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METAPLECTIC CORRESPONDENCE

by YUVAL Z. FLICKER *and* DAVID A. KAZHDAN

Shimura attached modular forms of even weight $k - 1$ to cusp forms of weight $k/2$, initiating the study of the metaplectic correspondence. Gelbart, Piatetski-Shapiro and Waldspurger extended his techniques, and the converse theta-series approach of Shintani, to the context of automorphic representations of the two-fold covering group of GL(2). [F] used the trace formula to establish the correspondence for the automorphic representations of the metaplectic n-fold covering of GL(r) when $r = 2$, for all $n \geqslant 2$. This gave a complete description of the representations of the metaplectic group locally and globally in terms of those of GL(2, $\mathbf{A}$). The purpose of the present work is to develop the last approach for any integer $r \geqslant 2$ in the local and global cases, continuing a program started in [KP], [KP'].

Let r, n be positive integers; F a number field containing the group μ_n of n-th roots of unity; F_v a completion of F at a place v; $\mathbf{A}$ the ring of adeles of F; $G = GL(r)$; $\widetilde{G}_v$ an n-fold covering group of $G_v = G(F_v)$ (by μ_n); $\widetilde{G}(\mathbf{A})$ a non-trivial central topological extension of $G(\mathbf{A})$ by μ_n, which splits over the group G(F) of F-rational points on G (see [Mo], [Mi], and (2)). We fix a character $\widetilde{\omega}$ of the center of $\widetilde{G}(\mathbf{A})$ (and $\widetilde{G}_v$) whose restriction to μ_n has order n, and deal only with the *genuine* representations $\widetilde{\pi}$ (or $\widetilde{\pi}_v$) of the metaplectic group, those with central character $\widetilde{\omega}$. If the restriction of $\widetilde{\omega}$ to μ_n has order n' dividing n, then $\widetilde{\pi}$ can be viewed as a representation of an n'-fold covering group of $\widetilde{G}(\mathbf{A})$ (or $\widetilde{G}_v$).

We shall first describe our local results. Let p denote the residual characteristic of F_v. Our aim is to develop a local theory relating admissible genuine $\widetilde{G}_v$-modules $\widetilde{\pi}_v$ with certain admissible G_v-modules π_v. In the case where p does not divide n, we study in (16), (17) the correspondence for representations which occur in the composition series of representations induced from unramified characters of a Borel subgroup. We show, generalizing a well-known result for G_v, that this category of representations consists of the $\widetilde{G}_v$-modules with a vector fixed under the action of an Iwahori subgroup I^* (I^* is a subgroup of $\widetilde{G}_v$ isomorphic (as in (2)) to an Iwahori subgroup I of G_v). Moreover, it is naturally isomorphic to the category of finite dimensional complex representations of the Hecke algebra $\widetilde{H}$ of $\widetilde{G}_v$ with respect to I^*. Thus $\widetilde{H}$ is the convolution algebra of complex-valued I^*-biinvariant functions on $\widetilde{G}_v$ which transform under the center by $\tilde{\omega}^{-1}$, and are compactly supported modulo the center. The isomorphism is given by $\tilde{V} \to \tilde{V}^{I^*}$, $\tilde{V}^{I^*}$ being the space of I^*-fixed vectors in $\tilde{V}$. Thus our aim is to define *an isomorphism from the category of* $\tilde{H}$*-modules to that of* H*-modules*. In fact we construct *an explicit isomorphism of the algebras* H *and* $\tilde{H}$. We also verify that the properties of being square-integrable (= discrete-series) or tempered are preserved under this isomorphism of modules. The proof is based on exhibiting a presentation of $\tilde{H}$ by means of generators and relations, generalizing the one given by Iwahori-Matsumoto [IM] in the case of $H(n = 1)$. It will be interesting to extend this geometric description of the correspondence to the categories of all algebraic representations. We define the notion of local correspondence for general admissible representations by means of character relations; see below.

To study the correspondence in the context of the categories of admissible representations locally, and to develop a global theory of correspondence, we use the trace formula. All our local results, and most of our global results, rely only on the simple trace formula, which is proven in (18). Before we describe the results which depend on the trace formula, note that *they are proven only in the case when* $(n, N) = 1$. Here N is the least common multiple of all composite (non-prime) positive integers $r' \leqslant r$. Our proofs reduce the general case (any n, r) to a statement (see Assertion 12) concerning algebraic groups only. It relates orbital integrals of unit elements in the Hecke algebras with respect to a maximal compact subgroup in G_v and H_v, when $(n, p) = 1$ (see below, (12) and [K]), where $H_v = GL(r', E_v)$, E_v is an extension of F_v with $r'[E_v : F_v] = r$.

We say that a genuine admissible $\tilde{G}_v$-module $\tilde{\pi}_v$ *corresponds*, or *lifts*, to an admissible G_v-module π_v, if they satisfy a character identity, see (26.1), relating the value of the character $\chi(\tilde{\pi}_v)$ of $\tilde{\pi}_v$ at a good element x^* (see (4)), with a certain sum of values of $\chi(\pi_v)$ at the "n-th roots" x in G_v of x^*. The image of the correspondence consists of π_v whose central character ω is determined by $\tilde{\omega}$ and the relation $\omega(z) = \tilde{\omega}(s(z^n))$. In particular the restriction of ω to the subgroup μ_n of $F_v^\times$ is trivial. To describe the image of the correspondence we say that an irreducible π_v is *metic* (for met(eplect)ic) if it is equivalent to a G_v-module unitarily induced from an $M = \Pi_i M_i$-module $\Pi_i \sigma_i \nu^{s_i}$, where $M_i = GL(r_i)$, the s_i are real, and the σ_i are square-integrable M_i-modules whose central character is trivial on μ_n for all i. Our main local theorem asserts that *the correspondence relation defines a bijection from the set of genuine tempered* $\tilde{G}_v$*-modules* $\tilde{\pi}_v$ *to the set of metic tempered* G_v*-modules* π_v. *It commutes with induction, bijects square-integrables with square-integrables, irreducibles with irreducibles. If* $|n|_v = 1$ *it maps unramified* $\tilde{\pi}_v$ *to unramified* π_v, *and coincides with the correspondence of* (16), (17). In fact, for global purposes we introduce in (27.2) the notion of *relevant* representations, and Theorem (27.3) asserts that *the correspondence bijects genuine relevant* $\tilde{\pi}_v$ *with metic relevant* π_v. The relevant representations are induced from square-integrables which are twisted "only a little" ($|s_i| < 1/2$). Tempered $\tilde{\pi}_v$ are relevant. Each component of a cuspidal (automorphic) $\tilde{G}(\mathbf{A})$-module which lifts (see below) to a cuspidal $G(\mathbf{A})$-module is relevant.

The first step in the proof is the square-integrable case. This is applied in the proof of Proposition (27) which asserts that *a $\widetilde{G}_v$-module unitarily induced from a tempered irreducible (in particular square-integrable) representation of a Levi subgroup is irreducible.* This in turn is used to show in Theorem (27.2) that a *relevant $\widetilde{G}_v$-module is irreducible.*

It is clear from the character relation that if $\widetilde{\pi}_v$ lifts to a supercuspidal π_v then $\widetilde{\pi}_v$ is supercuspidal; but a supercuspidal $\widetilde{\pi}_v$ may lift to a non-supercuspidal π_v. This occurs already in the well-known case of $r = 2$ and even n, when $\widetilde{\pi}_v$ is a Weil representation and π_v is an odd special representation (see, e.g., [F]). The character relation yields a formula for the number of Whittaker vectors of $\widetilde{\pi}_v$; see (22) and [KP], p. 99.

The definition of metic local π_v which is not necessarily relevant is given in (27.2). The case of the non-tempered unitary G_v-module π_v which is dual, in the sense of [Z], to a metic (generalized) Steinberg representation π'_v, is particularly interesting. For example, π_v can be a one-dimensional representation, a case studied by [KP]. In (29) we show that for such a representation π_v there exists a matching unitary $\widetilde{\pi}_v$ so that π and $\widetilde{\pi}_v$ satisfy the character identity (26.1), possibly up to a sign. Since the character of π_v occurs in (26.1) as a weighted sum, the weights being roots of unity, we may have that a non-tempered π_v is matched with a discrete-series, and even supercuspidal $\widetilde{\pi}_v$. This phenomenon occurs already in the case of $r = 2$ (see, e.g., [F]). Such $\widetilde{\pi}_v$ can be viewed as a generalization of the Weil representation.

To describe our global results we say that the genuine representation $\widetilde{\pi} = \bigotimes \widetilde{\pi}_v$ of $\widetilde{G}(\mathbf{A})$ *lifts* to the automorphic representation $\pi = \bigotimes \pi_v$ of $G(\mathbf{A})$ if $\widetilde{\pi}_v$ corresponds to π_v for *all* places v. Our global results are described in (28). A characteristic special case which uses only the simple trace formula of (18) asserts the following. *Suppose that $\widetilde{\pi}$ is a cuspidal genuine $\widetilde{G}(\mathbf{A})$-module whose components $\widetilde{\pi}_u$, $\widetilde{\pi}_{u'}$ at two places u, u' are supercuspidal, and $\widetilde{\pi}_u$ lifts to a supercuspidal G_u-module π_u. Then there exists a unique metic cuspidal $G(\mathbf{A})$-module π such that $\widetilde{\pi}$ lifts to π. Moreover, if $\widetilde{\pi}'$ is a cuspidal genuine $\widetilde{G}(\mathbf{A})$-module whose components at u, u' are also $\widetilde{\pi}_u$, $\widetilde{\pi}_{u'}$, and $\widetilde{\pi}'_v$ is equivalent to $\widetilde{\pi}_v$ for almost all v, then $\widetilde{\pi}'$ is equal to $\widetilde{\pi}$.* The last statement combines the *rigidity* (strong multiplicity one) theorem for $\widetilde{G}(\mathbf{A})$, with *multiplicity one* theorem, for such representations of $\widetilde{G}(\mathbf{A})$. The components of a cuspidal π are relevant (by [BZ], [B]). It follows from Theorem (28) that all components of a cuspidal $\widetilde{\pi}$ as above are also relevant.

In (29) we deal with those automorphic $\widetilde{\pi}$ which correspond (= lift) to discrete-series non-cuspidal π, of a certain type (these can be conjectured to be all the discrete-series non-cuspidal π). This includes the case of the one-dimensional π, studied in [KP]. The phenomenon which occurs here is that *there are cuspidal $\widetilde{\pi}$ with supercuspidal components, which* (lift to these π. Consequently the $\widetilde{\pi}$) *have non-tempered local components which are not relevant for almost all places.* This is the global analogue of the local statement noted above that supercuspidal $\widetilde{\pi}_v$ match sometimes with non-tempered π_v. Such examples occur already in the case of $r = 2$; see [F].

To apply the trace formula we show that corresponding spherical functions f_v

and $\widetilde{f}_v$ on G_v and $\widetilde{G}_v$ (see (11)) are matching, namely have matching orbital integrals (see (8)). The case of the unit element of the Hecke algebra is given in (12). It is due to [KP′], and relies on the results of [K]. However the methods of [K] apply only in the case specified in Theorem (12). This is the reason why our results are proven completely only when $(n, N) = 1$, as explained in Corollary (12). From this we deduce the case of general spherical functions in (19) using a new technique which is based on the usage of the " *regular functions* " introduced in (15). These are not spherical functions. They are essentially functions in the Hecke algebra with respect to an Iwahori subgroup, which isolate the representations with a vector fixed by the action of an Iwahori subgroup, and whose support can be conveniently controlled. Here we use our work on the Iwahori algebra, in particular Proposition (17). But it is clear from the proof of (19) that we could work with a congruence subgroup instead of an Iwahori subgroup. Since our technique does not require detailed knowledge of representation theory, it may be applicable in the study of transfer of orbital integrals of spherical functions for arbitrary groups; this was the main motivation for us to develop our technique; see [″F] for the rank one case of the Symmetric Square lifting.

We also use the transfer of a supercusp form $\widetilde{f}_v$ to a matching function f_v on G_v, which is carried out in (13), again using [K] (hence we need $(n, N) = 1$), and the theory of Harish-Chandra [H] and [K′], relating orbital integrals, characters and Fourier transforms of nilpotent measures, locally.

Finally we note the analogy between the metaplectic correspondence and the base-change lifting. While the second is a reflection of the norm map of field extensions, the first reflects extraction of n-th roots.

The work is presented in three parts. Chapter I consists of §§1-13, Chapter II of §§14-20, and Chapter III of §§21-29.

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I. — ORBITAL INTEGRALS

1. Notations. — Let $r \geqslant 2$, $n \geqslant 1$ be integers, and F a local or global field of characteristic 0 which contains the group μ_n of n-th roots of 1. If $n \geqslant 3$ and F is global then F is totally imaginary. If F is global and v is a place of F, we write F_v for the completion of F in the valuation $|\ |_v$, normalized as usual so that the product formula holds. If v is non-archimedean, we put p for the residual characteristic of F_v; $R = R_v$ for the ring of integers; π for a uniformizer; q for the cardinality of the field $R/\pi R$. Then $|\ | = |\ |_v$ satisfies $|\pi|^{-1} = q$. We denote by **Z**, **R**, **C** the rings of integral, real and complex numbers.

Put $G = GL(r, F)$, denote by A the diagonal subgroup and by N the group of unipotent upper triangular matrices. The Weyl group $W = W(G, A)$ of A in G is identified with the group of matrices in G with a single non-zero entry 1 in each row and column. The roots of A in G are denoted by pairs $\alpha = (ij)$ $(1 \leqslant i \neq j \leqslant r)$ and $\alpha(a) = a_i/a_j$ for $a = (a_1, \ldots, a_r)$ in A. The root $\alpha = (ij)$ is positive if $i < j$. If e_α is the matrix with entry 1 at the place α and 0 elsewhere, then we denote by N_α the group of matrices $n = I + xe_\alpha$ (x in F). Note that $ana^{-1} = I + \alpha(a)\, xe_\alpha$. The group W acts on A by $a^w = w^{-1}aw$, and on the set of roots $\Phi = \Phi(A, G)$ by $(w\alpha)(a) = \alpha(a^w)$. Then $e_{w\alpha} = we_\alpha w^{-1}$.

The n-th Hilbert symbol $(\ ,\ )$ is a continuous bilinear map from $F^\times \times F^\times$ onto μ_n, with $(a, b)(b, a) = (a, -a) = (a, b)(-b/a, a+b) = 1$, which satisfies $(a, b) = 1$ for all b in $F^\times$ if and only if a lies in $F^{\times n}$.

By a two-cocycle on a locally compact group H we mean a map β from $H^\times \times H^\times$ onto μ_n with

$$\beta(xx', x'')\,\beta(x, x') = \beta(x, x'x'')\,\beta(x', x'') \qquad \text{and} \qquad \beta(e, x) = \beta(x, e) = 1$$

for all x, x', x'' in H; e denotes the unit of H. It is said to be non-trivial if there is no map s from H to μ_n so that $\beta(x, x') = s(x)\, s(x')/s(xx')$ for all x, x' in H. An n-fold covering group $\widetilde{H}$ of H is a central extension

$$1 \to \mu_n \xrightarrow{i} \widetilde{H} \underset{s}{\overset{p}{\rightleftarrows}} H \to 1.$$

i is an injection of μ_n into the center of $\widetilde{H}$; we identify μ_n with $i(\mu_n)$. The map s is a section; in other words, $p \circ s = 1_H$, so that the multiplication in $\widetilde{H}$ is given by the two-cocycle β; thus $s(x)\, s(x') = s(xx')\,\beta(x, x')$.

Recall that a torus T is called *elliptic* if T/Z is compact. An element t is called *elliptic* if it lies in an elliptic torus, and *regular* if it has distinct eigenvalues. If t is regular elliptic its centralizer G_t in G is an elliptic torus T.

2. Covering groups. — A covering $\widetilde{A}$ of A is given by the two-cocycle $\sigma(a, a') = \Pi_{i<j}(a_i, a'_j)$. Note that

(2.1) $$s(a)\, s(b)\, s(a)^{-1} = s(b)\ (\det a, \det b)/\Pi_j(a_j, b_j).$$

Extend σ to $Y = AW = WA$ by $\sigma(w, w') = 1$, $\sigma(a, w) = 1$, and

$$\sigma(w, a) = (\det w, \det a)\ \Pi_{(i,j) \text{ in } \Phi(w)}(-1, a_i/a_j)/(a_i, a_j),$$

where $\Phi(w) = \{\alpha > 0;\ w\alpha < 0\}$. Then

$$\sigma(aw, a' w') = \sigma(a, a'^{w^{-1}})\ \sigma(w, a')$$

defines a cocycle on Y and a corresponding covering $\widetilde{Y}$.

The map $r : G \to Y$, $r(nyn') = y$, is well-defined by the Bruhat decomposition $G = NWAN$. Put $X = \{(g, \widetilde{y});\ g \text{ in } G, \widetilde{y} \text{ in } \widetilde{Y},\ r(g) = p(\widetilde{y})\}$. Consider the group L(X) of automorphisms of X generated by $\lambda(n)$ (n in N), $\lambda(\widetilde{a})$ ($\widetilde{a}$ in $\widetilde{A}$) and $\lambda(t)$ (t is a simple reflection in W, namely there exists a unique $\alpha > 0$ with $t\alpha < 0$), where

$$\lambda(n)\,(g, \widetilde{y}) = (ng, \widetilde{y}), \qquad \lambda(\widetilde{a})\,(g, \widetilde{y}) = (p(\widetilde{a})g, \widetilde{a}\widetilde{y}),$$

and $$\lambda(t)\ (g, \widetilde{y}) = (tg, s[r(tg)\ r(g)^{-1}]\,\widetilde{y}).$$

L(X) acts transitively on X, and so does the group R(X) of automorphisms of X generated by $\lambda^*(n)$, $\lambda^*(\widetilde{a})$, $\lambda^*(t)$ where

$$(g, \widetilde{y})\ \lambda^*(n) = (gn, \widetilde{y}), \qquad (g, \widetilde{y})\ \lambda^*(\widetilde{a}) = (gp(\widetilde{a}), \widetilde{y}\widetilde{a}),$$

and $$(g, \widetilde{y})\ \lambda^*(t) = (gt, \widetilde{y}s[r(gt)^{-1}\, r(g)]^{-1}).$$

Since $(gx)\, g^* = g(xg^*)$ (g in L(X), g^* in R(X), x in X) (see Milnor [Mi], §12), both L(X) and R(X) act simply transitively on X and R(X) is isomorphic to L(X). The fiber of the map $X \to G$, $(g, \widetilde{y}) \to g$ is μ_n. Hence L(X) is an extension of G by μ_n. The covering $\widetilde{Y}$ is a subgroup of L(X) which preserves $\{(p(\widetilde{y}), \widetilde{y});\ \widetilde{y} \text{ in } \widetilde{Y}\}$. We put $\widetilde{G} = L(X)$. With respect to the section $s : G \to \widetilde{G}$ defined by $s(nyn') = \lambda(n)\, s(y)\, \lambda(n')$ (n, n' in N; y in Y, $s(y)$ in $\widetilde{Y} \subset \widetilde{G}$), the covering group $\widetilde{G}$ is described by a cocycle σ extending the cocycle on Y defined above, and which satisfies

(2.2) $$\sigma(ng, g'\, n') = \sigma(g, g') \quad (n, n' \text{ in } N).$$

Other covering groups $\widetilde{G}_m$ ($0 \leqslant m < n$) are defined by the cocycles

$$\sigma_m(g, g') = \sigma(g, g')\,(\det g, \det g')^m.$$

Let $B = AN$ be the upper triangular minimal parabolic subgroup of G, $\widetilde{B}_m$ the subgroup of $\widetilde{G}_m$ covering B, and $\widetilde{Z}_m$ the pullback through $p : \widetilde{G}_m \to G$ of $Z_m = \{xI;\ x^{r-1+2rm} \text{ in } F^{\times n}\} \simeq F^{\times n/d}$, where $d = (n, r - 1 + 2rm)$. It follows from (2.1) that $\widetilde{Z}_m$ is the center of $\widetilde{B}_m$, hence of $\widetilde{G}_m$.

If F is non-archimedean and its ring of integers is denoted by R, then there exists (see Moore [Mo], pp. 54-56) an open compact subgroup K in GL(r, R) which splits σ_m. Note that $(\det k, \det k') = 1$ for k, k' in a sufficiently small K. Thus

$$\sigma_m(k, k') = \kappa(kk')/\kappa(k)\,\kappa(k')$$

for some function $\kappa : K \to \mu_n$. As $1 = \sigma(k, n) = \kappa(kn)/\kappa(k)\,\kappa(n)$, the restriction of κ to $K \cap N$ is a homomorphism, hence trivial. If $|n| = 1$ we can choose $K = GL(r, R)$. Consider the homomorphism $\kappa^* : K \to \widetilde{G}_m$, $k \to s(k)\,\kappa(k)$. It is not unique. But if κ_1^* is another such map then κ^*/κ_1^* is locally constant. Hence the topology on G defines a unique topology on $\widetilde{G}_m$, which makes κ^* a local homeomorphism. Then $\widetilde{G}_m$ is a locally compact totally disconnected Hausdorff topological group, and $p : \widetilde{G}_m \to G$ is a local homeomorphism.

We say that $\widetilde{G}_m$ splits over a subgroup H of G if there is a homomorphism $h : H \to \widetilde{G}_m$ whose composition with $p : \widetilde{G}_m \to G$ is the identity map on H. Whenever (h, H) are fixed, we identify H with $h(H)$. The map $s : N \to \widetilde{G}_m$ splits $\widetilde{G}_m$ over N. The map $\kappa^* : K \to \widetilde{G}_m$ splits $\widetilde{G}_m$ over K. We now extend κ to a map from G to μ_n.

If F is global and **A** is its ring of adeles, we define a global two-cocycle τ_m on $G(\mathbf{A}) = GL(r, \mathbf{A})$ by $\tau_m(x, x') = \prod_v \tau_{mv}(x, x')$, where $x = (x_v)$, $x' = (x'_v)$ are in $G(\mathbf{A})$. Here $\tau_{mv}(x, x') = \tau_{mv}(x_v, x'_v)$ is the cocycle $\sigma_{mv}(x_v, x'_v)\,\kappa_v(x_v)\,\kappa_v(x'_v)/\kappa_v(x_v\, x'_v)$ which is cohomologous to $\sigma_{mv}(x_v, x'_v)$ and obtains the value 1 on $K_v \times K_v$. The product ranges over all places v of F and it makes sense since $\tau_{mv}(x, x') = 1$ for almost all v. The product formula $\prod_v (a, b)_v = 1$ (a, b in $F^\times$) implies that $\sigma_m(x, x') = \prod \sigma_{mv}(x, x')$ is 1 for x, x' in G(F), hence that the map $x \to s(x)/\kappa(x)$ is a homomorphism from G(F) to $\widetilde{G}(\mathbf{A})$, where $\kappa(x) = \prod \kappa_v(x_v)$. Note that $\kappa_v(x_v) = 1$ for almost all v by [KP], Prop. 0.1.3. Hence $\widetilde{G}(\mathbf{A})$ splits over G(F), a fact which permits the development of a theory of automorphic representations on $\widetilde{G}(\mathbf{A})$.

3. Commutators. — Let x be a regular (distinct eigenvalues) element of G. The centralizer G_x of x in G is a torus T, and for any g in T we write $[x, g]$ for $\tilde{x}\tilde{g}\tilde{x}^{-1}\tilde{g}^{-1}$, where $\tilde{x}, \tilde{g}$ are elements of $\widetilde{G}_m$ which project to x, g. Note that $[x, g]$ depends only on x and g, but not on the lifts $\tilde{x}, \tilde{g}$ of x, g.

Proposition. — $[x, g] = 1$ *for all* g *in* T *if and only if* x *lies in* $Z_m T^n$.

Proof. — The torus T is a direct sum $\bigoplus F_j^\times$ of the multiplicative groups of field extensions F_j of F, with $\sum_j [F_j : F] = r$. Writing $x = (x_j)$, $g = (g_j)$ accordingly, we have

$$[x, g] = (\det x, \det g)^{1+2m}/\prod_j (x_j, g_j)_{F_j}$$

([F], p. 128, for $r = 2$; [KP], Prop. 0.1.5, all r). Here $(\ ,\ )_{F_j}$ is the n-th Hilbert symbol on F_j. Now $[x, g] = 1$ for all g in T if and only if for all j, and all g_j in $F_j^\times$, we have

$$1 = (\det x, N_{F_j/F}\, g_j)^{1+2m}/(x_j, g_j)_{F_j} = (\det x^{1+2m}/x_j, g_j)_{F_j}.$$

Hence x_j lies in $(\det x)^{1+2m} F_j^{\times n}$. Since $\det x = \Pi_j N_{F_j/F}\, x_j$ we have that $(\det x)^{(1+2m)r-1}$ is in $F^{\times n}$. It follows that x is of the form $y^n z$ with y in T and $z = (\det x)^{1+2m}$ in Z_m.

4. Definition of x^*. — We need to relate conjugacy classes on G and $\widetilde{G}$. If n is odd we put $x^* = s(x)^n$. Then $x^* = s(x^n)$ if x is diagonal. The map $x \to x^*$ preserves conjugacy classes ([F], Lemma 0.3.1). If n is even we put $x^* = s(x)^n u(x)$ for x in the subset G_0 of x in G such that $x_i + x_j \neq 0$ for any pair x_i, x_j of eigenvalues of x. Here u is a class function which has the property that $x^* = s(x^n)$ for any diagonal x in $G_0(F)$. Theorem 2.1 of [KP′] proves the existence of a continuous such function u on G_0 with

$$u(x_1, \ldots, x_t) = \prod_{j=1}^{t} u_j(x_j) \prod_{i<j} (\det x_i, \det x_j)_{2,F}$$

if $x = (x_1, \ldots, x_t)$ lies in a standard Levi subgroup of type $(r_1, \ldots, r_t)$ and u_j is the analogous function on $GL_0(r_j, F)$, and

$$u(x_1, \ldots, x_t) = \prod_{j=1}^{t} (x_j, (-1)^r P_x(-x_j)/2x_j)_{2,F_j} (-1, R(x))_{2,F}$$

if x_j is elliptic in $GL_0(r_j, F)$, generating an extension F_j of F. Here $(\ ,\ )_{2,E}$ signifies the 2nd Hilbert symbol of E, and x_j is regarded as an element of F_j. The polynomial $P_x(y) = \det(yI - x)$ is the characteristic polynomial of x, $R(x) = \Pi_{i<j}(x'_i + x'_j)$, where $x'_1, \ldots, x'_r$ are the r eigenvalues of x in G_0. Further, for z in $F^\times$ we have

$$u(zx) = u(x)\,(z, (-1)^{r(r-1)/2} \det x^{r-1})_2.$$

Hence we have $(zx)^* = s(z^n)\, x^*$. The case of $r = 2$ is in [F], Lemma 1.2.3.

5. Order. — The Jordan decomposition asserts that for any x in G there is a unique pair of a semi-simple element s and a unipotent element u in G so that $x = su = us$. Up to conjugacy in $G(\overline{F})$—where $\overline{F}$ is an algebraic closure of F—we have

$$s = (x_1 I_{r_1}, \ldots, x_t I_{r_t}),$$

where $x_1, \ldots, x_t$ are the distinct eigenvalues of x with multiplicities $r_1, \ldots, r_t$ and u is of the form $(u_1, \ldots, u_t)$ where u_i is an upper triangular unipotent $r_i \times r_i$ matrix. Such unipotent u_i consist of Jordan blocks of sizes $j_1, j_2, \ldots$, which we arrange so that $j_\alpha \geqslant j_{\alpha+1} \geqslant 0$. Note that u_i lies in the closure of the conjugacy class of u'_i if and only if $\Sigma_{\alpha=1}^{\beta} j'_\alpha \geqslant \Sigma_{\alpha=1}^{\beta} j_\alpha$ for all β $(= 1, 2, \ldots)$. We say that $x \leqslant x'$ if s, s' are conjugate and the u_i are in the closure of the conjugacy class of the u'_i for all i. Similarly we define $s(x) \leqslant s(x')$ in $\widetilde{G}$ if $x \leqslant x'$ in G.

