On the symmetric square: unstable local transfer

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Summary. We prove the "fundamental lemma" for spherical functions with respect to the natural (induction) lifting from $PGL(2)$ to $PGL(3)$ which appears as the unstable counterpart of the stable symmetric-square lifting from $SL(2)$ to $PGL(3)$ (see [IV] for an introduction to this project, and [VI] for the final results). Thus spherical functions on $PGL(2)$ and $PGL(3)$ which correspond to each other by satisfying an elementary representation theoretic relation are shown to have matching orbital integrals. The proof of this local statement is based on an application of the global trace formula.

Let $F$ be a local field. Put

$$G = PGL(3), \quad H_1 = PGL(2), \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\sigma \delta = J \delta^{-1} J$ for $\delta$ in $G(F)$. Fix an algebraic closure $\bar{F}$ of $F$. The elements $\delta, \delta'$ of $G(\bar{F})$ are called (stably) $\sigma$-conjugate if there is $g$ in $G(\bar{F})$ (resp. $G(F)$) with $\delta' = g^{-1} \delta \sigma(g)$. To state our theorems, we first recall the results of [I], §§1.2–1.6, concerning these classes. For any $\delta$ in $GL(3, F)$, $\delta \sigma(\delta)$ lies in $SL(3, F)$ and depends only on the image of $\delta$ in $G(F)$. The eigenvalues of $\delta \sigma(\delta)$ are $\lambda, 1, \lambda^{-1}$ (see [I], §1.4), with $[F(\lambda):F] \leq 2$; $\delta$ is called $\sigma$-regular if $\lambda \neq \pm 1$. In this case we write (as in [I], §1.5) $\gamma_i = N_1 \delta$ for the conjugacy class in $H_1(F)$ which corresponds to the conjugacy class with eigenvalues $\lambda, 1, \lambda^{-1}$ in $SO(3, F)$ under the isomorphism $H_1(F) = SO(3, F)$ (i.e., $\gamma_1$ is the image in $H_1(F)$ of a conjugacy class in $GL(2, F)$ with eigenvalues $a, b$ with $a/b = \lambda$). It is shown in [I], §1.5, that the map $N_1$ is a bijection from the set of stable regular $\sigma$-conjugacy classes in $G(F)$ to the set of regular conjugacy classes in $H_1(F)$ (clearly, we say that a conjugacy class $\gamma_1$ in $H_1(F)$ is regular if $\lambda = a/b \neq \pm 1$). The set of $\sigma$-conjugacy classes in the stable $\sigma$-conjugacy class of a $\sigma$-regular $\delta$ is (shown

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in [1], §1.5, to be) parametrized by $F^*/NK^*$, where $K$ is the field extension $F(\lambda)$ of $F$, and $N$ is the norm from $K$ to $F$. Explicitly, if the quotients of the eigenvalues of the regular element $\gamma_1$ are $\lambda$ and $\lambda^{-1}$, choose $x, \beta$ in $K$ with $\lambda = -x/\beta$ (for example with $\beta = 1$ if $K = F$, and with $\beta = x$ if $K + F$), let $a$ be an element of $GL(2, F)$ with eigenvalues $x, \beta$; put

$$e = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h_1 = \begin{pmatrix} x & 0 \\ 0 & 1 \\ z & 0 \\ t \end{pmatrix}$$

then $e = (uae)_1$ is a complete set of representatives for the $\sigma$-conjugacy classes within the stable $\sigma$-conjugacy class of the $\delta$ with $N_\sigma \delta$ equals $\gamma_1$, as $u$ varies over $F^*/NK^*$ (a set of cardinality one or two). In addition we associate (in $[1]$, §1.6) to $\delta$ a sign $\kappa(\delta)$, as follows: $\kappa(\delta) = 1$ if the quadratic form $x$ (in $F^3 \rightarrow x \delta J x$ (equivalently $x \rightarrow \frac{1}{2} [6] \delta J + (\delta J) x$) represents zero, and $\kappa(\delta) = -1$ if this quadratic form is anisotropic. It is clear that $\kappa(\delta)$ depends only on the $\sigma$-conjugacy class of $\delta$, but it is not constant on the stable $\sigma$-conjugacy class of $\delta$.

Denote by $f$ (resp. $f_1$) a complex-valued compactly-supported smooth (thus locally-constant if $F$ is non-archimedean) function on $G(F)$ (resp. $H_1(F)$). Fix Haar measures on $G(F)$ and on $H_1(F)$. Write $\Phi(\delta, f)$ for the twisted orbital integral

$$\int \{ f g \delta \sigma(g^{-1}) \} \, dg$$

(here $g$ ranges over $G(F)/G_2(F)$, where $G_2(F) = \{ g \in G(F) \mid g \delta \sigma(g^{-1}) = \delta \}$. For $f$ at $\delta$, and put

$$\Phi(\delta, f) = \sum_{g} \kappa(\delta) \Phi(\delta, f);$$

here $\delta'$ ranges over a set of representatives for the $\sigma$-conjugacy classes within the stable $\sigma$-conjugacy class of $\delta$. As usual, $\delta$ is $\sigma$-regular. If $\gamma_1$ is regular, we also put

$$\Phi(\gamma_1, f_1) = \int \{ f_1 g^{-1} \gamma_1 g \} \, dg$$

(g in $H_{1, \gamma_1}(F) \setminus H_1(F)$, where $H_{1, \gamma_1}(F)$ is the centralizer of $\gamma_1$ in $H_1(F)$). Note that when $\gamma_1 = N_\sigma \delta$ the groups $G_2^2(F), G_2^e(F), H_{1, \gamma_1}(F)$ are isomorphic to the centralizer of $\gamma_1$ in $H_1(F)$. We transfer Haar measures on them using these isomorphisms.

**Definition.** The functions $f$ and $f_1$ are called matching if they have matching orbital integrals, namely if $A(\delta) \Phi(\delta, f) = A_1(\gamma_1) \Phi(\gamma_1, f_1)$ for all $\delta$ with regular $\gamma_1 = N_\sigma \delta$.

