BERNSTEIN'S ISOMORPHISM AND GOOD FORMS

Yuval Z. Flicker

A. Statement of Main Result.

Let G be a p-adic reductive group, and σ an automorphism of G of finite order ℓ. A G-*module* is a representation $\pi : G \to \operatorname{Aut} V$ of the group G on a complex vector space V, which is *smooth* (terminology of Bernstein-Deligne-Kazhdan [BDK]; in Bernstein-Zelevinski [BZ1] this is termed algebraic): the stabilizer of any vector in V is an open subgroup of G. It will be denoted by (π, V), or simply π, or V. Put ${}^{\sigma}\pi(g) = \pi(\sigma^{-1}g)$. Then $({}^{\sigma}\pi, V)$ is a G-module. A G-module (π, V) is called σ-*invariant* if it is equivalent to $({}^{\sigma}\pi, V)$. Denote by $\operatorname{Aut}^{\sigma}_{G}\pi$ the set of vector space automorphisms $S : V \xrightarrow{\sim} V$ with $S\pi(g) = \pi(\sigma g)S$ for all $g \in G$ and $S^{\ell} = 1$. Then π is σ-invariant if and only if $\operatorname{Aut}^{\sigma}_{G}\pi$ is non-empty. In this case π extends to a $G^{\#}$-module by $\pi(\sigma) = S$, where $G^{\#}$ is the semi-direct product $G \rtimes <\sigma>$ of G with the group $<\sigma>$ generated by σ. When π is irreducible then S is uniquely determined up to an ℓth root ζ of unity in $\mathbb{C}$.

Let $\mathbb{M}(G)$ be the category of G-modules. An element E of $\mathbb{M}(G)$ is called *finitely generated* if for any filtered system of proper subobjects E_i in E, the subobject $\Sigma_i E_i$ is proper in E. Let $K(G)$ be the Grothendieck group of finitely generated G-modules, and $R(G)$ the Grothendieck group of G-modules of finite length. The group $K(G)$ coincides with the Grothendieck group of *projective* (i.e. the functor $E \mapsto \operatorname{Hom}(P, E)$ is exact) finitely generated G-modules P. Indeed, each finitely generated G-module has a projective resolution consisting of finitely generated G-modules, and this resolution is finite by virtue of the Theorem of Bernstein [B] recorded in the Appendix. This Theorem asserts that the category $\mathbb{M}(G)$ has finite cohomological dimension.

The group $R(G)$ is the free abelian group generated by the set $\operatorname{Irr} G$ of equivalence classes of irreducible G-modules. Denote by $\operatorname{Irr}^{\sigma}(G)$ the subset of σ-invariant elements in $\operatorname{Irr} G$. Let $R^{\sigma}(G)$ (resp. $K^{\sigma}(G)$) be the quotient of the free abelian group generated by the pairs (π, S) where π is a G-module of finite length (resp. projective finitely generated) and $S \in \operatorname{Aut}^{\sigma}_{G}\pi$, by the following relations.

(R_1) If $0 \to (\pi', S') \to (\pi, S) \to (\pi'', S'') \to 0$ is exact then $(\pi, S) \sim (\pi', S') + (\pi'', S'')$.

(R_2) If $\pi = \oplus_i \pi_i$ and for each i there is j such that $S\pi_i = \pi_j$, then $(\pi, S) \sim \Sigma_i(\pi_i, S_i)$, where the sum ranges over all i such that $S\pi_i = \pi_i$, and $S_i = S|\pi_i$ for such i.

The abelian group $R^{\sigma}(G)$ is generated by the pairs (π, S), $\pi \in \operatorname{Irr}^{\sigma} G$ and $S \in \operatorname{Aut}^{\sigma}_{G}\pi$ with $S^{\ell} = 1$. For any $\mathbb{Z}$-modules R and T put R_T for $R \otimes_{\mathbb{Z}} T$. The quotient of $R^{\sigma}(G)_{\mathbb{C}}$ by the relations $(\pi, \zeta S) \sim \zeta(\pi, S)$ for all $\pi \in \operatorname{Irr}^{\sigma}(G)$, $S \in \operatorname{Aut}^{\sigma}_{G}\pi$, $\zeta \in \mathbb{C}$ with $\zeta^{\ell} = 1$, is the free $\mathbb{C}$-module $\widetilde{R}^{\sigma}(G)_{\mathbb{C}}$ generated by $\operatorname{Irr}^{\sigma}(G)$.

Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210-1174; email: flicker@math.ohio-state.edu

Typeset by $\mathcal{AMS}$-TEX

Fix a minimal parabolic subgroup P_0 of G. Suppose that $\sigma P_0 = P_0$. If $P_0 = M_0U_0$ is a Levi decomposition, then $\sigma M_0 = u^{-1}M_0u$ for some u in (the unipotent radical) U_0 with $u\sigma(u)\cdots\sigma^{\ell-1}(u) = 1$. Since U_0 is an extension of additive groups, its first galois cohomology group is trivial, and there is $u' \in U_0$ with $u = u'\sigma(u')^{-1}$. Replacing M_0 by its conjugate by u' we may assume that the Levi subgroup M_0 is σ-invariant: $\sigma M_0 = M_0$. A *standard* Levi subgroup is a subgroup $M \supseteq M_0$ of G which is a Levi component of a parabolic subgroup $P = P_0M$; such P is called a *standard* parabolic subgroup. Notations: $M < G, P < G$. Since P has a unique Levi subgroup containing a fixed minimal one, if $\sigma P = P$ then $\sigma M = M$ for $M < G$.

For $M < G$, let $i_{GM} : \mathbb{M}(M) \to \mathbb{M}(G)$ be the functor of normalized induction. Given an M-module (ρ, E), the space $V = i_{GM}E$ consists of all smooth maps $f : G \to E$ with $f(mug) = \delta_P^{\frac{1}{2}}(m)\rho(m)f(g)$ $(m \in M, g \in G,\ u \in U(=$ unipotent radical of $P = MP_0))$, where $\delta_P(m) = |(\det : N \to N, n \mapsto m^{-1}nm)|$, and $\pi = i_{GM}\rho$ acts on $i_{GM}E$ by $(\pi(x)f)(g) = f(gx)$. If $M < N < G$ and $M = \sigma M, N = \sigma N$, and (ρ, E) is σ-invariant, then $(\pi = i_{NM}\rho, V = i_{NM}E)$ is σ-invariant: define $\pi(\sigma)$ by $(\pi(\sigma)f)(g) = (\rho(\sigma)f)(\sigma^{-1}g)$. Denote by $JH(E)$ the subset of $\mathrm{Irr}\, G$ consisting of all irreducible constituents of the G-module E. The automorphism σ of G defines a functor $\mathbb{M}(G) \to \mathbb{M}(G)$. It is easy to see that ${}^{\sigma}i_{GM}(\rho) = i_{G,\sigma M}({}^{\sigma}\rho)$, hence that $\pi \in JH(i_{GM}\rho)$ if and only if ${}^{\sigma}\pi \in JH(i_{G,\sigma M}({}^{\sigma}\rho))$.

Let $r_{MG} : \mathbb{M}(G) \to \mathbb{M}(M)$ be the normalized functor of coinvariants. If (π, V) is a G-module, then the space $V_U = r_{MG}V$ is the quotient of V by the span $V(U)$ of $\pi(u)v - v,\ v \in V,\ u \in U$ (= unipotent radical of $P = MP_0$). The action $r_{MG}\pi$ of M on $r_{MG}V$ is by $m : v + V(U) \mapsto \delta_U^{-1/2}(m)\pi(m)v + V(U)$ (note that $\pi(M)$ stabilizes $V(U)$). If $M < N < G,\ \sigma N = N,\ \sigma M = M$, and (π, V) is a σ-invariant N-module, then $r_{MN}\pi$ is σ-invariant, since $\pi(\sigma)(V(U)) = V(U)$. The functors i_{GM} and r_{MG} define homomorphisms $i_{GM} : R(M) \to R(G)$ and $r_{MG} : R(G) \to R(M)$, and $i_{GM} : R^{\sigma}(M) \to R^{\sigma}(G),\ r_{MG} : R^{\sigma}(G) \to R^{\sigma}(M)$, when $M = \sigma M$. Let $\overline{P}$ be the parabolic subgroup of G opposite to P (then $M = P \cap \overline{P}$), and let $\overline{r}_{MG}$ be the normalized functor of invariants defined using $\overline{P}$ instead of P. If $P = \sigma P$ then $\overline{P} = \sigma\overline{P}$.

The group $X(G)$ of complex-valued unramified characters of G is naturally isomorphic to $\mathbb{C}^{\times d}$ for some $d = d(G) \geq 0$, hence has a natural structure of a complex algebraic group. It acts on $\mathrm{Irr}\, G$ and $R(G)$ by $\psi : \pi \mapsto \psi\pi$. Let $X^{\sigma}(G)$ be the group of ψ in $X(G)$ which are fixed by σ. It is a subvariety of $X(G)$ which acts on $\mathrm{Irr}^{\sigma}(G)$ and $R^{\sigma}(G)$.

