecke algebra $\mathbb{H}'$ of the f'. The linear form $\mathbb{D}_{\pi'}(f')$ is G-biinvariant (thus $\mathbb{D}_{\pi'}({}^g f'^{\,h}) = \mathbb{D}_{\pi'}(f')$, where ${}^g f'^{\,h}(x) = f'(gxh)$) since $D_{\pi'} : V \to \mathbb{C}$ is G-invariant. It depends on π' only up to equivalence, and if $\pi'_1, \dots, \pi'_m$ are mutually inequivalent then the forms $\mathbb{D}_{\pi'_1}, \dots, \mathbb{D}_{\pi'_m}$ on $\mathbb{H}'$ are linearly independent. Denote by $\langle \cdot, \cdot \rangle$ the pairing $V \times V^* \to \mathbb{C}$. Then

$$\mathbb{D}_{\pi'}(f') = D_{\pi'}(\pi'(f')D_{\widetilde{\pi}'}) = \langle \pi'(f')D_{\widetilde{\pi}'}, D_{\pi'} \rangle = \langle D_{\widetilde{\pi}'}, {}^t\pi'(f')D_{\pi'} \rangle \,,$$

and

$$\mathbb{D}_{\sigma_{\pi'}}(f') = \langle D_{\pi'}, {}^t\pi'({}^\sigma f')D_{\widetilde{\pi}'} \rangle \,, \text{ since } {}^\sigma\pi' = \widetilde{\overline{\pi}}' \text{ and } D_{\overline{\pi}'} = D_{\pi'} \,.$$

Note that

$${}^t\pi'({}^\sigma f) = \int f'(\sigma g)\pi'({}^t g)dg = \int f'(J\overline{g}^{-1}J^{-1})\pi'(g)dg \,,$$

and by Proposition 10, for each g in G' there are h_g and h'_g in G with $J\overline{g}^{-1}J^{-1} = h_g \cdot g \cdot h'_g$. Since f' is locally constant, there is an open compact subgroup K_0 of $K' = GL(n, R_E)$ such that $f'(h_g \cdot g \cdot h'_g) = f'(k_1 h_g g h'_g k_2)$ for any k_1, k_2 in K_0. Denote by K_0 also the unit element in the algebra of K_0-biinvariant elements in $\mathbb{H}'$. Making the change $g \mapsto h_g^{-1}K_0 g K_0 {h'_g}^{-1}$ of variables, we get that

$$\langle D_{\pi'}, \int f'(J\overline{g}^{-1}J^{-1})\pi'(g)D_{\widetilde{\pi}'} \rangle = \langle D_{\pi'}, \int f'(h_g \cdot g \cdot h'_g)\pi'(g)D_{\widetilde{\pi}'} \rangle$$

is equal to

$$\int f'(g)\langle D_{\pi'}, \pi'(h_g^{-1})\pi'(K_0)\pi'(g)\pi'(K_0)\pi'({h'_g}^{-1})D_{\widetilde{\pi}'} \rangle dg$$
$$= \langle D_{\pi'}, \int f'(g)\pi'(g)D_{\widetilde{\pi}'} \rangle = \langle \pi'(f')D_{\widetilde{\pi}'}, D_{\pi'} \rangle = \mathbb{D}_{\pi'}(f') \,.$$

Then $\mathbb{D}_{{}^{\sigma}\pi'} \cong \mathbb{D}_{\pi'}$ implies that ${}^{\sigma}\pi' \cong \pi'$, as required.

Let $\psi' : N' \to \mathbb{C}$ be the character on the upper triangular unipotent subgroup N' of $G' = GL(n, E)$, defined by $\psi'((n_{ij})) = \psi'\left(\sum_{1\le i<n} n_{i,i+1}\right)$. A functional $W_{\pi',\psi'} : V \to \mathbb{C}$ satisfying $W_{\pi',\psi'}(\pi'(n)\phi) = \psi'(n)W_{\pi',\psi'}(\phi)$ $(\phi \in V,\ n \in N')$ is called a (ψ')-*Whittaker functional* on π'. The dimension of the space of such functionals on π' is at most 1 (this result holds in the context of any algebraic group, but not for metaplectic groups), and π' is called *non-degenerate* if the dimension is one, for some ψ'. For any f' in $\mathbb{H}'$, since $\pi'(f')$ is an operator of finite rank, the image $\pi'(f')W_{\widetilde{\pi}',\overline{\psi}'}$ of $W_{\widetilde{\pi}',\overline{\psi}'} \in \widetilde{V}^*$ lies in $\widetilde{\widetilde{V}} = V$, and we can define on $\mathbb{H}'$ the linear form

$$DW_{\pi',\psi'}(f') = D_{\pi'}(\pi'(f')W_{\widetilde{\pi}',\overline{\psi}'}) = \langle D_{\pi'}, \pi'(f')W_{\widetilde{\pi}',\overline{\psi}'}\rangle\,.$$