Here we put $A_1(\gamma_1) = |(a - b)^2/ab|^{1/2}$ if $a, b$ are the eigenvalues of a representative in $GL(2, F)$ of $\gamma_1$, and $A(\delta) = |(1 - \lambda^2)(1 - \lambda^{-2})|^{1/2}$ if $\lambda = a/b$. Thus

$$A_1(\gamma_1) = |(1 - \lambda)(1 - \lambda^{-1})|^{1/2}, \quad \text{and} \quad A(\delta)/A_1(\gamma_1) = |(1 + \lambda)(1 + \lambda^{-1})|^{1/2}.$$

Suppose that $F$ is non-archimedean; denote by $R$ its ring of integers. Put $K = G(R), K_1 = H_1(R)$. Let $\mathbb{H}$ (resp. $\mathbb{H}_1$) denote the convolution algebra of complex-valued compactly-supported $K$- (resp. $K_1$-)biinvariant functions on $G(F)$.
(resp. $H_1(F)$). The Haar measures are the same as those used in the definition of the orbital integrals. Denote by $f^0$ (resp. $f_1^0$) the unit element in $\mathbb{H}$ (resp. $\mathbb{H}_1$), namely the quotient by the volume $|K|$ (resp. $|K_1|$) of $K$ (resp. $K_1$) of the characteristic function of $K$ (resp. $K_1$). We prove below the following

**Theorem 1.** The functions $f^0$ and $f_1^0$ are matching.

This result is used in [VI] to complete the study of the symmetric square lifting, for all automorphic representations of $H=\text{SL}(2)$.

By a $G$-module $\pi$ (resp. $H_1$-module $\pi_1$) we mean an admissible representation of $G(F)$ (resp. $H_1(F)$) in a complex space. An irreducible $G$-module $\pi$ is called $\sigma$-invariant if it is equivalent to the $G$-module $^\sigma \pi$, defined by $^\sigma \pi(g) = \pi(\sigma g)$. In this case there is an intertwining operator $A$ on the space of $\pi$ with $\pi(g) A = A \pi(g)$ for all $g$. Since $\sigma^2 = 1$ we have $\pi(g) A^2 = A^2 \pi(g)$ for all $g$, and since $\pi$ is irreducible $A^2$ is a scalar by Schur's lemma. We choose $A$ with $A^2 = 1$.

This determines $A$ up to a sign, and when $\pi$ has a Whittaker model, [IV, §1.1.1] specifies a normalization of $A$ which is compatible with a global normalization. A $G$-module $\pi$ is called unramified if the space of $\pi$ contains a non-zero $K$-fixed vector. The dimension of the space of $K$-fixed vectors is bounded by one if $\pi$ is irreducible. If $\pi$ is $\sigma$-invariant and unramified, and $\nu_0 \neq 0$ is a $K$-fixed vector in the space of $\pi$, then $A \nu_0$ is a multiple of $\nu_0$ (since $\sigma K = K$), namely $A \nu_0 = c \nu_0$, with $c = \pm 1$. Replace $A$ by $c A$ to have $A \nu_0 = \nu_0$, and put $\pi(\sigma) = A$. As verified in [IV, §1.1.1], when $\pi$ is (irreducible) unramified and has a Whittaker model, both normalizations of the intertwining operator are equal.

For any $\pi$ and $f$ the convolution operator $\pi(f) = \int_{G(F)} f(g) \pi(g) \, dg$ has finite rank. If $\pi$ is $\sigma$-invariant put $\pi(f \times \sigma) = \int_{G(F)} f(g) \pi(g) \pi(\sigma g) \, dg$. Denote by $\text{tr} \, \pi(f \times \sigma)$ the trace of the operator $\pi(f \times \sigma)$. It depends on the choice of the Haar measure $dg$, but the (twisted) character $\chi_{\pi}$ of $\pi$ does not; $\chi_{\pi}$ is a locally-integrable complex-valued function on $G(F)$ (see [C], [H]) which is $\sigma$-conjugacy invariant and locally-constant on the $\sigma$-regular set, with $\text{tr} \, \pi(f \times \sigma) = \int_{G(F)} f(g) \chi_{\pi}(g) \, dg$ for all $f$.

If $f$ is spherical, namely it lies in $\mathbb{H}$, and $\pi$ is $\sigma$-invariant, then $\pi(f)$ (hence also $\pi(f \times \sigma)$) factorizes through the projection on the subspace of $K$-fixed vectors in $\pi$; thus $\text{tr} \, \pi(f \times \sigma) = 0$ for $f$ in $\mathbb{H}$ implies that $\pi$ is unramified. Similarly we introduce $\pi_1(f_1)$ and $\text{tr} \, \pi_1(f_1)$, and conclude that $\pi_1$ is unramified if $\text{tr} \, \pi_1(f_1) = 0$ for $f_1$ in $\mathbb{H}_1$.

A Levi subgroup of a maximal parabolic subgroup $P$ of $G(F)$ is isomorphic to $\text{GL}(2, F)$. Hence an $H_1(F)$-module $\pi_1$ extends to a $P$-module trivial on the unipotent radical $N$ of $P$. Let $\delta$ denote the character of $P$ which is trivial on $N$ and whose value at $\text{det} \, h = \pm m$ is $\text{det} \, h$ if $m$ corresponds to $h$ in $\text{GL}(2, F)$. Explicitly, if $P$ is the upper triangular parabolic subgroup of type $(2, 1)$, and $m \in M$ is represented in $\text{GL}(3, F)$ by $\begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$, then $\delta(m) = \text{det} \, m''$ (more lies in $\text{GL}(2, F)$, $m''$ in $\text{GL}(1, F)$). Denote by $I(\pi_1)$ the $G$-module $\pi_1 = \text{Ind}(\delta^{1/2} \pi_1; P, G)$ unitarily induced from $\pi_1$ on $P$ to $G$. It is clear from [BZ] that when $I(\pi_1)$ is irreducible then it is $\sigma$-invariant, and it is unramified if and only if $\pi_1$ is unramified.
Definition. The functions \( f_i \) in \( \mathcal{H} \) and \( f \) in \( \mathcal{H} \) are called corresponding if
\[
\text{tr} \, \pi_i(f_i) = \text{tr}(I(\pi_i))(f \times \sigma)
\]
for all unramified \( H_1 \)-modules \( \pi_i \), equivalently: for all \( H_1 \)-modules \( \pi_i \).

