A SIMPLE TRACE FORMULA[†]

By

YUVAL Z. FLICKER AND DAVID A. KAZHDAN

Department of Mathematics, Harvard University,
Science Center, One Oxford Street, Cambridge, MA 02138, USA

Abstract. The Selberg trace formula is of unquestionable value for the study of automorphic forms and related objects. In principal it is a simple and natural formula, generalizing the Poisson summation formula, relating traces of convolution operators with orbital integrals. This paper is motivated by the belief that such a fundamental and natural relation should admit a *simple and short* proof. This is accomplished here for test functions with a single supercusp-component, and another component which is spherical and "sufficiently-admissible" with respect to the other components. The resulting trace formula is then used to sharpen and extend the metaplectic correspondence, and the simple algebras correspondence, of automorphic representations, to the context of automorphic forms with a *single* supercuspidal component, over any global field. It will be interesting to extend these theorems to the context of all automorphic forms by means of a simple proof. Previously a simple form of the trace formula was known for test functions with two supercusp components; this was used to establish these correspondences for automorphic forms with two supercuspidal components. The notion of "sufficiently-admissible" spherical functions has its origins in Drinfeld's study of the reciprocity law for GL(2) over a function field, and our form of the trace formula is analogous to Deligne's conjecture on the fixed point formula in étale cohomology, for a correspondence which is multiplied by a sufficiently high power of the Frobenius, on a separated scheme of finite type over a finite field. Our trace formula can be used (see [FK′]) to prove the Ramanujan conjecture for automorphic forms with a supercuspidal component on GL(n) over a function field, and to reduce the reciprocity law for such forms to Deligne's conjecture. Similar techniques are used in ['F] to establish base change for GL(n) in the context of automorphic forms with a single supercuspidal component. They can be used to give short and simple proofs of rank one lifting theorems for *arbitrary* automorphic forms; see ["F] for base change for GL(2), [F′] for base change for $U(3)$, and ['F′] for the symmetric square lifting from SL(2) to PGL(3).

Let F be a global field, $\mathbf{A}$ its ring of adeles and $\mathbf{A}_f$ the ring of finite adeles, $\boldsymbol{G}$ a connected reductive algebraic group over F with center $\boldsymbol{Z}$. The group G of F-rational points on $\boldsymbol{G}$ is discrete in the adele group $G(\mathbf{A})$ of $\boldsymbol{G}$. Put $G' = G/Z$ and $G'(\mathbf{A}) = G(\mathbf{A})/Z(\mathbf{A})$. The quotient $G' \backslash G'(\mathbf{A})$ has finite volume with respect to the unique (up to a scalar multiple) Haar measure dg on $G'(\mathbf{A})$. Fix a *unitary* complex-valued character ω of $Z \backslash Z(\mathbf{A})$. For any place v of F let F_v be the completion of F at v, and $G_v = G(F_v)$ the group of F_v-points on $\boldsymbol{G}$. If F_v is non-archimedean, let R_v denote its ring of integers. For almost all v the group G_v is defined over R_v, quasi-split over F_v, split over an unramified extension of F_v, and

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$K_v = G(R_v)$ is a maximal compact subgroup. For an infinite set of places (of positive density) u of F, the group G_u is split (over F_u). A fundamental system of open neighbourhoods of 1 in $G(\mathbf{A})$ consists of the set $\Pi_{v\in V} H_v \times \Pi_{v\notin V} K_v$, where V is a finite set of places of F and H_v is an open subset of G_v, containing 1.

Let $L(G)$ denote the space of all complex-valued functions ϕ on $G \backslash G(\mathbf{A})$ which satisfy (1) $\phi(zg) = \omega(z)\phi(g)$ (z in $Z(\mathbf{A})$, g in $G(\mathbf{A})$), (2) ϕ is absolutely square-integrable on $G' \backslash G'(\mathbf{A})$. $G(\mathbf{A})$ acts on $L(G)$ by right translation: $(r(g)\phi)(h) = \phi(hg)$; $L(G)$ is unitary since ω is unitary. The function ϕ in $L(G)$ is called *cuspidal* if for each proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ over F with unipotent radical $\mathbf{N}$ we have $\int \phi(ng)dn = 0$ (n in $N \backslash N(\mathbf{A})$) for any g in $G(\mathbf{A})$. Let $L_0(G)$ denote the space of cuspidal functions in $L(G)$, and r_0 the restriction of r to $L_0(G)$. The space $L_0(G)$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $G(\mathbf{A})$-modules called *cuspidal* G-modules.

Let f be a complex-valued function on $G(\mathbf{A})$ with $f(g) = \omega(z)f(zg)$ for z in $Z(\mathbf{A})$, which is supported on the product of $Z(\mathbf{A})$ and a compact open neighborhood of 1 in $G(\mathbf{A})$, smooth as a function on the archimedean part $G(F_\infty)$ of $G(\mathbf{A})$, and bi-invariant by an open compact subgroup of $G(\mathbf{A}_f)$. Fix Haar measures dg_v on $G'_v = G_v/Z_v$ for all v, such that the product of the volumes $|K_v/Z_v \cap K_v|$ converges, Then $dg = \otimes\, dg_v$ is a measure on $G'(\mathbf{A})$. The convolution operator $r_0(f) = \int_{G'(\mathbf{A})} f(g)r_0(g)dg$ is of trace class; its trace is denoted by $\operatorname{tr} r_0(f)$. Then

$$\text{(1)} \qquad \operatorname{tr} r_0(f) = \sum{}' m(\pi) \operatorname{tr} \pi(f),$$

where Σ' indicates the sum over all equivalence classes of cuspidal representations π of $G(\mathbf{A})$, and $m(\pi)$ denotes the multiplicity of π in $L_0(G)$; each π here is unitary, and the sum is absolutely convergent.