If ${}^{g}f'^{\,n}(x) = f'(gxn)$, then $DW_{\pi',\psi'}({}^{g}f'^{\,n}) = \psi'(n)DW_{\pi',\psi'}(f')$. The form $DW_{\pi',\psi'}$ depends only on the equivalence class of π', and if $\pi'_1, \dots, \pi'_m$ are inequivalent, then $DW_{\pi'_1,\psi'}, \dots, DW_{\pi'_m,\psi'}$ are linearly independent linear forms on $\mathbb{H}'$.

Since $DW_{\pi',\psi'}(f')$ is left-G-invariant, it depends on f' only through $x \mapsto \int_{G/Z} f'(gx)dg$. Its behavior under right translation by N' implies that it depends on f' only through $x \mapsto \int_{N'} f'(xn)\overline{\psi}'(n)dn$. In summary, $DW_{\pi',\psi'}(f')$ depends on $f' \in \mathbb{H}'$ only through its relative orbital integral

$$\Phi(\gamma, f') = \Phi(\gamma, f'; G, N', \psi') = \int_{G\times N'/Z(\gamma)} f'(g\gamma n')\overline{\psi}'(n)dndg \qquad (\gamma \in G')\,,$$

where

$$Z(\gamma) = \{g \in G\,,\ n \in N'\,;\ g\gamma n = z\gamma \text{ for some } z = z(g,n) \text{ in } Z\}\,.$$

Of course $\Phi(\gamma, f') = 0$ if there are g, n, z with $g\gamma n = z\gamma$ and $\psi'(n) \ne 1$. If π' is distinguished and non-degenerate, then there is $f' \in \mathbb{H}'$ with $DW_{\pi',\psi'}(f') \ne 0$, hence $\Phi(f')$ is not identically zero on G'.

We shall make use of the following results of Bernstein [B] (Decomposition Theorem), whose proof relies on a study of the Bernstein center. Let G be a p-adic reductive group, and fix a Levi subgroup of a minimal parabolic subgroup. A *cuspidal pair* is a pair (M, ρ) consisting of a standard Levi subgroup M and a cuspidal $\rho \in \mathrm{Irr}M$ (= set of equivalence classes of smooth irreducible M-modules). Denote by $\Theta(G)$ the set of cuspidal pairs up to conjugation by G. An element θ of $\Theta(G)$ is called an *infinitesimal character* of G. The group $X(G)$ of unramified characters $\psi : G \to \mathbb{C}^\times$ of G acts on $\mathrm{Irr}G$ by $\psi : \pi \mapsto \psi\pi$. For any cuspidal pair (M, ρ) the image of the map $X(M) \to \Theta(G)$, $\psi \mapsto (M, \psi\rho)$, is called a *connected component* of $\Theta(G)$. This component has the natural structure of a complex algebraic affine variety as a quotient of $X(M)$ $(\cong \mathbb{C}^{\times d},\ d = d(M) \ge 0)$ by a finite group.

Then $\Theta(G)$ is a complex algebraic variety equal to the disjoint union of infinitely many connected components Θ; thus $\Theta(G) = \cup_\Theta \Theta$. For each $\pi \in \mathrm{Irr}G$ there exists a cuspidal pair (M, ρ) such that π is a subquotient of the induced G-module $i_{G,M}(\rho)$. The pair (M, ρ) is uniquely determined up to conjugation by G, hence defines a point $\chi(\pi) = \theta \in \Theta(G)$, called the *infinitesimal character* of π. The map $\chi : \mathrm{Irr}(G) \to \Theta(G)$ is onto and finite to one. For each connected component Θ consider the set $\chi^{-1}(\Theta) \subset \mathrm{Irr}G$, and the corresponding abelian subcategory

$$\mathbb{M}(\Theta) = \{E \in \mathbb{M}(G)\,;\; JH(E) \subset \chi^{-1}(\Theta)\}$$

of the category $\mathbb{M}(G)$ of smooth G-modules. Here $JH(E)$ is the set of irreducible constituents of E. Bernstein's Decomposition Theorem [B] asserts the following