6. Orbital integrals. — Suppose F is local. Fix unitary characters $\omega : Z \to \mathbf{C}^\times$ and $\widetilde{\omega} : \widetilde{Z} \to \mathbf{C}^\times$ with $\omega(z) = \widetilde{\omega}(s(z^n))$ so that the restriction of $\widetilde{\omega}$ to μ_n is injective. *Throughout* f and $\widetilde{f}$ denote smooth (this means locally constant in the non-archimedean case) complex-valued functions on G and $\widetilde{G}$, which satisfy $f(zx) = \omega(z)^{-1} f(x)$ (z in Z) and $\tilde{f}(\tilde{z}\tilde{x}) = \tilde{\omega}(\tilde{z})^{-1}\tilde{f}(\tilde{x})$ ($\tilde{z}$ in $\tilde{Z}$), and whose support is compact modulo the center. Let $\tilde{G}_x$ be the centralizer of $\tilde{x}$ in $\tilde{G}$. It depends only on $x = p(\tilde{x})$. Let $\tilde{Z}_x$ be the split component in the center of $\tilde{G}_x$. Similarly we have G_x, Z_x. For example $Z_x = Z$ if x is regular elliptic, for then G_x/Z is compact. Let dg, dt, dz, $\tilde{d}g$, $\tilde{d}t$, $\tilde{d}z$ denote Haar measures on G/Z, G_x/Z, Z_x/Z, $\tilde{G}/\tilde{Z}$, $\tilde{G}_x/\tilde{Z}$, $\tilde{Z}_x/\tilde{Z}$. For x in G, $\tilde{x}$ in $\tilde{G}$ with $p(\tilde{x}) = x$, such that $\tilde{x}\tilde{y} = \tilde{y}\tilde{x}$ whenever $xp(\tilde{y}) = p(\tilde{y})\,x$, the measures are related by $\tilde{d}g/\tilde{d}t = dg/dt$ via the isomorphism $G/G_x \simeq \tilde{G}/\tilde{G}_x$. For x in G, and $\tilde{x}$ in $\tilde{G}$, we put

$$\Phi(x,f) = \int_{G_x\backslash G} f(g^{-1}xg)\,\frac{dg}{dt}, \qquad \Phi''(x,f) = \int_{Z_x\backslash G} f(g^{-1}xg)\,\frac{dg}{dz},$$

$$\Phi(\tilde{x},\tilde{f}) = \int_{\tilde{G}_x\backslash\tilde{G}} \tilde{f}(g^{-1}\tilde{x}g)\,\frac{\tilde{d}g}{\tilde{d}t}, \qquad \Phi''(\tilde{x},\tilde{f}) = \int_{\tilde{Z}_x\backslash\tilde{G}} f(g^{-1}\tilde{x}g)\,\frac{\tilde{d}g}{\tilde{d}z}.$$

The convergence of these integrals for all x has been shown in [R]. Note that $\Phi(\tilde{x},\tilde{f}) = 0$ whenever there is g in $\tilde{G}$ with $g^{-1}\tilde{x}g = i(\zeta)\,\tilde{x}$ and $\zeta \neq 1$. Also let $D(x)$ be the discriminant of the characteristic polynomial of x in G, namely $D(x) = \Pi_{i<j}(x_i - x_j)^2$, where $x_1, x_2, \ldots$ are the distinct eigenvalues of x. Put $\Delta(x) = |D(x)|^{1/2}/|\det x|^{(r-1)/2}$. Hence

$$\Delta(x) = \left| \prod_{i<j} \frac{(x_i - x_j)^2}{x_i\,x_j} \right|^{1/2},$$

if x has distinct eigenvalues $x_1, \ldots, x_r$. Put

$$F(x,f) = \Delta(x)\,\Phi(x,f), \qquad F(\tilde{x},\tilde{f}) = \Delta(p(\tilde{x}))\,\Phi(\tilde{x},\tilde{f}).$$

Then $\Phi(f) : x \to \Phi(x,f)$ and $F(f)$ are functions on the space $X(G)$ of conjugacy classes in G, and $\Phi(\tilde{f})$ and $F(\tilde{f})$ are functions on $X(\tilde{G})$, the space of conjugacy classes in $\tilde{G}$. Similarly we define F'' using Φ''.

We shall deal only with functions f with the property $\tilde{\omega}(z)\,F(x,f) = F(x',f)$ for any x, x' in G with $zx^* = x'^*$ for some z in $\tilde{Z}$.

7. Change of variables. — From now on we denote by P a parabolic subgroup of G, with unipotent radical N and Levi subgroup M containing A. Denote by δ_P the modulus homomorphism on P, thus $d(ab) = \delta_P(a)\,db$ (a, b in P) for any right Haar measure db on P. There is a bijection between the sets of parabolic subgroups $\tilde{P} = \tilde{M}N$ of $\tilde{G}$ and $P = MN$ of G, given by $p(\tilde{P}) = P$, and $p(\tilde{M}) = M$. In (2) we identified N and $s(N)$. For $\tilde{m}$ in $\tilde{M}$ which projects to $m = p(\tilde{m})$ in M, put

$$\tilde{f}_N(\tilde{m}) = \delta_P(m)^{1/2} \int_K \int_N \tilde{f}(k^{-1}\,\tilde{m}nk)\,dn\,dk.$$

It depends on N, but its orbital integral at a regular element depends only on M. Also put

$$\begin{aligned} D_{G/M}(m) &= |\det(1 - \mathrm{Ad}(m))|_{\mathrm{Lie}\,G/\mathrm{Lie}\,M}| \\ &= |\det(1 - \mathrm{Ad}(m))|_{\mathrm{Lie}\,N + \mathrm{Lie}\,\bar{N}}| = |\det(1 - \mathrm{Ad}(m))|_{\mathrm{Lie}\,N}|^2\,\delta_P(m); \end{aligned}$$

here $\overline{N}$ is the unipotent radical of the parabolic subgroup $\overline{P} = M\overline{N}$ opposite to P; Lie H is the Lie algebra of a Lie group H. Note that $D_{G/M}(m^{-1}xm) = D_{G/M}(x)$ for m, x in M, and that the Jacobian of $N \to N$, $n \to m_0^{-1} n^{-1} m_0 n$, is

$$|\det(1 - \mathrm{Ad}(m_0))|_{\mathrm{Lie}\,N}| \quad (m_0 = m^{-1}xm).$$

We use the Iwasawa decomposition $\widetilde{G} = \widetilde{M}NK$; N embeds in $\widetilde{G}$ as $s(N)$. In particular $\widetilde{m}^{-1}\widetilde{n}^{-1}\widetilde{m}\widetilde{n} = s(m^{-1}n^{-1}mn)$ for any $\widetilde{n}$, $\widetilde{m}$ in $\widetilde{N}$, $\widetilde{M}$ which project to n, m. We put $\widetilde{f}^K(x) = \int \widetilde{f}(k^{-1}xk)\,dk$ (k in K), and note that $\widetilde{T}\backslash\widetilde{G} \simeq T\backslash G$, and $\widetilde{dg}/\widetilde{dt} = dg/dt$. For any x in a torus $\widetilde{T}$ contained in $\widetilde{M}$ we have

$$\begin{aligned}\int_{T\backslash G} \widetilde{f}(g^{-1}xg)\,\frac{dg}{dt} &= \int_{T\backslash M}\int_N\int_K \widetilde{f}(k^{-1}n^{-1}m^{-1}xmnk)\,dk\,dn\,\frac{dm}{dt}\\ &= \int_{T\backslash M}\int_N \widetilde{f}^K(n^{-1}m_0n)\,dn\,\frac{dm}{dt}\\ &= \int_{T\backslash M} D_{G/M}(x)^{-1/2}\,\delta_P(m^{-1}xm)^{1/2}\int_N \widetilde{f}^K(m^{-1}xmn)\,dn\,\frac{dm}{dt}\\ &= D_{G/M}(x)^{-1/2}\int_{T\backslash M}\widetilde{f}_N(m^{-1}xm)\,\frac{dm}{dt}.\end{aligned}$$

Define $\Delta_M(x)$ to be $\Pi\,\Delta_j(x_j)$ if $M = M_1 \times M_2 \times \dots$ and correspondingly $x = (x_1, x_2, \dots)$. Here Δ_j is the Δ-factor of M_j. Put

$$F^M(x, \widetilde{f}_N) = \Delta_M(x)\int_{T\backslash M}\widetilde{f}_N(m^{-1}xm)\,\frac{dm}{dt}.$$

We deduce

Proposition. — For any x in $\widetilde{M}$ regular in $\widetilde{G}$ we have $F(x, \widetilde{f}) = F^M(x, \widetilde{f}_N)$.

Proof. — The assertion follows from the relation $\Delta(x)\,D_{G/M}(x)^{-1/2} = \Delta_M(x)$. The analogous statement for f and G is the special case $n = 1$.

8. Germ expansion. — Recall that X(G) denotes the space of conjugacy classes on G. A germ in the stalk at x in X(G) (resp. X($\widetilde{G}$)) of the sheaf of complex valued functions on X(G) (resp. X($\widetilde{G}$)) is denoted here by $\widetilde{\sigma}_x$ (resp. σ_x). For convenience we say that $\widetilde{x}$ in $\widetilde{G}$ is *good* if $\widetilde{g}^{-1}\widetilde{x}\widetilde{g} = \widetilde{x}$ for any $\widetilde{g}$ in $\widetilde{G}$ with $g^{-1}xg = x$ ($x = p(\widetilde{x})$, $g = p(\widetilde{g})$). Namely $p(\widetilde{G}_x) = G_x$. The following result—which is a consequence of the uniqueness of the Haar measure—is due to Shalika, Harish-Chandra [H], Vigneras [V] and [KP′]. For the definition of $F(y, f)$ and $F(y, \widetilde{f})$ see (6).

Theorem. — At each good y in X($\widetilde{G}$) there is a germ $\widetilde{\sigma}_y$, with the following property. For any $\widetilde{f}$, the germ of F($\widetilde{f}$) at a semi-simple x in X($\widetilde{G}$) is given by

$$\sum_{y \geqslant x} F(y, \widetilde{f})\,\widetilde{\sigma}_y.$$

Conversely, suppose $\widetilde{\mathrm{H}}$ *is a function on* $\mathrm{X}(\widetilde{\mathrm{G}})$ *which transforms under the center by* $\widetilde{\omega}^{-1}$, *and which is supported on the projection by* $\widetilde{\mathrm{G}} \to \mathrm{X}(\widetilde{\mathrm{G}})$ *of the product of* $\widetilde{\mathrm{Z}}$ *and a compact set in* $\widetilde{\mathrm{G}}$. *If for each good* y *in* $\mathrm{X}(\widetilde{\mathrm{G}})$ *there is a complex number* $\widetilde{h}(y)$ *so that the germ of* $\widetilde{\mathrm{H}}$ *at each semi-simple* x *in* $\mathrm{X}(\widetilde{\mathrm{G}})$ *is of the form* $\Sigma_{y \geqslant x} \widetilde{h}(y)\, \widetilde{\sigma}_y$, *then there exists an* $\widetilde{f}$ *so that* $\widetilde{\mathrm{H}} = \mathrm{F}(\widetilde{f})$. *Moreover* $\widetilde{\mathrm{H}} = \widetilde{h}$.

Remarks.

(1) Here x is semi-simple; it need not be good. The germ expansion can be non-zero on the *closure* of the set of good elements. See the example below.

(2) If x is good and $x \leqslant y$ then y is good too, by Corollary 9 below.

(3) The analogous statement for the group G and the function f is contained in the above statement as the case of $n = 1$. In this context note that each x in G is good. Hence we have a germ σ_y at all y in X(G).

(4) If y is regular then we have $\sigma_y = 1$. If y is good and regular then we have $\widetilde{\sigma}_y = 1$. Namely $\mathrm{F}(f)$ and $\mathrm{F}(\widetilde{f})$ are smooth on the regular and good regular sets.

Definition. — The functions f and $\widetilde{f}$ are *matching* if $\mathrm{F}(x, f) = \mathrm{F}(x^*, \widetilde{f})$ for all x in G so that x^* is regular. A function f for which there exists a matching $\widetilde{f}$ is called *good.*

Example. — Consider the case $r = 2$, $n = 2$ and $x_0 = \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}$, where θ is a non-square in $\mathrm{F}^\times$. Hence x_0 lies in an elliptic torus T of G. The function $\mathrm{F}(f)$ is regular at x_0. At $\widetilde{x}$ in T near $x_0^* = s\left(\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\right)$ we have

$$\mathrm{F}(\widetilde{x}, \widetilde{f}) = \mathrm{F}(x_0^*, \widetilde{f})\, \widetilde{\sigma}_{x_0^*}(\widetilde{x}) + \mathrm{F}(\widetilde{x}_1, \widetilde{f})\, \widetilde{\sigma}_{\widetilde{x}_1}(\widetilde{x}) = \mathrm{F}(\widetilde{x}_1, \widetilde{f})\, \widetilde{\sigma}_{\widetilde{x}_1}(\widetilde{x}),$$

where $\widetilde{x}_1 = s\left(\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}\right)$, since $\mathrm{F}(x_0^*, \widetilde{f}) = 0$. Note that

$$\begin{aligned} \mathrm{F}(\widetilde{x}_1, f) &= |\theta|^{-1} \Phi(\widetilde{x}_1, \widetilde{f}) \\ &= |\theta|^{-1} \int_{\mathrm{F}^\times} \widetilde{f}^{\mathrm{K}}\left(s\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}^{-1} s\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix} s\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\right) |b|\, d^\times b \\ &= |\theta|^{-1} \int_{\mathrm{F}} \widetilde{f}^{\mathrm{K}}\left(s\begin{pmatrix} \theta & b \\ 0 & \theta \end{pmatrix} i((\theta, b))\right) db. \end{aligned}$$

It can be shown in several ways, one of which is [F], (1.3), that $\widetilde{\sigma}_{\widetilde{x}_1}$ is constant, so that $\mathrm{F}(\widetilde{x}, \widetilde{f})$ is locally constant at x_0^*. This is compatible with the statement that there exist matching $\widetilde{f}, f$.

In the case of G, it follows from [Ho], Lemma 5, [H], Theorem 5, and Theorem D of [K′], that the germs are locally constant on the elliptic set. Hence

Proposition. — Suppose $c(x)$ is a conjugacy invariant function on G *which is supported on the elliptic set, and its restriction to the elliptic set is locally constant. Then there exists an f with* $\Phi''(x, f) = c(x)$.

9. Good elements. — In view of Theorem 8 it will be illuminating to determine the good set of G. First we recall Proposition 1.1 of [KP'] which generalizes Lemma 3. Suppose x is a semi-simple element of G. The centralizer G_x of x is $\Pi\mathrm{GL}(r_i, F_i)$ with $\Sigma r_i f_i = r$, $f_i = [F_i : F]$. If $y \geqslant x$ then $y = xu = ux$ with unipotent u in G_x, hence $u = (u_i)$. Also we write $x = (x_i)$. Each u_i is conjugate to a product of Jordan blocks of sizes $r_{i1}, r_{i2}, \ldots$ with $\Sigma_j r_{ij} = r_i$. Let $d_i = d_i(y) = g.c.d_j\{r_{ij}\}$ be the greatest common divisor of $r_{i1}, r_{i2}, \ldots$

Lemma. — $s(y)$ is good if and only if x_i lies in $(\det x)^{1+2m} F_i^{\times n/(d_i, n)}$ *for all i.*

An element $y = xu$ (x semi-simple, u unipotent) of G is called regular (in the sense of Steinberg [St]) if there is no $y' > y$; namely the unipotent part u of y is as " large " as possible. In the next proposition, we say that y is good if $s(y)$ is good.

Proposition. — If x is elliptic regular then $y \geqslant x^n$ is good if and only if y is regular.

Proof. — If x is elliptic regular it generates a field extension E of F of degree r. Suppose that x^n generates the subfield E' of E over F. Since $n/(n, r)$ is prime to r, we have $x^{(r,n)}$ in $E'^{\times}$. Let r' be the minimal divisor of (r, n) so that $\theta = x^{r'}$ lies in $E'^{\times}$. Namely $[E : E'] = r'$ and the centralizer of $x^{r'}$ in G is $\mathrm{GL}(r', E')$ (and so is the centralizer of x^n). The $d = d(y)$, $y \geqslant x^n$, of the Lemma satisfies $d \leqslant r'$. But the Lemma implies that $y \geqslant x^n$ is good if and only if $x^n = \theta^{n/r'}$ lies in $E'^{\times n/(d,n)}$. Namely $n/(d, n)$ divides n/r' or r' divides (d, n). Since $d \leqslant r'$ we have $d = r'$, and y is regular.

Suppose x is semi-simple, M a minimal standard Levi subgroup containing a conjugate of x, and replace x by a conjugate to assume that x lies in M. Thus x is elliptic in M and we denote by x' a regular (in the sense of [St] as above) element in M with $x' \geqslant x^n$.

Corollary. — $y' \geqslant x^n$ is good if and only if $y' \geqslant x'$.

Remark. — Similarly we have that if x is good and $x \leqslant y$ then y is also good; and if $\tilde{x}$ is good and lies in the closure $\bar{O}(\tilde{y})$ of the conjugacy class $O(\tilde{y})$ of $\tilde{y}$ in $\tilde{G}$, then it does not lie in $\bar{O}(\zeta\tilde{y})$ for any $\zeta \neq 1$ in μ_n.

10. At the identity. — Denote by ρ a diagonal matrix in G with eigenvalues in μ_n. We assume that the centralizer G_ρ of ρ in G is the standard Levi subgroup $M(\rho) = M_1 \times M_2 \times \ldots \times M_n$, $M_i = \mathrm{GL}(r_i)$, $r_i = r_i(\rho) \geqslant 0$, $\Sigma_i r_i = r$, of a standard parabolic subgroup $P(\rho)$ with unipotent radical $N(\rho)$, and the eigenvalue of ρ in M_i is ζ^i for a fixed generator ζ of μ_n. For a large integer j, let U_j be the set of x in K (see (2)) so that the valuation of the entries of $x - 1$ are bounded by q^{-j}. Thus U_j is a compact open subgroup of G. A Haar measure on G/Z was fixed in (6). Denote by $|U_j|$ the volume of ZU_j/Z. Let $\varphi(\rho, j)$ be the function with the properties of f specified in (6) which is supported on $Z\rho U_j$, and takes the value $1/|U_j|$ on ρU_j. Let U_j^* be the group of x^*, where x lies in U_j. Let φ_j^* be the function with the properties specified in (6) which is supported on $\tilde{Z}U_j^*$, and takes the value $1/|U_j^*|$ on U_j^*. Here $|U_j^*|$ is the volume of $\tilde{Z}U_j^*/\tilde{Z}$ with respect to the measure of (6).

Throughout $\tilde{\pi}$, π denote admissible representations of $\tilde{G}$, G of finite length with central characters $\tilde{\omega}$, ω (see (6)). Put $\tilde{\pi}(\tilde{f}) = \int \tilde{f}(x)\,\tilde{\pi}(x)\,\tilde{d}x$ (x in $\tilde{G}/\tilde{Z}$). It is an operator of finite rank. Its trace is denoted by $\operatorname{tr}\tilde{\pi}(\tilde{f})$. The operator $\tilde{\pi}(\varphi_j^*)$ is a projection from the space of $\tilde{\pi}$ to the subspace of U_j^*-invariant vectors. The latter space is finite dimensional and $\operatorname{tr}\tilde{\pi}(\varphi_j^*)$ is its dimension. We now construct a matching φ_j.

Proposition. — For any large integer j put $r(\rho) = (r^2 - \Sigma_i r_i^2)/2$ *and* $\varphi_j = \Sigma_\rho q^{jr(\rho)}\varphi(\rho, j)$. *Then* $F(x, \varphi_j) = F(x^*, \varphi_j^*)$ *for all x with regular* x^*.

Proof. — The equalities below are valid up to constant multiples independent of j. If $F(x, \varphi(\rho, j)) \neq 0$ we may assume that x lies in ρU_j, and even in $\rho M(\rho)$, namely that $x = \rho u$ with u in $M(\rho) \cap U_j$. Consider the integral

$$F(\rho u, \varphi(\rho, j)) = \Delta_{M(\rho)}(\rho u)\int_{M(\rho)/M(\rho)_u}\int_{N(\rho)}\int_K \varphi(\rho, j)\,(km\rho um^{-1}nk^{-1}).$$

We may assume that mum^{-1} lies in U_j (and n in K). The integration over k has to be taken only over the subgroup U_j' of matrices k in K whose entries below $M(\rho)$ are bounded by q^{-j} in valuation. We obtain

$$\Delta_{M(\rho)}(u)\,q^{-jr(\rho)}\int_{M(\rho)/M(\rho)_u}\int_{N(\rho)}\int_K \varphi(1, j)\,(kmum^{-1}nk^{-1}) = q^{-jr(\rho)}F(u, \varphi(1, j)).$$

Since $\Delta\varphi(1, j)(u) = \Delta\varphi_j^*(u^*)$ for u in U_j, we have $F(u, \varphi(1, j)) = F(u^*, \varphi_j^*)$.

11. Satake transform. — Assume that ω is unramified. Let $\mu = (\mu_1, \ldots, \mu_r)$ be an unramified character of A whose restriction to Z is ω. It determines an r-tuple $\mu(\boldsymbol{\pi}) = (\mu_1(\boldsymbol{\pi}), \ldots)$ of complex numbers, or an element of the subset A′ of $A(\mathbf{C}) \simeq \mathbf{C}^{\times r}$ of elements with determinant $\omega(\boldsymbol{\pi})$. Let $I(\mu)$ denote the representation of G unitarily induced from the character μ of the upper triangular minimal parabolic subgroup B. Let $\mathbf{H}$ be the convolution algebra of K-biinvariant compactly supported modulo Z functions f on G transforming under Z by ω^{-1}. Here $K = G(R)$, R being the ring of integers of F. If f is spherical (lies in $\mathbf{H}$), and a is regular in A, then $F(a, f)\left(= \delta_{P_0}^{1/2}(a)\,|K|\int_N f(an)\,dn\right)$ depends on a in $A/A(R) \simeq \mathbf{Z}^r$. Thus if $a \equiv \boldsymbol{\pi}^\lambda$ (mod A(R)) for λ in $\mathbf{Z}^r$, we put $F(\lambda, f)$ for $F(a, f)$ and $\lambda(\mu(\boldsymbol{\pi}))$ for $\mu(a)$. The algebra $\mathbf{H}$

is isomorphic to the algebra $\mathbf{C}[A']^W$ of finite Laurent series on A' invariant under the action of the Weyl group $W = W(G, A)$ by the Satake isomorphism $f \to f^\vee$. Here

$$f^\vee(\mu(\pi)) = \operatorname{tr} I(\mu)(f) = \int_{A/Z} \mu(a)\, F(a,f)\, da$$
$$= |R^\times|^{r-1} \sum_{\lambda \in \mathbf{Z}^r/\mathbf{Z}} F(\lambda, f)\, \lambda(\mu(\pi)).$$

The first equality is our definition. The second equality follows by computing the character of the induced representation. We take the volume $|R^\times|$ of the multiplicative group $R^\times$ of R to be 1.

Now suppose that $(n, q) = 1$. An unramified character $\tilde{\mu}$ of $\tilde{A}^n \tilde{Z}$ whose restriction to $\tilde{Z}$ is $\tilde{\omega}$ determines an r-tuple $\tilde{\mu}(\pi^n) = (\tilde{\mu}_1(s(\pi^n)), \dots)$ of complex numbers. Here $\tilde{A}^n = p^{-1}(A^n)$; A^n is the group of a^n with a in A. Extend $\tilde{\mu}$ to a character of a maximal abelian subgroup $\tilde{A}_0$ of $\tilde{A}$, and set $\tilde{\mu} = 1$ on N. We denote by $I(\tilde{\mu})$ the representation of $\tilde{G}$ unitarily induced from $\tilde{\mu}$ on $\tilde{B}_0 = \tilde{A}_0 N$. As $|n| = 1$, K lifts to a subgroup $K^* = \kappa^*(K)$ of $\tilde{G}$. Hence we can define $\tilde{\mathbf{H}}$ to be the convolution algebra of K^*-biinvariant compactly supported modulo $\tilde{Z}$ functions $\tilde{f}$ on $\tilde{G}$ which transform under $\tilde{Z}$ by $\tilde{\omega}^{-1}$. $\tilde{\mathbf{H}}$ is isomorphic to $\mathbf{C}[A']^W$ by the Satake isomorphism $\tilde{f} \to \tilde{f}^\vee$, where

$$\tilde{f}^\vee(\tilde{\mu}(\pi^n)) = \operatorname{tr} I(\tilde{\mu}, \tilde{f}) = \int_{\tilde{A}^n \tilde{Z}/\tilde{Z}} t\tilde{\mu}(a)\, F(a, \tilde{f})\, \tilde{d}a$$
$$= \int_{A/Z} \tilde{\mu}(a^n)\, F(a^n, \tilde{f})\, da = \Sigma F(n\lambda, \tilde{f})\, \lambda(\tilde{\mu}(\pi^n)).$$

The sum ranges over $\lambda \in \mathbf{Z}^r/\mathbf{Z}$. The second equality is based on a character computation carried out in [F], p. 141, where t (erroneously omitted there) is the index in $\tilde{A}$ of a maximal abelian subgroup. We choose the measures $\tilde{d}a$, da to be so related that the third equality holds. We delay the discussion of measures to (24) below.

The relation $\mu(a) = \tilde{\mu}(s(a^n))$ defines an embedding of the variety of the characters $\tilde{\mu}$ into the variety of characters μ, the image being the subspace of characters μ which are the nth powers of characters on A. The map $\tilde{\mu} \to \mu$ defines a map $I(\tilde{\mu}) \to I(\mu)$ of induced representations. We define a dual map $\mathbf{H} \to \tilde{\mathbf{H}}$, $f \to f^*$, by

$$f^{*\vee}(\tilde{\mu}(s(\pi^n))) = f^\vee(\mu(\pi)).$$

The identity

$$\Sigma_\lambda F(n\lambda, f^*)\, \lambda(\tilde{\mu}(\tilde{s}(\pi^n))) = \Sigma_\lambda F(\lambda, f)\, \lambda(\mu(\pi))$$

implies $F(n\lambda, f^*) = F(\lambda, f)$ for all λ, or $F(a^*, f^*) = F(a, f)$ for all a in A with regular a^n in A.

In particular, if f^0 is the unit element of $\mathbf{H}$, whose support is KZ and which takes the value $|K|^{-1}$ on K, then the corresponding function f^* is the unit element $\tilde{f}^0$ of $\tilde{\mathbf{H}}$. The function $\tilde{f}^0$ is supported on $K^* \tilde{Z}$ and its value on K^* is $|K|^{-1}$.

12. Unit element. — The results of this section are due to [KP'], § 5. They are reproduced here for completeness.

The identity $F(x^*, \widetilde{f}^0) = F(x, f^0)$ can be proven in some cases using the results of [K], which we now recall. An element k of K is called K-*semi-simple* if $k^b = I$ for some $b \geqslant 1$ in $\mathbf{Z}$ with $(b, q) = 1$. Here q is the cardinality of the field $R/\pi R$. It is called K-*unipotent* if $k^{q^b} \to I$ as $b \to \infty$. In [K], Lemma 2, p. 226, an analogue of the Jordan decomposition—which we call the K-*decomposition*—is proven. It asserts that each k in K can be written uniquely in the form $k = su$, where s is K-semi-simple and u is K-unipotent, and s commutes with u.

Let F' be the unramified extension of degree m of F and ε a character of $F^\times$ whose kernel is $N_{F'/F} F'^\times$. Given $r' \geqslant 1$ let f'^0 be the characteristic function of $GL(r', R_{F'})$ in $GL(r', F')$ divided by the volume factor $|PGL(r', R_{F'})|$. Let $f^0_{F'}$ be the characteristic function of $GL(r' m, R)$ in $GL(r' m, F)$, divided by $|PGL(r' m, R)|$. We embed $GL(r', F')$ in $GL(r' m, F)$ so that $GL(r', R_{F'})$ lies in $GL(r' m, R)$.

Assertion (F', F, r'). — *For any regular elliptic* K-*unipotent* k *in* $K' = GL(r', R_{F'})$ *we have*

$$u(k)^{m-1} \Delta_{F'}(k) \int_{PGL(r', F')} f'^0(g^{-1} kg)\, dg$$
$$= \Delta_F(k) \int_{PGL(r' m, F)} \varepsilon(\det g)\, f^0(g^{-1} kg)\, dg.$$

Theorem. — *Assertion* (F', F, r') *is valid if* $r' = 1$ *or* $F' = F$.

Proof. — This is Theorem 1' of [K], p. 229. It is trivial if $F' = F$.

We would like to prove for each k in K with regular k^n in G that

$$(*) \qquad \Delta(k^n) \int_{PG} \widetilde{f}^0(g^{-1} k^* g)\, dg = \Delta(k) \int_{PG} f^0(g^{-1} kg)\, dg.$$

Here we put $PG = G/S = \widetilde{G}/\widetilde{S}$, where S is the split component of the centralizer T of k, and $\widetilde{S} = p^{-1}(S)$. We choose the same measure dg on both sides. It suffices to prove this with $G = GL(r'', F)$ for any $r'' \leqslant r$ and elliptic regular k in $K = G(R)$. Then $S = Z$.

Note that since $s = \lim k^{p^n}$ as $n \to \infty$, we have that $g^{-1} kg = k$ implies $g^{-1} sg = s$. Namely the centralizer G_k of k is contained in G_s, hence in G_{s^n}.

Now suppose that k is elliptic regular, and put $G_k = T$. Since s commutes with k it lies in T, hence it is elliptic and semi-simple. The centralizer G_{s^n} of s^n is then reductive and equals $\prod_{j=1}^{t} GL(r_j, F_j)$, where $f_j = [F_j : F]$ and $\Sigma_j f_j r_j = r$. This contains the elliptic torus T, hence $t = 1$ and $G_{s^n} = GL(r', F')$ with $r' f' = r$. In particular $\theta = s^n$ lies in F', and we can introduce the extension $F'' = F'(\theta^{1/n})$ of F' of degree m. It is clear that m divides n and r', so we write $r'' m = r'$.

Proposition. — *If* k^n *is elliptic regular, then Assertion* (F'', F', r'') *implies* $(*)$.

Corollary. — *If* N *is the least common multiple of all composite* $r^* \leqslant r$ *and* $(n, N) = 1$, *then* $F(k, f^0) = F(k^*, \widetilde{f^0})$ *for all* k *in* K *with regular* k^n.

Proof of Corollary. — By the Proposition we need Assertion (F'', F', r'') to hold for all $m = [F'' : F']$ and r'' with $mr'' \leqslant r$. If $m \neq 1 \neq r''$ then mr'' divides N, hence $(r'' m, n) = 1$. As m divides n it is 1 and we deduce that $m = 1$ or $r'' = 1$; but then Assertion (F'', F', r'') follows from the Theorem.

Proof of Proposition. — As the K-decomposition of k^n is $s^n u^n$, the integral on the left of $(*)$ can be taken only over PG' K, where $G' = G_{s^n}(F) = GL(r', F')$, by [K], Lemma 3.3, p. 226. The integral on the right ranges over PG'' K, where

$$G'' = G_s(F) = GL(r'', F''),$$

for the same reason.

Note that $k^* = s(k)^n u(k)^{m-1} = \kappa^*(s^n) \kappa^*(u^n) u(k)^{m-1}$, since $u(k) = 1$ if n is even and m is odd. By [KP'], Proposition 0.1.5, we have for g in $GL(r', F')$ that

$$g^{-1} \kappa^*(s^n) g = \kappa^*(s^n)\, i((\det s^n, \det g)_F^{1+2m}/(\theta, \det' g)_{F'}),$$

where det' is the determinant map of $GL(r', F')$. Since $\det s^n$ is an nth power in $F^\times$ we put $\varepsilon(g) = i((\theta, \det' g)_{F'}^{-1})$ to obtain

$$g^{-1} \kappa^*(s^n) g = \kappa^*(s^n)\, \varepsilon(g).$$

Since u is K-unipotent and $(n, q) = 1$ we have that $g^{-1} u^n g$ lies in $K' = GL(r', R_{F'})$ if and only if $g^{-1} ug$ lies in K'. Hence it remains to show that

$$u(k)^{m-1} \Delta(k^n) \sum_{\substack{g \in PG' \\ g^{-1}ug \in K'}} \varepsilon(g) = \Delta(k) \sum_{\substack{g \in PG'' \\ g^{-1}ug \in K''}} 1,$$

where $K'' = GL(r'', R_{F''})$. But since

$$\Delta(k^n) = \Delta(\theta u^n) = \Delta'(u^n) = \Delta'(u)$$

is the Δ-factor with respect to G', and $\Delta(k) = \Delta(su) = \Delta''(u)$ is the Δ-factor with respect to G'', the identity follows from Assumption (F'', F', r'').