The Selberg trace formula is an alternative expression for (1). To introduce it we recall the following

Definitions. Denote by $Z_\gamma(H)$ the centralizer of an element γ in a group H. A semi-simple element γ of G is called *elliptic* if $Z_\gamma(G'(\mathbf{A}))/Z_\gamma(G')$ has finite volume. It is called *regular* if $Z_\gamma(G'(\mathbf{A}))$ is a torus, and singular otherwise. Let γ be an elliptic element of G. The *orbital integral* of f at γ is defined to be

$$\Phi(\gamma, f) = \int_{G'(\mathbf{A})/Z_\gamma(G')} f(g\gamma g^{-1})dg.$$

Similarly, for any place v of F the element γ of G_v is called *elliptic* if $Z_\gamma(G'_v)$ has finite volume, and *regular* if $Z_\gamma(G_v)$ is a torus. If γ is an element of G and there is a place v of F such that γ is elliptic (resp. regular) in G_v, then γ is elliptic (resp. regular). The *orbital integral* of f_v at γ in G_v is defined to be

$$\Phi(\gamma, f_v) = \Phi(\gamma, f_v; d_\gamma) = \int_{G'_v/Z_\gamma(G'_v)} f_v(g\gamma g^{-1}) \frac{dg}{d_\gamma}.$$

It depends on the choice of a Haar measure d_γ on $Z_\gamma(G'_v)$.

Let $\{\phi_\alpha\}$ be an orthonormal basis for the space $L_0(G)$. The operator $r_0(f)$ is an integral operator on $G'(\mathbf{A})$ with kernel $K_f^0(x, y) = \Sigma_{\alpha,\beta}\, r(f)\phi_\alpha(x)\bar{\phi}_\beta(y)$. The operator $r(f)$ is an integral operator on $G'(\mathbf{A})$ with kernel $K_f(x, y) = \Sigma_\gamma f(x^{-1}\gamma y)$ (γ in G'). If $\boldsymbol{G}$ is anisotropic (namely $G' \backslash G'(\mathbf{A})$ is compact), then $L_0(G) = L(G)$ and $r = r_0$. Since $K_f^0(x, y) = K_f(x, y)$ is smooth in both x and y, we integrate over the diagonal $x = y$ in $G'(\mathbf{A})$, change the order of summation and integration as usual, and obtain the Selberg trace formula in the case of compact quotient, as follows.

Proposition. *If $\boldsymbol{G}$ is anisotropic, then for every function f on $G(\mathbf{A})$ as above we have*

$$\Sigma' m(\pi)\operatorname{tr}\pi(f) = \sum_{\{\gamma\}} \Phi(\gamma, f). \tag{2}$$

The sum on the left is the same as in (1). *The sum on the right is finite; it ranges over the conjugacy classes of elements in G'.*

Remark. If $\boldsymbol{G}$ is anisotropic, then each element γ in G is elliptic.

For a general group $\boldsymbol{G}$ we introduce the following

Definition. The function f is called *discrete* if for every x in $G(\mathbf{A})$ and γ in G we have $f(x^{-1}\gamma x) = 0$ unless γ is elliptic regular.

Changing again the order of summation and integration as usual we obtain the

Proposition. *If f is discrete, then*

$$\int_{G'(\mathbf{A})} \left[\sum_{\gamma\in G'} f(x^{-1}\gamma x)\right] dx = \sum_{\{\gamma\}} \Phi(\gamma, f). \tag{3}$$

The sum on the right is finite. It ranges over the set of conjugacy classes of elliptic regular elements in G'.

Remark. It is well known that the sum on the right is finite; for a proof see [FK], §18 (if $G = \mathrm{GL}(n)$), and [F], Prop. I.3 (in general).

Definition. The function f is called *cuspidal* if for every x, y in $G(\mathbf{A})$ and every proper F-parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$, we have $\int_{N(\mathbf{A})} f(xny)dn = 0$, where $\boldsymbol{N}$ is the unipotent radical of $\boldsymbol{P}$.

When f is cuspidal, the convolution operator $r(f)$ factorizes through the projection on $L_0(G)$, $r(f)$ is of trace class, $\operatorname{tr} r_0(f) = \operatorname{tr} r(f)$ and $K_f(x, y) = K_f^0(x, y)$, and we obtain

Corollary. *If f is cuspidal and discrete, then the equality* (2) *holds. The sum on the left is as in* (1). *The sum on the right is as in* (3).

For some applications we need to replace the requirement that f be discrete by a requirement on the orbital integrals of f (but not on f itself). The purpose of this work is to present such a requirement, and apply the resulting trace formula to extend some global lifting theorems, such as those of [FK].

Fix a non-archimedean place u of F such that G_u is split, and the component ω_u of ω at u is unramified (namely trivial on the multiplicative group $R_u^\times$ of R_u).