Example. The spherical functions \( f_0 \) and \( f_0' \) are corresponding.

It is shown in [IV, §2] that Theorem 1 implies

**Theorem 2.** If the spherical functions \( f, f_1 \) are corresponding, then they are matching.

However the argument given below establishes Theorem 2 directly, and Theorem 1 will follow as the special case of \( f = f_0, f_1 = f_0' \).

In [IV, §2] the following is proven:

**Theorem 0.** Suppose that \( \pi = I(\pi_1) \) where \( \pi_1 \) is an irreducible \( H_1(F) \)-module, and \( \delta, \delta' \) are \( \sigma \)-regular stably \( \sigma \)-conjugate but not \( \sigma \)-conjugate elements of \( G(F) \). Then
\[
\chi_\delta(\delta') = -\chi_\delta(\delta).
\]

Of course \( \delta \neq \delta' \) as in Theorem 0 exist only when \( F(\lambda) \neq F \), namely when \( N(\delta) \) is elliptic regular. Let \( \chi_\delta \) be the character of \( \pi_1 \); it is a locally-integrable complex-valued conjugacy-invariant function on \( H_1(F) \) which is smooth on the regular set and satisfies
\[
\text{tr} \, \pi_1(f_i) = \int_{H_1(F)} f_i(g) \chi_\delta(g) \, dg
\]
for all \( f_i \) on \( H_1(F) \). It is shown in [IV, §2] that Theorem 2 implies the following.

**Theorem 3.** If \( \pi = I(\pi_1) \) then \( \chi_\delta \) \( A(\delta) \chi_\delta(\delta') = A_i(\gamma_1) \chi_{\delta'}(\gamma_1) \) for all \( \delta \) with regular \( \gamma_1 = N(\delta) \).

In view of Theorem 0, it suffices to prove Theorem 3 only for one \( \sigma \)-conjugacy class within each stable \( \sigma \)-conjugacy class. It is clear that Theorem 3 implies Theorem 2 (see, e.g., proof of Proposition 27.3 in [FK]).

It is shown in [II, §2] that Theorem 3 is equivalent to the following

**Theorem 3'.** For any \( H_1(F) \)-module \( \pi_1 \) we have \( \text{tr}(I(\pi_1))(f \times \sigma) = \text{tr} \, \pi_1(f_i) \) for all pairs \( f, f_1 \) of matching functions on \( G(F) \) and \( H_1(F) \).

Our plan is to prove Theorem 3 directly only in the easiest case of the trivial representation \( \pi_1 \), and then use the global trace formula to deduce Theorem 2, hence also 1 and 3.3'. We emphasize that our method is to compare the representation theoretic sides of the trace formula in order to derive a comparison of orbital integrals. This is a new type of application of the trace formula.

To simplify our proof we now assume that \( F \) has characteristic zero and odd residual characteristic. We shall prove Theorem 2 for any such \( F \). Then Theorem 3 follows for every local \( F \) with characteristic zero by [IV, §2]. Our proof here then establishes Theorem 2 when \( F \) has residual characteristic two, and characteristic zero. By virtue of [K'] each of Theorems 2 and 3 holds also when \( F \) is local of positive characteristic. For example, Theorem 2 follows at once from a statement which we proceed to state; it is a corollary to [K'].

**Theorem A.** Suppose that \( G \) is a group as in [K'], §1, \( F \) is a local field, \( f \) is a locally constant measure on \( G(F) \), and \( U \) is a compact subset of \( G(F) \) consisting of regular elements. Clearly there exists a positive integer \( l \) such that
the function $f$, and the restriction $\Phi(f(x,f))$ to $U$ of the orbital integral $\Phi(x,f)$ of $f$: both lie in the Hecke algebra $H_1(G,F)$ of $K_1(F)$-biinvariant measures on $G(F)$. Theorem A of $[K']$ asserts that there exists $m \geq l$, such that for every local field $F$ which is $m$-close to $F$, the morphism $\Phi : H_1(G,F) \to H_1(G,F')$ (defined in $[K']$) is an algebra isomorphism. The statement which we require is that $\Phi(x,f') = \phi(\Phi(x,f))$ for every $x$ in $X'(F) = K_1(F) \backslash G(F)/K_1(F)$, where $x' = \phi(x), f' = \phi(f)$, and $U' = \phi(U)$. The analogous statement for twisted orbital integrals is equally valid. To deduce Theorem 2 for $F$ of positive characteristic we take $F'$ of characteristic zero.

We begin with the proof of

**Proposition 1.** If $\pi_1$ is the trivial $H_1(F)$-module, $\pi = I(\pi_1)$, and $\delta$ a $\sigma$-regular element of $G(F)$ with elliptic regular norm $\gamma_1 = N_1 \delta$, then $(A(\delta)/A_1(\gamma_1)) z_\delta(\delta) = \kappa(\delta)$.

**Proof.** To compute the character of $\pi$ we shall express $\pi$ as an integral operator in a convenient model, and integrate the kernel over the diagonal. Denote by $P$, trivial on $N$, by $\mu_\pi(p) = \mu((\det m)/m^{s-1})$ if $p = mn$ and $m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix}$ in $GL(2,F), m' \in GL(1,F)$. If $s = 0$, then $\mu_\pi = \delta^{1/2}$. Let $W_\pi$ be the space of complex-valued smooth functions $\psi$ on $G(F)$ with $\psi(pg) = \mu_\pi(p) \psi(g)$ for all $p$ in $P$ and $g$ in $G(F)$. The group $G(F)$ acts on $W_\pi$ by right translation: $\psi(g) = \psi(hg)$. By definition, $I(\pi_1)$ is the $G$-module $W_\pi$ with $s = 0$. The parameter $s$ is introduced for purposes of analytic continuation.