Theorem. *(1) The categories $\mathbb{M}(\Theta)$ and $\mathbb{M}(\Theta')$, for $\Theta \neq \Theta'$, are orthogonal, namely $Hom(E, E') = 0$ for every $E \in \mathbb{M}(\Theta)$ and $E' \in \mathbb{M}(\Theta')$. (2) $\mathbb{M}(G) = \prod_{\Theta \subset \Theta(G)} \mathbb{M}(\Theta)$; namely each G-module E has a unique decomposition $E = \bigoplus_\Theta E_\Theta = \prod_\Theta E_\Theta$, where $E_\Theta \in \mathbb{M}(\Theta)$.*

Consequently the G-module $\mathbb{H}(G)$ (= Hecke algebra of G) has the decomposition $\bigoplus_\Theta \mathbb{H}(G)_\Theta$ as a direct sum of the two sided ideals $\mathbb{H}(G)_\Theta$, and $E_\Theta = \mathbb{H}(G)_\Theta \cdot E$ for any (smooth) G-module E. If $f \in \mathbb{H}(G)$, write f_Θ for its component in $\mathbb{H}(G)_\Theta$; then $f = \sum_\Theta f_\Theta$, and for each f the sum is finite; $\pi(f) = 0$ if $f \in \mathbb{H}(G)_\Theta$ and $\chi(\pi) \in \Theta'$, and $\Theta \neq \Theta'$. Denote by $\Theta(\pi)$ the connected component which contains $\chi(\pi)$, $\pi \in \mathrm{Irr}G$. We conclude

13. Proposition. *If π' is a non-degenerate $G = GL(n, F)$-distinguished $G' = GL(n, E)$-module, then there exists f' in $\mathbb{H}'$ with $DW_{\pi',\psi'}(f') \neq 0$, $DW_{\pi'',\psi'}(f') = 0$ for every G'-module π'' with $\Theta(\pi'') \neq \Theta(\pi')$, and $\Phi(f') \neq 0$.*

Proof. In view of the discussion above, it suffices to note that for any $f' \in \mathbb{H}'$ we have $DW_{\pi',\psi'}(f') = DW_{\pi',\psi'}(f'_{\Theta(\pi')})$, and $f'_{\Theta(\pi')}$ has the required properties if we choose f' with $DW_{\pi',\psi'}(f') \neq 0$ (f' exists since π' is distinguished and non-degenerate).

14. Proposition. *Let E_u/F_u be a local quadratic separable extension, and ${\pi'_u}^0$ a G_u-distinguished non-degenerate G'_u-module, where $G = GL(n)$. Let E/F be a global quadratic extension such that at some place u of F the completion of E/F is our E_u/F_u. Then there exists a cuspidal $\mathbb{G}$-distinguished $\mathbb{G}'$-module π' whose component π'_u at u has infinitesimal character $\chi(\pi'_u)$ in the connected component $\Theta({\pi'_u}^0)$ of $\Theta(G'_u)$ defined by ${\pi'_u}^0$. Moreover, π' can be chosen to have as its component at finitely many places which split and are non-archimedean any preassigned supercuspidal representation.*

Remark. When $\mathbb{M}(G'_u)$ is defined to be the category of smooth G'_u-modules which transform under Z'_u according to a fixed central character (say $\omega'_u\kappa'_u$), then the minimal dimension of a component Θ of $\Theta(G'_u)$ is zero, in which case Θ is a point, and we call the G'_u-module π'_u *isolated* if $\Theta(\pi'_u)$ is a point. Proposition 14 implies that a non-degenerated distinguished isolated G'_u-module can be realized as a component of a cuspidal $\mathbb{G}$-distinguished $\mathbb{G}'$-module which has supercuspidal components at any finite set of finite split places.