Definition. A complex-valued compactly-supported modulo-center function f_u on G_u is called *spherical* if it is K_u-biinvariant. Let $\mathbf{H}_u$ be the convolution algebra of such functions. Of course $\mathbf{H}_u$ is empty unless the central character ω_u is unramified.

For any maximal (proper) F_u-parabolic subgroup $P_u = M_u N_u$ of G_u, where N_u is the unipotent radical of P_u and M_u a Levi subgroup, define an $F_u^\times$-valued character α_{P_u} of M_u by $\alpha_{P_u}(m) = \det(\operatorname{ad}(m) \mid L(N_u))$, where $L(N_u)$ denotes the Lie algebra of N_u, and $\operatorname{ad}(m) \mid L(N_u)$ denotes the adjoint action of m in M_u on $L(N_u)$. Let $\operatorname{val}_u : F_u^\times \to \mathbf{Z}$ be the normalized additive valuation. Let A_u be a maximally split torus in G_u. For any non-negative integer n let $A_u^{(n)}$ be the set of a in A_u such that $|\operatorname{val}_u(\alpha_{P_u}(a))| < n$ for some maximal F_u-parabolic subgroup P_u containing A_u of G_u.

Definition. A spherical function f_u is called *n-admissible* if the orbital integral $\Phi(a, f_u)$ is zero for every regular a in $A_u^{(n)}$.

Let $\mathbf{A}^u$ denote the ring of F-adeles without u-component. Put $G^u = G(\mathbf{A}^u)$. Write $f = f_u f^u$ if f is a function on $G(\mathbf{A})$, f_u on G_u, f^u on G^u, and $f(x, y) = f_u(x) f^u(y)$ for x in G_u and y in G^u. We choose the place u such that the central character ω is unramified at u.

Theorem 1. *Let f^u be a function on G^u which is compactly supported modulo $Z^u = Z(\mathbf{A}^u)$ and vanishes on the G^u-orbit of any singular γ in G. Then there exists a positive integer $n_0 = n_0(f^u)$ such that for every spherical n_0-admissible function f_u there is a function f_u' on G_u with* (1) *$\Phi(x, f_u') = \Phi(x, f_u)$ for all regular x in G_u, and* (2) *$f' = f_u' f^u$ is discrete.*

Proof. For every maximal F-parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$ and every place $v \neq u$ of F there exists a non-negative integer $C_{v,P}$ which depends on f^u, with

$C_{v,P}=0$ for almost all v, such that if γ lies in a Levi subgroup M of P and $f^u(x^{-1}\gamma x)\neq 0$ for some x in G^u, then

$$(4)_v \qquad |\mathrm{val}_v(\alpha_P(\gamma))| \leqq C_{v,P}.$$

Put $C_{u,P}=\Sigma_{v\neq u}C_{v,P}$. Since γ is rational (in G), the product formula $\Sigma_v \mathrm{val}_v(\alpha_P(\gamma))=0$ on $F^\times$ implies that the inequality $(4)_v$ remains valid also for $v=u$. Choose $n_0>C_{u,P}$ for all (of the finitely many conjugacy classes of) P. Let f_u be any spherical n_0-admissible function. Put $f=f_uf^u$. It is well known (for a proof see [F], Prop. I.3) that there are only finitely many rational conjugacy classes γ in G' such that f is not zero on the $G'(\mathbf{A})$-orbit of γ. Note that f is zero on the $G(\mathbf{A})$-orbits of all singular γ in G by assumption. Let γ_i $(1\leqq i\leqq m)$ be a set of representatives for the regular non-elliptic rational conjugacy classes in G such that f is non-zero on their $G(\mathbf{A})$-orbits. Since γ_i is non-elliptic, it lies in a Levi subgroup M_i of a maximal parabolic subgroup P_i of G. Since f_u is n_0-admisible, the relation $\Phi(\gamma_i,f_u)\neq 0$ implies that $|\mathrm{val}_u(\alpha_{P_i}(\gamma_i))|>n_0$. This contradicts $(4)_u$. Hence $\Phi(\gamma_i,f_u)=0$ for all i. Let S_i denote the characteristic function of the complement in G_u of a sufficiently small open closed neighborhood of the orbit of γ_i in G_u. Since γ_i is regular non-elliptic, we may and do take S_i to be one on the elliptic set of G_u. Put $f'_u=f_u\Pi_{i=1}^m S_i$. Then f'_u is zero on the orbit of γ_i $(1\leqq i\leqq m)$, and $\Phi(\gamma,f'_u)=\Phi(\gamma,f_u)$ for every regular γ in G_u. Since $f'=f'_uf^u$ vanishes on the $G(\mathbf{A})$-orbit of each rational γ in G which is not elliptic-regular, the theorem follows.

Since both sides of (2) are invariant distributions, we conclude the immediate

Corollary. *Suppose that $f=f_uf^u$ is a cuspidal function which vanishes on the $G(\mathbf{A})$-orbit of every singular γ in G, and f_u is a spherical n_0-admissible function with $n_0=n_0(f^u)$. Then the equality* (2) *holds, where the sum on the left is as in* (1), *while the sum on the right is as in* (3).

Definition. A G_u-module π_u is called *unramified* if it has a non-zero K_u-fixed vector.

For applications such as those given in Theorem 3 below, we need to show that the set of n-admissible functions is sufficiently large in the following sense.