We prefer to work in another model $V_\pi$ of the $G$-module $W_\pi$. Let $V$ denote the space of column 3-vectors over $F$. Let $V_\pi$ be the space of smooth complex-valued functions $\phi$ on $V - \{0\}$ with $\phi(\lambda v) = \mu_\pi(\lambda^{s-1}) \phi(v)$. The expression $\mu((\det g)/g^{s-1})$, which is initially defined for $g$ in $GL(3,F)$, depends only on the image of $g$ in $G(F)$. The group $G(F)$ acts on $V_\pi$ by $(\tau_\pi(g))(v) = \mu((\det g)/g^{s-1}) \phi(v)$. Let $v_0 = 0$ be a vector of $V$ such that the line $\{ \lambda v_0 : \lambda \in F \}$ is fixed under the action of $P$. Explicitly, we take $v_0 = (0,0,1)$. It is clear that the map $V_\pi \to V$, $\phi \to \psi = \phi_\psi$, where $\psi(g) = (\tau_\pi(g))(v_0) = \mu((\det g)/g^{s-1}) \phi_\psi(g_{v_0})$, is a $G$-module isomorphism, with inverse $\psi \to \phi = \phi_\psi$. $\phi(v) = \mu((\det g)/g^{s-1}) \psi(g)$ if $v = g v_0$. $G$ acts transitively on $V - \{0\}$.

For $v = (x,y,z)$ in $V$ put $\|v\| = \max(|x|, |y|, |z|)$. Let $V^0$ be the quotient of the set of $v$ in $V$ with $\|v\| = 1$ by the equivalence relation $v \sim w$ if $\alpha$ is a unit in $R$. Denote by $PV$ the projective space of lines in $V - \{0\}$. If $\Phi$ is a function on $V - \{0\}$ with $\Phi(0) = 0$, $\Phi(0) = z^{-s} \Phi(0)$ and $dv = dx \, dy \, dz$, then $\Phi(v) \, dv$ is homogeneous of degree zero. Define

$$\int_{V^0} \Phi(v) \, dv = \int_{V^0} \Phi(v) \, dv = \int_{V^0} \Phi(v) \, dv = |\det g| \int_{V^0} \Phi(gv) \, dv.$$

Clearly we have

$$\int_{PV} \Phi(v) \, dv = \int_{PV} \Phi(gv) \, dv = |\det g| \int_{PV} \Phi(gv) \, dv.$$

Put $v(x) = |x|$ and $m = 3(s - 1)/2$. Note that $v/\mu_\pi = \mu_{-s}$. Put $\langle v, w \rangle = \langle v, w \rangle = \langle v, w \rangle$. Then

$$\langle g v, \sigma(g) w \rangle = \langle v, w \rangle.$$
Lemma 1. The operator $T_s: V_q \rightarrow V_{-s}$,

$$(T_s \phi)(v) = \int_{\mathbb{P}^V} \phi(w) \langle w, v \rangle^s dw,$$

converges when $\text{Re } s > 2/3$ and satisfies $T_s \tau_s(g) = \tau_{-s}(\sigma g) T_s$ for all $g$ in $G(F)$ where it converges.

Proof. We have

$$(T_s(\tau_s(g)\phi))(v) = \int (\tau_s(g)\phi)(w) \langle w, Jv \rangle^s dw = \mu(\det g) \int \phi(gw) \langle gw, Jw \rangle^s dw$$

$$= \mu(\det g) \int \phi(w) \langle g^{-1}w, Jg^{-1}Jw \rangle^s dw$$

$$= \mu(\det g) \int \phi(w) \langle w, \sigma(g)w \rangle^s dw = \mu(\mu(\det \sigma g)) \int (T_s \phi)(\sigma(g)v)$$

$$= \mu(v) \mu(\sigma(g)) \langle \tau_s(\sigma g)(T_s \phi)(v) \rangle,$$

as required.

The spaces $V_q$ are isomorphic to the space $W$ of locally-constant complex-valued functions on $V^0$, and $T_s$ is equivalent to an operator $T^0_s$ on $W$. The proof of Lemma 1 implies also

Corollary 1. The operator $T^0_s(\tau_s(g^{-1}))$ is an integral operator with kernel

$$(\mu/v)(\det \sigma g) \langle w, \sigma(g^{-1})v \rangle^s (v, w \in V^0)$$

and trace

$$\text{tr}[T^0_s(\tau_s(g^{-1}))] = \int_{V^0} \langle v, gJv \rangle^s dv.$$  

Remark. (1) In the domain where the integral converges, it is clear that $\text{tr}[T^0_s(\tau_s(g^{-1}))]$ depends only on the $\sigma$-conjugacy class of $g$ if (and only if) $s = 0$.

(2) We evaluate below this integral at $s = 0$ in a case where it converges for all $s$, and no analytic difficulties occur. However, in the context of the Remark following the proof of our proposition, we claim that to compute the trace of the analytic continuation of $T^0_s(\tau_s(g^{-1}))$ it suffices to compute this trace for $s$ in the domain of convergence, and then evaluate the resulting expression at the desired $s$. Indeed, for each compact open $\sigma$-invariant subgroup $K$ of $G$ the space $W_K$ of $K$-biinvariant functions on $W$ is finite dimensional. Denote by $p_K: W \rightarrow W_K$ the natural projection. Then $T^0_s(\tau_s(g^{-1})) \cdot p_K$ acts on $W_K$, and the trace of the analytic continuation of $T^0_s(\tau_s(g^{-1})) \cdot p_K$ is the analytic continuation of the trace of $T^0_s(\tau_s(g^{-1})) \cdot p_K$. Since $K$ can be taken to be arbitrarily small the claim follows.

Next we normalize the operator $T = T_s$ so that it acts trivially on the one-dimensional space of $K$-fixed vectors in $V_q$. This space is spanned by the function $\phi_0$ in $V_q$ with $\phi_0(v) = 1$ for all $v$ in $V^0$. Fix a local uniformizer $\pi$ in $R$. Let $q$ be the cardinality of the quotient field of $R$. Normalize the valuation $|\cdot|$ by $|\pi| = q^{-1}$. Normalize the measure $dx$ by $\int_{|x|<1} dx = 1$, so that $\int_{|x|=1} dx = 1 - q^{-1}$.

In particular, the volume of $V^0$ is $(1 - q^{-3})(1 - q^{-1}) = 1 + q^{-1} + q^{-2}$.