Proof. Apply Proposition 13 with $\pi' = {\pi'_u}^0$ to produce a function $f'_u \in \mathbb{H}'_u$ with the properties listed in Proposition 13. In particular $\Phi(\gamma, f'_u)$ is not identically zero. Let $u_1, \dots, u_m$ be finitely many finite places of F which split in E, and fix supercuspidal G'_{u_i}-modules ${\pi'_{u_i}}^0$ which are G_{u_i}-distinguished; this means that there are supercuspidal G_{u_i}-modules π_{u_i}, and ${\pi'_{u_i}}^0 = \pi_{u_i} \otimes \widetilde{\pi}_{u_i}$. Denote by f'_{u_i} the elements of $\mathbb{H}'_{u_i}$ with the properties listed in Proposition 13, where π' is ${\pi'_{u_i}}^0$; this Proposition obviously holds in the split case too. Consider a global function $f' = \otimes f'_v$ whose components at $u, u_1, \dots, u_m$ are as specified already. We inquire when $f'(g\delta n) \neq 0$ for g in $\mathbb{Z}\backslash\mathbb{G}$, δ in $Z'G\backslash G'$, and n in $N'\backslash\mathbb{N}'$. Since the union $\bigcup_\delta \mathbb{G}\delta$ is disjoint, and each coset $\mathbb{G}\delta$ is open and closed, since the homogeneous space $N'\backslash\mathbb{N}'$ is compact and so is the image of the support $\mathrm{Supp} f'$ of f' in $\mathbb{Z}'\backslash\mathbb{G}'$, we conclude that there is a compact subset C_1 in $\mathbb{Z}\backslash\mathbb{G}$, and a finite subset E_1 of $Z'G\backslash G'$, such that $f'(g\delta n) \neq 0$ implies $g \in C_1$ and $\delta \in E_1$.

Consider the relative orbital integral

$$\Phi(\delta, f') = \iint_{(\mathbb{Z}\backslash\mathbb{G}\times\mathbb{N}')/Z(\gamma)} f'(g\delta n)\psi'(n)dgdn\,.$$

It is equal to the product over all places v of the local relative orbital integrals

$$\Phi(\delta, f'_v) = \iint_{(Z_v\backslash G_v\times N'_v)/Z_v(\gamma)} f'_v(g\delta n)\psi'_v(n)dgdn\,.$$

Since the f'_v are locally constant, and $\Phi(\delta, f'_v)$ is not identically zero ($v = u, u_1, \dots, u_m$), we may assume that there exists a δ_0 in $Z'G\backslash G'$, such that all entries of $\overline{\delta}_0^{-1}\delta_0$ under the diagonal are non-zero, and $\Phi(\delta_0, f'_v) \neq 0$ ($v = u, u_1, \dots, u_m$). Since f'_v ($v \neq u, u_1, \dots, u_m$) can be chosen arbitrarily, we take them to satisfy $\Phi(\delta_0, f') \neq 0$. Recall from (1) of the proof of Proposition 10 that the map $g \mapsto \overline{g}^{-1}g$, $Z'G\backslash G' \to S/Z$, where $S = \{g \in G'\,;\ g\overline{g} = 1\}$ and $Z = Z' \cap S$, is a bijection. As noted above, there exists a finite set E_2 in $Z'G\backslash G'/N'$, depending only on the support of f', such that $\Phi(\delta, f') \neq 0$ implies that δ lies in E_2. The set E_2 contains δ_0. Choose some finite place v, and a small neighborhood U_v of the orbit $Z'_vG_v\delta_0N'_v$, which does not intersect $G_v\delta N'_v$ for any $\delta \neq \delta_0$ in the finite set E_2. Replace f'_v by its product with the characteristic function of U_v, in f'. With this revised f', we have $\Phi(\delta_0, f') \neq 0$ as before, but $\Phi(\delta, f')$ is non-zero for a rational δ in $Z'G\backslash G'/N'$ only if δ is represented by δ_0. We conclude that the geometric side

$$\int_{\mathbb{G}/\mathbb{Z}G}\int_{N'\backslash\mathbb{N}'}\sum_{\delta\in Z'\backslash G'} f'(g\delta n)\psi'(n)dgdn$$

of the relative trace formula

$$\int_{\mathbb{G}/\mathbb{Z}G}\int_{N'\backslash\mathbb{N}'} K_{f'}(g,n)\psi'(n)dgdn = \int_{\mathbb{G}/\mathbb{Z}G}\int_{N'\backslash\mathbb{N}'} K_{f',0}(g,n)\psi'(n)dgdn$$

for the convolution operator $\rho(f')$ on $L^2_{\omega'\kappa',0}(G')$, is equal to $\Phi(\delta_0, f') \neq 0$. Note that only the kernel $K_{f',0}$ on the space of cusp forms occurs on the right, spectral side of this formula, since f'_{u_1} can be (and is) chosen to be a matrix coefficient of a supercuspidal G'_{u_1}-module, so that the operator $\rho(f')$ factorizes through the orthogonal projection on the space of cusp forms.