Theorem 2. *Let $\{\pi_i;\, i\geqq 0\}$ be a sequence of inequivalent unitary unramified G_u-modules, and c_i complex numbers, such that $\Sigma_i c_i \operatorname{tr}\pi_i(f_u)$ is absolutely convergent for every spherical function f_u. Suppose that there is a positive integer n_0 such that $\Sigma_i c_i \operatorname{tr}\pi_i(f_u)=0$ for all n_0-admissible f_u. Then $c_i=0$ for all i.*

Proof. This is delayed to the end of this paper.

Remark. The notion of n-admissible functions is suggested by Drinfeld [D], at least in the case of $G = \mathrm{GL}(2)$. For a general G the Corollary is a representation theoretic analogue of Deligne's conjecture on the Grothendieck–Lefschetz fixed point formula for the trace of a finite flat correspondence on a separated scheme of finite type over a finite field, which is multiplied by a sufficiently high power of the Frobenius morphism. We hope to explain this analogy in more detail in our work (in preparation) on the geometric Ramanujan conjecture for $\mathrm{GL}(n)$ (see also [FK′]).

In the proofs of Theorem 2 and Theorem 3 below we shall use some results concerning unramified representations and spherical functions (see [C]), and regular functions. These will be recalled now in order to be able to give an uninterrupted exposition of the proof of Theorem 3.

Let G be a split p-adic reductive group with minimal parabolic subgroup $B = AN$, where N is the unipotent radical of B and the Levi subgroup A is a maximal (split) torus. Let $X^* = X^*(A)$ be the lattice of rational characters on A, and let $X_* = X_*(A)$ be the dual lattice. If A^0 is the maximal compact subgroup of A then $X_* \simeq A/A^0$. Let $T = X^*(\mathbf{C})$ denote the complex torus $\mathrm{Hom}(X_*, \mathbf{C}^\times)$. The Weyl group W of A in G acts on A, X^*, X_* and T. Each t in T defines a unique $\mathbf{C}^\times$-valued character of B which is trivial on N and on A^0. The G-module $I(t) = \mathrm{Ind}(\delta^{1/2}t; B, G)$ normalizedly induced from the character t of B is unramified and has a unique unramified irreducible constituent $\pi(t)$. We have $\pi(t) \simeq \pi(t')$ if and only if $t' = wt$ for some w in W. The map $t \to \pi(t)$ is a bijection from the variety T/W to the set of unramified irreducible G-modules. Put $t(\pi)$ for the t associated with such a π. Let $\boldsymbol{\alpha}_i$ $(1 \leqq i \leqq m)$ be a set of simple (with respect to N) roots in the vector space $X^* \otimes \mathbf{R} = \mathrm{Hom}(X_*, \mathbf{R})$, and α_i the corresponding character of A, defined as usual by $\alpha_i(a) = \mathrm{ad}(a) \mid L(N_i)$, where $\mathrm{ad}(a)$ denotes the adjoint action of A on the Lie algebra $L(N_i)$ of the root subgroup N_i of $\boldsymbol{\alpha}_i$ in N. Denote by $\boldsymbol{\alpha}_i^\vee$ $(1 \leqq i \leqq m)$ the corresponding set of coroots in the dual space $X_* \otimes \mathbf{R}$, and by $\alpha_i^\vee$ the corresponding set of characters of the torus $T = X^*(\mathbf{C}) = \mathrm{Hom}(X_*, \mathbf{C}^\times)$, defined as usual by $\alpha_i^\vee(\exp T) = \exp\langle \boldsymbol{\alpha}_i^\vee, T\rangle$ for all T in $X^* \otimes \mathbf{C} = \mathrm{Hom}(X_*, \mathbf{C})$; here $\langle .\,, .\rangle$ is the pairing between X_* and X^*. There exists $\boldsymbol{q} = \boldsymbol{q}(G) > 1$ such that if π is (irreducible, unramified and) unitary, then (1) $\boldsymbol{q}^{-1} < |\alpha_i^\vee(t)| < \boldsymbol{q}$ for all i $(1 \leqq i \leqq m)$, and (2) the complex conjugate $\bar{t}$ of t is equal to wt^{-1} for some w in W.

If f is a spherical function then the value of the normalized orbital integral $F(a, f) = \Delta(a)\Phi(a, f)$ at a regular a in A depends only on the W-orbit of the image $\boldsymbol{x}$ of a in X_*; it is denoted by $F(\boldsymbol{x}, f)$. Let $\mathbf{C}[X_*]^W$ be the algebra of W-invariant elements in the group ring $\mathbf{C}[X_*]$. The Satake transform $f \to f^\vee = \Sigma_{\boldsymbol{x} \in X_*} F(\boldsymbol{x}, f)\boldsymbol{x}$ defines an algebra isomorphism from the convolution algebra $\mathbf{H}$ of spherical functions, to $\mathbf{C}[X_*]^W$. For each $\boldsymbol{x}$ in X_*, let $f(\boldsymbol{x})$ be the element of $\mathbf{H}$ with $f(\boldsymbol{x})^\vee = \Sigma_{w \in W} w\boldsymbol{x}$. Then $f(\boldsymbol{x})$ is n_0-admissible if $|\mathrm{val}\, \alpha_P(w(a(\boldsymbol{x})))| \geqq n_0$ for every w

in W and parabolic subgroup P containing A; $a(\boldsymbol{x})$ is an element of A which corresponds to $\boldsymbol{x}$ under the isomorphism of A/A^0 with X_* fixed above. We have $\operatorname{tr}(\pi(t))(f) = \operatorname{tr}(I(t))(f) = f^\vee(t)$ for every f in $\mathbf{H}$ and t in T, where $f^\vee(t) = \Sigma_{\boldsymbol{x}\in X_*} F(\boldsymbol{x}, f)t(\boldsymbol{x})$.