13. Supercusp forms. — Suppose $\widetilde{\pi}$ is an admissible irreducible representation of $\widetilde{G}$. Harish-Chandra [H] proved that there exists a locally integrable conjugation invariant function on $\widetilde{G}$, denoted here by $\chi_{\widetilde{\pi}}$ or $\chi(\widetilde{\pi})$ and called the *character* of $\widetilde{\pi}$. It is smooth on the regular set, transforms under $\widetilde{Z}$ by $\widetilde{\omega}$, and satisfies $\operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \int \chi_{\widetilde{\pi}}(x) \widetilde{f}(x)\, \widetilde{dx}$ (x in $\widetilde{G}/\widetilde{Z}$). A *matrix-coefficient* of $\widetilde{\pi}$ is a function $\widetilde{c}$ on $\widetilde{G}$ of the form $\widetilde{c}(x) = (\widetilde{\pi}(x)\, v, v')$, where v is in V, v' is in the dual space, (,) is the inner product and $v, v' \neq 0$. A *supercusp form* is a function $\widetilde{f}$ with the properties specified in (6), such that $\int_N \widetilde{f}(xny)\, dn = 0$

for all proper parabolic subgroups P of G, and all x, y in $\widetilde{G}$. An irreducible representation $\widetilde{\pi}$ is called *supercuspidal* if one (hence all) of its matrix coefficients is a supercusp form. If $d(\widetilde{\pi})$ denotes the formal degree [HD] of a supercuspidal representation $\widetilde{\pi}$ and $(v, v') = 1$, then the matrix coefficient $\widetilde{f}(x) = d(\widetilde{\pi})(\widetilde{\pi}(x)v, v')$ satisfies (i) $\Phi(x, \widetilde{f}) = 0$ if x is regular non-elliptic element of $\widetilde{G}$ (Selberg's principle; [H'], Theorem 29) ; (ii) $\Phi''(x, \widetilde{f}) = \chi(\widetilde{\pi})(x)$ if x is regular elliptic. Also, the irreducible $\widetilde{\pi}$ is called *square-integrable* or *discrete series* if its (not necessarily compactly supported) matrix coefficients (one, hence all) are absolutely square-integrable on $\widetilde{G}/\widetilde{Z}$. Such representations can be realized in the space $L^2(\widetilde{G})$ of functions on $\widetilde{G}$ which transform under $\widetilde{Z}$ by the character $\widetilde{\omega}$, and are absolutely square-integrable modulo $\widetilde{Z}$.

In the remainder of this section we prove

Theorem. — *If* $(N, n) = 1$ *(see Corollary* 12) *and* $\widetilde{f}$ *is a matrix coefficient of a supercuspidal representation* $\widetilde{\pi}$, *then there exists a matching* f *(see* (8)).

In fact we show that Assertion (F'', F', r'') of (12) implies the theorem for arbitrary (N, n). As this Assertion is proven only in the cases treated in Theorem 12, we put the restriction $(N, n) = 1$ as in Corollary 12.

Remark. — (i) By virtue of [K'], Theorems C, D and K, the proof given below shows that for every pseudo-coefficient $\widetilde{f}$ (resp. f) of a square-integrable $\widetilde{\pi}$ (resp. π), there exists a matching f (resp. $\widetilde{f}$). (ii) Using Theorem 27.3 and the theorem of [BDK] we can further conclude that for each $\widetilde{f}$ there exists a matching f. This fact is discussed in Corollary 27.3. The space of f with matching $\widetilde{f}$ is easily characterized; this is not needed here, but see Proposition 27.3.

The proof relies on the results of [K], and the following

Proposition. — a) *Let* ε *be a semi-simple element of* $\widetilde{G}$ *and* $G_\varepsilon = p(\widetilde{G}_\varepsilon)$, *where* $\widetilde{G}_\varepsilon$ *is the centralizer of* ε *in* $\widetilde{G}$. *For the character* $\widetilde{\chi}$ *of a representation* $\widetilde{\pi}$ *there exist complex numbers* $c(\xi, \widetilde{\pi})$ *so that*

$$\widetilde{\chi}(\varepsilon \exp Y) = \Sigma_\xi\, c(\xi, \widetilde{\pi})\, \hat{\nu}_\xi(Y)$$

for any regular Y *near zero in the Lie algebra* Lie G_ε. *Here* ξ *runs over all nilpotent* G_ε*-orbits in* Lie G_ε, ν_ξ *is the* G_ε*-invariant measure on* Lie G_ε *corresponding to* ξ *and* $\hat{\nu}_\xi$ *is the Fourier transform of* ν_ξ *on* Lie G_ε.

b) *Conversely, for each* ξ *there exist representations* π_ε *of* G_ε *and complex numbers* $c(\xi, \pi_\varepsilon)$ *so that for any elliptic regular* Y *in* Lie G_ε *sufficiently close to zero we have*

$$\hat{\nu}_\xi(Y) = \Sigma_{\pi_\varepsilon}\, c(\xi, \pi_\varepsilon)\, \chi_{\pi_\varepsilon}(\exp Y).$$

Proof. — a) is [H], Theorem 5. b) follows from [K'], Theorem 2.1 (b), Proposition 3.1 (a), and Theorem 4.1.

Corollary. — *Given* $\widetilde{\pi}$, *and a semi-simple* ε *in* $\widetilde{G}$, *there exist representations* π_ε *of* G_ε, *and complex numbers* $c(\widetilde{\pi}, \pi_\varepsilon)$, *so that for any elliptic regular* g *in a small neighborhood of* 1 *in* G_ε, *we have*

$$\chi_{\widetilde{\pi}}(\varepsilon g) = \Sigma_{\pi_\varepsilon} c(\widetilde{\pi}, \pi_\varepsilon)\, \chi_{\pi_\varepsilon}(g).$$

Proof of theorem. — Note that $p(\widetilde{G}_\varepsilon) = G_\varepsilon$ is contained in $G_{p(\varepsilon)}$. Suppose that $p(\varepsilon)$ is elliptic, with determinant in $F^{\times n}$. Its centralizer $G_{p(\varepsilon)}$ in G is of the form $GL(r', F')$, where $[F' : F] = f'$ and $r'f' = r$. Since (see the proof of Proposition 12) the commutator $[x, \varepsilon]$ of ε and x in $s(G_{p(\varepsilon)})$ is equal to $\varepsilon'(x) = (\det' x, p(\varepsilon))_{F'}$, where $\det' x$ is the determinant from $G_{p(\varepsilon)}$ to F', we deduce that G_ε is the kernel of ε' on $G_{p(\varepsilon)}$. Since $p(\varepsilon)$ lies in F', its nth root defines an extension $F'' = F'(p(\varepsilon)^{1/n})$ of F' of degree m dividing n. In particular the character ε' is of order m, and $G_{p(\varepsilon)}/G_\varepsilon$ is isomorphic to $F'^{\times}/N_{F''/F'} F''^{\times}$ via the determinant map. Note that for x in $G_{p(\varepsilon)}$ we have

$$(*) \qquad \chi_{\widetilde{\pi}}(\varepsilon x g x^{-1}) = \varepsilon'(x)\, \chi_{\widetilde{\pi}}(x \varepsilon g x^{-1}) = \varepsilon'(x)\, \chi_{\widetilde{\pi}}(\varepsilon g).$$

Here we use the fact that characters are invariant under conjugation. We identify μ_n with a subgroup of $\widetilde{Z}$ (resp. $\mathbf{C}^\times$) by means of i (resp. $\widetilde{\omega}$).

The group $G_{p(\varepsilon)}$ acts on the irreducible representation π_ε of G_ε by conjugation, thus $\pi_\varepsilon^g(x) = \pi_\varepsilon(gxg^{-1})$. If $G(\pi_\varepsilon)$ is the maximal subgroup of $G_{p(\varepsilon)}$ which fixes π_ε, and ε'' is a non-trivial character of $G_{p(\varepsilon)}/G(\pi_\varepsilon)$ (ε'' is some power of ε'), then linear independence of characters on G_ε implies that the span of the characters χ of π_ε^x (x in $G_{p(\varepsilon)}/G(\pi_\varepsilon)$) is equal to the span of

$$X_i = \Sigma_{x \text{ in } G_{p(\varepsilon)}/G_\varepsilon} \varepsilon''(x)^i \chi_{\pi_\varepsilon^x} \qquad (1 \leqslant i \leqslant \text{order } \varepsilon'').$$

Hence the Corollary implies that there is a neighborhood V of 1 in G_ε so that $\chi_{\widetilde{\pi}}(\varepsilon g)$ is equal to a finite linear combination of the expressions $X_i(g)$ on the set of elliptic regular g in V. Hence $(*)$ implies by linear independence of characters on G_ε that $\chi_{\widetilde{\pi}}(\varepsilon g)$ is a combination of $X_i(g)$'s with $\varepsilon''(x)^i = \varepsilon'(x)$, on the set of elliptic regular g in V.

So we have to study the sum

$$X = \Sigma \varepsilon'(x)\, \chi_{\pi_\varepsilon^x} \qquad (x \text{ in } G_{p(\varepsilon)}/G_\varepsilon)$$

for a given irreducible representation π_ε of G_ε with $G(\pi_\varepsilon) = G_\varepsilon$. It satisfies the relation $(*)$. Hence, if $X(g) \neq 0$, then g lies in the subgroup $H = GL(r'', F'')$ of $G_{p(\varepsilon)}$. According to [K], Lemma 1, p. 216, the sum $X(g)$ is equal to $\chi_\pi(g \times \varepsilon')$ (the character of $\pi(f) \circ A_\pi$ in the notation of [K]), where π is the representation of $G_{p(\varepsilon)}$ induced from π_ε on G_ε. Using Assertion (F'', F', r'') of (12), and the separation argument of (19) below, the proof of [K], § 4, implies that up to a scalar there exists a representation ρ_ε of H, so that $X(g)$ is of the form

$$(**) \qquad u_\varepsilon(g)\, \frac{\Delta''(g)}{\Delta'(g)}\, \Sigma_\sigma\, \chi_{\rho_\varepsilon}(\sigma(g)).$$

Here Δ' is the Δ-factor of $G_{p(\varepsilon)} = GL(r', F')$; Δ'' is the Δ-factor of H. The sum ranges over the Galois group $\mathrm{Gal}(F''/F') \simeq F'^{\times}/NF''^{\times} \simeq G_{p(\varepsilon)}/G_{\varepsilon}$. $u_{\varepsilon}(g)$ is $\varepsilon'(\widetilde{\Delta}(g)\, g^0)^{n(n-1)/2}$ ([K], p. 211, l. 9); it is equal to $u(x)^{n-1}$ (notation of (4)) if x is elliptic with $x^* = \varepsilon g u(x)^{n-1}$.

We can now complete the proof of the theorem. We are given a matrix coefficient $\widetilde{f}$ of a supercuspidal representation $\widetilde{\pi}$. According to the comments preceding the statement of the theorem, $F(x, \widetilde{f}) = 0$ unless x is elliptic, and $\Phi''(x, \widetilde{f}) = \chi_{\widetilde{\pi}}(x)$ for elliptic regular good x. In view of (3) we need to consider only $\chi_{\widetilde{\pi}}(x^*)$, and by virtue of (8) we need to concentrate only at those x^* which are close to a singular element. Namely we are interested in x^* of the form $\varepsilon g u(x)^{n-1}$ with regular elliptic g near 1 in $\widetilde{G}_{\varepsilon}$. This is the situation considered above. Note that g is elliptic, and the character of ρ_{ε} is locally constant on the elliptic set. In fact, for elliptic g near 1 as here the value of $\chi_{\rho_{\varepsilon}}(\sigma g)$ is constant. Since for our g we have $\Delta(\varepsilon g) = \Delta'(g)$ and $\Delta''(g) = \Delta''(\varepsilon^{-1} x^n) = \Delta(x)$ up to a scalar, it follows that $c(x) = \Delta\Phi''(x^*, \widetilde{f})/\Delta(x)$ is a function on G which satisfies the requirements of Proposition 8. Hence the matching function f exists, as required.

II. — REGULAR FUNCTIONS

14. Jacquet modules. — Let $\tilde{P} = \tilde{M}N$ be a parabolic subgroup of $\tilde{G}$ with unipotent radical N and Levi subgroup $\tilde{M}$. Suppose $\tilde{\pi}$ is an admissible representation of finite length of $\tilde{G}$ on a Hilbert space V. Let V_N be the span of $\tilde{\pi}(n)\,v - v$ (n in N, v in V). The space V_N is stabilized by $\tilde{M}$ since $\tilde{M}$ normalizes N. Let $\tilde{\pi}'_N$ denote the representation of $\tilde{M}$ on V/V_N. Recall (7) that δ_P is the modulus character of P. We call $\tilde{\pi}_N = \delta_P^{-1/2}\,\tilde{\pi}'_N$ the *Jacquet module* of $\tilde{\pi}$ with respect to $\tilde{N}$. Jacquet modules have been studied by Jacquet, Harish-Chandra, Casselman, Bernstein and Zelevinsky [BZ]. Some of their results are the following. $\tilde{\pi}_N$ is admissible of finite length. The functor $\tilde{\pi} \to \tilde{\pi}_N$ is exact. Let $I_{\tilde{M}}(\tilde{\rho})$ denote the representation $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}(\delta_P^{1/2}\,\tilde{\rho})$ induced to $\tilde{G}$ from $\delta_P^{1/2}\,\tilde{\rho} \otimes 1$ on $\tilde{P} = \tilde{M}N$; it does not depend on N. Then $\mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, I_{\tilde{M}}(\tilde{\rho})) = \mathrm{Hom}_{\tilde{M}}(\tilde{\pi}_N, \tilde{\rho})$ for all admissible representations $\tilde{\rho}$ of $\tilde{M}$ and $\tilde{\pi}$ of $\tilde{G}$. Hence $\tilde{\pi}_N \neq 0$ implies that $\tilde{\pi}$ is a constituent of $I_{\tilde{M}}(\tilde{\pi}_N)$. Further, the representation $\tilde{\pi}$ is supercuspidal if and only if $\tilde{\pi}_N$ is zero for all proper parabolic subgroups P of G.

Let $\Phi = \Phi(A, G)$ be the set of roots of A in G, and Δ the subset of simple roots. For a subset θ of Δ, put $A_\theta = \cap \ker \alpha$ (α in θ). Put A'' for the set of x in A with $|\alpha(x)| \leqslant 1$ for all α in Δ. Let T be a (maximal) torus with maximal split subtorus $'A$, and maximal anisotropic subtorus S. Thus T is isogenous to $'A \times S$. Namely, there is a positive integer m such that for each t in T we have $t^m = as$ (a in $'A$, s in S). There exists y in G with $a' = yay^{-1}$ in A''. Let Ω be the set of α in Δ with $|\alpha(a')| = 1$, and define $P_t = M_t N_t$ to be $y^{-1} P_\Omega y$, where P_Ω is the upper triangular parabolic subgroup whose Levi subgroup is the centralizer of A_Ω.

Theorem (Casselman [C]). — *Let t be a regular element in $\tilde{G}$ with $P_t = P$. Then $\chi(\tilde{\pi})(t) = \chi(\tilde{\pi}'_N)(t)$. Since $\Delta(t) = \Delta_M(t)\,\delta_P(t)^{-1/2}$, we have $(\Delta\chi(\tilde{\pi}))(t) = (\Delta_M\,\chi(\tilde{\pi}_M))(t)$ for such t.*

If $\tilde{\pi}$ is ramified (does not have a $\tilde{K} = \kappa^*(K)$ fixed vector) and $\tilde{f}$ is spherical, then $\tilde{\pi}(\tilde{f}) = 0$. Indeed, for any v in the space of $\tilde{\pi}$ the vector

$$\tilde{\pi}(\tilde{f})\,v = \int \tilde{f}(g)\,\tilde{\pi}(g)\,v\,\tilde{d}g = \int \tilde{f}(k^{-1}g)\,\tilde{\pi}(g)\,v\,\tilde{d}g = \tilde{\pi}(k)\,\tilde{\pi}(f)\,v$$

is fixed by $\tilde{K}$. Here $\tilde{d}g$ is a Haar measure on $\tilde{G}/\tilde{Z}$.

Denote by $\widetilde{G}_e$ the elliptic set of $\widetilde{G}$, by $[W(T)]$ the cardinality of the Weyl group $W(T)$ of a (maximal) torus $\widetilde{T}$ in $\widetilde{G}$, and by $\widetilde{dt}$ a Haar measure on $\widetilde{T}/\widetilde{Z}$. Put

$$(14.1) \qquad \langle \chi_{\widetilde{\pi}}, \widetilde{f} \rangle_e = \int_{\widetilde{G}_e/\widetilde{Z}} \chi_{\widetilde{\pi}}(g) \widetilde{f}(g)\, \widetilde{dg} = \sum_{\widetilde{T}} [W(\widetilde{T})]^{-1} \int_{\widetilde{T}/\widetilde{Z}} \Delta\chi_{\widetilde{\pi}}\,(t)\, F(t, \widetilde{f})\, \widetilde{dt}.$$

The sum ranges over the conjugacy classes of elliptic tori $\widetilde{T}$ in $\widetilde{G}$.

15. Regular functions. — As in (11) we now assume that $(n, q) = 1$ and ω is unramified. Let λ_i $(1 \leqslant i \leqslant r)$ be integers with $\lambda_r = 0$ and $\lambda_i > \lambda_{i+1}$. The function $\widetilde{f}$ of (6) is called *λ-regular* if $F(x, \widetilde{f})$ is zero for all x in $\widetilde{G}$ unless there is g in $\widetilde{G}$ and z in $\widetilde{Z}$ such that $p(zg^{-1}xg)$ is equal to $u^n \pi^{n\lambda}$ with u in $A(R)$ in the notations of (11), in which case we require that $F((u\pi^\lambda)^*, \widetilde{f})$ be equal to one. Since the support S of $F(x, \widetilde{f})$ is open and closed, we assume, as we may, that the support of $\widetilde{f}$ lies in S. Note that by definition of λ and $\widetilde{f}$, the group M_t associated in (14) with t in $\widetilde{G}$ such that $F(t, \widetilde{f}) \neq 0$, is A.

For λ-regular $\widetilde{f}$ the Weyl integration formula (14.1) implies that

$$\operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \int \widetilde{\chi}(x) \widetilde{f}(x)\, \widetilde{dx}$$

is equal to

$$(r!)^{-1} \int_{\widetilde{A}^n\widetilde{Z}/\widetilde{Z}} t(\Delta\widetilde{\chi})\,(a)\; F(a, \widetilde{f})\, \widetilde{da},$$

which, by definition of $\widetilde{f}$ and Theorem 14, is equal to

$$(r!)^{-1} \int_{A(R)/Z(R)} \chi(\widetilde{\pi}_N)\,(a^n \pi^{n\lambda})\; F(a^n \pi^{n\lambda}, f)\, da.$$

The restriction of the character $\chi(\widetilde{\pi}_N)$ of the $\widetilde{A}$-module $\widetilde{\pi}_N$ to $\widetilde{A}^n$ is a sum of characters (homomorphisms) of $\widetilde{A}^n$. Our integral vanishes unless $\chi(\widetilde{\pi}_N)$ contains a character $\widetilde{\mu}$, such that there exists an unramified character μ of A with $\widetilde{\mu}(a^n) = \mu(a)$ (a in A). In this case, assuming that $\widetilde{\pi}$ is irreducible, we note that by Frobenius reciprocity (14) all exponents in $\chi(\widetilde{\pi}_N)$ are of the form $w\widetilde{\mu}$, where w lies in the Weyl group $W(A)$, and $(w\widetilde{\mu})\,(a) = \widetilde{\mu}(w^{-1}aw)$. As in (11) we write $\lambda((w\mu)(\pi))$ for $(w\widetilde{\mu})(\pi^{n\lambda})$. We conclude the following

Proposition. — *Suppose $\widetilde{\pi}$ is irreducible, $\widetilde{f}$ is λ-regular. Then* $\operatorname{tr}\widetilde{\pi}(\widetilde{f})$ *vanishes unless* $\widetilde{\pi}$ *lies in the composition series of the induced, unramified representation* $I(\widetilde{\mu})$ *introduced in* (11), *and in this case there is a subset* $W(\widetilde{\pi})$ *of* $W(A)$ *depending on* $\widetilde{\pi}$ *and* $\widetilde{\mu}$ *such that*

$$(15.1) \qquad \operatorname{tr}\widetilde{\pi}(\widetilde{f}) = (r!)^{-1} \Sigma_w\, \lambda((w\mu)\,(\pi)) \qquad (w \textit{ in } W(\widetilde{\pi})).$$

In the special case $n = 1$ we obtain λ-regular functions f on G and conclude that $\operatorname{tr} \pi(f)$ vanishes for irreducible π unless π_N is unramified, in which case we denote by μ an exponent of π_N and obtain

$$(15.2) \qquad \operatorname{tr} \pi(f) = (r!)^{-1} \Sigma_w \lambda((w\mu)(\pi)) \qquad (w \text{ in } W(\pi))$$

for some subset $W(\pi)$ of $W(A)$ depending on π and μ.

Thus regular functions have several useful properties. They are supported on the regular set. If $\operatorname{tr} \tilde{\pi}(\tilde{f})$ is non-zero then (1) it is easily computable by the above explicit formula, (2) $\tilde{\pi}$ is a constituent of the unramified induced $I(\tilde{\mu})$. (2) implies that $\tilde{\pi}$ has a vector fixed by the action of an Iwahori subgroup, once Proposition 17 below is proven. As these properties are fundamental for our study of spherical functions in (19) we now pause and study in detail the $\tilde{G}$-modules with a vector fixed by the action of an Iwahori subgroup.

16. Iwahori algebra. — Let F be non-archimedean with $|n| = 1$. As in (2) we identify the maximal compact subgroup $K = GL(r, R)$ of $G = GL(r, F)$ with a subgroup K^* of $\tilde{G}$ by means of the injection $\kappa^* : K \to \tilde{G}$ of (2). Let A be the diagonal subgroup of G and $\tilde{A} = p^{-1}(A)$. Let μ, $\tilde{\mu}$ be characters of A, $\tilde{A}$ whose restrictions to Z, $\tilde{Z}$ are ω, $\tilde{\omega}$ (see (6)), related (as in (11)) by $\mu(a) = \tilde{\mu}(s(a^n))$ (a in A), such that μ is trivial on the maximal compact subgroup $A \cap K$ of A. As in (11) we let $I(\mu)$, $I(\tilde{\mu})$ be the representations of G, $\tilde{G}$ unitarily induced from the characters μ, $\tilde{\mu}$ of $B = AN$, $\tilde{B}_0 = \tilde{A}_0 N$; N is the unipotent upper triangular subgroup. In this section we prove

Theorem. — There is a natural bijection, preserving in particular Jacquet modules with respect to N, *between the irreducible constituents of* $I(\mu)$ *and of* $I(\tilde{\mu})$.

This gives a special case of the metaplectic correspondence, where the proof is independent of the trace formula. In particular we do not need to impose here the restriction $(N, n) = 1$ (see Corollary 12).

The proof is based on the study of H-modules, where H is the Hecke algebra with respect to an Iwahori subgroup I, and the analogous situation for the metaplectic group. Let I be the group of matrices in K whose entry below the diagonal has valuation less than one. Then $I \cap B = K \cap B$, $A \cap I = A \cap K$. Let I^* be the image of I under the isomorphism κ^* from K to K^*. We introduce the convolution algebra $\tilde{H} = C_c[I^* \backslash \tilde{G}/I^*]_{\tilde{\omega}}$ of complex valued I^*-biinvariant functions on $\tilde{G}$ which transform under $\tilde{Z}$ by $\tilde{\omega}^{-1}$ and are compactly supported modulo $\tilde{Z}$. In the case $n = 1$ the notation specializes to $H = C_c[I \backslash G/I]_{\omega}$. The study of H-modules is based on some properties of the algebra H, which follow from the presentation of H by means of generators and relations, due to Iwahori-Matsumoto [IM], Proposition 3.8. This we recall in Lemma 16.2. In Lemma 16.3 we establish an analogous description for the algebra $\tilde{H}$.

This implies that the study of H-modules generalizes to the metaplectic case; see Propositions 17, 17.1. Lemmas 16.2, 16.3 imply the following Proposition 16.1, and Theorem 16 is a consequence of Propositions 16.1, 17, 17.1. So we claim

Proposition (**16.1**). — *The algebras* $\widetilde{\mathrm{H}}$ *and* H *are isomorphic.*

As the central character ω is unramified, we assume that $\omega = 1$. So we need to study the Iwahori algebra of the projective group. To simplify the notation, we now denote G/Z by G, A/Z by A, $\mathrm{I}/\mathrm{I} \cap \mathrm{Z}$ by I. With this convention, we now describe (in Lemma 16.2) the Iwahori algebra $\mathrm{H} = \mathrm{C}_c[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ by means of generators and relations, as in [IM], Proposition 3.8. We use the Bruhat decomposition $\mathrm{G} = \mathrm{IYI}$ ([IM], Theorem 2.16), where $\mathrm{Y} = \mathrm{WA}$ is the normalizer of A in G (as in (2)). The affine Weyl group $\mathrm{W}' = \mathrm{Y}/\mathrm{Y} \cap \mathrm{I}$ is the semi-direct product of the Weyl group $\mathrm{W} = \mathrm{Y}/\mathrm{A}$, and the free group $\mathrm{P} = \mathrm{A}/\mathrm{A} \cap \mathrm{I} \simeq \mathbf{Z}^{r-1}$, where W acts on P by permutations. We identify W′ with the group of matrices which have in each row and column a single non-zero entry, which is an integral power of the uniformizer $\boldsymbol{\pi}$. Theorem 2.16 of [IM] asserts that G is the disjoint union of $\mathrm{I}y\mathrm{I}$ (y in W′). Hence each member of the convolution algebra H is a linear combination over $\mathbf{C}$ of the functions $'\mathrm{T}_y$ (y in W′) which are supported on $\mathrm{I}y\mathrm{I}$, and attain the value $1/|\mathrm{I}|$ there; a Haar measure was fixed in (6). It is clear that $'\mathrm{T}_y$ does not depend on the choice of a representative of y in G.

Let s_i be the transposition $(i, i+1)$ in W, for $1 \leqslant i < r$. Denote by s_0 $(= s_r)$ the matrix in W′ whose entries are 0 outside the anti-diagonal, and whose non-zero entries are $\boldsymbol{\pi}^{-1}$ on the top row, $\boldsymbol{\pi}$ on the bottom row, and 1 otherwise. Then $s_i^2 = 1$ $(0 \leqslant i < r)$. Also denote by τ the member (a_{ij}) of W′ whose non-zero entries are $a_{i,i+1} = 1$ $(1 \leqslant i < r)$ and $a_{r1} = \boldsymbol{\pi}$. Then $\tau^r = 1$, and $\tau s_{i+1} = s_i \tau$ $(0 \leqslant i < r)$. Note that W′ is generated by τ and s_i $(1 \leqslant i < r)$. Let S′ be the set $\{s_i\ (0 \leqslant i < r)\}$, and W″ the subgroup of W′ generated by S′. Then (W″, S′) is a Coxeter group ([BN]; IV, §1). Hence it has a length function ℓ, which assigns w in W″ the minimal integer m so that $w = t_1 \dots t_m$ (t_i in S′). In particular $\ell(1) = 0$, and $\ell(w) = 1$ if and only if $w = s_i$. The length function ℓ extends to W′ by $\ell(\tau w) = \ell(w)$ (w in W″), as for each w' in W′ there are unique i, w in W″, with $w' = \tau^i w$.

We now return to the functions $'\mathrm{T}_w$, and put $\mathrm{T}_w = q^{-\ell(w)/2}\, '\mathrm{T}_w$. In particular $\mathrm{T}_{s_i} = q^{-1/2}\, '\mathrm{T}_{s_i}$, and we put $\mathrm{T}_i = \mathrm{T}_{s_i}$ $(0 \leqslant i < r)$. Put τ for $\mathrm{T}_\tau = {}'\mathrm{T}_\tau$, and 1 for the unit in the algebra H, namely the function supported on I whose value there is $1/|\mathrm{I}|$.

Lemma (**16.2**) ([IM], Prop. 3.8). — *The convolution algebra* H *is an algebra with identity* 1, *with the presentation by means of generators* τ, T_i $(1 \leqslant i < r)$, *and relations*

(i) $\tau^r = 1$,
(ii) $\tau \mathrm{T}_{i+1} = \mathrm{T}_i \tau$ $(1 \leqslant i \leqslant r-2)$,
(iii) $\mathrm{T}_i^2 = 1 + \rho \mathrm{T}_i$ $(1 \leqslant i < r)$, *where* $\rho = q^{1/2} - q^{-1/2}$,
(iv) $\mathrm{T}_i \mathrm{T}_j \mathrm{T}_i = \mathrm{T}_j \mathrm{T}_i \mathrm{T}_j$ *if* $(s_i s_j)^3 = 1$ (*or* $i = j \pm 1$ *when* $r \geqslant 3$),
(v) $\mathrm{T}_i \mathrm{T}_j = \mathrm{T}_j \mathrm{T}_i$ *if* $(s_i s_j)^2 = 1$ (*or* $i \neq j \pm 1$ *when* $r \geqslant 4$).