The spectral side is equal to the sum over all distinguished cusp forms π', of the distributions

$$DW_{\pi',\psi'}(f') = \sum_{\Lambda\in\pi'} D(\pi'(f')\Lambda)\overline{W}_{\Lambda,\psi'}(e)\,.$$

Here

$$D(\Lambda) = \int_{\mathbb{Z}G\backslash\mathbb{G}} \Lambda(g)dg\,, \qquad W_{\Lambda,\psi'}(g) = \int_{N'\backslash\mathbb{N}'} \Lambda(ng)\overline{\psi}'(n)dn\,,$$

and the sum ranges over an orthonormal basis $\{\Lambda\}$ of π'. The non-vanishing of the geometric side implies that the spectral side is non-zero, hence the existence of a cuspidal $\mathbb{G}'$-module π' with $DW_{\pi',\psi'}(f') \neq 0$.

The uniqueness of the ψ'_v-Whittaker model for any non-degenerate G'_v-module π'_v, and the uniqueness of the G_v-invariant linear form on (a distinguished) π'_v (Proposition 11), imply that there exists at most one – up to scalar – form $DW_{\pi'_v,\psi'_v}$ on $\mathbb{H}'_v$ satisfying $DW_{\pi'_v,\psi'_v}({}^g f'^{\,n}_v) = \psi'_v(n)DW_{\pi'_v,\psi'_v}(f'_v)$, where ${}^g f'^{\,n}_v(x) = f'_v(gxn)$. We normalize $DW_{\pi'_v,\psi'_v}$ when π'_v is unramified and ψ'_v has conductor 0 to attain the value one at the unit element $f'^{\,0}_v$ of the Hecke algebra $\mathbb{H}'_v$. Hence there is a constant $c(\pi',\psi')$ with

$$DW_{\pi',\psi'}(f') = c(\pi',\psi')\prod DW_{\pi'_v,\psi'_v}(f'_v) \text{ if } f' = \otimes f'_v\,.$$

Since $DW_{\pi'_v,\psi'_v}(f'_v) \neq 0$ for $v = u, u_1, \ldots, u_m$, where f'_v are chosen to satisfy the conclusion of Proposition 13, it follows that π'_{u_i} are the supercuspidal $\pi'^{\,0}_{u_i}$ and the infinitesimal character $\chi(\pi'_u)$ lies in the connected component $\Theta(\pi'^{\,0}_u)$ specified by $\pi'^{\,0}_u$. Since π' is also distinguished, the proposition follows.

Proof of Theorem 7. Every G'_v-module π'_v with central character $\omega'_v\kappa'_v$ which is square-integrable or of the form $I(\mu_{1v},\mu_{2v})$, with $\mu_{iv} : E^\times_v/NE^\times_v \to \mathbb{C}^\times$, is isolated. If it is G_v-distinguished, then by Proposition 14 it is a component of a cuspidal $\mathbb{G}$-distinguished $\mathbb{G}'$-module π'. By Theorem 1, such π' is the unstable base-change lift (via b_κ) of some non-degenerate cuspidal $\mathbb{U}$-module π. By [F3] the local component π'_v is then the (unstable base change) lift of the local component π_v, as required.

Suitably modified, the proof of Proposition 14 implies the following Relative Density Theorem; it is analogous to Kazhdan's density theorem for usual characters ([K], Appendix).

15. Proposition. *If $f'_u \in \mathbb{H}'_u$ satisfies $DW_{\pi'_u,\psi'_u}(f'_u) = 0$ for all π'_u, then $\Phi(f'_u) \equiv 0$.*

Proof. (Sketch) If $\Phi(\delta, f'_u)$ is not identically zero, we can choose a rational δ_0 in $Z'G\backslash G'/N'$ with $\Phi(\delta_0, f'_u) \neq 0$, and $f'^{\,u} = \bigotimes_{v \neq u} f'_v$ whose component at a finite u_1 which splits in E is a matrix coefficient of a supercuspidal distinguished G'_{u_1}-module, with $\Phi(\delta_0, f'^{\,u}) \neq 0$. Moreover the components f'_v, $v \neq u, u_1$, can be chosen to have the property that $\Phi(\delta, f') \neq 0$, $f' = f'_u \otimes f'^{\,u}$, $\delta \in GZ'\backslash G'/N'$, implies that $\delta = \delta_0$. For such f' the relative trace formula $\iint K_{f'}(g,n)\psi'(n)dgdn = \iint K_{f',0}(g,n)\psi'(n)dgdn$ holds. The geometric side is equal to $\Phi(\delta_0, f') \neq 0$. The spectral side is the sum over the cuspidal distinguished π' of the products $c(\pi',\psi')\prod DW_{\pi'_v,\psi'_v}(f'_v)$, each of which is zero since $DW_{\pi'_u,\psi'_u}(f'_u)$ is zero for all π'_u. The resulting contradiction implies that $\Phi(f'_u)$ vanishes identically, as required.