Let $\mathbb{H}_G$ be the Hecke algebra of (locally-constant complex-valued compactly-supported measures on) G. Then $\mathbb{H}_G = C_c^{\infty}(G)dg$, where dg is a Haar measure. The automorphism σ acts on $\mathbb{H}_G$ by $\sigma(h\, dg) = {}^{\sigma}h dg$, where ${}^{\sigma}h(g) = h(\sigma^{-1}g)$. Put $\mathbb{H}_G^{\#}$ for the semi-direct product $\mathbb{H}_G \rtimes < \sigma >$. A measure h in $\mathbb{H}_G$ defines a linear form $F_h : R(G) \to \mathbb{C}$ by $F_h(\pi) = \mathrm{tr}\ \pi(h)$, and $F_h^{\sigma} : R^{\sigma}(G) \to \mathbb{C}$ by $F_h^{\sigma}((\pi, S)) = \mathrm{tr}\ \pi(h\sigma)$; here $\pi(h\sigma) = \pi(h)S$, and $\pi(h)$ is the convolution operator $\int_G h(g)\pi(g)$. This $\pi(h)$ is of finite rank on $V = V_\pi$ since π is admissible (smooth of finite length, see [BZ1]), hence $\pi(h\sigma)$ is of trace class. Note that $F_h^{\sigma}((\pi, \zeta S)) = \zeta F_h^{\sigma}((\pi, S))$ if $\zeta^{\ell} = 1$. It is useful to note that $\mathbb{H}_G$ is the tensor product with $\mathbb{C}$ over Q of the rational Hecke algebra of Q-valued measures with the above properties. A similar comment applies to $\mathbb{H}_K$ of §**B** below.

Let $R^*_\sigma(G) = \mathrm{Hom}_{\mathbb{Z},\zeta}(R^\sigma(G),\mathbb{C})$ $(= \mathrm{Hom}_{\mathbb{C}}(\widetilde{R}^\sigma(G)_{\mathbb{C}},\mathbb{C}))$ be the space of $\mathbb{C}$-valued linear forms F on $R^\sigma(G)$ which are "genuine", namely satisfy $F((\pi,\zeta S)) = \zeta F((\pi,S))$ for all $\zeta \in \mathbb{C}$ with $\zeta^\ell = 1$. Let $R^*_\sigma(G)_{\mathrm{tr}}$ be the subspace of the forms F^σ_h, $h \in \mathbb{H}_G$. A form in this subspace is called a *trace form.* Any trace form F is genuine and it satisfies:

(i) There exists a σ-invariant open compact subgroup K of G which *dominates* F. Namely $F((\pi,S)) = 0$ if π is a G-module which has no non-zero K-fixed vector, or alternatively $F((\pi,S))$ depends only on the space π^K of K-fixed vectors in π, and the restriction of S to π^K.

(ii) For any standard Levi subgroup $M = \sigma M < G$ and $\rho \in \mathrm{Irr}^\sigma(M)$, the function $\psi \mapsto F((i_{GM}(\psi\rho), i_{GM}(\rho(\sigma))))$ is a regular function on the complex algebraic variety $X^\sigma(M)$.

Denote by $R^*_\sigma(G)_{\mathrm{good}}$ the space of F in $R^*_\sigma(G)$ which satisfy (i), (ii); such forms will be called *good*.

Let $\tau_\sigma(\mathbb{H}_G)$ be the quotient of $\mathbb{H}_G$ by the linear span $[\mathbb{H}_G\sigma,\mathbb{H}_G]\sigma^{-1}$ of the commutators $f\sigma(h) - hf$ in $\mathbb{H}_G$. Then $\tau_\sigma(\mathbb{H}_G) \simeq \mathbb{H}_G\sigma/[\mathbb{H}_G\sigma,\mathbb{H}_G]$, where $[\mathbb{H}_G\sigma,\mathbb{H}_G]$ is the linear span (in $\mathbb{H}^\#_G$) of all commutators $f\sigma\cdot h - h\cdot f\sigma$; $f,h \in \mathbb{H}_G$. Note that $[\mathbb{H}_G\sigma,\mathbb{H}_G] = \mathbb{H}_G\sigma \cap [\mathbb{H}^\#_G,\mathbb{H}^\#_G]$.

Main Theorem. *The map* $\Psi : \mathbb{H}_G \to R^*_\sigma(G), h \mapsto F^\sigma_h$ *, yields an isomorphism* $\tau_\sigma(\mathbb{H}_G) \xrightarrow{\sim} R^*_\sigma(G)_{\mathrm{good}}$.

In the special case where $\ell = 1$ and $\sigma =$ identity, one has $R^*(G) = \mathrm{Hom}_{\mathbb{Z}}(R(G),\mathbb{C}) = \mathrm{Map}(\mathrm{Irr}\, G,\mathbb{C})$ and its subspaces $R^*(G)_{\mathrm{good}} \supset R^*(G)_{\mathrm{tr}}$. Put $\tau(\mathbb{H}_G) = \mathbb{H}_G/[\mathbb{H}_G,\mathbb{H}_G]$. The assertion that the map $\mathbb{H}_G \to R^*(G)_{\mathrm{good}}$ is surjective, namely that $R^*(G)_{\mathrm{tr}} = R^*(G)_{\mathrm{good}}$, is called the *trace Paley-Wiener theorem*; it is the main result of [BDK]. It is an analogue of the classical Paley-Wiener theorem which characterizes the image of the Fourier transform. The main ingredients in extending the proof of [BDK] to the twisted case, where σ is non-trivial, are explained in [F; I, §7]. As the twisted analogue requires only minor changes to the exposition of [BDK], it is noted in [F] that there is no need to reproduce the entire proof of [BDK] in the twisted setting.