Lemma 2. We have $(T\phi_0)(v_0) = (1 - q^{-3}(1 + 1/2))(1 - q^{-1} - 3^1/2) \cdot \phi_0(v_0)$. When $s = 0$ the constant is $-q^{-1/2}(1 + q^{-1/2} + q^{-1})$. 

Proof:

\[ \int \phi_0(t) J T_0^w J v_0^{(n)} dV = \int_{|x| \leq 1} |x|^n dx \, dy \, dz = (1 - q^{-3(s + 1)/2}) \int_{|x| \leq 1} |x|^n dx \int_{|x| = 1} dx, \]

as required.

To complete the proof of the proposition we have to compute \( \text{tr}[T \circ T_1(\delta^{-1})] \).

Let \( T = T_0^w \). Put \( a = \begin{pmatrix} \alpha & 1 \\ \theta & u \end{pmatrix} \) with \( \alpha \neq 0 \) in \( F \) and \( \theta \) in \( F - F^2 \) with \( |\theta| = 1 \) or \( |\theta| = q^{-1} \).

Put

\[ \delta = \delta_a = u^{-1} a e_1 \begin{pmatrix} -\alpha & 0 \\ 0 & u \alpha \end{pmatrix}, \]

where \( u \) ranges over a set of representatives in \( F^* \) for \( F^*/NK^* \), where \( K = F(\theta^{1/2}) \). Then \( \delta = u(\theta - x^2) \). The eigenvalues of \( \delta \sigma(\delta) = (-\det a)^{-1} a^2 \) are \( \mu, \lambda, \lambda^{-1} \) where \( \lambda = -(\alpha + \theta^{1/2})(\alpha - \theta^{1/2}) \). We have

\[ (1 + \lambda)(1 + \lambda^{-1}) = \left( 1 - \frac{\alpha + \theta^{1/2}}{\alpha - \theta^{1/2}} \right) \left( 1 - \frac{\alpha - \theta^{1/2}}{\alpha + \theta^{1/2}} \right) = -4 \theta/\alpha^2 - \theta, \]

hence

\[ (\nu/\mu) (\det \delta) A(\delta)/A(\gamma_1) = |u(\alpha^2 - \theta)|^{(1 - s)/2} |4 \theta/(\alpha^2 - \theta)|^{1/2} = |4 \alpha \theta|^{1/2} |u(\alpha^2 - \theta)|^{-s/2}. \]

Further,

\[ \delta J = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & u & 0 \\ \alpha & 0 & -\theta \end{pmatrix}, \]

hence \( \text{tr} J = x^2 + u y^2 - \theta z^2 \). Consequently

\[ \frac{A(\delta)}{A(\gamma_1)} \text{tr}[T \circ T_1(\delta^{-1})] = |4 \alpha \theta|^{1/2} |u(\alpha^2 - \theta)|^{-s/2} \int_{|x| \leq 1} |u y^2 + x^2 - \theta z^2|^{3s - 1/2} dx \, dy \, dz. \]

We are interested in the value of this expression at \( s = 0 \). When \( \kappa(\delta) = 1 \) the quadratic form \( u y^2 + x^2 - \theta z^2 \) represents zero. Then the integral converges only for \( s > 2/3 \), but not at \( s = 0 \). At \( s = 0 \) the integral can be evaluated by analytic continuation. However when \( \kappa(\delta) = -1 \) the quadratic form \( u y^2 + x^2 - \theta z^2 \) is anisotropic, hence reaches a non-zero minimum (in valuation) on the compact set \( \|v\| = 1 \). Consequently the integral converges for all values of \( s \), and we may restrict our attention to the case of \( s = 0 \). Here the character depends only on the \( \sigma \)-conjugacy class of \( \delta \), and we may take \( |u| = 1 \) if \( |\theta| = q^{-1} \), and \( |u| = q^{-1} \) if \( |\theta| = 1 \). Then \( |u \theta|^{1/2} = q^{-1/2} \) and

\[ \int_{|x| \leq 1} |u y^2 + x^2 - \theta z^2|^{3s - 1/2} dx \, dy \, dz = (1 + q^{-1/2} + q^{-1}) \int_{|x| \leq 1} dx. \]

We conclude that

\[ \frac{A(\delta)}{A(\gamma_1)} \text{tr}[T \circ T_1(\delta^{-1})] = \kappa(\delta) \text{tr}(T \phi_0) (v_0) \]
when $\kappa(\delta) = -1$, hence for all $\sigma$-regular $\delta$ with elliptic $\gamma_1 = N_1 \delta$, by Theorem 0. Since $t_\delta(\delta) = \text{tr}[T_\delta(\delta) - T]/(T_\delta(\delta))$, the proposition follows.

**Remark.** It is clear that when $\kappa(\delta) = 1$ the proof of Proposition 1 can be completed without using Theorem 0 on computing $\text{tr}[T_\delta(\delta)^{-1}]$ by analytic continuation, namely first for large $\Re s$ and then on evaluating the resulting expression at $s = 0$.

To prove Theorem 2 we have to take corresponding spherical functions $f$ and $f_0$, and show that $A(\delta) \Phi(\delta, f) = \Delta(\gamma_1) \Phi(\gamma_1, f_0)$ for all $\sigma$-regular $\delta$ with $N_1 \delta$. When $\gamma_1$ is split (its centralizer in $H_1(F)$ is conjugate to the diagonal torus), then the stable $\sigma$-conjugacy class of $\delta$ consists of a single $\sigma$-conjugacy class, $\kappa(\delta) = 1$ and the required relation follows formally from the definition of $f_0, f$ being corresponding (see [11, §1]). Hence we have to prove the equality when $\gamma_1$ is elliptic regular (the quotient $\lambda$ of its eigenvalues generates a quadratic extension of $F$).