In particular, $T_0 = \tau T_1 \tau^{-1} = \tau^{-1} T_{r-1} \tau$ satisfies (iii) since T_1 does, and also (iv), (v). In fact, [IM] include T_0 among the generators. Our presentation of H simplifies theirs. Equivalently, H has the presentation by means of the generators T_w (w in W′), and the relations (iii) and: (vi) $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$. Relation (vi) implies that $\mathrm{I}w\mathrm{I}w'\mathrm{I} = \mathrm{I}ww'\mathrm{I}$, and (iii) that $\mathrm{I}s\mathrm{I}s\mathrm{I} = \mathrm{I} \cup \mathrm{I}s\mathrm{I}$ (disjoint union, s in S′). Note that these relations stand for convolution of functions.

The Hecke algebra $\widetilde{\mathrm{H}}$ of the covering group $\widetilde{\mathrm{G}}$ of $\mathrm{G} = \mathrm{GL}(r)$, with respect to the subgroup I^*, has a similar presentation. For $\widetilde{y}$ in $\widetilde{\mathrm{Y}} = p^{-1}(\mathrm{Y})$ (notations of (2)), denote by $\mathrm{ch}(\mathrm{I}^*\widetilde{y}\mathrm{I}^*)$ an element of $\widetilde{\mathrm{H}}$ which is supported on $\widetilde{\mathrm{Z}}\mathrm{I}^*\widetilde{y}\mathrm{I}^*$. Put $y = p(\widetilde{y}) = wx$ (w in W, x in P). It is clear from the definition (2) of $\widetilde{\mathrm{G}}$ that if this function is non-zero, then x lies in $p(\widetilde{\mathrm{Z}})\,\mathrm{P}^n$, where $\mathrm{P}^n \simeq n\mathbf{Z}^r$. Indeed, if ε' lies in $\mathrm{I}^* \cap \widetilde{\mathrm{A}}$, then $\varepsilon' w = w\varepsilon''$ for some ε'' in $\mathrm{I}^* \cap \widetilde{\mathrm{A}}$. Further, if $x = (x_1, \dots, x_r)$ and $j \neq k$, take $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_r)$ with $\varepsilon_i = 1$ if $i \neq j, k$; and $\varepsilon_j = \varepsilon$, $\varepsilon_j \varepsilon_k = 1$; ε is a unit in $\mathrm{F}^\times$. Then by (2.1) the commutator of x and ε' is $(\varepsilon, x_j/x_k)$. Since the unit ε is arbitrary, x_j/x_k must be an nth power for each pair (j, k) if the (genuine) function $\mathrm{ch}(\mathrm{I}^*\widetilde{y}\mathrm{I}^*)$ does not vanish, as asserted.

Thus we replace $\widetilde{\mathrm{G}}$ by $\widetilde{\mathrm{G}}/s(\mathrm{Z}_m)$ (notation of (2)), put $\widetilde{s}_i = \kappa^*(s_i)$ $(1 \leqslant i < r)$, and write $\widetilde{\tau}$ for $x_1 \widetilde{s}_{r-1} \dots \widetilde{s}_2 \widetilde{s}_1$; x_1 is the element $s(\boldsymbol{\pi}_1^n)$ of $\widetilde{\mathrm{A}}$, where $\boldsymbol{\pi}_1 = (1, \dots, 1, \boldsymbol{\pi})$. Let $\widetilde{\mathrm{W}}'$ be the group generated by $\widetilde{s}_i$ $(1 \leqslant i < r)$ and $\widetilde{\tau}$. There is a natural isomorphism ψ from $\widetilde{\mathrm{W}}'$ to W′, given by $\psi(\widetilde{s}_i) = s_i$ $(1 \leqslant i < r)$ and $\psi(\widetilde{\tau}) = \tau$. Indeed, $\widetilde{\tau}$ satisfies $\widetilde{\tau}^r = 1$, and $\widetilde{\tau}\widetilde{s}_{i+1} = \widetilde{s}_i\widetilde{\tau}$ $(1 \leqslant i \leqslant r-2)$, by the definition of σ in (2), and these are the only relations which $\widetilde{\tau}$ satisfies. In particular, the length function ℓ is defined on $\widetilde{\mathrm{W}}'$ by $\ell(w) = \ell(\psi(w))$ (w in $\widetilde{\mathrm{W}}'$). We use below the fact that for $\widetilde{a}$ in $\widetilde{\mathrm{A}}$ with $a = p(\widetilde{a})$, we have $\widetilde{a}^{-1}\kappa^*(n)\,\widetilde{a} = \kappa^*(a^{-1}na)$ if both n and $a^{-1}na$ lie in $\mathrm{I} \cap \mathrm{N}$. Indeed, since $\kappa \mid \mathrm{K} \cap \mathrm{N} = 1$ (see (2)), this follows from $\sigma(n, a) = 1$ and $\sigma(a, a^{-1}na) = 1$ (see (2)).

Put $'\widetilde{\mathrm{T}}_w$ (w in $\widetilde{\mathrm{W}}'$) for the member of $\widetilde{\mathrm{H}}$ which is supported on $\mathrm{I}^* w \mathrm{I}^*\,\mathrm{Z}/s(\widetilde{\mathrm{Z}}_m)$, and takes the value $1/|\mathrm{I}|$ at w. Put $\widetilde{\mathrm{T}}_w = q^{-\ell(w)/2}\,'\widetilde{\mathrm{T}}_w$, $\widetilde{\mathrm{T}}_i$ for $\widetilde{\mathrm{T}}_w$, where $w = \widetilde{s}_i$, and $\widetilde{\tau}$ for $\widetilde{\mathrm{T}}_{\widetilde{\tau}} = {}'\widetilde{\mathrm{T}}_{\widetilde{\tau}}$. It is clear from the above comments that if the $\widetilde{\mathrm{T}}_w$ exist, then the algebra $\widetilde{\mathrm{H}}$ is generated by the $\widetilde{\mathrm{T}}_w$ (w in $\widetilde{\mathrm{W}}'$).

We use below the Iwahori decomposition $\mathrm{I} = (\mathrm{I} \cap \mathrm{N}^-)(\mathrm{I} \cap \mathrm{A})(\mathrm{I} \cap \mathrm{N})$ of [IM], Theorem 2.5, and write accordingly $i = n^- an$; here N^- is the lower triangular unipotent subgroup ${}^t\mathrm{N}$ of G.

To show that the $\widetilde{\mathrm{T}}_w$ exist it suffices to show that $x_1^{-1} i x_1$ (all product below are in $\widetilde{\mathrm{G}}$) lies in K^* for all i in $\mathrm{I} \simeq \mathrm{I}^*$ for which $p(x_1^{-1} i x_1)$ lies in K, namely that if $i x_1 w i' = \zeta x_1 w$ (ζ in μ_n; i, i' in I) then $\zeta = 1$. By (2.1) and (2.2) we may assume that i lies in N^-. Let u denote the anti-diagonal reflection in W. Then $u^{-1} i u$ lies in $\mathrm{K}^* \cap \mathrm{N} = s(\mathrm{K} \cap \mathrm{N})$. Put $x = u^{-1} x_1 u$. It lies in $\widetilde{\mathrm{A}}$. Hence $x^{-1} u^{-1} i u x$ lies in $\mathrm{K} \cap \mathrm{N}$ by (2.1) and (2.2). Hence $x_1^{-1} i x_1 = u x^{-1} u^{-1} i u x u^{-1}$ lies in K^*, as required.

Lemma **(16.3)**. — *The convolution algebra* $\widetilde{\mathrm{H}}$ *has a presentation by means of the generators* $\widetilde{\tau}$, $\widetilde{\mathrm{T}}_i$ $(1 \leq i < r)$ *and the relations* (16.2) (i)-(v), *with* $\widetilde{\tau}$ *replacing* τ *and* $\widetilde{\mathrm{T}}_i$ *replacing* T_i.

Proof. — The relations (iii), (iv), (v) follow at once from the case $n = 1$ and the isomorphism $K \simeq K^*$. To prove the lemma, it suffices to show that (a) for any $\tilde{w}$ in $\tilde{W}'$ and $s = \tilde{s}_i$ $(1 \leqslant i < r)$ with $\ell(s\tilde{w}) = 1 + \ell(\tilde{w})$, we have $\tilde{T}_{s\tilde{w}} = \tilde{T}_i \tilde{T}_{\tilde{w}}$, and (b) $\tilde{\tau}\tilde{T}_{\tilde{w}} = \tilde{T}_{\tilde{\tau}\tilde{w}}$ for any $\tilde{w}$ in $\tilde{W}'$.

To prove (a) we need to show that $sI^*\tilde{w} \subset I^* s\tilde{w}I^*$. It is clear that $sn^- as^{-1}$ lies in I. On the other hand, n in N can be written as a product $n = n_i(1 + xe_i)$. Here n_i is a matrix (n_{jk}) in N with $n_{i,i+1} = 0$, and e_i is the matrix (a_{jk}) with $a_{jk} = 0$ if $(j, k) \neq (i, i+1)$, and $a_{i,i+1} = 1$. If n lies in $N \cap I$ then n_i lies in $N \cap I$ and $sn_i s^{-1}$ lies in I; also in this case $|x| \leqslant 1$. So it remains to show that $\tilde{w}^{-1}(1 + xe_i)\tilde{w}$ lies in I^*. For that, write the image $\psi(\tilde{w})$ of $\tilde{w}$ under the isomorphism ψ in the form wp, with w in W and p in P. Then $w^{-1}(1 + xe_i)w = 1 + xe_{jk}$, where $e_{jk} = (a_{uv})$ with $a_{uv} = 0$ if $(u, v) \neq (j, k)$, and $a_{jk} = 1$. Since $\ell(swp) = 1 + \ell(wp)$, (16.2) (vi) implies that $p^{-1}w^{-1}(1 + xe_i)wp = 1 + \pi^\alpha xe_{jk}$ with $\alpha \geqslant 0$ if $j < k$, and $\alpha > 0$ if $j > k$. Put $p' = wpw^{-1}$. Then $p'^{-1}(1 + xe_i)p' = 1 + \pi^{\alpha n} xe_i$; this is an equality in $\tilde{G}$ by (2.2), and $1 + \pi^{\alpha n} xe_i$ lies in $s(N \cap K) = (N \cap K)^*$ (since $\kappa \mid N \cap K = 1$, see (2)). Hence $\tilde{w}^{-1}(1 + xe_i)\tilde{w} = 1 + \pi^{\alpha n} xe_{jk}$ (equality in $\tilde{G}$) lies in I^*, as required.

It remains to show (b). Again we use the decomposition $i = nan^-$, and note that $\tilde{\tau}na\tilde{\tau}^{-1}$ lies in I^*. Moreover, n^- can be written as $n''n'$, with n', n'' in N^-, so that $n'' = (n_{jk})$ with $n_{k1} = 0$ for $k > 1$, and $n' = (n_{jk})$ with $n_{jk} = 0$ for $j > k > 1$. It is clear that $\tilde{\tau}n''\tilde{\tau}^{-1}$ lies in I^*.

We now study $\tilde{\tau}n'\tilde{\tau}^{-1}$. For that it is useful to note that $1 + xe_{j1}$ and $1 + ye_{k1}$ $(j, k > 1; x, y \text{ in F})$ commute. We denote by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_i$ the matrix (a_{jk}) in G whose non-zero entries are $a_{jj} = 1$ $(j \neq i, r)$, $a_{ii} = a$, $a_{rr} = d$, $a_{ir} = b$, $a_{ri} = c$. We denote by n'_j the matrix obtained from n' on replacing $n_{j+1,1}$ by 0. Then $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}_{j+1}$ commutes with n'_j. Put $p = \pi^n$. If $|x/p| > 1$, we have (in $\tilde{G}$, not only in G)

$$(*) \qquad \tilde{\tau}(1 + xe_{j+1,1})\tilde{\tau}^{-1} = 1 + (x/p)\, e_{jr}$$
$$= \begin{pmatrix} 1 & 0 \\ p/x & 1 \end{pmatrix}_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_j \begin{pmatrix} p/x & 0 \\ 0 & x/p \end{pmatrix}_j \begin{pmatrix} 1 & 0 \\ p/x & 1 \end{pmatrix}_j.$$

It is clear from the first equality here that $\tilde{\tau}n'\tilde{\tau}^{-1}$ lies in I^* if $|n_{j1}/p| \leqslant 1$ for all $j > 1$. So suppose there is $j > 1$ so that $x = n_{j1}$ satisfies $|x| > |p|$, namely $x = \pi^\alpha y$, where y is a unit, and $0 < \alpha < n$. Note that $I^*\tilde{\tau}n'\tilde{w}I^*$ depends only on α, but not on y. In this case $\tilde{\tau}n'\tilde{w}$, hence $\tilde{\tau}i\tilde{w}$, lies in the I^*-double coset

$$D = I^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_j \begin{pmatrix} \pi^{1-\alpha} & 0 \\ 0 & \pi^{\alpha-1} \end{pmatrix}_j \begin{pmatrix} 1 & 0 \\ p/x & 1 \end{pmatrix}_j \tilde{\tau}n'_j\tilde{w}I^*.$$

Since the member $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}_{j+1} = \tilde{\tau}^{-1}\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}_j \tilde{\tau}$ of I^* (ε is a unit) commutes with n'_j, $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}_j$ and $\begin{pmatrix} 1 & 0 \\ p/x & 1 \end{pmatrix}_j$ lie in I^*, and we can choose ε so that the commutator of $\begin{pmatrix} p/x & 0 \\ 0 & x/p \end{pmatrix}_j$

and $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}_j$ is not 1 in $\widetilde{G}$, the function ch(D) vanishes. Hence the convolution $\int \widetilde{\tau}(xy^{-1})\,\widetilde{T}_{\widetilde{w}}(y)$ is 0 unless x lies in $I^*\widetilde{\tau}\widetilde{w}I^*$. Its values at $x = \widetilde{\tau}\widetilde{w}$ is $1/|I|$, as the above argument can be modified to show that the function $\widetilde{\tau}$ is supported on $\widetilde{\tau}I^*$ (we showed that it is supported on $I^*\widetilde{\tau}$); hence the integral ranges over y in $I^*\widetilde{w} \simeq I^*$. From this it follows that $\widetilde{\tau}T_{\widetilde{w}} = T_{\widetilde{\tau}\widetilde{w}}$, which is the required assertion (b). The relation (b) implies (16.2) (i), and (a), (b) imply (16.2) (ii) as well as (vi), and the lemma follows.

Proposition 16.1 now follows at once, since the map $\psi : \widetilde{W}' \to W'$ induces an isomorphism from $\widetilde{H}$ to H.

Corollary. — For each w in $\widetilde{W}'$, the element $\widetilde{T}_w$ in $\widetilde{H}$ is invertible.

Proof. — Let $w = t_1 \dots t_m$ be a reduced expression for w in terms of the generators $\widetilde{\tau}$, s_i $(1 \leqslant i < r)$ of $\widetilde{W}'$. Then $\widetilde{T}_w = \widetilde{T}_{t_1} \dots \widetilde{T}_{t_m}$, and each $\widetilde{T}_t$ is invertible.

As usual, $(\widetilde{\pi}, \widetilde{V})$ denotes an admissible representation of $\widetilde{G}$ of finite length and central character $\widetilde{\omega}$. Indicate by $\widetilde{V}^{I^*}$ the $\widetilde{H}$-module of I^*-fixed vectors in $\widetilde{V}$, and by $(\widetilde{V}_N)^{I^* \cap \widetilde{A}}$ the space of $I^* \cap \widetilde{A}$-fixed vectors in the Jacquet module $\widetilde{V}_N$ of $\widetilde{V}$. It is a $C_c[I^* \cap \widetilde{A}\backslash\widetilde{A}/I^* \cap \widetilde{A}]_{\widetilde{\omega}}$-module, or an $\widetilde{A}^n$-module. We use

17. *Proposition. — The canonical projection from $\widetilde{V}^{I^*}$ to $(\widetilde{V}_N)^{I^* \cap \widetilde{A}}$ is a linear isomorphism. Hence $(\widetilde{\pi}, \widetilde{V})$ is an irreducible admissible representation of $\widetilde{G}$ with $\widetilde{V}^{I^*} \neq 0$ if and only if $(\widetilde{\pi}, \widetilde{V})$ is a subquotient of an unramified induced representation $I(\widetilde{\mu})$.*

Proof. — The first claim is Lemma 4.7 of [Bo]. The second follows from Frobenius reciprocity (see (14)). In the case $n = 1$ we use the notations (π, V), V^I, $(V_N)^{I \cap A}$. The work of [Bo] is formulated in this context only, but the proofs apply in the case of general n too, in view of Lemma 16.3. Indeed, beyond the Iwahori decomposition, [Bo], (4.7) uses only the fact that the T_w are invertible ([Bo], (3.6)).

Further, we use the following result of Bernstein, Borel ([Bo], Theorem 4.10), Matsumoto.

Proposition (**17.1**). — *The map $\widetilde{V} \to \widetilde{V}^{I^*}$ is a bijection from the (isomorphism classes of) irreducible admissible representations $(\widetilde{\pi}, \widetilde{V})$ with $\widetilde{V}^{I^*} \neq 0$, to the set of irreducible (finite dimensional) complex representations of the Hecke algebra $\widetilde{H}$ of $\widetilde{G}$ with respect to I^*.*

The theorem follows at once from the Propositions. More precisely, the bijection of the Theorem is described by

$$(\pi, V) \leftrightarrow (\pi \mid V^I, V^I) \leftrightarrow (\widetilde{\pi} \mid \widetilde{V}^{I^*}, \widetilde{V}^{I^*}) \leftrightarrow (\widetilde{\pi}, \widetilde{V}),$$

where the middle arrow is defined by Proposition 16.1, and the others by Propositions 17 and 17.1.

Lemma. — *Let* $\widetilde{A}_+$ *denote the subset of* a *in* $\widetilde{W}'$ *such that* $p(a)$ *lies in* A *and has the form* $(a_1, a_2, \ldots)$ *with* $|a_1| \leqslant |a_2| \leqslant \ldots$ *Then the action of* $\widetilde{T}_a$ (a *in* $\widetilde{A}_+$) *commutes with the map* $\widetilde{V}^{I^*} \to (\widetilde{V}_N)^{I^* \cap \widetilde{A}}$ *of* (17).

Proof. — We have to show for each v in $\widetilde{V}^{I^*}$ with image $\bar{v}$ that the image of $\widetilde{T}_a . v$ is $\widetilde{\pi}_N(a)\,\bar{v}$. Recall that $\widetilde{T}_a$ is the convolution operator $|I|^{-1}\widetilde{\pi}(I^* a I^*)$ (up to a scalar $\delta^{-1/2}(a)$ which appears also in $\widetilde{\pi}_N(a)$). Since a lies in $\widetilde{A}_+$, the Iwahori decomposition $I = (N \cap I)(A \cap I)(N^- \cap I)$ implies that $I^* a I^* = s(N \cap I)\, a I^*$. As v lies in $\widetilde{V}^{I^*}$, the operator $|I|^{-1}\widetilde{\pi}(I^*)$ acts trivially on v. Hence

$$\widetilde{T}_a v = \widetilde{\pi}(s(N \cap I))\,\widetilde{\pi}(a)\,v = \widetilde{\pi}(a)\,v + [\widetilde{\pi}(s(N \cap I))\,\widetilde{\pi}(a)\,v - \widetilde{\pi}(a)\,v]$$

maps to $\widetilde{\pi}_N(a)\,\bar{v}$, as asserted.

Remark. — (1) In the case $n = 1$ we denote $\widetilde{A}_+$ by A_+. (2) Since each w in $\widetilde{W}' \cap \widetilde{A}$ can be expressed in the form $a^{-1}a'$ with a, a' in $\widetilde{A}_+$, it is possible to define a $\widetilde{W}' \cap \widetilde{A}$-module structure on $\widetilde{V}^{I^*}$ by putting $\widetilde{T}''_w = \widetilde{T}_a^{-1}\widetilde{T}_{a'}$. The Lemma shows that the map $v \to \bar{v}$ of (17) commutes with the $\widetilde{W}' \cap \widetilde{A}$-structure. The Hecke algebra $\widetilde{H}$ has a presentation, due to Bernstein (see Lusztig [L], (4.4)), by means of the generators $\widetilde{T}''_w$ (w in $\widetilde{W}' \cap \widetilde{A}$) and $\widetilde{T}'_w$ (w in $\widetilde{W} = \widetilde{W}' \cap K^*$), and suitable relations. Using this presentation together with the Lemma, it can be seen that induction commutes with the functor $\widetilde{V} \to \widetilde{V}^{I^*}$. Namely if $\widetilde{\sigma}$ is an $\widetilde{M}$-module ($\widetilde{M}$ is a Levi subgroup of $\widetilde{G}$), then the $\widetilde{H}$-module $I(\widetilde{\sigma}^{I^* \cap \widetilde{M}}) = \mathrm{Hom}_{\widetilde{H}_{\widetilde{M}}}(\widetilde{H}, \widetilde{\sigma}^{I^* \cap \widetilde{M}})$ induced from the $\widetilde{H}_{\widetilde{M}}$-module $\widetilde{\sigma}^{I^* \cap \widetilde{M}}$ is the $\widetilde{H}$-module $I(\widetilde{\sigma})^{I^*}$.

Corollary (**17.2**). — *The map of the Theorem bijects square-integrable representations of* $\widetilde{G}$ *and* G.

The same statement is valid for tempered representations.

Proof. — This follows from Harish-Chandra's criterion ([S], Theorem 4.4.4; see (21) below) for square-integrability. Indeed, for each irreducible factor (character) $\widetilde{\chi}$ in the composition series of the finite dimensional $\widetilde{A}^n$-module $(\widetilde{V}_N)^{I^* \cap \widetilde{A}}$, there exists an irreducible factor χ in the composition series of $(V_N)^{I \cap A}$, and vice versa, so that $\chi(a) = \widetilde{\chi}(s(a^n))$ (a in A). The characters χ and $\widetilde{\chi}$ are the central exponents (see (21)) of π and $\widetilde{\pi}$ with respect to N. The criterion asserts that π is square-integrable if and only if for each such χ we have $|\chi(a)| < 1$ for all $a = (a_1, \ldots, a_r)$ in A with $|a_i| \leqslant |a_{i+1}|$ $(1 \leqslant i < r)$ and $|a_1| < |a_r|$, and $|\chi(a)| = 1$ for the a with $|a_r| = |a_1|$. The analogous statement holds for $\widetilde{\pi}$, hence the corollary.

The same criterion asserts that π is tempered if and only if the above requirement is satisfied with the weak inequality $|\chi(a)| \leqslant 1$ replacing the strict $|\chi(a)| < 1$ of the square-integrable case. Hence the statement concerning the tempered representations.

Corollary (**17.3**). — *A representation of* $\widetilde{G}$ *unitarily induced from a tempered one with an Iwahori fixed vector is irreducible. Namely any tempered representation of* $\widetilde{G}$ *which has a vector fixed by the Iwahori subgroup* I *is equal to a representation unitarily induced from a discrete series.*

Proof. — A representation $(\widetilde{\pi}, \widetilde{V})$ of $\widetilde{G}$ is tempered if and only if it is a direct summand of a representation $I(\widetilde{\sigma})$ unitarily induced from a discrete series representation $\widetilde{\sigma}$ of a Levi subgroup $\widetilde{L}$ (see [BW]). Then if $\widetilde{V}^{I^*}$ is non-zero, it is a submodule of $I(\widetilde{\sigma})^{I^*}$. The representation $\widetilde{\sigma}$ corresponds by Corollary 17.2 to a discrete series σ. The H-module $I(\sigma)^I$ is irreducible since the conclusion of our corollary holds in the case of G $(n = 1)$ by [BZ]. It suffices to show that the $\widetilde{H} \simeq$ H-module $I(\widetilde{\sigma})^{I^*}$ is equivalent to $I(\sigma)^I$; for then $I(\widetilde{\sigma})^I$ is irreducible, $\widetilde{V}^{I^*} = I(\widetilde{\sigma})^{I^*}$, and as $\widetilde{V}^{I^*}$ generates $\widetilde{V}$ we have $\widetilde{V} = I(\widetilde{\sigma})$ as required. Thus it remains to note that

$$(I(\widetilde{\sigma})_N)^{I^*\cap\widetilde{A}} \simeq \Sigma((\widetilde{\sigma}^w)_N)^{I^*\cap\widetilde{A}} \simeq \Sigma((\sigma^w)_N)^{I\cap A} \simeq (I(\sigma)_N)^{I\cap A},$$

where the sums range over the Weyl group of L in G. The isomorphisms commute with the action of $A_+ \simeq \widetilde{A}_+$ by Lemma 17.1. But the G-module $I(\sigma)$ is uniquely determined by the A-module $I(\sigma)_N$, since $I(\sigma)$ is irreducible. Hence the H-module corresponding to the $\widetilde{H}$-module $I(\widetilde{\sigma})^{I^*}$ is $I(\sigma)^I$, as required.

A representation $\widetilde{\pi}$ is called *elliptic* if its character is not identically zero on the elliptic regular set. A representation $\widetilde{\pi}$ with an I^*-fixed vector is called *Steinberg* if it corresponds by the Theorem to a Steinberg subquotient $\sigma(\mu)$ of $I(\mu)$; it is denoted by $\sigma(\widetilde{\mu})$. We denote by $\pi(\mu)$ the one-dimensional constituent of $I(\mu)$, and by $\pi(\widetilde{\mu})$ the corresponding representation of $\widetilde{G}$. We deduce

Corollary (**17.4**). — *Suppose* $\widetilde{\pi}$ *is tempered, elliptic and has an Iwahori fixed vector. Then it is* $\sigma(\widetilde{\mu})$.

Recall that the modulus function δ takes the value $\Pi\,|\,\alpha(a)\,|$ at a in A; the product ranges over the (positive) roots α of A in N. Let $P = LU$ be an upper triangular parabolic subgroup. The Levi subgroup L is a product of $L_i = GL(r_i)$. Note that when $I(\mu)$ contains $\sigma(\mu)$, then $I(\mu) = \mu \otimes I(1)$, where μ is regarded as a character of $F^\times$. To simplify notation we now assume that $\mu = 1$, and denote $\sigma(\mu)$ by σ, $\pi(\mu)$ by π, etc. We denote the Steinberg representation of L_i by σ_i, and its modulus function by δ_i. Since the Jacquet module is transitive $(\pi_N = (\pi_U)_{N\cap L})$, we have

$$\sigma_U = \delta^{-1/2}\,\Pi_i\,\delta_i^{1/2}\,\sigma_i.$$

Employing analogous notation (see also Remark (26) below) we also have

$$\widetilde{\sigma}_U = \delta^{-1/2}\,\Pi_i\,\delta_i^{1/2}\,\widetilde{\sigma}_i,$$

and

$$(\widetilde{V}_U)^{\widetilde{L}\cap I^*} \simeq (\widetilde{V}_N)^{\widetilde{A}\cap I^*} \simeq \widetilde{V}^{I^*} \simeq V^I \simeq (V_N)^{A\cap I} \simeq (V_U)^{L\cap I}$$

is one-dimensional. We shall now show

Corollary **(17.5)**. — *If* $L \neq A$ *then* $\widetilde{\sigma}_U$ *does not have a* $K^* \cap \widetilde{L}$*-fixed vector.*

Proof. — In view of the above it suffices to show that $\widetilde{\sigma}$ does not have a K^*-fixed vector, namely the $\widetilde{H}$-module $\widetilde{\sigma}^{I^*}$ does not have a vector fixed under the action of all $\widetilde{T}_i$ ($1 \leqslant i \leqslant r$). This is clear in the case $n = 1$. If (π, V) is the one-dimensional constituent in the composition series of the induced $I(\mu)$ which contains σ, then $I(\mu)$ has a single K-fixed vector, and it spans V. Hence σ has no K-fixed vector. Also $I(\widetilde{\mu})$ has a unique K^*-fixed vector, and it lies in the subspace $\widetilde{V}^{I^*}$ of the representation $(\widetilde{\pi}, \widetilde{V})$ matching (π, V), since $\widetilde{V}^{I^*} \simeq V^I$ has a $K^* \simeq$ K-fixed vector. Hence $\widetilde{\sigma}$ has no K^*-fixed vector, as required.

An alternative proof is given on noting that the special representation σ corresponds to the one-dimensional representation of H which assigns -1 to each generator T_i ($1 \leqslant i < r$; [Bo], Prop. 3.4), and therefore has no K-fixed vector.

18. Trace formula. — Let F be a global field, and suppose $f = \otimes f_v$, $\widetilde{f} = \otimes \widetilde{f}_v$ is a smooth function on $G(\mathbf{A})$, $\widetilde{G}(\mathbf{A})$, compactly supported modulo $Z(\mathbf{A})$, $\widetilde{Z}(\mathbf{A})$, where $f_v = f_v^0$, $\widetilde{f}_v = \widetilde{f}_v^0$ for almost all v, and $f, \widetilde{f}$ transform by a character ω^{-1}, $\widetilde{\omega}^{-1}$ of $Z(\mathbf{A})/Z(F)$, $\widetilde{Z}(\mathbf{A})/\widetilde{Z}(F)$ on the center (see (6) and (11)). Here $\mathbf{A}$ denotes the ring of adeles of F. The following is a case ([F'']) of the " simple " trace formula. Let r be the right representation of $\widetilde{G}(\mathbf{A})$ in the space $L^2(\widetilde{G})$ of smooth functions on $G(F)\backslash\widetilde{G}(\mathbf{A})$ which transform under $\widetilde{Z}(\mathbf{A})$ by $\widetilde{\omega}$ (see (6)) and are square-integrable on $\widetilde{Z}(\mathbf{A})\, G(F)\backslash\widetilde{G}(\mathbf{A})$. By a discrete series $\widetilde{\pi}$ we mean an irreducible constituent of r which appears in the discrete part of the spectral decomposition of $L^2(\widetilde{G})$.