The injectivity of the map $\tau(\mathbb{H}_G) \to R^*(G)$ implies the following *density theorem.* If $h \in \mathbb{H}_G$ satisfies $\mathrm{tr}\,\pi(h) = 0$ for all π in $R(G)$ then all orbital integrals $\Phi_h(\gamma) = \int h(g^{-1}\gamma g)\ (g \in Z_G(\gamma)\backslash G)$ of h at the regular elements γ, are zero. The density theorem is proven in Kazhdan [K1; Appendix] in characteristic zero, and subsequently in [K2; Theorem B], in positive characteristics. The proof of [K1] is global (it uses the trace formula) and requires non-trivial galois-cohomological constructions. The main ingredients in establishing a twisted analogue of the density theorem along the lines of the proof of [K1; Appendix], are explained in [F; I, §4].

The assertion of isomorphism in the Main Theorem above combines surjectivity (trace Paley-Wiener theorem) and injectivity (density theorem). The proof given here is due to J. Bernstein (in the case of $\sigma =$ identity). Its advantage over that of [BDK] is in proving injectivity simultaneously to surjectivity. The proof is purely local, using neither the trace formula nor galois cohomology, and it applies with any characteristic . The new tool is the theory of "dévissage (unscrewing)" which is applied to a certain generalization (σ-cocenter of the category $\mathbb{M}(G)$) of the Grothendieck group $K^\sigma(G)$. Thus we work with finitely generated G-modules which are not necessarily of finite length, and study their support on the variety $\Theta(G)$ of infinitesimal characters. For completeness we reproduce here those parts of [BDK] which we need.

I wish to express my very deep gratitude to Joseph Bernstein for explaining his proof to me. My minor contribution is in carrying out the generalization to the twisted case, where σ is arbitrary. Since the present proof seems to be quite satisfactory, it is attempted here to supply all details, also in the twisted case. Further, we refer to Bernstein's fundamental lecture notes [B]. However, those results of [B] which we use can be found already in the preliminary work [BD], with the exception of the "second adjointness theorem": i is left adjoint to $\overline{r}$; see §**F**.

I wish to thank J.-L. Colliot-Thélène, Bill Jacob, Wayne Raskind and Alex Rosenberg, for an instructive and enjoyable summer school. Nato-grant CRG-921232 is gratefully acknowledged.

B. Categorical center.

A *cuspidal pair* is a pair (M,ρ) consisting of a standard Levi subgroup $M<G$ and the equivalence class $\rho\in \operatorname{Irr} M$ of a supercuspidal irreducible M-module. Denote by $\Theta(G)$ the set of all cuspidal pairs up to conjugation by G. It is the disjoint union of infinitely many sets $\Theta=\Theta(M,\rho)$, each of which is the image of the map $X(M)\to\Theta(G),\ \psi\mapsto(M,\psi\rho)/G$, for some cuspidal pair (M,ρ). Each such Θ is called a *connected component* of $\Theta(G)$ and has the natural structure of a complex affine algebraic variety as the quotient of $X(M)$ by a finite group. Then $\Theta(G)=\cup\Theta$ has the structure of a complex algebraic variety consisting of infinitely many connected components.

For any $\pi\in \operatorname{Irr} G$ there is a unique up to conjugation by G cuspidal pair (M,ρ) such that π is a constituent of $i_{GM}(\rho)$. The image θ of (M,ρ) in $\Theta(G)$ is called the *infinitesimal character* of π, and the map $\chi:\operatorname{Irr} G\to\Theta(G)$, $\chi(\pi)=\theta$, is onto and finite to one (see [BZ1]). Note that χ is $X(G)$-equivariant, where $X(G)$ acts on $\Theta(G)$ by $\psi:(M,\rho)\mapsto(M,\psi|M\cdot\rho)$.