The proof of Theorem 2 which is now to follow is global and uses the trace formula. We shall use the notations of [IV, §1] without much ado. That is, we fix a totally imaginary global field $F$ such that its completion at a place $u$ is our local field, which is now denoted by $F_u$. Let $\phi_u$ be a pseudo-coefficient of the Steinberg $H_1(F_u)$-module $St_{1u}$. Here $St_{1u}$ is the complement of the trivial $H_1(F_u)$-module in the $H_1(F_u)$-module $I_{1u}(\gamma_1^{(1)}_{u})$ unitarily induced from the character $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \rightarrow |u/a|^{1/2}$ of the upper triangular subgroup of $H_1(F_u)$. By definition, $\text{tr} St_{1u}(\phi_{1u}) = 1$ and $\text{tr} \pi_{1u} (\phi_{1u}) = 0$ for any tempered irreducible $H_1(F_u)$-module $\pi_{1u}$ inequivalent to $St_{1u}$. By [K], Theorem K, the orbital integral $\Phi(\gamma_1, \phi_{1u})$ is zero if the quotients of the eigenvalues of $\gamma_1$ lie in $F_u^*$ but are different from $1$ or $-1$, and $\Phi(\gamma_1, \phi_{1u})$ is equal to $-1$ if the quotients of the eigenvalues of $\gamma_1$ do not lie in $F_u^*$. Note that $\text{tr} \pi_{1u} (\phi_{1u}) = -1$ if $\pi_{1u}$ is the trivial $H_1(F_u)$-module, and $\text{tr} \pi_{1u} (\phi_{1u}) = 0$ for any other non-tempered representation.

Let $\phi_u$ be a function on $G(F_u)$ matching $\phi_u$ such that $\Phi(\delta, \phi_u) = -\Phi(\delta', \phi_u)$ if $\delta, \delta'$ are $\sigma$-regular stably $\sigma$-conjugate but not $\sigma$-conjugate elements. The existence of such a function is proven in [I, §3] by a local elementary proof. If $\pi_{1u}$ is induced then the (twisted) character $\chi_{\pi_{1u}}$ of $\pi_{1u} = I(\gamma_1^{(1)}_{u})$ is supported on the $\sigma$-split set (see [11], §1), hence $\text{tr} \pi_{1u}(\phi_u \times \sigma) = 0$. By Proposition 1 for $\sigma$-elliptic regular $\delta$ we have

$$\kappa(\delta) A(\delta) \otimes (\gamma_1^{(1)}_{u}) (\Phi(\delta, \phi_u) = A(\gamma_1, \phi_u) \Phi(\gamma_1, \phi_{1u}) = A(\delta, \phi_u) \Phi(\delta, \phi_u) \kappa(\delta),$$

hence $2 \Phi(\delta, \phi_u) = \chi_{\pi_{1u}}(\delta, \phi_u)$, and by [II, §3] we conclude that $\text{tr} (I(\pi_{1u})) (\phi_u \times \sigma) = 1$, hence $\text{tr} (I(\pi_{1u})) (\phi_u \times \sigma) = -1$ if $\pi_{1u}$ is the trivial $H_1(F_u)$-module. If $\pi_{1u}$ is a square-integrable $H_1(F_u)$-module inequivalent to $St_{1u}$ then $I(\pi_{1u})$ and $I(\pi_{1u})$ are not relatives in the terminology of [K], hence their characters are orthogonal by [K], Theorem G, as stated in [II, §3]. Note that although the work of [K] is formulated in the non-twisted case only, the twisted analogue follows by the same proof on noting that the twisted analogue of [K], Theorem 0, is available (in our case it is given in [IV, §1]). In particular we have $\text{tr} (I(\pi_{1u})) (\phi_u \times \sigma) = 0$ for all tempered $\pi_{1u}$ inequivalent to $St_{1u}$. Moreover, since the residual
characteristic of \( F \) is odd, all \( \sigma \)-invariant elliptic \( G(F_u) \)-modules not mentioned above are of the form \( I(\pi_\sigma(\theta) \otimes \chi_a) \) by [IV, §1] and in the notations of [IV, §1] (thus \( \chi_a \) is a quadratic character and \( \theta \) is a character of the quadratic extension of \( F \) determined by \( \chi_a \) and class field theory). Their characters are \( \sigma \)-stable by [IV, §2], hence \( \text{tr} \pi(\phi_a \times \sigma) = 0 \) for such \( \pi \). In summary we have

**Proposition 2.** There exist matching functions \( \phi_a \) and \( \phi_{1\sigma} \) on \( G(F_u) \) and \( H_1(F_u) \) with

1. \( \text{tr}(I(\pi_{1\sigma})) \phi_{1\sigma} \times \sigma = \text{tr} \pi_{1\sigma}(\phi_{1\sigma}) \) for all \( \pi_{1\sigma} \);
2. \( \text{tr} \pi_{1\sigma}(\phi_{1\sigma}) = -1 \) if \( \pi_{1\sigma} \) is trivial, \( \text{tr} \pi_{1\sigma}(\phi_{1\sigma}) = 1 \) if \( \pi_{1\sigma} = \text{St}_{1\sigma} \) and \( \text{tr} \pi_{1\sigma}(\phi_{1\sigma}) = 0 \) otherwise;
3. \( \text{tr} \pi_{1\sigma}(\phi_{1\sigma}) = 0 \) unless \( \pi_{1\sigma} \) is \( I(\text{St}_{1\sigma}) \) or \( I(\pi_{1\sigma}) \) with trivial \( \pi_{1\sigma} \); (4) \( \Phi(\gamma, \phi_{1\sigma}) \) is \(-1\) on the regular elliptic set and zero on the regular split set;
5. \( \Phi(\delta, \phi_{1\sigma}) = -\Phi(\delta', \phi_{1\sigma}) \) if \( \delta, \delta' \) are \( \sigma \)-elliptic regular stably \( \sigma \)-conjugate non-\( \sigma \)-conjugate elements of \( G(F_u) \).