Definition. Consider $\boldsymbol{x}$ in X_* with val $\alpha(a(\boldsymbol{x})) \neq 0$ for each root α of A on N. A complex-valued locally-constant function f with $f(zg)\omega(z) = f(g)$ for all g in G and z in Z which is compactly supported modulo Z is called *x-regular* if $f(g)$ is zero unless there is z in Z such that zg is conjugate to an element a in A whose image in X_* is $\boldsymbol{x}$, in which case the normalized orbital integral $F(g, f)$ is equal to $\omega(z)^{-1}$. If f is $\boldsymbol{x}$-regular then we denote it by $f_{\boldsymbol{x}}$. A *regular* function is a linear combination with complex coefficients of $\boldsymbol{x}$-regular functions.

Remarks. (1) Any regular function vanishes on the singular set; in fact it is supported on the regular split set by definition.

(2) If π is an admissible G-module with central character ω, then the normalized module π_N of coinvariants [BZ] is an A-module; its character is denoted by $\chi(\pi_N)$. If $f_{\boldsymbol{x}}$ is an $\boldsymbol{x}$-regular function, then a simple application of the Weyl integration formula and the theorem of Deligne–Casselman [CD] implies that

$$\operatorname{tr}\pi(f_{\boldsymbol{x}}) = [W]^{-1} \int_{A/Z} (\Delta\chi(\pi_N))(a)F(a, f_{\boldsymbol{x}})da.$$

If $\operatorname{tr}\pi(f_{\boldsymbol{x}})$ is non-zero, then there exists (i) t in T such that π is a constituent of $I(t)$ (by Frobenius reciprocity), and (ii) a subset $W(\pi, t)$ of W such that

$$\operatorname{tr}\pi(f_{\boldsymbol{x}}) = \sum_w t(w\boldsymbol{x}) \qquad (w \text{ in } W(\pi, t)).$$

(3) Each constituent of $I(t)$, including π, has a non-zero vector fixed by the action of an Iwahori subgroup (see Borel [B], (4.7), in the case of a reductive group, and [FK], §17, for the case of the metaplectic groups considered below).

(4) Regular functions play a crucial role in the study of orbital integrals of spherical functions; see [F″].

We shall now use the Corollary, Theorem 2 and the results concerning spherical and regular functions, to extend the global correspondence results of [FK] (resp. [BDKV] and [F]) which deal with cuspidal representations of metaplectic groups (resp. inner forms) of GL(n). The definitions and proofs which are not given in the following discussion are detailed in these references. Put $G = \mathrm{GL}(n)$. Let $\tilde{G}$ be either a metaplectic group of G, or the multiplicative group of a simple algebra central of rank n over F. The cuspidal G-module $\pi = \otimes\, \pi_v$ and the cuspidal (genuine) $\tilde{G}$-module $\tilde{\pi} = \otimes\, \tilde{\pi}_v$ are called *corresponding* if π_v and $\tilde{\pi}_v$ correspond for each place v of F, where the notion of local correspondence is defined by means of character relations (see [FK], §27; [F; III], §1). Fix a non-archimedean place u'. Let A be the set of equivalence classes of cuspidal G-modules π with a supercuspidal component at u', such that each component of π is obtained by the local correspondence. Let $\tilde{A}$ be the set of equivalence classes of cuspidal $\tilde{G}$-modules $\tilde{\pi}$ whose component at u' corresponds to a supercuspidal $G_{u'}$-module (then $\tilde{\pi}_{u'}$ is necessarily supercuspidal).

Theorem 3. *The correspondence defines a bijection between the sets A and $\tilde{A}$. The multiplicity of each $\tilde{\pi}$ of $\tilde{A}$ in the cuspidal spectrum $L_0(\tilde{G})$ is one.*

Remark. (1) In [FK], §28; and [BDKV]; [F; III], §8; this is proven for the set of π in A with two supercuspidal components, and the corresponding subset of $\tilde{A}$.

(2) Theorem 3 can be extended from the context of A, $\tilde{A}$ to the context of all cusp forms on G, $\tilde{G}$ by known techniques; it will be interesting to establish such an extension by *simple* means.