Remark. (1) When D is a compact-modulo-its-center Z subgroup of an ℓ-group G', then a G'-module π' is D-distinguished if and only if its space contains a non-zero D-fixed vector. Indeed, if $L \neq 0$ is a D-invariant form on π', there is $w \in \pi'$ with $L(w) \neq 0$, and $u = \int_{Z\backslash H} \pi'(h)wdh$ is a D-fixed vector with $L(u) \neq 0$, and so $u \neq 0$. In the opposite direction, given a D-fixed vector $u \neq 0$ in π', choose a positive definite bilinear form $\langle\cdot,\cdot\rangle$ on π', and consider $L(w) = \int_{Z\backslash D}\langle\pi'(h)w, u\rangle dh$. Then $L(u) \neq 0$, and L is a D-invariant non-zero linear form on π'.

(2) The anisotropic inner form of $G = GL(2,F)$ can be realized as the subgroup $D = \left\{\begin{pmatrix} a & b\varepsilon \\ \overline{b} & \overline{a}\end{pmatrix}\ ;\ a \in E\,,\ b \in E\,,\ a\overline{a} \neq \varepsilon b\overline{b}\right\}$ of $G' = GL(2,E)$, where $\varepsilon \in F - NE$. Since D acts transitively on the projective line, we have $G' = B'D$. The induced G'-module $I(\mu_1,\mu_2)$ consists of all smooth $\varphi : G' \to \mathbb{C}$ with

$$\varphi\left(\begin{pmatrix} a & * \\ 0 & b\end{pmatrix} g\right) = \mu_1(a)\mu_2(b)|a/b|^{1/2}\varphi(g)\,.$$

If φ is a D-fixed vector in $I(\mu_1,\mu_2)$, then it is determined by its value at e since $G' = B'D$. Such vector should satisfy $\varphi(g) = \varphi(e)$ for $g \in B' \cap D = \left\{\begin{pmatrix} a & 0 \\ 0 & \overline{a}\end{pmatrix}\ ;\ a \in E^\times\right\}$. Since $\varphi\begin{pmatrix} a & 0 \\ 0 & \overline{a}\end{pmatrix} = \mu_1(a)\mu_2(\overline{a})\varphi(e)$, we conclude that: *if $I(\mu_1,\mu_2)$ is D-distinguished then $\mu_1 = \mu$, $\mu_2 = \overline{\mu}^{-1}$.* But *$I(\mu,\overline{\mu}^{-1})$ is distinguished*, since $\varphi\left(\begin{pmatrix} a & * \\ 0 & b\end{pmatrix} h\right) = \mu(a/\overline{b})|a/b|^{1/2}$ defines a non-zero D-fixed vector in its space. The D-invariant form on $I(\mu,\overline{\mu}^{-1})$ is given by $L(\varphi) = \int_{B'\cap D\backslash D}\varphi(h)dh$.

Similarly, *the square-integrable submodule $sp(\mu)$ of the induced G'-module $I(\mu\nu^{1/2},\mu\nu^{-1/2})$, where $\nu(z) = |z|_E$, is D-distinguished if and only if the kernel of the restriction of μ to $F^\times$ is the index-two subgroup $NE^\times$.* Indeed $\varphi \in I(\mu\nu^{1/2},\mu\nu^{-1/2})$ satisfies $\varphi\left(\begin{pmatrix} a & * \\ 0 & b\end{pmatrix} g\right) = \mu(ab)|a/b|_E\varphi(g)$, and it lies in the submodule $sp(\mu)$ precisely when $\int_{Z\backslash D}\varphi(h)\mu(\det h)^{-1}dh = 0$, since $G' = B'D$. If $\varphi \in sp(\mu)$ is D-invariant, then it is determined by its value at e, and it satisfies $\varphi(e) = \varphi\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \mu(a\overline{a})\varphi(e)$ (as $B' \cap D = \left\{\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}\ ;\ a \in E^\times\right\}$). Moreover

$$0 = \int_{Z\backslash D} \varphi(h)\mu(\mathrm{det} h)^{-1} dh = \varphi(1)\int_{Z\backslash D} \mu(\mathrm{det} h)^{-1} dh$$

(and $\mu(\mathrm{det} z) = \mu(z^2) = 1$, $z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in Z \cong F^\times$), and $\mathrm{det} D = F^\times$, implying that $sp(\mu)$ is D-distinguished precisely when $\mu \mid F^\times$ is non-trivial, and $\mu \mid NE^\times = 1$.