For each connected component Θ in $\Theta(G)$ consider the set $\chi^{-1}(\Theta)\subset \operatorname{Irr} G$, and the corresponding abelian subcategory

$$\mathbb{M}(\Theta)=\{E\in\mathbb{M}(G);\ JH(E)\subset\chi^{-1}(\Theta)\}\text{ of }\mathbb{M}(G).$$

The Decomposition Theorem of [B] asserts that for $\Theta\neq\Theta'$ the categories $\mathbb{M}(\Theta)$ and $\mathbb{M}(\Theta')$ are orthogonal, namely $\operatorname{Hom}(E,E')=0$ for $E\in\mathbb{M}(\Theta), E'\in\mathbb{M}(\Theta')$. Moreover, we have $\mathbb{M}(G)=\Pi_\Theta\mathbb{M}(\Theta)$, where the product ranges over all connected components Θ in $\Theta(G)$. Thus each G-module E has a unique decomposition $E=\oplus_\Theta E_\Theta=\Pi_\Theta E_\Theta$ with $E_\Theta\in\mathbb{M}(\Theta)$. In particular $\mathbb{H}_G$ is a G-module under the left action of G, and so

$\mathbb{H}_G$ decomposes as a direct sum $\oplus_\Theta \mathbb{H}_\Theta$ of two sided ideals $\mathbb{H}_\Theta$, and $E_\Theta = \mathbb{H}_\Theta E$ for any G-module E.

The *central algebra* $\mathcal{Z}(\mathbb{M})$ of an abelian category $\mathbb{M}$ is the algebra $\mathrm{End}\,(Id_{\mathbb{M}})$ of endomorphisms of the identity functor $Id_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}$. Thus $z \in \mathcal{Z}(\mathbb{M})$ is a set of endomorphisms $\{z_E : E \to E; E \in \mathbb{M}\}$ such that for any morphism $\alpha : E \to F$ in $\mathbb{M}$ we have $z_F \circ \alpha = \alpha \circ z_E$. Put $\mathcal{Z}(G)$ for $\mathcal{Z}(\mathbb{M}(G))$.

A ring $\mathbb{H}$ is called an *id-ring* if for any finite set $h_1, \cdots, h_n$ in $\mathbb{H}$ there is an idempotent e in $\mathbb{H}$ with $eh_i = h_i = h_i e$. Any id-ring can be presented as $\lim\limits_{\rightarrow}{}_{\mathbf{I}} \mathbb{H}_i$, where $\mathbf{I}$ is an ordered filtered set (for any i, j in $\mathbf{I}$ there is k in $\mathbf{I}$ with $i < k, j < k$), and where $\{\mathbb{H}_i (i \in \mathbf{I})\}$ is a directed system of rings with identity, but the morphisms $\mathbb{H}_i \to \mathbb{H}_j (i < j)$ are not assumed to map the identity of $\mathbb{H}_i$ to that of $\mathbb{H}_j$. For example, $\mathbb{H}_G$ is an id-algebra (algebra which is an id-ring), $\mathbf{I}$ is the set of compact open subgroups K of G, and $\mathbb{H}_K$ the convolution algebra of K-biinvariant measures in $\mathbb{H}_G$. Note that the subset $\mathbf{I}^\sigma$ of σ-invariant K in $\mathbf{I}$ is cofinal in $\mathbf{I}$, hence $\mathbb{H}_G = \lim\limits_{\rightarrow} \mathbb{H}_K (K \in \mathbf{I}^\sigma)$.

A module E over an id-ring $\mathbb{H}$ is called *non-degenerate* if $\mathbb{H}E = E$, equivalently if $E = \lim\limits_{\rightarrow} eE$, where the limit ranges over the set of idempotents in $\mathbb{H}$. From now on by an $\mathbb{H}$-module we shall mean a non-degenerate $\mathbb{H}$-module. Denote by $\mathbb{M}(\mathbb{H})$ the category of (non-degenerate) $\mathbb{H}$-modules. Note that $\mathbb{M}(\mathbb{H}_G) = \mathbb{M}(G)$, and $\mathbb{M}(\mathbb{H}_\Theta) = \mathbb{M}(\Theta)$ for each connected component Θ of $\Theta(G)$. Write $\mathcal{Z}(\mathbb{H})$ for $\mathcal{Z}(\mathbb{M}(\mathbb{H}))$. If $\mathbb{H}$ is an id-ring, the morphism $z \mapsto z_{\mathbb{H}}$ identifies $\mathcal{Z}(\mathbb{H})$ with the algebra $\mathrm{End}_{\mathbb{H} \times \mathbb{H}^{opp}}(\mathbb{H})$ of endomorphisms of $\mathbb{H}$ which commute with right and left multiplication. In particular, if $\mathbb{H}$ has an identity then $\mathcal{Z}(\mathbb{H})$ is isomorphic to the center of $\mathbb{H}$. For example, $\mathcal{Z}(\mathbb{H}_K)$ is the center of $\mathbb{H}_K$.