Proof of Theorem 3. Fix corresponding supercuspidal $G_{u'}$ and $\tilde{G}_{u'}$-modules $\pi_{u'}$ and $\tilde{\pi}_{u'}$, and matrix coefficients $f_{u'}$ and $\tilde{f}_{u'}$ thereof. Then $f_{u'}$ and $\tilde{f}_{u'}$ are matching (see [FK], §7; [F; III], §1), namely have matching orbital integrals. For any functions $f^{u'}$ on $G^{u'}$ and $\tilde{f}^{u'}$ on $\tilde{G}^{u'}$, the functions $f = f_{u'}f^{u'}$ and $\tilde{f} = \tilde{f}_{u'}\tilde{f}^{u'}$ are cuspidal (see, e.g., [F], Lemma I.3). Fix two distinct non-archimedean places u and u'' of F, other than u', with sufficiently large residual characteristics. Put $\tilde{G}^{u,u',u''} = \tilde{G}(\mathbf{A}^{u,u',u''})$, where $\mathbf{A}^{u,u',u''}$ is the ring of F-adeles without u, u', u'' components. Similarly we have $G^{u,u',u''}$, $G^{u,u'}$, etc. Let $\tilde{f}^{u,u',u''}$ be any function on $\tilde{G}^{u,u',u''}$, and $f_{u''}$ any regular function on $G_{u''}$. Let $f^{u,u',u''}$ be a matching function on $G^{u,u',u''}$, and $f_{u''}$ a matching regular function on $G_{u''}$. Put $f^u = f^{u,u',u''}f_{u'}f_{u''}$ and $\tilde{f}^u = \tilde{f}^{u,u',u''}\tilde{f}_{u'}\tilde{f}_{u''}$. Put $n_0 = \max\{n_0(f^u), n_0(\tilde{f}^u)\}$. Let $\tilde{f}_u$ and f_u be matching spherical n_0-admissible functions. Since $f_{u''}$ and $\tilde{f}_{u''}$ are zero on the singular set, the functions $f = f_u f^u$ and $\tilde{f} = \tilde{f}_u\tilde{f}^u$ are zero on the $G(\mathbf{A})$ and $\tilde{G}(\mathbf{A})$-orbits of any singular element γ in G and $\tilde{G}$ (respectively); hence they are discrete. Since f and $\tilde{f}$ are matching, the right sides of the trace formulae (2) for G and for $\tilde{G}$, namely $\Sigma\,\Phi(\gamma, f)$ and $\Sigma\Phi(\gamma^*, \tilde{f})$ (see [FK], §4), are equal. By the Corollary to Theorem 1, the left sides are equal, namely $\Sigma'\, m(\pi)\mathrm{tr}\,\pi(f) = \Sigma'\, m(\tilde{\pi})\mathrm{tr}\,\tilde{\pi}(\tilde{f})$. By virtue of the choice of $f_{u'}$ and $\tilde{f}_{u'}$, the π and $\tilde{\pi}$ are cuspidal, with the supercuspidal components $\pi_{u'}$ and $\tilde{\pi}_{u'}$ at u'. Hence $m(\pi) = 1$ (by multiplicity one theorem for the cuspidal representations of $\mathrm{GL}(n)$), and each component π_v of π is relevant (see [FK], §27; [F; III], §7; for definition and proof). Since $\mathrm{tr}\,\pi_v(f_v) \neq 0$ for f_v matching an $\tilde{f}_v$, and π_v is relevant, the main local correspondence theorem ([FK], §27; [F; III], §8) implies that π_v corresponds to some $\tilde{\pi}_v(\pi_v)$, for each v. Since f_u and $\tilde{f}_u$ are spherical, if $\mathrm{tr}\,\pi_u(f_u)$ and $\mathrm{tr}\,\tilde{\pi}_u(\tilde{f}_u)$ are non-zero then π_u and $\tilde{\pi}_u$ are unramified, and so is $\tilde{\pi}_u(\pi_u)$. We write our equality in the form

$$\sum \left[\sum\nolimits_1 m(\tilde\pi)\operatorname{tr} \tilde\pi^u(\tilde f^u) - \sum\nolimits_2 m(\pi)\operatorname{tr}(\tilde\pi^u(\pi^u))(\tilde f^u)\right] \operatorname{tr} \tilde\pi_u(\tilde f_u) = 0.$$

The sum Σ ranges over all equivalence classes of unramified unitary (genuine) $\tilde G_u$-modules $\tilde\pi_u$. Σ_1 ranges over the equivalence classes of $\tilde G^u$-modules $\tilde\pi^u$ such that $\tilde\pi = \tilde\pi_u \otimes \tilde\pi^u$ appears in (2). Σ_2 ranges over the $\pi^u = \otimes_{v\neq u} \tilde\pi_v$ such that there is a cuspidal $\pi = \otimes \pi_v$ with $\tilde\pi_v = \tilde\pi_v(\pi_v)$ for all v. Since all sums and products in the trace formula are absolutely convergent, and all the representations which appear there are unitary, Theorem 2 implies that $\Sigma_1 = \Sigma_2$ for each $\tilde\pi_u$. We write this identity in the form

$$\sum \left[\sum\nolimits^1 m(\tilde\pi)\operatorname{tr} \tilde\pi_{u'}(\tilde f_{u'}) - \sum\nolimits^2 m(\pi)\operatorname{tr}(\tilde\pi_{u'}(\pi_{u'}))(\tilde f_{u'})\right] \operatorname{tr} \tilde\pi^{u,u''}(\tilde f^{u,u''}) = 0.$$

Here Σ ranges over all equivalence classes of irreducible $\tilde G^{u,u''}$ modules $\tilde\pi^{u,u''}$. Σ^1 ranges over all irreducible $\tilde G_{u''}$-modules $\tilde\pi_{u''}$ such that $\tilde\pi^u = \tilde\pi_{u''}\tilde\pi^{u,u''}$ appears in Σ_1, and Σ^2 is over the $\tilde\pi_{u''}$ such that the resulting $\tilde\pi^u$ occurs in Σ_2. Since the function $\tilde f^{u,u',u''}$ is arbitrary, all sums here are absolutely convergent and all representations are unitary, a standard argument of linear independence of characters implies that $\Sigma^1 = \Sigma^2$, for every $\tilde\pi^{u''} = \tilde\pi_u\tilde\pi^{u,u''}$.