For any place \( v \) of \( F \), let \( \mu_v \) be a character of \( F_v^* \) and \( \pi_{1v} \), the \( H_1(F_v) \)-module \( I(\mu_v) \) unitarily induced from the character \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \rightarrow \mu_v(a/c) \) of the upper triangular subgroup. Then the character of \( \pi_{1v} \) is supported on the split set, the character of \( I(\pi_{1\sigma}) \) is supported on the \( \sigma \)-split set, they are easily computable and comparable (see [II, §1]), and we have \( \text{tr}(I(\pi_{1v})) \phi_{1v} \times \sigma = \text{tr} \pi_{1v}(\phi_{1v}) \) for all matching \( f_v \) and \( f_{1v} \). When \( v \) is an archimedean place of \( F \) then the completion \( F_v \) is \( \mathbb{C} \) since \( F \) is totally imaginary. Since any unitary non-induced \( H_1(\mathbb{C}) \)-module is one-dimensional, and its character is the difference of the characters of two induced \( H_1(\mathbb{C}) \)-modules, we conclude

**Proposition 3.** If \( F_v = \mathbb{C} \) and \( f_v, f_{1v} \) are matching functions then \( \text{tr}(I(\pi_{1v}))(f_v \times \sigma) = \text{tr} \pi_{1v}(f_{1v}) \) for all \( H_1(F_v) \)-modules \( \pi_{1v} \).

This is a trivial case of Theorem 3', where \( F_v = \mathbb{C} \).

Let \( u_0, u_1 \) be two finite distinct places of \( F \), different from \( u_v \), of odd residual characteristic. Denote by \( \mathcal{A} \) the ring of adeles of \( F \). Let \( f_1 = \bigotimes f_{1v} \) be a function on \( H_1(\mathcal{A}) \) with \( f_{1v} = f_{1v}^* \) for almost all (finite) \( v \), with \( f_{1v} \) in the Hecke algebra \( H_{1v} \) for all finite \( v \neq u, u_0, u_1 \), and with \( f_{1u} = \phi_{1u}, f_{1u_0} = \phi_{1u_0} \), and \( f_{1u_1} = \phi_{1u_1} \).

Let \( f = \bigotimes f_v \) be a function on \( G(\mathcal{A}) \) such that

1. \( f_v = \phi_v, f_{1u} = \phi_{1u}, f_{1u_0} = \phi_{1u_0} \), and \( f_{1u_1} = \phi_{1u_1} \),
2. at each finite \( v \neq u, u_0, u_1 \) the component \( f_v \) lies in \( H_{1v} \) and \( f_{1v}, f_{1v} \) are corresponding (in particular, \( f_v \) is \( f_v^* \) for almost all \( v \)),
3. at each archimedean place \( v \) the functions \( f_v \) and \( f_{1v} \) are matching. Then \( \text{tr} \pi_{1v}(f_{1v}) = \text{tr}(I(\pi_{1v}))(f_v \times \sigma) \) for every \( H_1(\mathbb{A}) \)-module \( \pi_{1v} \), for every place \( v \), and therefore \( \text{tr} \pi_1(f_{1v}) = \text{tr}(I(\pi_{1v}))(f_v \times \sigma) \) for every \( H_1(\mathbb{A}) \)-module \( \pi_1 \). In particular, we obviously have the following

**Proposition 4.** We have

\[
\sum_{\pi_1} \text{tr} \pi_1(f_{1v}) = \sum_{\pi_{1v}} \text{tr} (I(\pi_{1v}))(f_v \times \sigma).
\]

Here both sums range over all discrete-series (cuspidal or one-dimensional) automorphic \( H_1(\mathbb{A}) \)-modules \( \pi_1 \). Of course the \( \pi_1 \) which contribute a non-zero term have a Steinberg or trivial component at the places \( u, u_0, u_1 \), while all of their other finite components are unramified.

Choose a component \( f_{1v} \) at an archimedean place \( v \) to vanish on the set of non-\( \sigma \)-regular elements \( \delta \) in \( G(F_v) \). Since at \( v = u_0, u_1 \) the components \( f_v \)
(resp. $f_{1,s}$) have orbital integrals which vanish on the $\sigma$-regular-split (resp. regular split) sets, the trace formula for $H_1$ asserts the following

**Proposition 5.** We have

$$\sum_{\gamma_1} c(\gamma_1) A_1(\gamma_1) \Phi(\gamma_1, f_1) = \sum_{\gamma_1} \text{tr} \pi_1(f_1).$$

Here $\gamma_1$ ranges over the set of regular elliptic conjugacy classes in $H_1(F)$, $c(\gamma_1)$ is a volume factor, and

$$A_1(\gamma_1) \Phi(\gamma_1, f_1) = \Phi(\gamma_1, f_1)$$

is the product $\prod_\pi A_\pi(\gamma_1) \Phi(\gamma_1, f_1).$

Since the stable twisted orbital integrals of $f_{1,n}$ (and $f_n$) are zero, the twisted trace formula for $G$ in its stabilized form (see [III, § 3]), asserts the following

**Proposition 6.** We have

$$\sum_{\gamma_1} c(\gamma_1) A(\gamma_1) \Phi(\gamma_1, f_1) = \sum_{\gamma_1} \text{tr} \pi(\gamma_1) (f \times \sigma).$$

The sum over $\gamma_1$ and the volume factors $c(\gamma_1)$ are the same as above (as noted in [III, § 1]), $\delta$ signifies (a representative of) the stable $\sigma$-conjugacy class in $G(F)$ with $\gamma_1 = N_1 \delta$, and $A(\delta) \Phi^{s\sigma}(\delta, f) = \Phi^{s\sigma}(\delta, f)$ the product $\prod_\pi A_\pi(\delta) \Phi^{s\sigma}(\delta, f_\pi).$

We briefly sketch the stabilization argument on which the proof of Proposition 6 is based. The sum over $\sigma$-conjugacy classes in the twisted trace formula can be expressed as a sum over the set of stable $\sigma$-conjugacy classes $\delta_0$ in $G(F)$, of the sums $\sum \Phi(\delta, f)$, where $\delta$ ranges over the set $D(\delta_0/F)$ of $\sigma$-conjugacy classes in $G(F)$ within the stable $\sigma$-conjugacy class of $\delta_0$. Since $\delta_0$ is $\sigma$-regular elliptic for our $f$, the set $D(\delta_0/F)$ is isomorphic to $F^* / NK^*$. Hence we have:

$$\sum_{\delta \in D(\delta_0/F)} \Phi(\delta, f) = \frac{1}{2} \sum_{\delta \in D(\delta_0/F)} \Phi(\delta, f) + \frac{1}{2} \sum_{\delta \in D(\delta_0/F)} \kappa(\delta) \Phi(\delta, f).$$

Since the stable orbital integral $\Phi^{s\sigma}(\delta, f_{\sigma}) = \Phi(\delta, f_{\sigma}) + \Phi(\delta', f_{\sigma})$ is assumed to be zero, we have that

$$\sum_{\delta \in D(\delta_0/F)} \Phi(\delta, f) = \prod_v \Phi^{s\sigma}(\delta_0, f_v) = \Phi^{s\sigma}(\delta_0, f)$$

is zero, implying the desired equality

$$\sum_{\delta \in D(\delta_0/F)} \Phi(\delta, f) = \frac{1}{2} \Phi^{s\sigma}(\delta_0, f)$$

for our $f$, from which Proposition 6 follows.