The orthogonal decomposition $\mathbb{M}(G) = \Pi_\Theta \mathbb{M}(\Theta)$ implies that $\mathcal{Z}(G) = \Pi_\Theta \mathcal{Z}(\Theta)$, where $\mathcal{Z}(\Theta)$ is the center of $\mathbb{M}(\Theta)$. A theorem of [B] asserts that $\mathcal{Z}(\Theta)$ is naturally isomorphic to the algebra ofregular (polynomial) functions on the variety Θ. Hence $\mathcal{Z}(G) = \mathcal{Z}(\mathbb{H}_G)$ is the algebra of regular functions on $\Theta(G)$. In particular $z \in \mathcal{Z}(G)$ acts on $\pi \in \mathrm{Irr}\,(G)$ by multiplication by the scalar $z(\theta)$, where $\theta = \chi(\pi)$.

For any compact open subgroup K of G put $\mathrm{Irr}^K(G) = \{E \in \mathrm{Irr}\, G; E^K \neq 0\}; E^K$ is the space of K-fixed vectors in $E \in \mathbb{M}(G)$. By a Proposition of [B] the subset $\chi(\mathrm{Irr}^K(G))$ of $\Theta(G)$ is a union of finitely many components, and for any component Θ of $\Theta(G)$ there is $K = K_\Theta$ such that $\chi^{-1}(\Theta) \subset \mathrm{Irr}^K(G)$. The open compact subgroup K of G is called *special* if $\mathrm{Irr}^K(G)$ is equal to a union of pullbacks $\chi^{-1}(\Theta)$ of components Θ. Put $\mathbb{M}_K(G) = \{E \in \mathbb{M}(G); E$ is generated by $E^K\}$, and $\mathbb{M}_K^\perp(G) = \{E \in \mathbb{M}(G); E^K = 0\}$. If K is special then $\mathbb{M}(G)$ is the direct sum of the abelian subcategories $\mathbb{M}_K(G)$ and $\mathbb{M}_K^\perp(G)$, and $\mathbb{M}_K(G) = \mathbb{M}(\mathbb{H}_K)$, by a theorem of [B]. Consequently $\mathcal{Z}(\mathbb{M}_K(G)) = \mathcal{Z}(\mathbb{H}_K)$ is the ring of regular functions on the union Θ_K of finitely many connected components Θ of $\Theta(G)$ with $\chi^{-1}(\Theta) \subset \mathrm{Irr}^K(G)$. Moreover, the algebra $\mathbb{H}_K$ decomposes as $\oplus_{\Theta \subset \Theta_K} \mathbb{H}_\Theta$. By [B] the algebra $\mathbb{H}_\Theta$ is finitely generated $\mathcal{Z}(\Theta)$-module, and $\mathbb{H}_K$ is a finitely generated $\mathcal{Z}(\Theta_K)$-module (and $\mathcal{Z}(G)$-module). Finally, it is shown in [B] that K is special if it has an Iwahori decomposition for each $M < G$ (thus $K = K \cap \overline{U} \cdot K \cap M \cdot K \cap U$ where $M = P \cap \overline{P}$ is the intersection of the standard parabolic subgroup $P = M_0 = MU$ and its opposite parabolic $\overline{P} = M\overline{U}$), and there exists a compact subgroup K_0 which normalizes K and satisfies $G = K_0 P_0$. Congruence subgroups and Iwahori subgroups are special.

For any standard Levi subgroup $M < G$ the morphism $i_{GM} : \Theta(M) \to \Theta(G)$ defined by $(N,\rho) \mapsto (N,\rho)$ is finite. It is not injective since cuspidal pairs conjugate under G may be non-conjugate under M. Denote the adjoint morphism by $i^*_{GM} : \mathcal{Z}(G) \to \mathcal{Z}(M)$. Then $\mathcal{Z}(M)$ is a finitely generated $\mathcal{Z}(G)$-module. Put $z_M = i^*_{GM} z \in \mathcal{Z}(M)$ for $z \in \mathcal{Z}(G)$. Then by a Propoposition of [B], for each M-module ρ we have $i_{GM}(z_M) = z$ on $i_{GM}\rho$, and for each G-module π we have $r_{MG} z = z_M$ on $r_{MG}\pi$.

Recall that $\pi \in JH(i_{GN}\rho)$ if and only if ${}^\sigma\pi \in JH(i_{G,\sigma N}({}^\sigma\rho))$. Hence the morphism $\sigma : \Theta(G) \to \Theta(G)$ defined by $(N,\rho) \mapsto (\sigma N, {}^\sigma\rho)$ satisfies $\sigma(\chi(\pi)) = \chi({}^\sigma\pi)$. Denote by σ also the dual map $\sigma : \mathcal{Z}(G) \to \mathcal{Z}(G),\ \sigma z(\theta) = z(\sigma^{-1}\theta)$.