We now use the fact that $\tilde f_{u''}$ is an arbitrary regular function. If $\operatorname{tr} \tilde\pi_{u''}(\tilde f_{u''}) \neq 0$ then $\tilde\pi_{u''}$ has a non-zero vector fixed by an Iwahori subgroup. Hence the sum Σ^1 is finite by a theorem of Harish-Chandra (see [BJ]) which asserts that there are only finitely many cuspidal $\tilde G$-modules with fixed infinitesimal character and fixed ramification at all finite places. The sum Σ^2 consists of at most one term, by the rigidity theorem for cuspidal G-modules.

Recall that $\operatorname{tr} \tilde\pi_{u''}(f_{u''})$ is a linear combination of characters (of the form $t \to t(wx)$, where t lies in $T = \{(z_i) \text{ in } \mathbf{C}^{\times n}; \Pi_i z_i = 1\}$, and $x = (x_i)$ varies over $X_* = \mathbf{Z}^n/\mathbf{Z}$, and $(z_i)(wx) = \Pi_i z_{w(i)}^{x_i}$). Applying linear independence of finitely many characters it is clear that Σ^1 is empty if Σ^2 is empty, and that $m(\tilde\pi) = 1$ and $\operatorname{tr} \tilde\pi_{u''}(\tilde f_{u''}) = \operatorname{tr} \pi_{u''}(f_{u''})$ for all matching regular $f_{u''}$ and $\tilde f_{u''}$ otherwise. Since the Hecke algebras of $G_{u''}$ and $\tilde G_{u''}$ with respect to an Iwahori subgroup are isomorphic (by [FK], §17, in the metaplectic case), we conclude that $\pi_{u''}$ and $\tilde\pi_{u''}$ correspond, and Theorem 3 follows.

Proof of Theorem 2. Fix $q \geqq 1$. Let $T' = T'(q)$ be the set of t in T with $\bar t = wt^{-1}$ for some w in W (w depends on t) and $q^{-1} \leqq |\alpha^\vee(t)| \leqq q$ for every root α of A on N. The quotient $\tilde T = \tilde T(q)$ of T' by W is a compact Hausdorff space. Let $\mathbf{C}(\tilde T)$ be the algebra of complex-valued continuous functions on $\tilde T$. Let n_0 be a non-negative integer. The element x of X_* is called *n_0-admissible* if $|\operatorname{val} \alpha_P(a(x))| \geqq n_0$ for every maximal parabolic subgroup P of G. This condition means that there are finitely many walls, determined by the α_P, in the lattice X_*, such that x is called n_0-admissible if it is sufficiently far (the distance depends on

n_0) from these walls. The function $P_x(t) = \Sigma_w t(wx)$ (w in W) is a function on $\tilde{T}$ which depends only on the image of x in X_*/W. Note that $f(x)^\vee = P_x$, and in particular $\mathrm{tr}(\pi(t))(f_x) = P_x(t)$. Let $C(n_0)$ be the $\mathbf{C}$-span of all $P_x(t)$ with n_0-admissible x. It is a subspace of $\mathbf{C}(\tilde{T})$, but it is not multiplicatively closed, unless $n_0 = 0$. An element of $\mathbf{C}(\tilde{T})$ is called *n_0-admissible* if it lies in $C(n_0)$.

Lemma. *The space $C(0)$ is dense in $\mathbf{C}(\tilde{T})$.*

Proof. This follows from the Stone–Weierstrass theorem, since (1) the space $\tilde{T}$ is compact and Hausdorff, and (2) $C(0)$ is a subalgebra of $\mathbf{C}(\tilde{T})$ which separates points, contains the scalars and the complex-conjugate of each of its elements.

Theorem 2 follows from the special case where $G = \mathrm{GL}(n)$ and $c_i(t) = 0$ for all i in the Proposition below. The general form with non-zero $c_i(t)$ is used in [F′] when $G = \mathrm{GL}(3)$ to give a short and simple proof of the trace formulae identity for the base-change lifting from $U(3)$ to $\mathrm{GL}(3, E)$ for an arbitrary test function f.

Proposition. *Fix $n_0 \geqq 0$. Let t_i ($i \geqq 0$) be elements of $\tilde{T}$; c_i complex numbers; $\tilde{T}_j$ ($j \geqq 0$) compact submanifolds of $\tilde{T}$; and $c_j(t)$ complex valued functions on $\tilde{T}_j$ which are measurable with respect to a bounded measure dt on $\tilde{T}_j$. Suppose that*

$$\beta = \sum_i |c_i| + \sum_j \sup_{t \in \tilde{T}_j} |c_j(t)| + \sum_j \int_{\tilde{T}_j} |c_j(t)|\ |dt|$$

is finite, and that for any n_0-admissible x in X_* we have

$$\sum_{i \geq 0} c_i P_x(t_i) = \sum_{j \geq 0} \int_{\tilde{T}_j} c_j(t) P_x(t) |dt|. \tag{5}$$

Then $c_i = 0$ for all i.

Proof. We begin with a definition. Let ε be a positive number. The points t and t' in $\tilde{T}$ are called *ε-close* if there are representatives $\mathbf{t}$ and $\mathbf{t}'$ of t and t' in T' such that $|\alpha^\vee(\mathbf{t}) - \alpha^\vee(\mathbf{t}')| < \varepsilon$ for every root $\alpha^\vee$ on T(= coroot on X_*). Denote by $\tilde{T}_\varepsilon(t)$ the ε-neighborhood of t in $\tilde{T}$. The quotient by ε of the volume of $\tilde{T}_\varepsilon(t)$ is bounded uniformly in ε.