Theorem. — *Suppose that at two finite places* $v = u, u'$ *the component* $\widetilde{f}_v$ *of* $\widetilde{f}$ *satisfies* $F(x, \widetilde{f}_v) = 0$ *for all regular non-elliptic* x *in* $\widetilde{G}(F_v)$, *and at a place* u'' *the component* $\widetilde{f}_{u''}$ *vanishes on the* x *in* $G(F)$ *so that* x^r *is singular. If, moreover,* $\widetilde{f}_u$ *is a supercusp form, then*

$$\Sigma_{\widetilde{\pi}} \operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \Sigma_x \,|\, \widetilde{G}_x(\mathbf{A})/\widetilde{G}_x(F)\, Z(\mathbf{A}) |\, F(x, \widetilde{f}). \tag{18.1}$$

On the left the sum is over the discrete series $\widetilde{\pi}$ *of* $\widetilde{G}(\mathbf{A})$. *On the right the sum ranges over the conjugacy classes of good elliptic regular* x *in* $G(F)$ *modulo* $Z(F)$. *Hence we can replace* $F(x, \widetilde{f})$ *by* $F(x^*, \widetilde{f})$ *in* (18.1) *if we replace the sum on the right by a sum over the classes of elliptic* x *in* $G(F)$ *modulo* $Z(F)$ *with regular* x^*. *Both sums are absolutely convergent. The sum on the right is finite.*

Proof. — Since the component of $\widetilde{f}$ at u is supercuspidal, the operator $r(\widetilde{f})$ annihilates the non-cuspidal spectrum. Hence it is of trace class, and tr $r(\widetilde{f})$ is given by the sum on the left of (18.1). On the other hand, $r(\widetilde{f})$ is an integral operator with kernel $\Sigma_x \widetilde{f}(hxg^{-1})$ (sum over x in $G(F)/Z(F)$; g, h in $\widetilde{G}(\mathbf{A})/\widetilde{G}(F)\, Z(\mathbf{A})$) and the trace is obtained by integrating the kernel over the diagonal $g = h$.

We first claim that the sum $\Sigma_x \widetilde{f}(gxg^{-1})$ ranges over finitely many conjugacy classes of x. Indeed, let X be the space of semi-simple conjugacy classes in $G(\mathbf{A})/Z(\mathbf{A})$. Mapping g in $G(\mathbf{A})$ to the set $(a_1, \ldots, a_r)$ of coefficients in its characteristic polynomial, X is isomorphic to the quotient of $\mathbf{A}^{r-1} \times \mathbf{A}^\times$ by $\mathbf{A}^\times$, where

$$(a_1, \ldots, a_r) \simeq (a_1 z, a_2 z^2, \ldots, a_r z^r).$$

If $\widetilde{f}(g^{-1} xg) \neq 0$, then the projection of $p(x)$ in X lies in a compact subset, depending only on the support of $\widetilde{f}$. On the other hand, the image of the rational $p(x)$ lies in a discrete subset $F^{r-1} \times F^\times/F^\times$ of X, as required.

Now suppose y is a representative of one of the finitely many rational conjugacy classes with $\widetilde{f}(gyg^{-1}) \neq 0$ for some g in $\widetilde{G}(\mathbf{A})$. Then y^r is regular by the condition at u''. In particular y is regular and its conjugacy class in $\widetilde{G}(F_{u'})$ is closed. Suppose that y is not elliptic over F. Then it is not elliptic over $F_{u'}$, and our assumption at u' implies that the orbital integral $\Phi(y, \widetilde{f}_{u'})$ of $\widetilde{f}_{u'}$ at x is 0. Since the group $\widetilde{G}(F_{u'})$ acts transitively on the orbit of y there is a unique (up to scalar) Haar measure on this orbit. Consequently there exists a unique invariant distribution on the space of locally-constant compactly-supported complex-valued functions on the orbit of y, and its kernel is spanned by the functions $h - h^g$ (h as above, g in $\widetilde{G}(F_{u'})$, $h^g(u) = h(g^{-1} ug)$). Since $\Phi(y) : h \to \Phi(y, h)$ is a non-zero invariant distribution on the orbit of y, and $\widetilde{f}_{u'}$ is in its kernel, we conclude that there are finitely many h_i, g_i such that *if* $h' = \Sigma_i(h_i - h_i^{g_i})$, then $\widetilde{f}_{u'} = h'$ on the orbit of y. Since the orbit of y is closed in $\widetilde{G}(F_{u'})$ we can extend h to locally-constant compactly-supported functions on $\widetilde{G}(F_{u'})$ which vanish outside a small neighborhood of the orbit of y. We now replace in $\widetilde{f}$ the component $\widetilde{f}_{u'}$ by its difference with h'. We have not changed the value of the left side of (18.1), since tr $\widetilde{\pi}$ is an invariant distribution. We have not changed the value of $\widetilde{f}(gxg^{-1})$ on any rational orbit x other than y, since h' is supported on a small neighborhood of the orbit of y in $\widetilde{G}(F_{u'})$. However $\widetilde{f}(gyg^{-1})$ is zero for all g, so that the orbit of y can be omitted from $(*)$ $\Sigma \widetilde{f}(gxg^{-1})$. Repeating this argument to the finitely many rational regular non-elliptic orbits of x in the sum we conclude that we may assume that only elliptic regular x appear in $(*)$, without changing the value of the integral of $(*)$ over $\widetilde{G}(\mathbf{A})/\widetilde{G}(F)\, Z(\mathbf{A})$. But now that the x are all elliptic regular we may change the order of integration and summation in the usual way, and arrive at the right side of (18.1).

Corollary. — Suppose F *is a global field. Fix* $\widetilde{f} = \otimes \widetilde{f}_v$ *such that its components* (13) *at the finite places* $v = u, u'$ *satisfy* $F(x, \widetilde{f}_v) = 0$ *for all regular non-elliptic* x *in* $G(F_v)$, *and* $\widetilde{f}_{u''}$ *vanishes on the singular set. Let* $f = \otimes f_v$ *be a matching function on* $G(\mathbf{A})$. *Suppose that each of* $\widetilde{f}$ *and* f *has a supercuspidal component. Then*

$$\Sigma \operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \Sigma \operatorname{tr} \pi(f).$$

Both sums are absolutely convergent, and range over the discrete series representations of $G(\mathbf{A})$ *or* $\widetilde{G}(\mathbf{A})$ *which have elliptic components at* u, u'.

Remark. — We take finite u, u', u''. In (19) we take supercuspforms $\widetilde{f}_u$, $f_{u'}$, and $\widetilde{f}_{u''}$ supported on the regular elliptic set. From (20) on, we take regular $f_{u''}$, $\widetilde{f}_{u''}$ as in (15), and supercusp forms $\widetilde{f}_u$, $\widetilde{f}_{u'}$. This suits all our needs up to (27).

19. *Theorem.* — *Suppose* f_w *in* $\mathbf{H}_w$ *and* f_w^* *in* $\widetilde{\mathbf{H}}_w$ *are related by the map* $f_w \to f_w^*$ *of* (11). *Then* $F(x^*, f_w^*) = F(x, f_w)$ *for all* x *in* $G(F_w)$ *with regular* x^*.

Proof. — Choose a global field F whose completion at a place w is our local field. It can be chosen to be totally imaginary, to simplify the work at infinity. But we can deal also with the real places (see below). We choose three distinct non-archimedean places u, u', u'', supercusp forms $f_{u'}$, $\widetilde{f}_u$ on $G(F_{u'})$, $\widetilde{G}(F_u)$, matching functions $\widetilde{f}_{u'}$, f_u on $\widetilde{G}(F_{u'})$, $G(F_u)$, and matching functions $f_{u''}$, $\widetilde{f}_{u''}$ supported on the elliptic regular sets of $G(F_{u''})$, $\widetilde{G}(F_{u''})$. Given x_v in $G(F_v)$ ($v = w, u, u', u''$) and a small neighborhood U_v of x_v in $G(F_v)$ such that $\Phi(x, f_v)$ is constant on U_v for $v = u, u', u''$, there exists y in $G(F)$ with y in U_v for all $v = w, u, u', u''$. Hence we need to prove the assertion for y in $G(F)$, regular and elliptic, with $\Phi(y, f_v) \neq 0$ for $v = u, u', u''$. Then we take functions f, $\widetilde{f}$ matching as in Corollary 18, whose components at u, u', u'' are as above, with $\Phi(y, f^{w,\infty}) \neq 0$, where $f^{w,\infty} = \bigotimes f_v$ (product over all finite places v other than w), and such that the components f_w, $\widetilde{f}_w$ at w are regular matching functions as defined in (15), or the unit elements f_w^0, $\widetilde{f}_w^0$ in the respective Hecke algebras (this is permitted by (12)). We then have the identity $\Sigma \operatorname{tr} \pi(f) = \Sigma \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ of Corollary 18. Although we do not use the following comment, note that after Theorem 26 is proven it is possible to take $u' = u$.

Next we match orbital integrals in the case where F is archimedean. This we do using the Paley-Wiener theorem [CD]. Given $\widetilde{f}$ on $\widetilde{G}$, we define a function $F(\pi) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ on the space of G-modules π induced from $\sigma \otimes \mu$, where σ is a discrete-series M-module and μ is a character of M, and M is a " cuspidal " Levi subgroup. If F is complex, M must be the diagonal subgroup, which is the only Levi subgroup carrying discrete-series, and then $\sigma = 1$. If F is real, M is a product of GL(2)'s and GL(1)'s, and it is known [F] how to lift M-modules to $\widetilde{M}$-modules in this case. Hence it is clear how to lift such π to $\widetilde{\pi}$, which appear in the definition of our function F. The function F so defined satisfies the conditions of the Theorem of [CD], hence there is a function f on G as in (6), with $\operatorname{tr} \pi(f) = F(\pi)$ for the above π. A simple application of the Weyl integration formula, as in Proposition 27.1 below, shows that $\widetilde{f}$ and f are matching, since $\operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ for our π.

Now we return to our previous notation; thus F is a global field. Since $f_{u'}$ is a supercusp form, the π in our sum have a supercuspidal component at u', hence they are cuspidal, have Whittaker vectors, and all their local components are non-degenerate. At ∞, such $G(F_\infty)$-modules are " large ", or have maximal Gelfand-Kirillov dimension ([Vo], p. 98); by [Vo], Theorem 6.2 (f) they are of the form $\pi_\infty = I(\sigma \otimes \mu)$ described above, which are known, as noted above, to lift to $\widetilde{G}(F_\infty)$-modules $\widetilde{\pi}_\infty$. Hence we write our identity $\Sigma \operatorname{tr} \pi(f) = \Sigma \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ in the form

$$\Sigma \operatorname{tr} \pi^\infty(f^\infty) \operatorname{tr} \widetilde{\pi}_\infty(\widetilde{f}_\infty) = \Sigma \operatorname{tr} \widetilde{\pi}^\infty(\widetilde{f}^\infty) \operatorname{tr} \widetilde{\pi}_\infty(\widetilde{f}_\infty).$$

On the left, the sum is over the cuspidal $\pi = \pi^\infty \otimes \pi_\infty$, and $\widetilde{\pi}_\infty$ is the lift of π_∞; π_∞ is the component of π at ∞, π^∞ is the component of π outside ∞. We are now in a position to apply linear independence of characters on $\widetilde{G}(F_\infty)$, as $\widetilde{f}_\infty$ is arbitrary.

We conclude that given irreducible representations π_∞, $\widetilde{\pi}_\infty$ of $G(F_\infty)$, $\widetilde{G}(F_\infty)$ with $\operatorname{tr} \pi_\infty(f_\infty) = \operatorname{tr} \widetilde{\pi}_\infty(\widetilde{f}_\infty)$ for all matching $f_\infty, \widetilde{f}_\infty$, it follows from the linear independence of characters that

$$\Sigma \operatorname{tr} \widetilde{\pi}^\infty(\widetilde{f}^\infty) = \Sigma \operatorname{tr} \pi^\infty(f^\infty).$$

The sums are over all representations $\widetilde{\pi}^\infty = \bigotimes \widetilde{\pi}_v \ (v \neq \infty)$, $\pi^\infty = \bigotimes \pi_v \ (v \neq \infty)$ such that $\widetilde{\pi} = \widetilde{\pi}^\infty \otimes \widetilde{\pi}_\infty$, $\pi = \pi^\infty \otimes \pi_\infty$ appear in the identity of Corollary 18.3. This can be expressed in the form

$$\textbf{(19.1)} \qquad \Sigma_{\widetilde{\pi}_w} c(\widetilde{\pi}_w) \operatorname{tr} \widetilde{\pi}_w(\widetilde{f}_w) = \Sigma_{\pi_w} c(\pi_w) \operatorname{tr} \pi_w(f_w).$$

Here $c(\pi_w)$ is the sum of $\operatorname{tr} \pi^{w,\infty}(f^{w,\infty})$ over all $G(\mathbf{A}^{w,\infty})$-modules $\pi^{w,\infty}$ such that $\pi_w \otimes \pi^{w,\infty}$ occurs in the previous sum over π^∞. The sums are over the components (necessarily with an Iwahori fixed vector) $\widetilde{\pi}_w$, π_w of the $\widetilde{\pi}^\infty$, π^∞. Applying (15.1) and (15.2) we obtain

$$\Sigma_{\widetilde{\pi}_w} c(\widetilde{\pi}_w) \Sigma_{s \operatorname{in} W(\widetilde{\pi}_w)} \lambda((s\mu(\widetilde{\pi}_w))(\boldsymbol{\pi}_w)) = \Sigma_{\pi_w} c(\pi_w) \Sigma_{s \operatorname{in} W(\pi_w)} \lambda((s\mu(\pi_w))(\boldsymbol{\pi}_w)).$$

Recall that $\mu(\widetilde{\pi}_w)$, $\mu(\pi_w)$ are diagonal complex matrices, $W(\widetilde{\pi}_w)$ and $W(\pi_w)$ are subsets of the Weyl group $W(A)$, and λ is any character of $\mathbf{C}^r$ determined by $\lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda_i > \lambda_{i+1}$ and $\lambda_r = 0$.

A theorem of Harish-Chandra [BJ] asserts that there are only finitely many automorphic representations of $\widetilde{G}(\mathbf{A})$ with a given infinitesimal character, or component $\widetilde{\pi}_\infty$, and a given $\widetilde{K}' = \kappa^*(K')$-type, where K' denotes a sufficiently small compact open subgroup of $\widetilde{G}(\mathbf{A}_f)$ ($\mathbf{A}_f$ denotes the ring of finite adeles). Hence for a fixed choice of $\widetilde{f}_v$ and f_v $(v \neq w)$ the sums over $\widetilde{\pi}^\infty$, π^∞, hence those over $\widetilde{\pi}_w$, π_w, are finite.

Since the sums over $\widetilde{\pi}_w$, π_w are finite, we can apply linear independence of characters, and using the fact that the set of characters λ is sufficiently large we conclude the following. Given an irreducible π'_w which has an Iwahori fixed vector, the equality (19.1) remains true if we sum only over the $\widetilde{\pi}_w$, π_w with $\operatorname{tr} \widetilde{\pi}_w(\widetilde{f}_w) = \operatorname{tr} \pi'_w(f_w)$

and $\operatorname{tr} \pi_w(f_w) = \operatorname{tr} \pi'_w(f_w)$. Namely the sums of (19.1) are now taken over a subset of the set of irreducible subquotients of $I(\mu'_w)$ and $I(\widetilde{\mu}'_w)$, the induced representations determined by π'_w. However, precisely one irreducible subquotient of $I(\mu'_w)$, and $I(\widetilde{\mu}'_w)$, is unramified, and we are permitted to apply (19.1) with $\widetilde{f}_w$, f_w equal to the unit element $\widetilde{f}^0_w$, f^0_w of the respective Hecke algebra. We conclude that $c(\widetilde{\pi}_u) = c(\pi_w)$ for any unramified π_w, $\widetilde{\pi}_w$ related by the relation $\mu_w(a) = \widetilde{\mu}_w(s(a^n))$ discussed in (11). We conclude that (19.1) remains true when we replace $\widetilde{f}_w$, f_w by any spherical functions f^*_w, f_w in $\widetilde{\mathbf{H}}_w$, $\mathbf{H}_w$ related by $f_w \to f^*_w$. This is then true for the sums over $\widetilde{\pi}^\infty$, π^∞, and the sums over $\widetilde{\pi}$, π. Hence we have

$$\Sigma c(x) \, F(x^*, \widetilde{f}) = \Sigma c(x) \, F(x, f).$$

Both sums range over the conjugacy classes of x in $G(F)$ modulo $Z(F)$ with elliptic regular x^* and $c(x)$ denotes a volume factor. The two sums are finite, as explained in the proof of Theorem 18. We can choose the components f_∞, $\widetilde{f}_\infty$ of f, $\widetilde{f}$ at ∞ to satisfy $F(y^*, \widetilde{f}_\infty) \neq 0$ and $F(y, f_\infty) \neq 0$. Moreover, since the sums are finite, we can reduce the support of f_∞, $\widetilde{f}_\infty$ so that y is the only term in each of the two sums. We conclude that $F(y, f) = F(y^*, \widetilde{f})$, and since $F(y, f^w) = F(y^*, \widetilde{f}^w) \neq 0$ by choice of f^w, $\widetilde{f}^w$ and y, we conclude that $F(y, f_w) = F(y^*, \widetilde{f}_w)$, as required.

Theorem **(19.2)**. — *If* $\operatorname{tr} \widetilde{\pi}_w(\widetilde{f}'_w) = 0$ *for all tempered* $\widetilde{\pi}_w$ *then* $F(x, \widetilde{f}'_w) = 0$ *for all regular* x *in* $\widetilde{G}(F_w)$.

Proof. — As this is analogous to the proof of Theorem 19 we shall be brief. By [BW], XI, (2.11), the Grothendieck group of $\widetilde{G}(F_w)$ is spanned by $\widetilde{G}(F_w)$-modules induced from $\widetilde{M}(F_w)$-modules $\widetilde{\rho} \otimes \xi$, where $\widetilde{\rho}$ is tempered and ξ is an unramified character of the center of the Levi subgroup $\widetilde{M}(F_w)$. Hence we may assume that $\operatorname{tr} \widetilde{\pi}_w(\widetilde{f}'_w) = 0$ for all admissible $\widetilde{\pi}_w$. We choose F with completion F_w, supercuspidal $\widetilde{f}_u$ and $\widetilde{f}_{u''}$ supported on the elliptic regular set, approximate x by a global elliptic regular y as in (18), conclude that $\Sigma c(y) \, \Phi(g^*, \widetilde{f})$ is 0 by the assumption, and reduce the support of $\widetilde{f}_\infty$ to make this finite sum range over y only. As $\Phi(y^*, \widetilde{f}^w) \neq 0$ by construction, the claim follows.

Remark. — Harish-Chandra [H], Theorem 10, shows that if $F(x, \widetilde{f}) = 0$ for all regular x, then $D(\widetilde{f}) = 0$ for any invariant distribution D on the space of the functions $\widetilde{f}$.

20. Approximation. — Theorem 19 and a standard approximation argument imply

Proposition. — *Let* V *be a finite set of places containing the archimedean places,* u, u', u'' *and those* v *with* $|n|_v < 1$. *For each* v *outside* V *fix an unramified representation* π_v *and a cor-*

responding $\tilde{\pi}_v$ *(see* (11)). *For each v in* V *choose matching* $f_v, \tilde{f}_v$, *so that the conditions at* u, u', u'' *of Corollary* 18 *are held. Then*

$$\textbf{(20.0)} \qquad \sum_{\pi} \prod_{v \in V} \operatorname{tr} \pi_v(f_v) = \sum_{\tilde{\pi}} \prod_{v \in V} \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v).$$

Each sum ranges over all discrete series $\tilde{\pi}$, π *whose components at* u, u' *are elliptic, and at any* v *outside* V *are the given* $\tilde{\pi}_v$ *and* π_v.

We use the Proposition with a cuspidal π and a set V as above so that π_u, $\pi_{u'}$ are elliptic, and $\pi_{u''}$ is unramified. The rigidity (strong multiplicity one) theorem for cusp forms of G(**A**) [JS] implies that π is the unique entry on the left if we choose $\{\pi_v\}$ to be the components outside V of π. At each archimedean place v denote by $\tilde{\pi}_v$ the representation corresponding to π_v. Using the arguments of the proof of Theorem 19 at the archimedean places, we obtain the statement of the Proposition with V replaced by its subset of finite places.

Denote by $\tilde{\pi}^0_{u''}$ the representation corresponding to $\pi_{u''}$ by Theorem 16. Since $\pi_{u''}$ is unramified and non degenerate, $\tilde{\pi}^0_{u''}$ is equal to an irreducible representation induced from the Borel subgroup ([Z], Theorem 9.7). The argument of the proof of Theorem 19, which uses regular functions and linear independence of characters, as permitted by Harish-Chandra's finiteness theorem and fixing the components at infinity, implies that the Proposition holds with the set V replaced by V-$\{u''\}$, provided the sum on the right ranges over $\tilde{\pi}$ whose component at u'' is $\tilde{\pi}^0_{u''}$. Then the identity takes the form

$$\textbf{(20.1)} \qquad \prod \operatorname{tr} \pi_v(f_v) = \sum m(\{\tilde{\pi}_v\}) \prod \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v);$$

products over v in a finite set V, and sum over equivalence classes of $(\{\tilde{\pi}_v\})$ with positive integral multiplicities m.

Thus from now on we use (20.1) with a finite set V of finite places, including u, u' but not u''; this identity is valid for arbitrary matching functions whose components at u, u' satisfy the requirements of Corollary 18. We use it in the case where π is cuspidal, and the components at u, u' of some $\tilde{\pi}'$ which appears on the right of (20.1) are supercuspidal. We construct such π in the Lemma below. Fix a place w in V. For each $\tilde{\pi}_w$ we write $c(\tilde{\pi}_w) = \sum \prod \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v)$ (v in V, $v \neq w$). The sum ranges over the $\tilde{\pi}$ of the Proposition whose component at w is $\tilde{\pi}_w$. Let $\tilde{f}_u$, $\tilde{f}_{u'}$ be matrix coefficients of the supercuspidal components at u, u' of $\tilde{\pi}'$, so that $\operatorname{tr} \tilde{\pi}'_v(\tilde{f}_v) = 1$ for $v = u, u'$. For each $v \neq w, u, u'$ in V choose a large j, and let $\tilde{f}_v$ be the characteristic function φ^*_{jv} of (10). Then $\operatorname{tr} \tilde{\pi}_v(\tilde{f}_v)$ is a non-negative integer (see (10)), and each $c(\tilde{\pi}_w)$ is a non-negative integer. Also we put $c = \prod \operatorname{tr} \pi_v(f_v)$ (v in V, $v \neq w$). Then for all matching $f_w, \tilde{f}_w$ we obtain

$$\textbf{(20.2)} \qquad c \operatorname{tr} \pi_w(f_w) = \sum c(\tilde{\pi}_w) \operatorname{tr} \tilde{\pi}_w(\tilde{f}_w).$$

To construct π as in the discussion above, let F be a global field; w, u, u' distinct finite places of F; V a finite set of finite places, including u, u', excluding w, of cardinality at least three. As usual (see (10)) we denote by π (resp. $\tilde{\pi}$) representations of G (resp. $\tilde{G}$) with central character ω (resp. $\tilde{\omega}$) (see (6)).

Lemma. — Suppose we have (1) *a square-integrable representation* π_w; (2) *supercuspidal representations* $\tilde{\pi}_v$ *at* $v = u, u'$; (3) *supercuspidal representations* π_v *at all* $v \neq u, u'$ *in* V. *Then there exist cuspidal representations* π, π'' *of* G, *such that*: (i) *the component at* w *of* π *is as in* (1) *and that of* π'' *is unramified*; (ii) *their components at all finite places* $v \neq w$ *outside* V *are unramified*; (iii) *at any* $v \neq u, u'$ *in* V *the components of* π *and* π'' *are as in* (3); (iv) *at* $v = u, u'$ *their components* $\pi_v = \pi''_v$ *satisfy* $\operatorname{tr} \pi_v(f_v) \neq 0$, *where* f_v *matches* (*see* (8)) *a matrix coefficient* $\tilde{f}_v$ *of the* $\tilde{\pi}_v$ *of* (2).

Proof. — Let G′ be the multiplicative group of a central division algebra over F of rank r which is anisotropic at u, u', w, and unramified outside $V \cup \{w\}$. For $v = u, u'$ let π'_v be a representation of $G'(F_v)$ with $\operatorname{tr} \pi'_v(f'_v) \neq 0$, where f'_v is a function on $G'(F_v)$ matching (see [DKV] or [F″]) a function f_v as specified in (iv). For $v \neq u, u'$ in $V \cup \{w\}$, denote by π'_v a representation of $G'(F_v)$ corresponding (see [DKV] or [F″]) to π_v. Let f' be a function on $G'(\mathbf{A})$ such that: (a) its component f'_v at v in $V \cup \{w\}$ is a matrix coefficient of π'_v; (b) at all other finite places we take $f'_v = f^0_v$ (see (11)); (c) $f'(1) \neq 0$; (d) for γ in PG′(F) and x in $PG'(F)\backslash PG'(\mathbf{A})$ we have $f'(x^{-1}\gamma x) = 0$ unless $\gamma = 1$. Here we put PG′ for G′/Z.

It is clear that there exists an f' which satisfies (d) in addition to (a), (b), (c). Indeed, the finiteness argument in the proof of Theorem 18 implies that there are only finitely many conjugacy classes γ with $f'(x^{-1}\gamma x) \neq 0$. To obtain (d) we take the archimedean components of f' to be supported on a sufficiently small set.

Let r be the right representation on the space $L^2(G')$ of smooth functions on $G'(F)\backslash G'(\mathbf{A})$ which transform under $Z(\mathbf{A})$ according to ω. The kernel of the operator $r(f')$ is given by $\Sigma f'(x^{-1}\gamma y)$ (γ in PG′(F)). Its trace is obtained on integrating the kernel over the diagonal $x = y$ in $PG'(F)\backslash PG'(\mathbf{A})$. By (d), the trace of $r(f')$ is the product of $f'(1)$ ($\neq 0$ by (c)) and the volume of $PG'(F)\backslash PG'(\mathbf{A})$. It is non-zero. Hence there exists an automorphic representation π' of G′ which is unramified by (b) outside $V \cup \{w\}$. Denote by π the representation of G corresponding (by [DKV] or [F″]) to π'. It is cuspidal, as it has supercuspidal components (by (a)), and has the properties required by the lemma.

To produce π'' we repeat the above steps but with G′ which splits at w, and f' whose component at w is f^0_w. The π'' which is so obtained is cuspidal, hence non-degenerate. Its component π''_w at w is unramified, and non-degenerate, hence equal (by [Z], Theorem 9.7) to a representation induced from a minimal parabolic subgroup. It is clearly a lift of some such induced representation $\tilde{\pi}''_w$. The lemma follows.

III. — APPLICATIONS

21. Square-integrable. — For brevity we now omit the index w in (20.2). Our aim is to show that the $\widetilde{\pi}$ in (20.2) are square-integrable if π is. So we consider a small compact open congruence subgroup C^* of $K^* = \kappa^*(K)$, with $C^* = (C^* \cap \overline{N})(C^* \cap \widetilde{M})(C^* \cap N)$ for any standard parabolic $P = MN$. Here $\overline{N}$ denotes the unipotent radical of the opposite parabolic $\overline{P} = M\overline{N}$. We take t in A so that $\widetilde{P} = \widetilde{P}_{t^*}$ (see (14)) is standard. It suffices to consider only t in the center of M. Then $t^*(C^* \cap \widetilde{M})\, t^{*-1}$ is in $C^* \cap \widetilde{M}$, $t^*(C^* \cap N)\, t^{*-1}$ in $C^* \cap N$, and $t^{*-1}(C^* \cap \overline{N})\, t^*$ is in $C^* \cap \overline{N}$. Namely Lemma 2.1 of [C] holds with t^* replacing g and C^* replacing K_i. Let $\widetilde{f} = \widetilde{f}_{t^*}$ be the function with the properties of (6), supported on $\widetilde{Z}C^*\, t^*\, C^*$, with the value $\delta_P^{-1/2}(t^*)/|\, C^*\, t^*\, C^*/C^* \cap \widetilde{Z}\,|$ on $C^*\, t^*\, C^*$. Let $\widetilde{\varphi} = \widetilde{\varphi}_{t^*}$ be the function on $\widetilde{M}$ supported on $\widetilde{Z}t^*(C^* \cap \widetilde{M})$ whose value on $t^*(C^* \cap \widetilde{M})$ is $|\, C^* \cap \widetilde{M}/C^* \cap \widetilde{Z}\,|^{-1}$, and which transforms under $\widetilde{Z}$ as usual. Put $\widetilde{\varphi}_1$ for $\widetilde{\varphi}$ when $t = 1$.

Lemma. — (i) *We have* $\operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \operatorname{tr} \widetilde{\pi}_N(\widetilde{\varphi})$. (ii) *The orbital integral* $F(x, \widetilde{f})$ *of* $\widetilde{f}$ *at the regular element* x *is* 0 *unless* x *is conjugate to a member of* $\widetilde{M}$. *For a regular* x *in* $\widetilde{M}$ *we have* $F(x, \widetilde{f}) = \Sigma_w F^M(wxw^{-1}, \widetilde{\varphi})$; *the sum ranges over the quotient* W(M, G) *of the normalizer of* M *in* G *by* M.

Proof. — (i) follows as in [C], Lemma 5.1, which deals with the case $n = 1$. For (ii) we note that by the characterization (8) of orbital integrals, there exists a function $\widetilde{f}'$ on $\widetilde{G}$ which satisfies the assertion of (ii). But then it follows from Weyl integration formula, and Theorem 14, that $\operatorname{tr} \widetilde{\pi}(\widetilde{f}') = \operatorname{tr} \widetilde{\pi}_N(\widetilde{\varphi})$ for any $\widetilde{\pi}$. Hence $\operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \operatorname{tr} \widetilde{\pi}(\widetilde{f}')$ for all $\widetilde{\pi}$, and $F(x, \widetilde{f}') = F(x, \widetilde{f})$ for all regular x in $\widetilde{G}$ by Theorem 15.1, as required.

In the proof of the following Proposition we use Harish-Chandra's criterion for square-integrability ([C'], Theorem 4.4.6; [S], Theorem 4.4.4). To state it, we define a *central exponent* of π with respect to M to be the central character of an irreducible constituent of π_N. We say that the central exponent ω of π with respect to M_θ (θ in Δ, as in (14)) *decays* if $|\,\omega(a)\,| < 1$ for every a in A_θ with $|\,\alpha(a)\,| \leqslant 1$ for all α in $\Delta - \theta$, with $|\,\alpha(a)\,| < 1$ for some α. The criterion asserts that π is discrete series if and only if its central character is unitary, and its central exponents with respect to any proper Levi subgroup all decay.