(3) In his thesis, J. Hakim [H] stated that a square-integrable G'-module π' is G-distinguished if and only if it is D-distinguished. In fact he assumed that the central character of π' is trivial, but this restriction can be removed. This statement is proven in [FH] by means of a "simple relative trace formula", in the context of $GL(n)$ and any of its inner forms (π' is supercuspidal if $n \neq 2$).

The following is also shown in [FH]. Let $\mathbb{D}$ be an inner form of $\mathbb{G} = GL(2, \mathbb{A}_F)$, and $\mathbb{D}' = \mathbb{D} \otimes_F E$. Let π' be a discrete series $\mathbb{G}'$-module which corresponds to a cuspidal $\mathbb{D}'$-module π^D. Denote by V the finite set of F-places which stay prime in E and where D is ramified. Then π^D is $\mathbb{D}$-distinguished if and only if π is $\mathbb{G}$-distinguished and at each v in V the component $\pi_v = \pi_v^D (D'_v = G'_v)$ is not of the form $I(\mu_1, \mu_2)$, where μ_i are characters of $E_v^\times$ trivial on $F_v^\times$. In [FH] this is used to prove Tate's conjecture on algebraic cycles for some new proper surfaces over Q.

A comparison of $\mathbb{G}$-distinguished and $\mathbb{D}$-distinguished cuspidal $\mathbb{G}'$-modules and $\mathbb{D}'$-modules has been carried out in [F5] for cuspidal $\mathbb{G}'$-modules π' with a supercuspidal and a square-integrable components at two distinct places of F, and corresponding $\mathbb{D}'$-modules π^D, where $\mathbb{G}' = GL(n, \mathbb{A}_E)$, $\mathbb{G} = GL(n, \mathbb{A}_F)$ and $\mathbb{D}$ is an inner form of $\mathbb{G}$ such that v splits in E at each place where $\mathbb{D}$ is ramified, and $\mathbb{D}' = \mathbb{D} \otimes_F E$. In [FH] this restriction on $\mathbb{D}$ is removed. As noted in [F5], the restriction that π' has a square-integrable component at u' (not u as erroniously misprinted on p. 421, l. -4, there) can be removed on applying further techniques.

(4) Let E/F be a quadratic extension, and U the unitary group $U(n, E/F) = \{g \in G(E);\ \sigma g = g\}$, where $G = GL(n)$ and $\sigma g = J{}^t\overline{g}^{-1}J^{-1}$. Then $\mathbb{U} = U(\mathbb{A}_F)$ is a subgroup of $\mathbb{G}' = G(\mathbb{A}_E)$, and at a place v of F which splits in E, $U_v = U(n, E_v/F_v) = G(F_v) = G_v$ embeds as the group of $\{(g, \sigma g);\ g \in G_v\}$, $\sigma g = J{}^t g^{-1}J^{-1}$, in $G'_v = G(E_v) = G_v \times G_v$. Now if $\pi' = \otimes \pi'_v$ is a $\mathbb{U}$-distinguished cuspidal $\mathbb{G}'$-module, then each of its components π'_v (v is a place of F) is a U_v-distinguished G'_v-module. In particular, if v splits into the places v', v'' in E, then $\pi'_v = \pi'_{v'} \times \pi'_{v''}$ is $U_v (\cong G_v)$-distinguished only when $\pi'_{v'} \cong \pi'_{v''}$. Writing $\overline{\pi}'_v$ for the representation $\overline{\pi}'_v(g) = \pi'_v(\overline{g})$, where $\overline{g} = (\overline{a}_{ij})$ if $g = (a_{ij})$, and $a \mapsto \overline{a}$ is the non-trivial automorphism of E_v/F_v (it maps (x, y) to (y, x) when $E_v = F_v \oplus F_v$), we conclude that $\overline{\pi}'_v \cong \pi'_v$ for almost all v, and hence $\overline{\pi}' \cong \pi'$ by virtue of rigidity and multiplicity one theorem for $GL(n)$. By the theory of base-change for $GL(n)$, such a $\mathbb{U}$-distinguished π' is the base-change lift of a cuspidal $\mathbb{G} = G(\mathbb{A}_F)$-module