(i) Suppose that $c_0 \neq 0$. Multiplying by a scalar we assume that $c_0 = 1$. The Lemma implies that for every $\varepsilon > 0$ there is $P = P_\varepsilon$ in $C(0)$ with $P(t_0) = 1$, $|P(t)| \leqq 2$ for all t in $\tilde{T}$, and $|P(t)| < \varepsilon$ unless t is ε^2-close to t_0. Such a polynomial P is called below an ε-approximation of the delta function at t_0, or simple a *"delta function"* at t_0. Since β is finite, for every $\varepsilon > 0$ there exists $N > 0$ such that

$$\sum_{i > N} |c_i| + \sum_{j > N} \int_{\tilde{T}_j} |c_j(t)|\ |dt| < \varepsilon.$$

Take $\varepsilon = 1/4(1+\beta)$. Substituting P for P_x in (5), if $n_0 = 0$ then we obtain a contradiction to the assumption that $c_0 = 1$. Hence the proposition is proven in the case of $n_0 = 0$. It remains to deal with a general n_0.

(ii) Let x be an n_0-admissible element of X_*. Put $k' = 2\max_P |\operatorname{val} \alpha_P(a(x))|$. For any x' in X_*, $x + k'x'$ is n_0-admissible. Since $P_x(t)P_{x'}(t^{k'}) = \Sigma_{w\in W} P_{x+k'wx'}(t)$, we have that (5) applies with $P_x(t)$ replaced by $P_x(t)P_{x'}(t^{k'})$. For a fixed x (and k'), x' is arbitrary. Replacing q by $q^{k'}$ in the definition of $\tilde{T}$ we argue as in (i) and conclude that for every $r \geqq 0$ we have

$$\sum_i c_i P_x(t_i) = 0; \tag{6}$$

here the sum ranges over all i with $t_i^{k'} = t_r^{k'}$ (equality in $\tilde{T}$). Take $r = 0$. We conclude that the equality (6) holds also for any n_0-admissible x, provided that the sum ranges over the set I of all i for which there is $k = k(i)$ with $t_i^k = t_0^k$. It remains to prove the following

Lemma. *Suppose that c_i ($i \geqq 0$) are complex numbers such that $\beta = \Sigma_i |c_i|$ is finite, and t_i are elements of T' whose images in $\tilde{T} = T'/W$ are distinct, such that for each i there is $k = k(i)$ with $t_i^k = t_0^k$. If $\Sigma_i c_i P(t_i) = 0$ for every n_0-admissible P then $c_i = 0$ for all i.*

Proof. We may and do assume that $c_0 = 1$ in order to derive a contradiction. If $\eta = 1/4(1+\beta)$ there is $N > 0$ such that $\Sigma_{i>N} |c_i| < \eta$, and a W-invariant polynomial $P(t) = \Sigma_x b(x)P_x(t)$ with $P(t_0) = 1$, $|P(t)| \leqq 2$ on T' and $|P(t_i)| < \eta$ for i ($1 \leqq i \leqq N$). This P is a "delta function", and if $n_0 = 0$ then we are done. If $n_0 \neq 0$ then the "delta function" P is not necessarily n_0-admissible. Our aim is to replace P by an n_0-admissible "delta function" on multiplying P with a suitable admissible polynomial Q which (depends on P and) attains the value one at t_0, while remaining uniformly bounded (by $2[W]$) at each t_i ($i \geqq 1$). For this purpose note that our assumption (that for each i there is k with $t_i^k = t_0^k$) implies that t_i/t_0 lies in the maximal compact subgroup of T for all i. Hence for every x in X_*, the absolute value $|t_i(x)|$ of the complex number $t_i(x)$ is independent of i. Take any one-admissible μ in X_*, such that $|t_i(\mu)| \geqq |t_i(w\mu)|$ for all w in W. Then $|P_\mu(t_i^s)| \leqq [W]|t_0(m)|^s$ for every positive integer s (and all i). Put $u_w = t_0(w\mu)/|t_0(w\mu)|$ (w in W), and

$$s_0 = 2n_0 + 2\max\{|\operatorname{val} \alpha_P(a(x))|;\ \text{all } P \supset A, \text{ all } x \text{ with } b(x) \neq 0\}.$$

For every $\varepsilon > 0$ there is $s > s_0$ such that $|u_w^s - 1| < \varepsilon$ for all w in W, and the choice of a sufficiently small ε guarantees that $|P_\mu(t_0^s)| \geqq \frac{1}{2}|t_0(\mu)|^s$. Hence the W-invariant polynomial $Q_s(t) = P_\mu(t^s)/P_\mu(t_0^s)$ on T' satisfies $Q_s(t_0) = 1$ and $|Q_s(t_i)| \leqq 2[W]$ for all i. The polynomial $Q(t) = P(t)Q_s(t)$ lies in $C(n_0)$, hence it satisfies the relation $\Sigma_i c_i Q(t_i) = 0$. Since Q is a delta function at t_0, we obtain a contradiction to the assumption that $c_0 \neq 0$. This proves the lemma, and completes the proof of Theorem 2.

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