Proposition. — *Suppose* π *in* (20.2) *is square-integrable. Then so are all of the* $\widetilde{\pi}$.

Proof. — Fix an open compact C^* as above, and a proper standard parabolic P. Take any t^* in the center of $\widetilde{M}$ with $\widetilde{P}_{t^*} = \widetilde{P}$. Since $\widetilde{f}$ (of the Lemma) is C^*-invariant, the sum in (20.2) is finite (by the theorem of Harish-Chandra [BJ], mentioned in (19) and (20)), independently of t^*.

We have $\operatorname{tr} \widetilde{\pi}(\widetilde{f}) = \operatorname{tr} \widetilde{\pi}_N(\widetilde{\varphi})$ by (i) of the Lemma. If $\widetilde{\rho}$ is an irreducible constituent of $\widetilde{\pi}_N$, denote its character by $\widetilde{\chi}_\rho$ and its central character by $\widetilde{\omega}_\rho$. Then

$$\operatorname{tr} \widetilde{\rho}(\widetilde{\varphi}) = \int \widetilde{\chi}_\rho(t^* x)\, \widetilde{\varphi}(t^* x)\, dx = \widetilde{\omega}_\rho(t^*) \operatorname{tr} \widetilde{\rho}(\widetilde{\varphi}_1),$$

where $\operatorname{tr} \widetilde{\rho}(\widetilde{\varphi}_1)$ is the (non-negative integral) multiplicity of the trivial representation of $C^* \cap \widetilde{M}$ in $\widetilde{\rho}$.

The function $\widetilde{\varphi}$ defined prior to the Lemma is supported on a neighborhood of the identity in $\widetilde{M}$, multiplied by the central element t^*. Proposition 10 asserts that there exists a function φ on M matching $\widetilde{\varphi}$. The characterization (8) of orbital integrals asserts that there exists f on G which matches φ in the sense of (ii) in the above Lemma. Hence there exists f matching $\widetilde{f}$, so that (20.2) can be applied. By the Weyl integration formula and Theorem 14 we have

$$\begin{aligned}\operatorname{tr} \pi(f) = \Sigma_T \int (\Delta\chi(\pi))(x)\, F(x,f)\, dx &= \Sigma_T \int (\Delta\chi(\pi))(x)\, F(x^*,\widetilde{f})\, dx \\ &= \Sigma_{T \text{ in } M} \int (\Delta_M\, \chi(\pi_N))(x)\, F(x^*, \widetilde{\varphi})\, dx.\end{aligned}$$

As t lies in the center of M, changing variables $x \to tx$ we obtain

$$\Sigma_{\rho \text{ in } \pi_N}\, \omega_\rho(t)\, \Sigma_T \int (\Delta_M\, \chi(\rho))(x)\, F(x^*, \widetilde{\varphi}_1)\, dx = \Sigma_\rho\, \omega_\rho(t) \operatorname{tr} \rho(\varphi_1),$$

where φ_1 matches $\widetilde{\varphi}_1$ as in (10). Note that $\operatorname{tr} \rho(\varphi_1)$ is a complex number independent of t.

We now conclude that the identity (20.2) implies the identity

$$\Sigma_\rho\, c_\rho\, \omega_\rho(t) = \Sigma_{\widetilde{\rho}}\, n(\widetilde{\rho})\, \widetilde{\omega}_{\widetilde{\rho}}(t^*)$$

for every t in the center of M with $P_t = P$. On the left the sum is over the irreducible constituents ρ in π_N, and the c_ρ are complex numbers. On the right the $\widetilde{\rho}$ are the irreducible constituents of the $\widetilde{\pi}_N$ for the finitely many $\widetilde{\pi}$ which occur. The coefficients $n(\widetilde{\rho})$ are all positive integers, hence there are no cancellations on the right. Since t is sufficiently arbitrary, and ω_ρ all decay, linear independence of characters implies that the $\widetilde{\omega}_{\widetilde{\rho}}$ all decay. Harish-Chandra's criterion then implies that all $\widetilde{\pi}$ with a C^*-fixed vector, which appear in (20.2), are square-integrable. However, we can take C^* to be as small as we like. Hence every $\widetilde{\pi}$ in (20.2) is square-integrable, as required.

22. Asymptotics. — As usual, π and $\widetilde{\pi}$ are admissible and irreducible. Denote by $[W(\widetilde{\pi})]$ the dimension of the space of Whittaker vectors (see [BZ], [KP] (p. 74) for a definition) of $\widetilde{\pi}$ with respect to a fixed non-trivial additive character ψ. This $[W(\widetilde{\pi})]$ is independent of ψ, and $[W(\pi)] \leqslant 1$. The representation $\widetilde{\pi}$ is called non-degenerate if $[W(\widetilde{\pi})] \neq 0$. The functions $\varphi(\rho, j)$, φ_j and φ_j^* are defined in (10). Put

$$A(\widetilde{\pi}) = \lim_{j\to\infty} \operatorname{tr} \widetilde{\pi}(\varphi_j^*)\, q^{-jr(r-1)/2}, \qquad A(\pi) = \lim_{j\to\infty} \operatorname{tr} \pi(\varphi_j)\, q^{-jr(r-1)/2}.$$

Theorem. — a) *The limits* $A(\pi)$ *and* $A(\widetilde{\pi})$ *exist and are finite.* b) *There exists* $c > 0$ *so that* $A(\widetilde{\pi}) = c[W(\widetilde{\pi})]$; *in fact* $c = 1$.

Proof. — Denote the character of π by χ. We have

$$\operatorname{tr} \pi(\varphi(\rho, j)) = \int_{G/Z} \chi(x)\, (\varphi(\rho, j))\, (x)\, dx = \int_{G/Z} \chi(\rho u)\, (\varphi(1, j))\, (u)\, du.$$

For a sufficiently large j, if u lies in U_j (see (10)), Proposition 13 implies that

$$\chi(\rho u) = \Sigma_\xi\, c(\xi, \pi)\, \hat{\nu}_\xi(u).$$

The sum ranges over the nilpotent orbits in the Lie algebra Lie $M(\rho)$ of the centralizer $M(\rho)$ of ρ in G. Hence it remains to study $\hat{\nu}_\xi(\varphi(1,j))$, and $\hat{\nu}_\xi(\varphi_j^*)$ for (b), for all nilpotent orbits ξ in Lie G.

Recall that the dimension of the nilpotent orbit ξ in Lie $M(\rho)$ is even. Put $\dim \xi = 2d$. If $\xi(\rho)$ is the regular nilpotent orbit in Lie $M(\rho)$, then its dimension is maximal, and equal to $2m(\rho)$. In the notations of (10), $m(\rho) = (\Sigma_i r_i^2 - r)/2$. Note that $r(\rho) + m(\rho) = r(r-1)/2$. In particular, $m(1) = r(r-1)/2$ is greater than $m(\rho)$ if $\rho \neq 1$.

Recall that the Fourier transform $\hat{\nu}_\xi(f) = \nu_\xi(\hat{f})$ is defined ([Ho]) for a function f supported on ZK by

$$\int_\xi \int_{\operatorname{Lie} G} f(1 + X)\, \psi(\operatorname{tr} XY)\, dY\, d\nu_\xi(X).$$

Since (10) the function $\varphi(1, j)$ is (for simplicity of notation we ignore the center) the quotient by the volume $|K_j|$ of the characteristic function of $U_j = 1 + K_j$, and K_j is the compact open subgroup of X in Lie G with $|X| \leqslant q^{-j}$, we have that

$$\begin{aligned}\hat{\nu}_\xi(\varphi(1,j)) &= |K_j|^{-1} \int_\xi \int_{K_j} \psi(\operatorname{tr} XY)\, dY\, d\nu_\xi(X) \\ &= q^{jd}\, |K_0|^{-1} \int_\xi \int_{K_0} \psi(\operatorname{tr} XY)\, dY\, d\nu_\xi(X) = q^{jd}\, \hat{\nu}_\xi(\varphi(1, 0)).\end{aligned}$$

Hence the contribution of any non-regular nilpotent orbit ξ disappears in the limit, and (a) follows. To prove (b) we note that $\hat{\nu}_\xi(\varphi_j^*) = \hat{\nu}_\xi(\varphi(1,j))$ (large j), and recall from [KP], p. 99, that $c(\xi(1), \widetilde{\pi}) = [W(\widetilde{\pi})]$. (b) follows once we establish the identity $\hat{\nu}_{\xi(1)}(\varphi(1, 0)) = 1$.

Thus we take $\rho = 1$, so that $M(\rho) = G$. Each nilpotent orbit ξ in Lie G determines a standard parabolic subgroup $P(\xi)$ of G. Denote by $\pi(\xi)$ the representation of G unitarily induced from the trivial representation of $P(\xi)$. By [Ho], Lemma 5, we have $\operatorname{tr} \pi(\xi)(f) = \hat{\nu}_\xi(f)$. But $\operatorname{tr} \pi(\xi)\, (\varphi(1, 0))$ is equal to 1, since $\pi(\xi)$ is unramified and has a unique vector fixed by $K = G(R)$. The theorem follows.

23. *Proposition.* — *If $\tilde{\pi}$ is square-integrable then it has a Whittaker vector.*

Proof. — Let $P \subset G$ be the stabilizer of a vector in F^r (as in [BZ]). Let N denote the maximal unipotent subgroup of G in P_r. We may assume that N is the group of unipotent upper triangular matrices. Denote by $\psi : F \to \mathbf{C}^\times$ a non-trivial additive character of F, and by $\psi : N \to \mathbf{C}^\times$ the character $\psi(n) = \psi\,(\sum_{i=1}^{r-1} n_{i,i+1})$ ($n = (n_{ij})$ in N). The map $p : \tilde{G} \to G$ splits over N. Hence we consider N as a subgroup of $\tilde{G}$. We have a natural isomorphism $\tilde{N} = \mu_n \times N$, where $\tilde{N} = p^{-1}(N)$. Let $\tilde{\psi} : \tilde{N} \to \mathbf{C}^\times$ be the character defined by $\tilde{\psi}(\zeta n) = \tilde{\omega}(\zeta)\,\psi(n)$; as usual, $\tilde{\omega}$ is the central character of $\tilde{\pi}$ (fixed in (6)). Put $\tilde{P} = p^{-1}(P)$. Let L be the space of complex valued functions φ on $\tilde{P}$ with $\varphi(\tilde{n}\tilde{p}) = \tilde{\psi}(\tilde{n})\,\varphi(\tilde{p})$ ($\tilde{n}$ in $\tilde{N}$, $\tilde{p}$ in $\tilde{P}$), with $|\varphi|$ in $L^2(\tilde{N}\backslash\tilde{P})$. Denote, by σ the representation of $\tilde{P}$ on L by right translation. It follows from Mackey's theory of induced representations that σ is irreducible. The number $[W(\tilde{\pi})]$ of Whittaker vectors of $\tilde{\pi}$ is determined by the restriction of $\tilde{\pi}$ to $\tilde{P}$. By Frobenius reciprocity [BZ] we have that $[W(\tilde{\pi})]$ is equal to the dimension of $\mathrm{Hom}(\tilde{\pi} \mid \tilde{P}, \sigma)$. Indeed, the Kirillov model can be realized in $L^2(\tilde{N}\backslash\tilde{P})$; see [B], Theorem 6.2, p. 78.

Let L′ be the space of complex-valued absolutely square-integrable (with respect to a right invariant measure) functions φ on $\tilde{P}$ with $\varphi(\zeta\tilde{p}) = \tilde{\omega}(\zeta)\,\varphi(\tilde{p})$ (ζ in μ_n, $\tilde{p}$ in $\tilde{P}$). Denote by ρ the representation of $\tilde{P}$ on L′ by right translations.

Lemma. — ρ *is a multiple of* σ.

Proof. — Induction on r; hence we denote P by P_r in the course of the proof. Let U be the unipotent radical of P_r, and G_{r-1} its Levi component. Put $\tilde{U} = p^{-1}(U)$ ($\simeq \mu_n \times U$), and $\tilde{G}_{r-1} = p^{-1}(G_{r-1})$. Denote by $\hat{U}$ the dual group (of characters) of U. For each λ in $\hat{U}$, we define the character $\tilde{\lambda}$ of $\tilde{U}$ by $\tilde{\lambda}(\zeta u) = \tilde{\omega}(\zeta)\,\lambda(u)$. Denote by $\tilde{P}^\lambda$ the stabilizer of $\tilde{\lambda}$ in $\tilde{P}_r$. It contains U, and $\tilde{P}^\lambda$ is conjugate to the subgroup $\tilde{P}_{r-1}U$ of $\tilde{P}_r$ for any non-trivial λ. Identify $\tilde{P}^\lambda/U$ with $\tilde{P}_{r-1}$, and denote by σ_{r-1} and ρ_{r-1} the corresponding representations of $\tilde{P}_{r-1}$. Extend λ to a character on $\tilde{P}^\lambda$ by $\lambda(\tilde{p}) = 1$ for $\tilde{p}$ in $\tilde{P}_{r-1}$.

Let $(\rho^\lambda, W^\lambda)$ be the representation $I(\tilde{\lambda}; \tilde{P}_r, \tilde{U})$ of $\tilde{P}_r$ induced from the character $\tilde{\lambda}$ of $\tilde{U}$. Since $\rho = \int_{\hat{U}} \rho^\lambda\, d\lambda$, it suffices to show that ρ^λ is a multiple of σ_r for all non-trivial λ. But ρ^λ is induced from the representation $I(\tilde{\lambda}; \tilde{P}^\lambda, \tilde{U}) \simeq \lambda \otimes \rho_{r-1}$ of $\tilde{P}^\lambda = U\tilde{P}_{r-1}$. The induction assumption asserts that ρ_{r-1} is a multiple of σ_{r-1}. Hence ρ^λ is a multiple of $I(\lambda \otimes \sigma_{r-1}; \tilde{P}_r, \tilde{P}^\lambda) \simeq \sigma_r$, as required.

Let τ be the right representation of G on the space of absolutely square-integrable complex valued functions φ on $\tilde{G}$ with $\varphi(\zeta\tilde{g}) = \tilde{\omega}(\zeta)\,\varphi(\tilde{g})$.

Corollary. — *The restriction of* τ *to* $\tilde{P}$ *is a multiple of* σ.

Proof. — The restriction of τ to $\widetilde{\mathrm{P}}$ is a multiple of ρ.

To complete the proof of the Proposition, it remains to note that $\widetilde{\pi}$ is a direct summand of τ, for then the Corollary asserts that the restriction of $\widetilde{\pi}$ to $\widetilde{\mathrm{P}}$ is a multiple of σ, as required.

Proposition **(23.1)**. — *Suppose that on the left of* (20.1) *there appears a cuspidal* π, *and for each* v *in* V *the component* π_v *is square-integrable. Then the sum over* $\widetilde{\pi}$ *is finite.*

Proof. — We use (20.1) with the functions φ_{jv} and φ_{jv}^* of (10) at all $v \neq u, u'$ in V, multiplied by a suitable power of q^j as specified in (22). For a sufficiently large j the left side is bounded by a fixed positive number, by the inequality (a) of Theorem 22. On the right all contributions are non-negative, as explained in (10). Each π_v is square-integrable by Proposition 21. Each non-zero entry is bounded from below by a fixed positive number, for a sufficiently large j, by Proposition 23 and the equality (b) of Theorem 22. The proposition follows.

24. Measures. — The following is (an alternative exposition to) a computation of [KP′]. Our aim is to choose Haar measures dx, $\widetilde{dx}$ so that when the characters χ, $\widetilde{\chi}$ below (which are independent of dx, $\widetilde{dx}$) are related in a certain way, then we have the relation $\operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ (which does depend on dx, $\widetilde{dx}$) for all matching f, $\widetilde{f}$. This amounts to choosing the scalar c' in the Corollary below.

Lemma. — *Suppose* π, $\widetilde{\pi}$ *with characters* χ, $\widetilde{\chi}$ *satisfy the identity* $\operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ *for all* f, $\widetilde{f}$ *with* $\mathrm{F}(x, f) = \mathrm{F}(x^*, \widetilde{f})$. *Then for each torus* T *there is a constant* $b(\mathrm{T})$ *with*

$$\Delta\widetilde{\chi}(x^*) = b(\mathrm{T}) \, \Sigma_{\{t;\, t^* z = x^*\}} \Delta\chi(t) \, \widetilde{\omega}(z). \tag{24.1}$$

The sum ranges over all t *in* T/Z *such that there is* z *in* $\widetilde{\mathrm{Z}}/\mathrm{Z}^*$ *with* $t^* z = x^*$; *here* $\mathrm{Z}^* = \{ z^*;\ z \text{ in } \mathrm{Z}\}$.

Proof.

$$\int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{Z}}} \widetilde{\chi}(x) \widetilde{f}(x) \, \widetilde{dx} = \Sigma_{\mathrm{T}}[\mathrm{W}(\mathrm{T})]^{-1} \int_{\widetilde{\mathrm{T}}/\widetilde{\mathrm{Z}}} \Delta\widetilde{\chi}(t) \, \mathrm{F}(t, \widetilde{f}) \, \widetilde{dt},$$

is equal to

$$\int_{\mathrm{G}/\mathrm{Z}} \chi(x) f(x) \, dx = \Sigma_{\mathrm{T}}[\mathrm{W}(\mathrm{T})]^{-1} \int_{\mathrm{T}/\mathrm{Z}} \Delta\chi(t) \, \mathrm{F}(t^*, \widetilde{f}) \, dt.$$

Recall that if t lies in $\widetilde{\mathrm{T}}$ but not in $\mathrm{T}^* \widetilde{\mathrm{Z}}$, then $\mathrm{F}(t, \widetilde{f})$ vanishes by Proposition 3. The lemma follows.

Lemma. — *We have* $[\mathrm{F}^\times : \mathrm{F}^{\times n}] = n^2/|n|$ *and* $[\mathrm{R}^\times : \mathrm{R}^{\times n}] = n/|n|$.

Proof. — This is [KP], Lemma 0.3.2.

Recall (2) that $p(\widetilde{K}) = K$, but $p(\widetilde{Z}) = Z'$ is the subgroup $F^{\times n/d}$ of $Z \simeq F^{\times}$. In particular Z/Z' has order $(n/d)^2/|n/d|$.

Let the measures dx, $\widetilde{dx}$ on G/Z, $\widetilde{G}/\widetilde{Z}$ be related by $c'\,|\widetilde{K}\widetilde{Z}/\widetilde{Z}|_{\widetilde{dx}} = |KZ/Z|_{dx}$. Here $|S|_{\omega}$ denotes the volume of a set S with respect to a measure ω.

Lemma. — $b(T)$ *is independent of* T. *It is equal to* $b = c'\,n\,|d|/d\,|n^r|$.

Proof. — Consider the natural epimorphism $e : \widetilde{G}/\widetilde{Z} \to G/Z$. We claim that the pullback $e^*(dx)$ of dx satisfies $c''\,\widetilde{dx} = e^*(dx)$, where $c'' = c'\,n/d\,|n/d|$. Indeed, since e restricts to an epimorphism $KZ'/Z' \to KZ/Z$, with kernel $K \cap Z/Z' \cap K \simeq R^{\times}/R^{\times n/d}$ of cardinality $[R^{\times} : R^{\times n/d}] = n\,|d|/d\,|n|$, we have

$$|\widetilde{K}\widetilde{Z}/\widetilde{Z}|_{dx} = (n/d\,|n/d|)\,|KZ/Z|_{dx} = c''\,|\widetilde{K}\widetilde{Z}/\widetilde{Z}|_{\widetilde{dx}}.$$

The Jacobian of the map $x \to x^*$, $G/Z \xrightarrow{*} \widetilde{G}/\widetilde{Z}$ is $|n^{r-1}|(\Delta(x^n)/\Delta(x))^2$. Hence the pullback via * of $\widetilde{dx}$ is $b^{-1}[\Delta(x^n)/\Delta(x)]^2\,dx$, where $b = c''\,|n^{1-r}|$.

For any $\widetilde{f}$ supported on the regular set, define $f(x) = (\Delta\widetilde{f})\,(x^*)/\Delta(x)$. Then

$$\begin{aligned}\int_{G/Z}\chi(x)\,f(x)\,dx &= \int_{G/Z}\chi(x)[(\Delta\widetilde{f})\,(x^*)/\Delta(x)]\,dx\\ &= b\int_{G/Z}(\Delta\chi)\,(x)[\widetilde{f}(x^*)/\Delta(x^*)]\;b^{-1}[\Delta(x^n)/\Delta(x)]^2\,dx\\ &= b\int_{\widetilde{G}/\widetilde{Z}}\Delta(x)^{-1}[\Sigma_{\{t \text{ in } G/Z;\; t^*z=x\}}(\Delta\chi)\,(t)\,\widetilde{\omega}(z)]\,\widetilde{f}(x)\,\widetilde{dx}.\end{aligned}$$

Since $\widetilde{f}$ is arbitrary, the lemma follows.

Lemma. — b *is equal to* $n/d\,|n^r/d|^{1/2}$.

Proof. — The required character relation (24.1) holds (by [F], p. 141) in the case of a representation induced from a Borel subgroup. Here T is the split torus A, and the sum over t consists of n^{r-1} equal terms. Hence $bn^{r-1} = [\widetilde{A} : \widetilde{A}_0] = n^r/d\,|n^r/d|^{1/2}$, where $\widetilde{A}_0$ is a maximal abelian subgroup of $\widetilde{A}$, and the lemma follows.

Corollary. — *If* χ, $\widetilde{\chi}$ *are related by* (24.1) *and* f, $\widetilde{f}$ *are any matching functions, then we have* $\operatorname{tr}\pi(f) = \operatorname{tr}\widetilde{\pi}(\widetilde{f})$ *provided that the measures* dx, $\widetilde{dx}$ *are related by* $c' = |n^r/d|^{1/2}$.

From now on we choose dx, $\widetilde{dx}$ to be so related.

Let $X(\widetilde{G})_e$ be the disjoint union of a set of representatives $\widetilde{T}$ for the conjugacy classes of elliptic tori in $\widetilde{G}$. Fix a Haar measure $\widetilde{dt}$ on $\widetilde{T}/\widetilde{Z}$. Then

$$\widetilde{dx} = \Sigma([W(T)]\,|\widetilde{T}/\widetilde{Z}|)^{-1}\,\Delta^2\,\widetilde{dt}$$

defines a measure on $X(\widetilde{G})_e/\widetilde{Z}$. Define the inner product

$$\langle\widetilde{\chi}, \widetilde{\chi}'\rangle = \int_{X(\widetilde{G})_e/\widetilde{Z}}\widetilde{\chi}(x)\,\overline{\widetilde{\chi}}'(x)\,\widetilde{dx} = \Sigma_T([W(T)]\,|\widetilde{T}/\widetilde{Z}|)^{-1}\int_{\widetilde{T}/\widetilde{Z}}\Delta\widetilde{\chi}(t)\,\Delta\overline{\widetilde{\chi}}'(t)\,\widetilde{dt}$$

on the space of integrable functions $\widetilde{\chi}$, $\widetilde{\chi}'$ on $X(\widetilde{G})_e$ which transform on $\widetilde{Z}$ by $\widetilde{\omega}$. The analogous definition applies to conjugacy invariant functions χ, χ' on G, which transform on Z by ω.

Proposition. — *Let* χ, χ' *and* $\widetilde{\chi}'$, $\widetilde{\chi}'$ *be pairs of characters related by* (24.1). *Then* $\langle \widetilde{\chi}, \widetilde{\chi}' \rangle = \langle \chi, \chi' \rangle$.

Proof. — If T is elliptic then it is isomorphic to the multiplicative group $E^\times$ of a field extension E of degree r of F. The map $x \to x^*$ from T/Z to $\widetilde{T}/\widetilde{Z}$ is injective. Indeed if $x'^n = x^n z$ (x, x' in $E^\times$, z in $F^{\times n/d}$), then $(x'/x)^d$ lies in $F^\times$; but $(r, d) = 1$ hence x'/x lies in $F^\times$. In particular, $E^{\times n} \cap F^{\times n/d} = F^{\times n}$. Hence $\widetilde{\chi}$ is supported on $T^* \widetilde{Z}$ and $\Delta\widetilde{\chi}(x^*) = b\, \Delta\chi(x)$, as a function on T. The corresponding term in $\langle\ ,\ \rangle$ is

$$|\widetilde{T}/\widetilde{Z}|^{-1} \int_{\widetilde{T}/\widetilde{Z}} \Delta\widetilde{\chi}(t)\, \Delta\bar{\widetilde{\chi}}'(t)\, \widetilde{dt}$$
$$= [\widetilde{T}/\widetilde{Z} : (T/Z)^n]^{-1} |T/Z|^{-1} \int_{T/Z} b\, \Delta\chi(t)\, b\, \Delta\bar{\chi}'(t)\, dt.$$

Here dt is the pullback of $\widetilde{dt}$ with respect to the map $x \to x^*$. But

$$[(\widetilde{T}/\widetilde{Z}) : (T/Z)^*] = [E^\times : E^{\times n} F^{\times n/d}] = [E^\times : E^{\times n}]/[E^{\times n} F^{\times n/d} : E^{\times n}]$$
$$= n^2 |n|_E^{-1}/[F^{\times n/d} : F^{\times n}] = n^2 |n^r|^{-1}/d^2 |d|^{-1} = b^2,$$

as asserted.

25. *Proposition.* — *Suppose that on the left of* (20.0) *there appears a single cuspidal* π, *and for each* v *in* V *the component* π_v *is square-integrable. Then there is a single* $\widetilde{\pi}$ *on the right side of* (20.1).

Proof. — For any v in V we obtain from (20.1) (single term on the left) the identity

$$\operatorname{tr} \pi_v(f_v) = \Sigma\, c(\widetilde{\pi}_v) \operatorname{tr} \widetilde{\pi}_v(\widetilde{f}_v),$$

on fixing the component of $\widetilde{f}$ at the places of V other than v. The sum is finite by Proposition 23.1. It is non-empty since the left side is non-zero. For each $\widetilde{\pi}$ which appears on the right of (20.1), the product $\Pi c(\widetilde{\pi}_v)$ (v in V) is equal to a positive integer by linear independence of (finitely many) characters. Since the sum is finite, the distributions $\operatorname{tr} \widetilde{\pi}_v$ are represented by functions which are smooth on the regular set, and $\widetilde{f}_v$ is an arbitrary function on the regular set. We deduce the character relation

$$b\, \Delta\chi_v(x) = \Sigma c(\widetilde{\pi}_v)\, \Delta\widetilde{\chi}_v(x^*)$$

for all x in G with elliptic regular x^* in $\widetilde{G}$; b is as in (24).

The orthonormality relations for square-integrable representations of [K'], Theorem K, imply that for any discrete series representations $\widetilde{\pi}$, $\widetilde{\pi}'$ we have that $\langle \widetilde{\chi}, \widetilde{\chi}' \rangle$ is 1 if $\widetilde{\pi}$, $\widetilde{\pi}'$ are equivalent, and 0 otherwise. For brevity we put $\widetilde{\chi} = \chi(\widetilde{\pi}_v)$, $\widetilde{\chi}' = \chi(\widetilde{\pi}'_v)$, $\chi = \chi(\pi_v)$. Thus

$$1 = \langle \chi, \chi \rangle = \langle \Sigma c(\tilde{\pi}_v)\, \tilde{\chi}, \Sigma c(\tilde{\pi}_v)\, \tilde{\chi} \rangle = \sum_{\tilde{\pi}_v, \tilde{\pi}'_v} c(\tilde{\pi}_v)\, \bar{c}(\tilde{\pi}'_v) \langle \tilde{\chi}, \tilde{\chi}' \rangle = \sum_{\tilde{\pi}_v} |c(\tilde{\pi}_v)|^2.$$

Hence $|c(\tilde{\pi}_v)| \leqslant 1$ for all local $\tilde{\pi}_v$ in (20.1). But $\Pi c(\tilde{\pi}_v)$ is integral, hence $|c(\tilde{\pi}_v)| = 1$ for all $\tilde{\pi}_v$ and the proposition follows.

25.1. Alternative proof. — The proof of Proposition 25 given above is, in some sense, elementary. It is based on the existence of Whittaker models (23), and their relations with characters (22) by means of the matching result of (10), near the identity. We shall now present a simpler proof, which is based on the results of [K′] about the existence and properties of pseudo-coefficients. It is independent of the work of (22), (23). When combined with the construction of Lemma 26 it can be made independent of (10) too. The $c(\tilde{\pi})$ of (21.1) become the multiplicities of cuspidal representations with a certain local behaviour.

We use the following Theorem K of [K′]. Recall that $\langle .,. \rangle$ is defined in (24), and Φ'' in (6). We denote the character of $\tilde{\pi}$ (resp. $\tilde{\pi}'$) by $\tilde{\chi}$ (resp. $\tilde{\chi}'$).

Theorem. — Given a square-integrable representation $\tilde{\pi}$, there exists a function $\tilde{f}$ such that $\Phi(x, \tilde{f}) = 0$ for any regular non-elliptic x in $\tilde{G}$, and $\Phi''(x, \tilde{f}) = \tilde{\chi}(x)$ for any regular elliptic x in $\tilde{G}$. Moreover, $\langle \tilde{\chi}, \tilde{\chi} \rangle = 1$, but $\langle \tilde{\chi}, \tilde{\chi}' \rangle = 0$ for all tempered irreducible $\tilde{\pi}'$ inequivalent to $\tilde{\pi}$. Hence $\operatorname{tr} \tilde{\pi}(\tilde{f}) = 1$, and $\operatorname{tr} \tilde{\pi}'(\tilde{f}) = 0$ for such $\tilde{\pi}'$.

Such an $\tilde{f}$ is called a *pseudo-coefficient* of $\tilde{\pi}$.