π. Given such a π', its central character ω' satisfies $\omega'(z) = \omega'(\overline{z})$ $(z \in \mathbb{A}_E^\times)$, and there is a character $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ with $\omega'(z) = \omega(z\overline{z})$. We may choose ω to be the central character of π. Now π is uniquely determined by π' up to tensoring with the non-trivial quadratic character $\chi_{E/F}$ of $\mathbb{A}_F^\times$ which is trivial on $F^\times N\mathbb{A}_E^\times$, and so the central character ω of π is determined by π' if n is even, but it might be both ω and $\omega\chi_{E/F}$ (for a suitable π) if n is odd. Proposition 0.1 asserts that a cuspidal $GL(2, \mathbb{A}_E)$-module π' with $\omega' = 1$ is distinguished by $Z(\mathbb{A}_E)GL(2, \mathbb{A}_F)$, namely the group of unitary similitudes, if and only if π' is the base-change of a cuspidal π with the central character $\omega = \chi_{E/F}$ (thus $\omega \neq 1$). It will be interesting to find out whether a cuspidal $GL(2, \mathbb{A}_E)$-module π' which is the base-change lift of a cuspidal $GL(2, \mathbb{A}_F)$-module π with the central character $\omega = 1$ is distinguished by the smaller group $Z(\mathbb{A}_E)U(2, \mathbb{A}_E/\mathbb{A}_F)$. Naturally, this question should be asked in the context of $GL(n)$.

Although we feel that the answer is positive, we shall not answer this question here, but only give an example to warn against making the (wrong) conjecture that a $U = U(2, E/F)$-distinguished $G' = GL(2, E)$-module π' (with a trivial central character $\omega' = 1$) is the base-change lift of a $G = GL(2, F)$-module π whose central character ω is $\chi_{E/F}$. By Theorem 7, a G-distinguished G'-module π' with $\omega' = 1$ is indeed the base-change lift of a G-module π with $\omega = \chi_{E/F}$. Suppose that χ is a quadratic character of $E^\times$ whose restriction to $F^\times$ is $\chi_{E/F}$. For example, suppose E/F and χ are unramified. Then $\chi(z/\overline{z}) = \chi(z\overline{z}) = 1$ $(z \in E^\times)$, and there is a character χ_1 of $F^\times$ with $\chi(z) = \chi_1(z\overline{z})$ $(z \in E^\times)$; thus $\chi_1^2 = \chi_{E/F}$ and $\chi_1^4 = 1$. The G'-module $\pi' \otimes \chi$ is U-distinguished, since $\chi(\det u) = 1$ for any $u \in U$, and it is the base-change lift of $\pi \otimes \chi_1$, a G-module with central character $\omega = \chi_{E/F}\chi_1^2 = 1$.

(5) Following the proof of Theorem 7, especially using Proposition 14, we deduce from Proposition 0.1 that a square-integrable $PGL(2, E)$-module π' is $G = GL(2, F)$-distinguished if and only if it is the base-change of a G-module π, necessarily square-integrable, whose central character is $\chi_{E/F}$. To answer a question of D. Prasad (email correspondence, Jan. 1990), let us spell this out in the case of special G'-modules $\pi' = sp(\chi') = sp(\chi'\nu_E^{1/2}, \chi'\nu_E^{-1/2})$. Here the central character $\omega' = \omega_{\pi'}$ is χ'^2, and it is assumed to be 1. If $sp(\chi')$ is G-distinguished then it is the base-change of the G-module π whose central character $\omega = \omega_\pi$ is $\chi_{E/F}$. By the theory of base-change for $GL(2)$ (see [F2]) π must be the special G-module $\pi = sp(\chi)$, whose central character is $\omega = \chi^2$, and χ, χ' are related by $\chi'(z) = \chi(z\overline{z})$ on $z \in E^\times$. To have consistency with Theorem 7, we need to show that the restriction of χ' to $F^\times$ is $\chi_{E/F}$. But for $f \in F^\times$ we have $\chi'(f) = \chi(f^2) = \omega(f) = \chi_{E/F}(f)$, as required.

(6) An even more speculative, ambitious and fascinating question can be asked in the context of a cubic extension E/F. Let D_4 denote the semi-simple group of type D_4 viewed as an algebraic F-group, $D_4(F)$ its group of F-points, and $D_4(E)$ its group of E-points. Let ${}^3D_4$ be the associated triality-D_4 group, viewed as an algebraic F-group, ${}^3D_4(F)$ its group of F-points, and ${}^3D_4(E)$ $(= D_4(E))$ its group of E-points. If π' is a cuspidal $D_4(\mathbb{A}_E)$-module which is distinguished by $D_4(\mathbb{A}_F)$, is it a base-change from ${}^3D_4(\mathbb{A}_F)$? If such π' is distinguished by ${}^3D_4(\mathbb{A}_F)$, is it a base-change from $D_4(\mathbb{A}_F)$?

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