Remark. Denote by Θ^σ, where Θ is a component of $\Theta(G)$, the subset of σ-fixed points of Θ. The subset Θ^σ is empty unless $\sigma\Theta = \Theta$, and it contains the infinitesimal characters of all σ-invariant G-modules π with $\chi(\pi) \in \Theta$ (however $\sigma\theta = \theta$ does not imply the existence of $\pi \in \operatorname{Irr} G$ with $\theta = \chi(\pi)$ and $\pi \simeq {}^\sigma\pi$). The set Θ^σ is a (closed) subvariety of Θ. Indeed, if Θ^σ is not empty then it contains a point represented by a cuspidal pair (M,ρ). Let $W_G = W(M_0,G) = \operatorname{Norm}(M_0,G)/M_0$ be the Weyl group of G. Then there is $s \in W_G$ with $(\sigma N, \sigma\rho) = (sN, s\rho)$. If $(N,\psi\rho), \psi \in X(N)$, represents any other point in Θ^σ, then there is s_ψ in W_G with $(\sigma N, \sigma(\psi\rho)) = (s_\psi N, s_\psi(\psi) s_\psi(\rho))$. Since we have $sN = s_\psi N$, there is $w_\psi \in W(N,G) = \operatorname{Norm}(N,G)/N$ such that $s_\psi = s w_\psi$. Hence $s w_\psi(\psi) \cdot s\rho \simeq \sigma\psi \cdot s\rho$, or $((s w_\psi)(\psi)/\sigma(\psi)) \otimes s\rho \simeq s\rho$, and $s w_\psi(\psi)/\sigma(\psi)$ lies in a fixed finite group depending only on ρ (and σ). Consequently Θ^σ is (Zariski) closed in Θ.

C. Discrete modules.

Put $R^\sigma_I(G) = \sum_{M=\sigma M \lneqq G} i_{GM}(R^\sigma(M))$. A G-module $\pi \in \operatorname{Irr}^\sigma(G)$ is called σ-*discrete* if it does not lie in $R^\sigma_I(G)$. An element θ of $\Theta(G)$ is called σ-*discrete* if it is equal to $\chi(\pi)$ for a σ-discrete $\pi \in \operatorname{Irr}^\sigma(G)$. Denote by $R^\sigma_\theta(G)$ the subgroup of $R^\sigma(G)$ generated by the G-modules with infinitesimal character θ. Denote by $\Theta^\sigma_{\text{disc}}(G)$ the subset of σ-discrete θ in $\Theta(G)$, and for each connected component Θ of $\Theta(G)$ put $\Theta^\sigma_{\text{disc}} = \Theta \cap \Theta^\sigma_{\text{disc}}(G)$.

Theorem 1. *For each connected component Θ of $\Theta(G)$, the set $\Theta^\sigma_{\text{disc}}$ is a union of finitely many $X^\sigma(G)$-orbits (and in particular is a subvariety of Θ).*

A main step in the proof of this Theorem is the following

Proposition 1.1. *For each Θ the set $\Theta^\sigma_{\text{disc}}$ is constructible (a finite union of locally closed, in the Zariski topology, subsets) in Θ.*

Proof. We begin with some preliminaries. Let $\mathbb{B}$ be a commutative algebra over $\mathbb{C}$. A $G \times \mathbb{B}$-*module* is a G-module E equipped with a homomorphism $\mathbb{B} \to \operatorname{End}_G E$. Such E is called a $\mathbb{B}$-*family* of G-modules if E is finitely generated as a $G \times \mathbb{B}$-module, and for each open compact subgroup K of G the $\mathbb{B}$-module E^K is finitely generated and projective. For any homomorphism $\mathbb{B} \to \mathbb{B}'$ of algebras write $E_{\mathbb{B}'} = \mathbb{B}' \otimes_{\mathbb{B}} E$ for the induced $\mathbb{B}'$-family of G-modules. If $\mathbb{B}$ is the algebra $k[X]$ of regular functions on a variety X, call E an X*-family* of G-modules. Given a morphism $X' \to X$, denote by $E_{X'}$ the induced X'-family of G-modules. In particular for any point s in X (thus $s : \text{Spec } \mathbb{C} \to X$) the corresponding G-module $E_s = \mathbb{C} \otimes_{k[X]} E$ is called the *specialization* of the X-family E at s.

Given an X-family of G-modules E define a function $\nu_E : X \to R(G)$ by $\nu_E(s) = E_s$, and a function $\overline{\nu}_E : X \to \overline{R}^\sigma(G)$ by $\overline{\nu}_E(s) = \overline{E}_s$, where $\overline{E}_s$ is the image of $E_s \in R(G)$ in the quotient $\overline{R}^\sigma(G)$ of $R^\sigma(G)$ by the relation $(\pi, \zeta S) \sim (\pi, S)$ if $\zeta^\ell = 1; \overline{R}^\sigma(G)$ is the free abelian group generated by $\text{Irr}^\sigma(G)$. A function $\nu : X \to \overline{R}^\sigma(G)$ is called *regular* if $\nu = \overline{\nu}_E$ for some X-family E of G-modules. A regular function $\nu : X \to \overline{R}^\sigma(G)$ is called *irreducible* if $\nu(X) \subset \text{Irr}^\sigma(G)$. Two irreducible functions ν, ν' are called *disjoint* if $\nu(s) \neq \nu(s')$ for every $s \neq s'$ in X.