Proof of Proposition 25. — The left side of (20.1) consists of a single term with square-integrable components. The right side contains at least one term, and all $\tilde{\pi}_v$ are square-integrable. Choose a $\tilde{\pi}'$ which appears, and let $\tilde{f}_v$ be a pseudo-coefficient of $\tilde{\pi}'_v$ for all v. With this choice of $\tilde{f}$, the right side of (20.1) becomes a sum of 1's. By the Theorem we have $\Phi''(\tilde{f}_v) = \tilde{\chi}'_v$ on the elliptic regular set (and 0 on the regular non-elliptic set). Theorem 13 implies that there exists a matching function f_v for all v. Then $\operatorname{tr} \pi_v(f_v) = \langle \chi_v, \chi'_v \rangle$, where χ'_v is the function on the elliptic set of G defined by $b\, \Delta\chi'_v(x) = \Delta\tilde{\chi}'_v(x^*)$. The Schwarz lemma implies that

$$|\langle \chi_v, \chi'_v \rangle|^2 \leqslant \langle \chi_v, \chi_v \rangle \langle \chi'_v, \chi'_v \rangle = \langle \tilde{\chi}'_v, \tilde{\chi}'_v \rangle = 1;$$

indeed, $\langle \chi, \chi \rangle = 1$ and $\langle \tilde{\chi}, \tilde{\chi} \rangle = 1$ for square-integrable π_v and $\tilde{\pi}_v$. Hence the left side of (20.1) is at most one, and the right side consists of a single term, as required.

Remark. — Since we have the equality $|\langle \chi_v, \chi'_v \rangle| = 1$, the Schwarz lemma implies that there exists a complex c with $|c| = 1$ and $\chi'_v = c\chi_v$ on the elliptic regular set.

26. Correspondence. — Let F be a local non-archimedean field. Recall (10) that by π, $\tilde{\pi}$ we mean representations of G, $\tilde{G}$ with central characters ω, $\tilde{\omega}$ related (in (6)) by $\omega(z) = \widetilde{\omega}(s(z^n))$. In particular (1) ω is trivial on the group of nth roots of unity in F, (2) $\widetilde{\pi}$ is *genuine*, namely the restriction of its central character to μ_n is injective.

Definition. — We say that $\widetilde{\pi}$ *corresponds*, or *lifts*, to π, if they satisfy the character relation

$$(\mathbf{26.1}) \qquad \Delta\chi(\widetilde{\pi})(x^*) = b\Sigma_{\{t \text{ in } T/Z;\ t^* z = x^*,\ z \text{ in } \widetilde{Z}\}} \widetilde{\omega}(z)\, \Delta\chi(\pi)(t)$$

for all x in G *with regular x^* in* $\widetilde{G}$, *or equivalently* $\operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ *for all matching* $f, \widetilde{f}$. *Here,* T *is the projection to* G *of the centralizer in* $\widetilde{G}$ *of* x^*.

Remark. — Recall that $(\Delta\chi(\widetilde{\pi}))(y) = 0$ if y is regular in $\widetilde{G}$ but $p(y)$ is not of the form $p(\widetilde{z})\, x^n$ for $\widetilde{z}$ in $\widetilde{Z}$, x in G.

Theorem. — *The correspondence defines a bijection from the set of square-integrable representations* $\widetilde{\pi}$ *of* $\widetilde{G}$ *to the set of square-integrable representations* π *of* G.

Proof. — Let F be a totally imaginary global field whose completion at w is our local field. Given a square-integrable π_u, we use Lemma 20 to construct π and π''. Proposition 25 and (20.1) imply that we obtain two equalities $\Pi\alpha_v = \Pi\beta_v$, where $\alpha_v = \operatorname{tr} \pi_v(f_v)$, and $\beta_v = \operatorname{tr} \widetilde{\pi}_v(\widetilde{f}_v)$. One ranges over V, the other over $V \cup \{w\}$. Hence $\alpha_w = \beta_w$, and $\widetilde{\pi}_w$ is as required.

Remark. — Had we argued that $\beta_v = 1$ $(v \neq w)$ at a pseudo-coefficient $\widetilde{f}_v$, we could as in Remark (25.1) conclude only that $|\alpha_v| = 1$, and that $c\alpha_w = \beta_w$ with $|c| = 1$.

To prove the opposite direction, namely that given a (local) $\widetilde{\pi}$ there exists a π as in the theorem, we can use Lemma 20, with (1) replaced by: π_w is a square-integrable with $\operatorname{tr} \pi_w(f_w) \neq 0$, where f_w is a function matching a pseudo-coefficient $\widetilde{f}_w$ of the given $\widetilde{\pi}_w$.

Remark. — Alternatively, we define a non-zero conjugacy class function

$$\chi(x) = b^{-1}(\chi(\widetilde{\pi}))(x^*)\, \Delta(x^n)/\Delta(x)$$

on the elliptic regular set of G. By the completeness of characters of representations of the anisotropic form of G with respect to the inner form $\langle\ ,\ \rangle$, and the correspondence of [DKV] (see [F'']), there exists a square-integrable π with $\langle \chi_\pi, \chi \rangle \neq 0$. But π corresponds to some $\widetilde{\pi}'$ as above, hence $\langle \widetilde{\chi}', \widetilde{\chi} \rangle \neq 0$, and $\widetilde{\pi}' \simeq \widetilde{\pi}$ by the orthogonality relations for characters of square-integrable representations.

The theorem follows, since the uniqueness of π or $\widetilde{\pi}$ in the theorem is clear from linear independence of characters.

Corollary. — *If* π *is supercuspidal then so is* $\widetilde{\pi}$.

This is obvious (14). But if $\widetilde{\pi}$ is supercuspidal the corresponding discrete series π need not be supercuspidal. As an example we take the case of $r = 2$ and even n, and consider the special representation σ in the composition series of the induced representation $I(\nu|\ |^{1/2}, \nu|\ |^{-1/2})$, where $\nu(\zeta) = -1$ for some ζ in μ_n. Since $\begin{pmatrix} a & 0 \\ 0 & \zeta b \end{pmatrix}^* = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^*$ it follows that the right side of (26.1) vanishes outside the maximal compact subgroup (modulo Z) of each torus; hence $\tilde{\pi}$ is supercuspidal (see (14)) which corresponds to a discrete series non-supercuspidal $\pi(=\sigma)$.

More generally, suppose that (1) $r = mr'$, (2) ρ is a square-integrable $GL(r', F)$-module, (3) the order of the restriction β of the central character of ρ to μ_n is divisible by m. Let $M = \Pi_i M_i$, $M_i = GL(r')$ $(1 \leqslant i \leqslant m)$, put $\nu(x) = |\det x|$. Write $I(\rho(\mathbf{s}))$ for the G-module induced from the M-module which is the product of the M_i-modules $\rho\nu^{((m+1)/2)-i}$. This $I(\rho(\mathbf{s}))$ has a unique square-integrable (called generalized Steinberg) subrepresentation $\sigma(\rho(\mathbf{s}))$. It is clear from (26.1), Theorem 14 and the Geometric Lemma (2.12) of [BZ], that the corresponding $\tilde{\sigma}$ is supercuspidal. Moreover, $I(\rho(\mathbf{s}))$ has a unique quotient (non-tempered) representation $\pi(\rho(\mathbf{s}))$, which is one-dimensional when ρ is one-dimensional (namely $r' = 1$), and the sum on the right of (26.1) for $\pi(\rho(\mathbf{s}))$ is equal to the sum for $\sigma(\rho(\mathbf{s}))$ perhaps up to a sign (which depends only on m and r').

26.2. Induction. — By a parabolic (resp. Levi) subgroup $\tilde{P}$ (resp. $\tilde{M}$) of $\tilde{G}$ we mean the pullback via $p : \tilde{G} \to G$ of such a subgroup P (resp. M) of G.

We say that the $\tilde{M}$-module $\tilde{\rho}$ lifts to an M-module ρ if (26.1) holds (for x in M). Suppose that $M = M_1 \times \ldots \times M_\ell$ with $M_i = GL(r_i)$. Thus we identify M_i with a subgroup of M. Each $\tilde{m}$ in $\tilde{M}$ can be written (not uniquely) in the form $\tilde{m}_1 \ldots \tilde{m}_\ell$ ($\tilde{m}_i$ in $\tilde{M}_i = p^{-1}(M_i)$).

Let B be a maximal subgroup of $F^\times$ with the property that $(b, b') = 1$ for all b, b' in $F^\times$. It contains $F^{\times n}$. Let $\tilde{M}_i^B$ be the group of m_i in $\tilde{M}_i$ with $\det p(\tilde{m}_i)$ in B, and $\tilde{M}^B$ the group of $\tilde{m}$ in $\tilde{M}$ with $\tilde{m}_i$ in $\tilde{M}_i^B$ for all i. Then $\tilde{m}_i \tilde{m}'_j = \tilde{m}'_j \tilde{m}_i$ for any $\tilde{m}, \tilde{m}'$ in $\tilde{M}^B$ and $i \neq j$, by (3). Let $\tilde{\rho}_i$ be an irreducible $\tilde{Z}\tilde{M}_i$-module which transforms under $\tilde{Z}$ by $\tilde{\omega}$. Its restriction $\tilde{\rho}_i''$ to $\tilde{Z}\tilde{M}_i^B$ is the sum of the conjugates $\tilde{\rho}_i'^m$ of some irreducible $\tilde{\rho}_i'$ by m in $\tilde{M}_i/\tilde{Z}\tilde{M}_i^B$. They are all inequivalent unless $M_i = GL(1)$. Let $\tilde{\rho}' = \otimes \tilde{\rho}_i'$ be their tensor product. It is irreducible. It follows from (3) that for any m in $\tilde{M} - \tilde{M}^B\tilde{Z}$ (we may assume that m is diagonal), $\tilde{\rho}'$ is not equivalent to its conjugate $\tilde{\rho}'^m$ by m. Hence the $\tilde{M}$-module $\tilde{\rho}$ induced from the $\tilde{M}^B\tilde{Z}$-module $\tilde{\rho}'$ is irreducible. It is clearly independent of the choice of the irreducible ρ_i'. Hence we write $\otimes \rho_i$ for $\tilde{\rho}$. The character of $\tilde{\rho}$ is supported on $\tilde{M}^n\tilde{Z}$, where $\tilde{M}^n$ is the group of $\tilde{m}$ in $\tilde{M}$ with $\det p(\tilde{m}_i)$ in $F^{\times n}$ for all i. It is a scalar multiple of the product of the characters of $\tilde{\rho}_i$, the scalar being independent of the $\tilde{\rho}_i$ (it depends on the M_i). Write $\tilde{\pi} = I(\tilde{\rho}) = I(\otimes \tilde{\rho}_i)$ for the $\tilde{G}$-module unitarily induced from $\tilde{\rho}$. A standard computation of a character of an induced representation (see [D]), easily adapted to the context of covering groups (see [F], p. 141, where, as noted in (11), lines 13, 14, -8, -7, -3 of page 141, and 2, 6, 7 of 142, should be multiplied by $t = [\tilde{A} : \tilde{A}_0]$ $(= bn^{r-1}$ of (24))), asserts

Proposition. — *If the irreducible* $\widetilde{Z}\widetilde{M}_i$*-modules* $\widetilde{\rho}_i$ *lift to the* ZM_i*-modules* ρ_i, *then* $I(\bigotimes \widetilde{\rho}_i)$ *lifts to* $I(\bigotimes \rho_i)$.

Proof. — As is clear from the construction of $\bigotimes \widetilde{\rho}_i$, we have that $I(\widetilde{\rho})$ lifts to $I(\rho)$ ($\widetilde{\rho} = \bigotimes \widetilde{\rho}_i$, $\rho = \bigotimes \rho_i$), at least up to a positive multiple which does not depend on ρ and $\widetilde{\rho}$. Since the proposition holds at least in the case of M being the diagonal subgroup A, it follows in general on applying induction in stages.

27. Tempered. — The bijection (26) of the set of discrete series $\widetilde{\pi}$ and the set of discrete series π (with central character trivial on μ_n), yields new results about the representations of the group $\widetilde{G}$.

Lemma. — *Suppose that the irreducible tempered representation* $\widetilde{\pi}$ *is elliptic. Then* $\widetilde{\pi}$ *is square-integrable.*

Proof. — Suppose the character $\widetilde{\chi}$ of $\widetilde{\pi}$ is non-zero at the elliptic regular element y^*. Let $\widetilde{f}$ be the characteristic function of a small neighborhood of y^* (modulo $\widetilde{Z}$), where $\widetilde{\chi}$ is constant. Then $\langle \widetilde{\chi}, \Phi''(\widetilde{f})\rangle \neq 0$. Since $\widetilde{f}$ is supported on the regular set, there is a matching f, with $F(x, f) = F(x^*, \widetilde{f})$. As follows from the completeness of characters on the anisotropic form of G, and the correspondence of [DKV] (see [F'']), there are finitely many discrete series π_i with characters χ_i, and complex numbers c_i, so that $\Phi''(x, f) = \Sigma_i c_i \chi_i(x)$ on the elliptic regular set. The correspondence of (26) implies that $\Phi''(x^*, \widetilde{f}) = \Sigma_i c_i \widetilde{\chi}_i(x^*)$, where $\widetilde{\chi}_i$ are the characters of the corresponding discrete series representations $\widetilde{\pi}_i$ of $\widetilde{G}$. Hence $\langle \widetilde{\chi}, \Phi''(\widetilde{f})\rangle = \Sigma_i c_i \langle \widetilde{\chi}, \widetilde{\chi}_i\rangle$. Since this is non-zero, we have $\langle \widetilde{\chi}_i, \widetilde{\chi}\rangle \neq 0$ for some i. But the orthogonality relations of Theorem 25.1 imply that $\widetilde{\pi}$ is equivalent to the discrete series $\widetilde{\pi}_i$, as required.

Proposition. — *Let* $\widetilde{\sigma}$ *be a square-integrable representation of a Levi subgroup* $\widetilde{M}$. *Then the (unitarily) induced representation* $\widetilde{\pi}'' = I(\widetilde{\sigma})$ *is irreducible.*

In particular, the same conclusion holds when $\widetilde{\sigma}$ is tempered.

Proof. — (1) We shall first show that $\widetilde{\pi}''$ is a multiple of an irreducible representation. By induction, we assume the assertion for all proper Levi subgroups of $\widetilde{G}$. Suppose that $\widetilde{\pi}''$ contains the irreducible representation $\widetilde{\pi}'$.

Suppose that $\widetilde{\pi}'$ does not lie in the space $R_I(\widetilde{G})$ of [K'], which is spanned over **C** by the properly induced representations. Then by Theorem D of [K'], $\widetilde{\pi}'$ is elliptic. Since it is also tempered, the Lemma implies that $\widetilde{\pi}'$ is square-integrable. The uniqueness Theorem (2.10) of [BW] implies that $\widetilde{\pi}' = \widetilde{\sigma}$, $\widetilde{M} = \widetilde{G}$ and so $\widetilde{\pi}' = \widetilde{\pi}''$, as required.

Suppose that $\widetilde{\pi}'$ does lie in $R_I(\widetilde{G})$. Then by Proposition 1.1 of [K'], there are finitely many proper Levi subgroup $\widetilde{L}_i$, irreducible tempered $\widetilde{L}_i$-modules $\widetilde{\rho}_i$, and complex numbers α_i, so that in the Grothendieck group $R(\widetilde{G})$ we have $\widetilde{\pi}' = \Sigma_i \alpha_i I(\widetilde{\rho}_i; \widetilde{G}, \widetilde{L}_i)$. Since $\widetilde{\rho}_i$ is tempered, there is a unique (up to conjugacy) pair $(\widetilde{\sigma}_i, \widetilde{R}_i)$, where $\widetilde{R}_i$ is a Levi subgroup of $\widetilde{L}_i$ and $\widetilde{\sigma}_i$ is a square-integrable $\widetilde{R}_i$-module, such that $\widetilde{\rho}_i$ is a direct summand of $I(\widetilde{\sigma}_i; \widetilde{L}_i, \widetilde{R}_i)$. Since $\widetilde{L}_i \neq \widetilde{G}$, the induction assumption implies that there is a positive integer β_i^{-1}, such that $I(\widetilde{\sigma}_i; \widetilde{L}_i, \widetilde{R}_i) = \beta_i^{-1} \widetilde{\rho}_i$. Hence $\pi' = \Sigma_i \alpha_i \beta_i I(\widetilde{\sigma}_i; \widetilde{G}, \widetilde{R}_i)$. The uniqueness of $\widetilde{\sigma}$ implies that either $\widetilde{\pi}'' = I(\widetilde{\sigma}_i; \widetilde{G}, \widetilde{R}_i)$, or $\widetilde{\pi}''$ and $I(\widetilde{\sigma}_i)$ are not relatives (in the terminology of [K']). Hence $\widetilde{\pi}' = \alpha\widetilde{\pi}''$, where α is the sum of $\alpha_i \beta_i$ over the i where $\widetilde{\pi}'' = I(\widetilde{\sigma}_i)$, as required.

(2) It remains to show that $I(\widetilde{\sigma})$ is in fact irreducible. For that we use the work of [S] which is stated for a connected reductive algebraic group, but whose proofs hold for the metaplectic group as well. First we consider the case where $M = p(\widetilde{M})$ is of rank one, thus $M = M' \times M''$, with $M' = GL(a)$, $M'' = GL(b)$. Then [S], Theorem 2.5.8 (p. 99), implies that $I(\widetilde{\sigma})$ is irreducible unless $a = b$, in which case its composition series has length bounded by the order of the Weyl group $W(A)$ of [S], p. 100, which is two. But if $I(\widetilde{\sigma})$ is the direct sum of k copies of an irreducible, its commuting algebra has dimension k^2, which is at least four, unless $k = 1$.

(3) Next we consider the general case. We shall express the $\widetilde{M}$-module $\widetilde{\sigma}$ as a product of square-integrables in the sense of (26.2). We may, upon rearranging the factors, assume that $\widetilde{\sigma} = (\widetilde{\sigma}_1 \times \dots \times \widetilde{\sigma}_1) \times (\widetilde{\sigma}_2 \times \dots \times \widetilde{\sigma}_2) \times \dots$, where each square-integrable $\widetilde{M}_i$-module $\widetilde{\sigma}_i$ $(M_i = GL(r_i))$ occurs t_i times, and $\widetilde{\sigma}_i, \widetilde{\sigma}_j$ are inequivalent if $i \neq j$. Then $r = \Sigma_i r_i t_i$. Put $t_0 = 0$. The center of $M = (M_1 \times \dots \times M_1) \times (M_2 \times \dots) \times \dots$ is $A = A_1^{t_1} \times A_2^{t_2} \times \dots$. Let $W(A)$ be the product $S_{t_1} \times S_{t_2} \times \dots$ of the symmetric groups S_{t_i} on t_i letters. The Harish-Chandra commuting algebra theorem ([S], §5.5.3) asserts that the commuting algebra of $I(\widetilde{\sigma})$ is spanned by the intertwining operators $R(w)$ (w in $W(A)$), subject to the relations $R(1) = 1$, $R(w) R(w') = R(ww')$. Hence it is generated by the $R(s(i))$, where $s(i)$ is a reflection of the form $(i, i+1)$ $(t_{j-1} < i \leqslant t_j)$. However, the operator $R(s(i))$ is induced (recall the induction is a functor) from the intertwining operator of the representation induced from $\widetilde{\sigma}$ on $\widetilde{M}$ $(M = M_1' \times \dots \times M_m')$ to $\widetilde{G}_i = p^{-1} G_i$, where G_i is $M_1' \times \dots \times M_{i-1}' \times X_i \times M_{i+2}' \times \dots \times M_m'$ and $X_i = GL(2r_j)$ if $M_i' = GL(r_j)$. It follows from the rank one case considered in (2) that $R(s(i))$ is a scalar. Hence the commuting algebra of $I(\widetilde{\sigma})$ consists of scalars, which proves that $I(\widetilde{\sigma})$ is irreducible, as required.

Remark. — (1) We do not discuss here the question of normalization of intertwining operators; see [KP], Theorem 1.2.6, for a special case.

(2) It is possible to complete the proof of irreducibility above, on further analyzing the proof of Chapter II of [S], without using the commuting algebra theorem of Chapter V, but we do not do it here.

27.1. Irreducibility. — Let $\widetilde{\rho}_i$ be supercuspidal $\widetilde{M}_i$-modules, where

$$M_i = p(\widetilde{M}_i) = GL(r_i), \quad \sum_{i=1}^{m} r_i = r;$$

ν is the character $\nu(\widetilde{x}) = |\det p(\widetilde{x})|^{1/n}$ of $\widetilde{M}_i$; s_i are real numbers; $\widetilde{I} = I(\widetilde{\rho}(\mathbf{s}))$ is the $\widetilde{G}$-module obtained by induction from the $\widetilde{M} = p^{-1}(\Pi_i M_i)$-module $\prod_i \widetilde{\rho}_i \nu^{s_i}$ constructed in (26.2). If P is a parabolic with Levi M, then the Jacquet module $\widetilde{I}_{\widetilde{P}}$ of $\widetilde{I}$ with respect to $\widetilde{P}$ consists (by [BZ], (2.12)) of composition factors of the form $\widetilde{\rho}_\sigma(\mathbf{s}) = \prod_i \widetilde{\rho}_{\sigma(i)} \nu^{s\sigma(i)}$, where σ ranges over the symmetric group S_m on m letters.

Definition. — (1) If $\widetilde{\pi}$ is a subquotient of $\widetilde{I}$, then its *support* is the set of $\widetilde{M}$-modules $\widetilde{\rho}_\sigma(\mathbf{s})$ which are constituents of $\widetilde{\pi}_{\widetilde{P}}$.

(2) $\widetilde{\pi}$ is called *multiplicity free* if each $\widetilde{\rho}_\sigma(\mathbf{s})$ occurs in $\widetilde{\pi}_{\widetilde{P}}$ at most once.

(3) A reflection in S_m of the form $s(i) = (i, i+1)$ is called *admissible* if $|s_{i+1} - s_i| \neq 1$ or $\widetilde{\rho}_{i+1}$ is inequivalent to $\widetilde{\rho}_i$. This term depends on $\widetilde{\rho}(\mathbf{s})$.

Lemma. — (1) *If* $m = 2$, *and* $|s_1 - s_2| \neq 1$ *or* $\widetilde{\rho}_1, \widetilde{\rho}_2$ *are inequivalent, then* $\widetilde{I} = I(\widetilde{\rho}_1 \nu^{s_1} \times \widetilde{\rho}_2 \nu^{s_2})$ *is irreducible.*

(2) *The support of* $\widetilde{\pi}$ *is invariant under the action of the set of admissible reflections.*

Proof. — (1) By Proposition 27, which deals with the tempered case $s_1 = s_2$, we may assume that $s_1 \neq s_2$, hence that $s_1 > s_2$ without loss of generality. The Jacquet module of $\widetilde{I}$ with respect to the parabolic of type (r_1, r_2) has two exponents, one increasing and one decaying. If $\widetilde{I}$ is reducible then its composition series has length two (by [S], Theorem 2.5.8, since the $\widetilde{\rho}_i$ are supercuspidal). One of the constituents has only decaying exponents, hence it is square-integrable by Harish-Chandra's criterion ([S], (4.4.4); [C'], (4.4.6)), quoted prior to Proposition 21. But this square-integrable should lift by Theorem 26 to a square-integrable constituent of the lift $I = I(\rho_1 \nu^{s_1} \times \rho_2 \nu^{s_2})$ of $\widetilde{I}$. As I is irreducible (by [BZ]), (1) follows.

(2) Suppose that $\widetilde{\rho}(\mathbf{s})$ lies in the support of $\widetilde{\pi}$; we have to show that so does $\widetilde{\rho}_{s(i)}(\mathbf{s}) = \widetilde{\rho}_1 \nu^{s_1} \times \ldots \times \widetilde{\rho}_{i+1} \nu^{s_{i+1}} \times \widetilde{\rho}_i \nu^{s_i} \times \ldots$. For that we consider the parabolic $\widetilde{Q}$ of type $(r_1, \ldots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \ldots)$, and its standard Levi subgroup $\widetilde{L}$. Since $\widetilde{\pi}_{\widetilde{P}} = (\widetilde{\pi}_{\widetilde{Q}})_{\widetilde{L} \cap \widetilde{P}}$, there is an irreducible $\widetilde{L}$-module $\widetilde{\tau}$ in the composition series of $\widetilde{\pi}_{\widetilde{Q}}$ such that $\widetilde{\tau}_{\widetilde{L} \cap \widetilde{P}}$ contains $\widetilde{\rho}(\mathbf{s})$. But part (1) implies that if $|s_{i+1} - s_i| \neq 1$ or $\widetilde{\rho}_i, \widetilde{\rho}_{i+1}$ are inequivalent, then $\widetilde{\tau}_{\widetilde{L} \cap \widetilde{P}}$ contains also $\widetilde{\rho}_{s(i)}(\mathbf{s})$, and (2) follows.

Proposition. — (1) *If* $\widetilde{\pi}$ *is multiplicity free, and the set of admissible transpositions acts transitively on the support of* $\widetilde{\pi}$, *then* $\widetilde{\pi}$ *is irreducible.*

(2) *Suppose that* $\widetilde{\sigma}'$ *and* $\widetilde{\sigma}''$ *are square-integrable, and* $|s| < 1/2$. *Then*

$$\widetilde{I} = I(\widetilde{\sigma}' \nu^s \times \widetilde{\sigma}'' \nu^{-s})$$

is irreducible.

Remark. — (2) here sharpens (1) of the lemma.

Proof. — (1) This is clear by (2) of the lemma, and the fact that each subquotient of $\widetilde{\pi}$ has a non-zero subquotient $\widetilde{\rho}_\sigma(\mathbf{s})$ in its Jacquet module $\widetilde{\pi}_{\widetilde{\mathrm{P}}}$.

(2) By Theorem 26, and the results of [BZ] for G, it follows that there exist super-cuspidal $\widetilde{\rho}'$ and $\widetilde{\rho}''$, which correspond to square-integrable $\mathrm{GL}(r')$- and $\mathrm{GL}(r'')$-modules, where $r = m'r' + m''r''$, such that the support of $\widetilde{\mathrm{I}}$ consists of all $m' + m''$ tuples $(a_{\sigma(i)})$ obtained from

$$(*) \qquad (a_i) = \left(\left(\widetilde{\rho}', \frac{m'-1}{2} + s\right), \left(\widetilde{\rho}', \frac{m'-3}{2} + s\right), \ldots, \left(\widetilde{\rho}', \frac{1-m'}{2} + s\right); \left(\widetilde{\rho}'', \frac{m''-1}{2} - s\right), \ldots\right),$$

where we put $(\widetilde{\rho}', s)$ for $\widetilde{\rho}'\nu^s$, on permuting by σ in $S_{m'+m''}$ which satisfies $\sigma(i) < \sigma(j)$ if $i < j \leqslant m'$ or $m' < i < j$. We may assume that $s \neq 0$ by Proposition 27. This set is multiplicity free, and the set of admissible transpositions act transitively if (1) $\widetilde{\rho}'$ is inequivalent to $\widetilde{\rho}''$; or, when $\widetilde{\rho}' = \widetilde{\rho}''$, if (2) $m' - m''$ is even, as $2|s| < 1$; or (3) $m' - m''$ is odd, unless $|s| = 1/4$. Hence the proposition follows from part (1), except that we have to deal with the case where $\widetilde{\rho}' = \widetilde{\rho}''$ (and $m' - m''$ is odd, $|s| = 1/4$). In this case we use the notation (*) for vectors in the support, omitting the reference to $\widetilde{\rho}', \widetilde{\rho}''$; namely from now on we deal with the case $\widetilde{\rho}' = \widetilde{\rho}''$.

By a segment we mean a vector (c_i) of real numbers with $c_i - c_{i+1} = 1$ for all i. The center of the segment $(c_1, \ldots, c_m)$ is $(c_1 - c_m)/2$. The vector (c_i) is called an L-vector if it has a partition $(\mathbf{b}_j)$ into segments $\mathbf{b}_j = (b_{ij})$ whose centers are non-decreasing. The description of tempered representations of G by [BZ], transferred to $\widetilde{\mathrm{G}}$ by Theorem 26 and Proposition 27, together with the classification theorems of [BW], IV, § 2, asserts that each irreducible $\widetilde{\mathrm{G}}$-module has (at least one) L-vector in its support. But it is easy to check that the support of our $\widetilde{\mathrm{I}}$, namely the set of $(a_{\sigma(i)})$ obtained from the (a_i) of (*), contains only one L-vector. Hence $\widetilde{\mathrm{I}}$ is irreducible, as required.

Corollary. — *Given any irreducible* $\widetilde{\rho}', \widetilde{\rho}''$, *and* $\widetilde{\sigma}', \widetilde{\sigma}'', s$ *as in* (2) *of the proposition,* $\mathrm{I}(\widetilde{\rho}' \times \widetilde{\sigma}'\nu^s \times \widetilde{\sigma}''\nu^{-s} \times \widetilde{\rho}'')$ *is equal to* $\mathrm{I}(\widetilde{\rho}' \times \widetilde{\sigma}''\nu^{-s} \times \widetilde{\sigma}'\nu^s \times \widetilde{\rho}'')$. *In particular, one of them is unitarizable if and only if so is the other.*

Proof. — This follows by induction in stages, since $\mathrm{I}(\widetilde{\sigma}'\nu^s \times \widetilde{\sigma}''\nu^{-s})$ is irreducible, hence equal to $\mathrm{I}(\widetilde{\sigma}''\nu^{-s} \times \widetilde{\sigma}'\nu^s)$.

27.2. Unitarity. — Let U be a finite dimensional complex vector space.

Lemma. — *Let* $\langle \cdot, \cdot \rangle_s$ *be a family of non-degenerate Hermitian forms on* U *depending continuously on a parameter* s *in a connected set. If* $\langle \cdot, \cdot \rangle_s$ *is positive definite for some value of* s, *then it is positive definite for all* s.

Proof. — The set of s where $\langle\cdot,\cdot\rangle_s$ is positive definite is clearly open and closed.

Proposition. — *The* $\widetilde{G}$*-modules* $\widetilde{I}(s) = I(\widetilde{\sigma}\nu^s \times \widetilde{\sigma}\nu^{-s})$ *of* 27.1 (2) *are unitary.*

Remark. — For brevity we say unitary for unitarizable.