Combining Propositions 4, 5 and 6 we obtain

**Proposition 7.** We have

$$\sum_{\gamma_1} c(\gamma_1) A_1(\gamma_1) \Phi(\gamma_1, f_1) = \sum_{\gamma_1} c(\gamma_1) A(\delta) \Phi^{s\sigma}(\delta, f).$$
Lemma. Both sums in (7) are finite.

Proof. Identifying $\text{PGL}(2)$ with the subgroup $SO(3)$ of $GL(3)$, we note that $\gamma_1$ is determined by the coefficients in its characteristic polynomial. These coefficients are rational (in $F$), and $f_1$ (or $f$) is compactly supported, whence the sums are finite.

Let $\gamma_1^0$ be a regular elliptic element of $H_1(F)$. Then there exists an element $\gamma_1^0$ of $H_1(F)$ which is elliptic regular in $H_1(F_a)$ and $H_1(F_u)$, and whose orbit in $H_1(F)$ is as close to that of $\gamma_1^0$ as desired. At each $v \neq u$, $u_0$, $u_1$, choose $f_{1,v}$ with $\Phi(\gamma_1^0, f_{1,v}) \neq 0$ such that $f_{1,v} = f_{1,u}$ for almost all $v$; this is clearly possible, since $\gamma_1^0$ is a rational element, in $H_1(F)$. For our $f_{1,v}$ which depends on $\gamma_1^0$, the sum on the left of (7) is finite, and includes $\gamma_1^0$. We now replace $f_{1,v}$ by its product with a smooth function which takes the value one on the orbit of $\gamma_1^0$ in $H_1(F_a)$ and vanishes outside of a small neighbourhood of this orbit; choosing a suitable replacement we may assume that $\gamma_1^0$ is the only class which contributes a non-zero term on the left of (7).

Next we denote by $\delta^0$ the stable $\sigma$-conjugacy class in $G(F)$ with $N_1 \delta^0 = \gamma_1^0$. The sum on the right of (7) is also finite. We can replace the component $f_{1,u}$ as above, by another function with the property that $\Phi^\sigma(\delta^0, f_{1,u})$ will not change, yet $\Phi^\sigma(\delta, f_{1,u})$ be zero at each of the finitely many (stable) classes $\delta + \delta^0$ which appear on the right of (7). We conclude that $f_{1,v}$ and $f_{1,u}$ can be chosen so that we obtain the following

Proposition 8. We have

$$\prod_{v \neq u} A_1(\gamma_1^0)(\gamma_1^0, f_{1,v}) = \prod_{v \neq u} A_1(\delta^0)(\delta^0, f_{1,v}).$$

(8)

In particular the right side here is non-zero.

Proof. It is clear that we have (8) where the product ranges over all places $v$. Since $\gamma_1^0$ is elliptic regular in $H_1(F_a)$ and $f_{1,u}$ is a pseudo-coefficient of the Steinberg $H_1(F_a)$-module, we have $\Phi(\gamma_1^0, f_{1,u}) = -1$. Further we have

$$A_1(\gamma_1^0, f_{1,u}) = A_1(\delta^0, f_{1,u})$$

since $f_{1,u}$ and $f_{1,u}$ are matching (by definition of $f_{1,u}$, which uses Proposition 2: here $f_{1,u} = \phi_{1,u}$ and $f_{1,u} = \phi_{1,u}$). Hence we can take the product to range only over $v \neq u$, as asserted.

We can now complete the proof of Theorem 2. Let $\phi$ and $\phi_1$ be corresponding elements of $H_1$ and $H_{1,u}$. We have to show that

$$A_1(\gamma_1^0, \phi_{1,u}, \phi_1) = A_1(\gamma_1^0, \phi_{1,u}, \phi_1)$$

for any regular elliptic $\gamma_1^0$ in $H_1(F_a)$ and $\delta^0$ with $\gamma_1^0 = N_1 \delta^0$. Since $\phi, \phi_1$ are locally constant it suffices to show the following

Proposition 9. We have

$$A_1(\gamma_1^0, \phi_1, \phi_1) = A_1(\delta^0, \phi)$$

for $\gamma_1^0$ as above.
Proof. Let \( f_1 = \oplus f_1^v \) and \( f' = \oplus f'_v \) be the functions obtained from \( f_v = \oplus f_v^v \) and \( f = \oplus f_v \) by replacing the components \( f_v^v \) and \( f_v \) by \( \phi \) and \( \phi \) (thus \( f_v^v = f_v \) for \( v \neq u \)). Repeating the discussion leading to (8) with \( f_1, f' \) instead of \( f_1, f \) we obtain
\[
\prod_v \mathcal{A}_v(\gamma_1^v) \Phi(\gamma_1^v, f_1^v) = \prod_v \mathcal{A}_v(\delta^0) \Phi^{\varepsilon}(\delta^0, f'_v).
\]
Since both sides of (8) are non-zero, (9) follows, and Theorem 2 is proven.

As explained above, this completes the proof of Theorems 1, 3 and \( 3' \) as well.

Remark. (1) The proof given above can be adapted to establish the analogous unstable twisted transfer of spherical functions from \( GL(3, E) \) to \( U(2) \), which is stated (but neither used nor proved) in Lemma 3.4 in [U1]; however we do not discuss this here. (2) The same method applies also in the study of the endo-lifting from \( GL(m, E) \) to \( GL(n, F) \), where \( E/F \) is a cyclic extension of degree \( n/m \); see [F].

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