Proof. — Let V_s be the space of the representation $\widetilde{I}(s)$. As a space V_s is independent of s, but the action of $\widetilde{G}$ does depend on s. Since $\widetilde{I}(s)$ is irreducible, it is equivalent to its contragredient $\widetilde{I}(s)' = I(\widetilde{\sigma}\nu^{-s}, \widetilde{\sigma}\nu^s)$. The choice of an isomorphism $\widetilde{I}(s) \to \widetilde{I}(s)' = \widetilde{I}(-s)$, which is unique up to a scalar, determines an Hermitian inner product $\langle\cdot,\cdot\rangle_s$ on V_s which is non-degenerate. We can choose the isomorphism, or the inner product $\langle\cdot,\cdot\rangle_s$, to vary continuously with s. For each compact open congruence subgroup C in $K \simeq K^*$, the isomorphism $V_s \to V_{-s}$ determines an isomorphism from the space V_s^C of C-fixed vectors in V_s, to the dual $(V_s^C)' = (V_s')^C = V_{-s}^C$. For each C we obtain a continuous family $\langle\cdot,\cdot\rangle_{s,C}$ of non-degenerate Hermitian inner products, which varies continuously with the parameter s in $-1/2 < s < 1/2$.

Now the tempered $\widetilde{I}(0)$ is unitary, being unitarily induced from a unitary representation $\widetilde{\sigma} \times \widetilde{\sigma}$. Hence $\langle\cdot,\cdot\rangle_{s,C}$ is positive-definite at $s = 0$. Consequently it is positive-definite for all s $(-1/2 < s < 1/2)$ and for all C (by the Lemma). As V_s is the union of V_s^C over all C, we conclude that $\langle\cdot,\cdot\rangle_s$ is positive-definite for all s, hence $\widetilde{I}(s)$ is unitary, as required.

Corollary. — *Let* $\widetilde{\sigma}_i$ $(1 \leqslant i \leqslant m)$ *be square-integrable, let* s_i $(1 \leqslant i \leqslant k; k \leqslant m)$ *be positive numbers with* $s_i < 1/2$, *and let* $\widetilde{\sigma}$ *denote the product* $\prod_{i=1}^{m} (\widetilde{\sigma}_i \nu^{s_i} \times \widetilde{\sigma}_i \nu^{-s_i}) \times \prod_{j=m+1}^{k} \widetilde{\sigma}_j$ *in the sense of* (26.2). *Then* $\widetilde{I} = I(\widetilde{\sigma})$ *is unitary for any choice of a parabolic subgroup or, equivalently, for any order of the factors* $\widetilde{\sigma}_i \nu^{s_i}, \widetilde{\sigma}_i \nu^{-s_i}, \widetilde{\sigma}_j$.

Proof. — $\widetilde{I}$ is independent of the choice of order of the factors by Corollary 27.1. Let $M = \prod_{i=1}^{m} (M_i \times M_i) \times \prod_j M_j$ be the Levi subgroup from which we induce. If $M_i = GL(r_i)$, put $L_i = GL(2r_i)$, and $L = \prod_i L_i \times \prod_j M_j$. Since

$$I(\widetilde{\sigma}; \widetilde{G}, \widetilde{M}) = I(I(\widetilde{\sigma}; \widetilde{L}, \widetilde{M}); \widetilde{G}, \widetilde{L}),$$

and $I(\widetilde{\sigma}; \widetilde{L}, \widetilde{M})$ is unitary by the Proposition, we conclude that $\widetilde{I}$ is unitary.

Theorem. — *Suppose that the* $\widetilde{\rho}_i$ $(0 \leqslant i \leqslant m)$ *are irreducible and tempered, and* s_i $(1 \leqslant i \leqslant m)$ *are distinct positive numbers with* $s_i < 1/2$. *Put* $\widetilde{\rho} = \prod_{i=1}^{m} (\widetilde{\rho}_i \nu^{s_i} \times \widetilde{\rho}_i \nu^{-s_i})$. *Then the induced representations* $\widetilde{I} = I(\widetilde{\rho})$ *and* $I(\widetilde{\rho}_0 \times \widetilde{\rho})$ *are irreducible.*

Proof. — We induce from the Levi $M = M_0 \times \prod (M_i \times M_i)$; here $M_0 = GL(r_0)$, $r_0 \geqslant 0$, and $r_0 = 0$ means that M_0 does not appear in M. There exists a parabolic subgroup $\widetilde{P}$ with Levi subgroup $\widetilde{M}$ such that the vector determined by $(0; s_1, -s_1; s_2, -s_2; \ldots)$ lies in the positive Weyl chamber (in the Lie algebra of the diagonal subgroup) determined by $\widetilde{P}$. Consequently $\widetilde{I}$ has a unique quotient $\widetilde{J}$ (see [BW], IV, (4.6), p. 127). On the other hand, the Corollary implies that $\widetilde{I}$ is unitary. As each constituent of a unitary representation is a direct summand of it, the unique quotient $\widetilde{J}$ has to be $\widetilde{I}$ itself, and we conclude that $\widetilde{I}$ is irreducible.

Definition. — A $\widetilde{G}$-module $\widetilde{\pi}$ is called *relevant* if it is equivalent to $I(\widetilde{\rho}_0 \times \widetilde{\rho})$ or $I(\widetilde{\rho})$ as in the Proposition. In the case $n = 1$ this definition applies to G.

We have just seen that the relevant representations are irreducible.

The motivation for this definition is the fact that each component of any cuspidal automorphic $G(\mathbf{A})$-module is unitary and non-degenerate, hence relevant by [B], Lemma 8.9, p. 94, and [Z], Theorem 9.7 (b).

27.3. We say that f is *good* if there exists an $\widetilde{f}$ matching f. An admissible π is called *metic* (= met(aplect)ic) if for each subquotient π' of π (not necessarily irreducible) there is a good f so that $\operatorname{tr} \pi'(f) \neq 0$. If π is induced from an irreducible elliptic representation $\rho = (\rho_i)$ of a Levi subgroup $M = \prod_i M_i$, $M_i = GL(r_i)$, then it is metic if and only if the central character of each ρ_i is trivial on μ_n.

Corollary 26.2 now implies that Theorem 26 extends to all relevant, in particular tempered, representations, from the case of square-integrables, by induction.

Theorem. — *The correspondence relation (see Definition (26)) defines a bijection between the set of genuine relevant $\widetilde{\pi}$ and the set of relevant metic π. It commutes with induction, bijects square-integrables with square-integrables and tempered with tempered.*

Proposition. — *Suppose that f and $\widetilde{f}$ satisfy $\operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ for all corresponding tempered π and $\widetilde{\pi}$, and $\operatorname{tr} \pi(f) = 0$ for the irreducible tempered π not obtained from any $\widetilde{\pi}$. Then f and $\widetilde{f}$ are matching.*

Proof. — Induction on the Levi subgroup M. If ρ, $\widetilde{\rho}$ are corresponding tempered representations of M, $\widetilde{M}$, we have $\operatorname{tr} \rho(f_M) = \operatorname{tr} \widetilde{\rho}(\widetilde{f}_M)$. Hence

$$F(x, f) = F^M(x, f_M) = F^M(x^*, \widetilde{f}_M) = F(x^*, \widetilde{f}),$$

for all x in M with regular x^*, by induction. It remains to establish this relation for elliptic x with regular x^*. Fix such a pair x, x^*. Let U be a sufficiently small compact neighborhood of y, and $\widetilde{f}'$ a function on $\widetilde{G}$ as in (6), supported near $y^* \widetilde{Z}$, whose orbital integral $'\Phi(\widetilde{f}')$ is the characteristic function of $\widetilde{Z}U^{*\widetilde{G}}$. Let f' be a matching function on G; it exists by Remark (i) following Theorem 13. Now $'\Phi(\widetilde{f}')$ is a finite linear combination of the characters of square-integrable $\widetilde{\pi}_i$ with coefficients c_i, by [K], Theorem K. Then $'\Phi(f')$ is the corresponding combination of the characters of the π_i which correspond to the $\widetilde{\pi}_i$ by Theorem 26. Since U^* is small, the Weyl integration formula implies that $\Sigma c_i \operatorname{tr} \widetilde{\pi}_i(\widetilde{f})$ is equal to $\int_{T/Z} F(t^*, \widetilde{f}') F(t^*, \widetilde{f})\, dt$, where T is the centralizer of y in G. The assumption of the proposition implies that $\Sigma c_i \operatorname{tr} \widetilde{\pi}_i(\widetilde{f})$ is equal to $\Sigma c_i \operatorname{tr} \pi_i(f)$. But this is $\int_{T/Z} F(t, f') F(t, f)\, dt$. We take U to be so small that both $F(t^*, \widetilde{f})$ and $F(t, f)$ are constant on U. The desired equality $F(x^*, \widetilde{f}) = F(x, f)$ now follows from the choice of f' and $\widetilde{f}'$, which guarantees that $F(t, f') = F(t^*, \widetilde{f}')$.

Corollary. — For each $\widetilde{f}$ there is a matching f.

Proof. — Given $\widetilde{f}$ we define the function F on the space of tempered π as follows. If π is a lift of a (tempered) $\widetilde{\pi}$, we put $F(\pi) = \operatorname{tr} \widetilde{\pi}(\widetilde{f})$. Otherwise we put $F(\pi) = 0$. This is a function in F_{good} (see (1.2) of [BDK]), hence a trace function by Theorem 1.3 of [BDK]. Namely, there is an f with $F(\pi) = \operatorname{tr} \pi(f)$ for all tempered π. The corollary now follows from the Proposition.

28. Global lifting. — From now on we denote by π and $\widetilde{\pi}$ (global) discrete series representations (see (18)), with elliptic components at the two places u, u'. We say that the genuine representation $\widetilde{\pi} = \otimes \widetilde{\pi}_v$ of $\widetilde{G}(\mathbf{A})$ *(quasi-)lifts* to $\pi = \otimes \pi_v$ on $G(\mathbf{A})$ if $\widetilde{\pi}_v$ corresponds (see (26)) to π_v for (almost) all places v. We say that π is *metic* if there exists a good $f = \otimes f_v$ (that is, with a matching $\widetilde{f} = \otimes \widetilde{f}_v$), so that $\operatorname{tr} \pi(f) \neq 0$. Namely, its components are all metic.

In what follows π is not required to be cuspidal. But when π does not have a supercuspidal component, we have to use the following

Assertion. — The conclusion of Theorem 18 remains valid under the same assumptions, except that $\widetilde{f}_u$ is not required anymore to be a supercusp form. This is likely to follow from work in progress of Arthur. If $\widetilde{f}$ has no supercusp component, the simple form of the left side of (18.1) holds for $\widetilde{G}$ by virtue of Theorem 27.3. It is not so for $SL(r)$.

Arthur expresses the right side, called the O-expansion, in terms of invariant distributions under no condition on $\widetilde{f}$. These invariant distributions are then expressed as sums of products of local invariant distributions. The assumption of Theorem 18 at u, u' implies the vanishing of all terms indexed by elements with non-elliptic semi-simple part. At the remaining elements we obtain orbital integrals of the local components $\widetilde{f}_v$. These vanish because of the assumption at u''.

The Assertion implies that Corollary 18 holds also when $\widetilde{f}$ and f do not have supercuspidal components.

Proposition. — For each metic π there exists a genuine $\widetilde{\pi}$ which quasi-lifts to π. If $\widetilde{\pi}$ has two supercuspidal components it quasi-lifts to a metic π.

Proof. — Given π, we use the identity (20.1), and the theorem of Jacquet-Shalika [JS] which asserts that if π appears on the left, then it is the only term there, up to multiplicity. If π is cuspidal, this is the rigidity (strong multiplicity one) theorem for cusp forms of G. This proves the existence of $\tilde{\pi}$, since the left side is non-zero.

In the opposite direction, given $\tilde{\pi}^0$ we form (20.1). If π does not exist then $\Sigma \Pi \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v) = 0$. At the two places where $\tilde{\pi}_v^0$ is supercuspidal we let $\tilde{f}_v$ be a matrix coefficient. It suffices to let $\tilde{f}_v$ be the characteristic function φ_{vj}^* of (10) at all other finite v, so that $\operatorname{tr} \tilde{\pi}_v(\tilde{f}_v)$ is a non-negative integer, to deduce a contradiction.

Theorem. — *If π is metic cuspidal with elliptic components at the three places u, u', u'', then there is a unique discret-series representation $\tilde{\pi}$ of $\tilde{G}(\mathbf{A})$ which lifts to π. Any discrete-series $\tilde{\pi}'$ which quasi-lifts to π is equal to $\tilde{\pi}$.*

Remark. — (1) Omitting the assumption at u'' we can conclude the existence of $\tilde{\pi}$ such that $\tilde{\pi}_v$ lifts to π_v for all $v \neq u, u'$.

(2) The last claim of the Theorem combines the multiplicity one theorem for $\tilde{G}(\mathbf{A})$ and the rigidity theorem for $\tilde{G}(\mathbf{A})$, at least for the $\tilde{\pi}$ which appear in the Theorem.

(3) Our proof generalizes to deal with all cuspidal π once an identity of trace formulae is available with no restriction on u, u'.

Proof. — We use the identity (20.1), where on the left appears π, with multiplicity 1. Since π is cuspidal, it has a Whittaker model, and its local components are all non-degenerate. Hence its components at u, u' are discrete series, and correspond to discrete series $\tilde{\pi}_v$, $v = u, u'$. Let $\tilde{f}_v$ be their pseudo-coefficients. With this choice of function at u, u', we need to show that there is an entry $\tilde{\pi}''$ on the right of (20.1) whose components $\tilde{\pi}_v''$ correspond to the components π_v of π at each $v \neq u, u'$ in V. For such v denote π_v by π'. It is non-degenerate. Hence, by [Z], Theorem 9.7 (b), it is equal to a representation $I(\rho' \nu^{s'})$ induced from a discrete series representation ρ' of a Levi subgroup M, tensored by an unramified quasi-character $\nu^{s'}$ of M. Being a component of an automorphic representation, π' is unitarizable. Hence Lemma 8.9 of [B], p. 94, implies that π' is relevant; see Definition 27.2. This fact motivated our study of relevant representations in (27.2). By Theorem 27.3 this relevant π' corresponds to the relevant representation $\tilde{\pi}' = I(\tilde{\rho}' \nu^{s'})$ induced from the product of the discrete series $\tilde{\rho}'$ which lifts to ρ', and the corresponding character.

According to Corollary 27, for any $\tilde{f}$ there is a matching f so that our $\pi', \tilde{\pi}'$ satisfy $\operatorname{tr} \pi'(f) = \operatorname{tr} \tilde{\pi}'(\tilde{f})$. This argument applies at all $v \neq u, u'$ in V. Hence (20.1) becomes

$$\Pi_v \operatorname{tr} \tilde{\pi}_v'(\tilde{f}_v) = \Sigma\, c(\tilde{\pi})\, \Pi_v \operatorname{tr} \tilde{\pi}_v(\tilde{f}_v).$$

The product is over all $v \neq u, u'$ in V and $c(\tilde{\pi})$ is a natural number. This holds for arbitrary $\{\tilde{f}_v; v \neq u, u' \text{ in } V\}$. The theorem now follows from linear independence of characters. Note that the component at u'' of each $\tilde{\pi}$ on the right must lift to $\pi_{u''}$. But u'' can be replaced by u and u'. This determines the components of $\tilde{\pi}''$ at all places, as required.

29. Duals. — We shall now discuss the local representations which are dual—in the sense of [Z]—to the square-integrable non-supercuspidal representations, and their global analogues. These include the one-dimensional representations. Fix a global field F, with places $v, w, \ldots$

Suppose $r = r'm$, and ρ is a cuspidal representation of $\mathrm{GL}(r', \mathbf{A})$ with a central character θ. We write $\rho(m)$ for the representation $\rho \times \ldots \times \rho$ (m times) of

$$M = \Pi_i M_i, \qquad M_i = \mathrm{GL}(r') \quad (1 \leqslant i \leqslant m).$$

If $\nu(x) = |\det x|$, and $\mathbf{s} = ((m-1)/2, (m-3)/2, \ldots, -(m-1)/2)$, write $\nu(\mathbf{s})$ for the product of the characters $\nu^{((m+1)/2)-i}$ of M_i, and $\rho(\mathbf{s})$ for $\rho(m) \otimes \nu(\mathbf{s})$. The automorphic induced representation $I(\rho(\mathbf{s}))$ of $G(\mathbf{A})$ has an irreducible quotient $\pi(\rho(\mathbf{s}))$, obtained as the image of the intertwining operator $T(w_0, X)$ of [J], Proposition, p. 189. This quotient $\pi(\rho(\mathbf{s}))$ is a discrete series representation ([J], end of (2.4), p. 191). If π is a discrete series representation with $\pi_v \simeq \pi(\rho(\mathbf{s}))_v$ for almost all v, then this holds for all v ([JS]). The local component $\pi(\rho(\mathbf{s}))_v$ is the unique irreducible quotient of $I(\rho_v(\mathbf{s}))$ ([J], Proposition, p. 189). When ρ_v is square-integrable, then $I(\rho_v(\mathbf{s}))$ has a square-integrable irreducible subrepresentation $\sigma(\rho_v(\mathbf{s}))$ ([Z]), which is dual, in the sense of [Z], to the quotient $\pi(\rho_v(\mathbf{s}))$. Note that we follow the convention of [J], where the components of $\mathbf{s} = (s_1, s_2, \ldots)$ satisfy $s_i - s_{i+1} = 1$. The convention of [Z] is the opposite: $s_{i+1} - s_i = 1$. Hence our sub (and quotient) are quotient (and sub) in [Z].

Suppose that the order of θ on μ_n is m' dividing m. Define $\mathbf{s}'$ by the expression for $\mathbf{s}$, with m' replacing m, and also $\mathbf{s}''$, with m'' replacing m. Here $m'm'' = m$. It is then clear that $\rho(\mathbf{s}) = \rho(\mathbf{s}')(m'') \otimes \nu(m'\mathbf{s}'')$. Also, $\pi(\rho(\mathbf{s}))$ is a quotient of the representation $I(\pi(\rho(\mathbf{s}'))(m'') \otimes \nu(m'\mathbf{s}''))$, induced to G from the Levi subgroup of type $(r'm', \ldots, r'm')$.

We deal first with the case of $m = m'$, or $m'' = 1$.

Theorem. — *Suppose ρ_w is square-integrable and its central character has order m on μ_n. Let σ_w be the square-integrable subrepresentation of $I(\rho_w(\mathbf{s}))$, and $\tilde{\sigma}_w$ the supercuspidal representation which lifts to σ_w; see the example following Corollary 26. Then the quotient $\pi_w = \pi(\rho_w(\mathbf{s}))$ satisfies $\operatorname{tr} \pi_w(f_w) = (-1)^{m-1} \operatorname{tr} \tilde{\sigma}_w(\tilde{f}_w)$ for all f_w with matching $\tilde{f}_w$.*

Remark. — It is possible to give a local proof of this claim, by expressing the character of σ_w as an alternating sum of certain induced representations analogous to π_w (the case $r' = 1$ is in [C'], § 8), and concluding that $\operatorname{tr} \pi_w(f_w) = (-1)^{m-1} \operatorname{tr} \sigma_w(f_w)$ for the f_w as above. Indeed, the properly induced representations in this sum are not metic, due to the requirement on the central character of ρ_w. The following proof is global.

Proof. — As in Lemma 20, we construct a cuspidal representation ρ, whose component at w is our ρ_w; whose archimedean components are spherical; whose components at two finite places u, u' are supercuspidal, with central character of order m on μ_n. Consider the trace identity (20.1), where the quotient $\pi(\rho(\mathbf{s}))$ is the unique term on the left (up to multiplicity). At u (and u') let f_u be a pseudo-coefficient of $\sigma_u = \sigma_u(\rho(\mathbf{s}))$, and $\widetilde{f}_u$ a (matching) pseudo-coefficient of the square-integrable $\widetilde{\sigma}_u$ which lifts to σ_u. It is clear (by considering the Jacquet modules) that $\widetilde{\sigma}_u$ is supercuspidal. Then $\operatorname{tr} \pi_u(f_u) = (-1)^{m-1}$, and $\operatorname{tr} \widetilde{\pi}_u(\widetilde{f}_u) = 1$ on the right side of (20.1). This eliminates u, u' from the set V of (20.1).

To eliminate the archimedean components of π, one can try to show that the archimedean components of π are lifts of $\widetilde{G}_v$-modules. For the special case of the one dimensional representation see [KP'], Proposition 5.9 (ii). But we do not do it here. Instead, we note that spherical functions at infinity can be matched ([KP'], proof of Prop. 5.9 (iii)), and use linear independence with respect to the Hecke algebra to fix the components at infinity. At the remaining finite $v \neq w$ in V we use the functions of (10) to obtain the identity (20.2). Again w is omitted, π is $\pi_w(\rho(\mathbf{s}))$, and the $c(\widetilde{\pi})$ are positive integers.

Now, the condition on the central character of ρ implies the following. The Jacquet module of π with respect to any proper Levi subgroup $M' = \prod_i M'_i$ consists of irreducibles $\rho' = (\rho'_i)$, such that the central character of ρ'_i is non-trivial on μ_n (all i). Hence the ρ' satisfy $\operatorname{tr} \rho'(f_{w,M'}) = 0$ for each f_u matching a $\widetilde{f}_w$.

As in (21) we observe that all $\widetilde{\pi}$ are supercuspidal. The proof of (25.1) implies that there is a single term in (20.1), and (20.2) reduces to $c \operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\sigma}(\widetilde{f})$, where c has absolute value 1. But $\widetilde{\sigma}$ is supercuspidal and lifts to a discrete-series representation σ of G. Hence $c \operatorname{tr} \pi(f) = \operatorname{tr} \sigma(f)$. Hence σ is the square-integrable constituent in the composition series of $I(\rho(\mathbf{s}))$, and comparing the characters of σ and π on the elliptic set we find that $c = (-1)^{m-1}$, as required.

The Theorem exhibits a special phenomenon. The unitary non-tempered local $\pi(\rho(\mathbf{s}))$ can be viewed as a lift, up to a sign if m is even, of a supercuspidal $\widetilde{\pi}$, which lifts, according to Theorem 27, to the discrete series $\sigma(\rho(\mathbf{s}))$ which is dual to $\pi(\rho(\mathbf{s}))$. Hence there are global metic representations $\pi(\rho(\mathbf{s}))$ which are discrete series but not cuspidal, which are quasi-lifts of cuspidal representations $\widetilde{\pi}$ with supercuspidal components. Moreover, almost all components of such $\widetilde{\pi}$ are non-tempered. Such $\widetilde{\pi}$ are cuspidal representations which do not satisfy the so called generalized Ramanujan conjecture.

Next we deal with the general local case, where $m = m' m''$. It has already been noted that $\pi(\rho(\mathbf{s}))$ is a quotient of $I(\pi(\rho(\mathbf{s}'))(m'') \otimes \nu(m' \mathbf{s}''))$. The Theorem implies that $\operatorname{tr} \pi(\rho(\mathbf{s}'))(f) = (-1)^{m'-1} \operatorname{tr} \sigma(\rho(\mathbf{s}'))(f)$ for f on $GL(r' m')$ matching to a $\widetilde{f}$.

Theorem 26 (see the example following Corollary 26) asserts that there is a supercuspidal $\widetilde{\sigma}(\rho(\mathbf{s}'))$ which lifts to $\sigma(\rho(\mathbf{s}'))$; Theorem 27 and Corollary 26.2 imply that $\mathrm{I} = \mathrm{I}(\sigma(\rho(\mathbf{s}'))(m'') \otimes \nu(m'\,\mathbf{s}''))$ corresponds to the representation $\widetilde{\mathrm{I}}$ induced from $\widetilde{\sigma}(\rho(\mathbf{s}'))(m'') \otimes \widetilde{\nu}(m'\,\mathbf{s}'')$ on the Levi subgroup of type $(r'\,m', \ldots, r'\,m')$. Moreover, I has the square-integrable subrepresentation $\sigma(\rho(\mathbf{s}))$, and this is a lift of the square-integrable $\widetilde{\sigma} = \widetilde{\sigma}(\rho(\mathbf{s}))$. Comparing the Jacquet modules of $\widetilde{\sigma}$ and $\sigma(\rho(\mathbf{s}))$, we conclude that $\widetilde{\sigma}$ is the square-integrable subquotient of $\widetilde{\mathrm{I}}$. Put $\iota = (-1)^{(m'-1)m''}$. For simplicity we now assume that ρ is supercuspidal, and m divides n.

Theorem (**29.1**). — *The representation* $\widetilde{\mathrm{I}}$ *has a subquotient* $\widetilde{\pi}(\sigma(\rho(\mathbf{s})))$, *which satisfies* $\operatorname{tr} \pi(\rho(\mathbf{s}))(f) = \iota \operatorname{tr} \widetilde{\pi}(\sigma(\rho(\mathbf{s})))(\widetilde{f})$ *for matching* f *and* $\widetilde{f}$.

Proof. — We consider our local representation $\pi(\rho(\mathbf{s}))$ as the component at w o a global representation $\pi(\rho(\mathbf{s}))$, constructed using a cuspidal representation ρ whose component at w is our local ρ_w. Our assumption is that the order of the restriction to μ_n of the central character θ_w of ρ_w is m'. Note that we can construct ρ so that at two other places $v = u, u'$, the component ρ_v is square-integrable with central character θ_v whose restriction to μ_n has order m. Then $\pi(\rho_v(\mathbf{s}))$ corresponds, in the sense of Theorem 29, to supercuspidal $\widetilde{\sigma}_v$ for $v = u, u'$. We apply the trace formula (20.1) with the global $\pi(\rho(\mathbf{s}))$, and take $\widetilde{f}_v$ to be a coefficient of $\widetilde{\sigma}_v$ at $v = u, u'$, to obtain the identity (20.2), with positive $c(\widetilde{\pi})$. We now return to local notations.

Let M′ be the Levi subgroup of type $(r'\,m', \ldots, r'\,m')$. Consider matching f and $\widetilde{f}$ with $\mathrm{F}(x, f) = 0$ unless, up to conjugacy, M_x of (14) (and [C]) is contained in M′. We conclude from (20.2) and Theorem 14 the identity $c \operatorname{tr} \pi_{\mathrm{N}'}(f_{\mathrm{N}'}) = \Sigma c(\widetilde{\pi}) \operatorname{tr} \widetilde{\pi}_{\mathrm{N}'}(\widetilde{f}_{\mathrm{N}'})$, where N′ is the unipotent radical of the standard parabolic with Levi subgroup M′. But the Jacquet module $\pi_{\mathrm{N}'}$ is the representation $\pi(\rho(\mathbf{s}'))(m'') \otimes \nu(m'\,\mathbf{s}'')$. Denote by $\widetilde{\sigma}(\rho(\mathbf{s}'))$ the supercuspidal representation matching $\pi(\rho(\mathbf{s}'))$ in the sense of Theorem 29. Since the function $\widetilde{f}_{\mathrm{N}'}$ is an arbitrary function on $\widetilde{\mathrm{M}}'$ (see the proof of Proposition 21), we conclude by linear independence of characters on $\widetilde{\mathrm{M}}'$, that, in (20.2), there appears a single irreducible representation $\widetilde{\pi}''$ whose Jacquet module with respect to M′ is non-zero. Further, $\pi = \pi(\rho(\mathbf{s}))$ satisfies $\iota \operatorname{tr} \pi_{\mathrm{N}'}(f_{\mathrm{N}'}) = \operatorname{tr} \widetilde{\pi}''_{\mathrm{N}'}(\widetilde{f}_{\mathrm{N}'})$ (we have m'' factors, each yielding $(-1)^{m'-1}$ by Theorem 29). In particular, $\widetilde{\pi}''$ is a subquotient of $\widetilde{\mathrm{I}}$.

By induction, we assume that the relation $\iota \operatorname{tr} \pi_{\mathrm{N}'}(f_{\mathrm{N}'}) = \operatorname{tr} \widetilde{\pi}''_{\mathrm{N}'}(\widetilde{f}_{\mathrm{N}'})$ holds for all $\mathrm{M}' \neq \mathrm{G}$. Hence we have $\iota \operatorname{tr} \pi(f) = \operatorname{tr} \widetilde{\pi}''(\widetilde{f}) + \Sigma c(\widetilde{\pi}) \operatorname{tr} \widetilde{\pi}(\widetilde{f})$ for any matching $f, \widetilde{f}$, where the $\widetilde{\pi}$ are all supercuspidal. Choose $\widetilde{f}$ to be a coefficient of one of the $\widetilde{\pi}$. For a matching f we have $\operatorname{tr} \sigma(f) = 0$, where σ is the square-integrable quotient $\sigma(\rho(\mathbf{s}))$ determined by $\pi = \pi(\rho(\mathbf{s}))$, so that the characters of σ and π are equal on the elliptic regular set up to a sign. Indeed, suppose $\operatorname{tr} \sigma(f) \neq 0$. Since σ is the lift of a square-integrable, necessarily a subquotient $\tilde{\sigma}$ of $\tilde{I}$ by (26.1) and Frobenius reciprocity, we obtain by the orthogonality relations that $\tilde{\sigma}$ is equivalent to the supercuspidal $\tilde{\pi}$, which is impossible. Hence $c(\tilde{\pi}) = 0$ for the supercuspidal $\tilde{\pi}$, as required.

It is clear that the above local Theorem can be used as in (28) to obtain global lifting results. But we do not elaborate on this.

Finally, note that in the case of $r = 2$ the local Theorem(s) 26(29) give(s) a complete description of the local correspondence. In this case the full trace formulae for G and $\tilde{G}$ are easily computed (as in [F], but note that in (5) of [F], p. 159, there should appear a sum over x in $F^{\times n/2}/F^{\times n/d}$ (with $d = 1$ since $m = 0$ in [F]), and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has to be replaced by its multiple by x). They can be compared (with no restriction on f_u) due to the explicit comparison of weighted orbital integrals for matching spherical functions in [F], p. 170. The complicated regularity argument of [F], p. 160, can be replaced by the correction argument of [F′], p. 59. The local results and the identity of trace formulae for arbitrary $f, \tilde{f}$ imply at once the full global correspondence in the case of GL(2) ([F], § 5).

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