Lemma 1.1.1. *Given a regular function $\nu : X \to \overline{R}^\sigma(G)$ there exists a dominant étale morphism $\phi : X_1 \to X$, finitely many irreducible disjoint regular functions $\lambda_j : X_1 \to \overline{R}^\sigma(G)$, and positive integers n_j, such that $\nu \circ \phi = \Sigma_j n_j \lambda_j$.*

Proof. Let E be an X-family of G-modules such that $\nu = \overline{\nu}_E$. Then there is an open compact σ-invariant subgroup K of G such that E is generated by E^K as a G-module. The subgroup K can be chosen to be special, and then any non-zero subquotient E' of E is generated by its subspace E'^K (which is non-zero) by a theorem of [B]. Consequently it suffices to prove the lemma with finitely generated $k[X]$-families of $\mathbb{H}_K$-modules E^K, instead of finitely generated $k[X]$-families of G-modules E.

It suffices to prove the lemma with X replaced by an irreducible component. Hence we assume that X is irreducible. Write $k(X)$ for the fraction field of $k[X]$. The $\mathbb{H}_K \times k[X]$-module E^K is finitely generated as a $k[X]$-module; hence $k(X) \otimes_{k[X]} E^K$ is a finite dimensional vector space over the field $k(X)$. Over an algebraic closure $\overline{k(X)}$ of $k(X)$ there is an $\mathbb{H}_K$-stable flag $0 = \overline{E}'_0 \underset{\neq}{\subset} \overline{E}'_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} \overline{E}'_r$ of $\overline{k(X)}$-vector spaces in $\overline{E}'_r = \overline{k(X)} \otimes_{k[X]} E^K$, such that each $\overline{E}_j = \overline{E}'_j / \overline{E}'_{j-1}$ is an irreducible $\mathbb{H}_K$-module over $\overline{k(X)}$. Since $k(X) \otimes_{k[X]} E^K$ is finite dimensional over $k(X)$, there exists a finite extension $k(X)'$ of $k(X)$ in $\overline{k(X)}$, namely a finite étale dominant morphism $X' \to X$, such that the $\mathbb{H}_K$-module $k(X') \otimes_{k[X]} E^K$ is completely reducible. Thus there is an $\mathbb{H}_K$-stable flag $0 = E'_0 \underset{\neq}{\subset} E'_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} E'_r$ of $k(X')$-vector spaces in $E'_r = k(X') \otimes_{k[X]} E^K$, such that each $\tilde{E}_j = E'_j / E'_{j-1}$ is an irreducible $\mathbb{H}_K$-module over $k(X')$. In particular $\mathbb{H}_K$ spans $\text{End}_{k(X')} \tilde{E}_j$ over $k(X')$.

Choose a basis B_j of $\tilde{E}_j$ over $k(X')$. Then $L'_j = (\mathbb{H}_K \times k[X'])B_j$ is a finitely generated projective $\mathbb{H}_K \times k[X']$-module, and $k(X') \otimes_{k[X']} L'_j = \tilde{E}_j$. Hence $\text{End}_{k[X']} L'_j$ is a ring of matrices over $k[X']$ of size $|B_j|$. Since $\text{End}_{k(X')} \tilde{E}_j$ is $\mathbb{H}_K \times k(X')$, there exists an open subset X'' of X' such that $\text{End}_{k[X'']} L''_j$, where $L''_j = k[X''] \otimes_{k[X']} L'_j$, is equal to $\mathbb{H}_K \times k[X'']$. Hence L''_j is an irreducible $\mathbb{H}_K \times k[X'']$-module, and $L''_{j,s} = \mathbb{C} \otimes_{k[X'']} L''_j$ is an irreducible $\mathbb{H}_K$ - module for every s in X''. In $R(G)$ we then have $E_s = \sum_j L''_{j,s}$ for all $s \in X''$, and so $\nu_E \circ \phi = \sum_j \nu_{L''_j}$ on X'', where ϕ is the morphism $X'' \to X$. The regular functions $\nu_{L''_j}$ are irreducible.

Write λ_j for the distinct functions among the $\nu_{L''_j}$; then $\nu = \sum_j n_j \lambda_j$ for some $n_j \geq 1$. Replacing X'' by an open subset we may assume that the λ_j are disjoint; indeed, the set of $s \in X''$ with $\lambda_j(s) = \lambda_{j'}(s)$ is closed in the Zariski topology.

Denote by J the set of j such that the irreducible $\mathbb{H}_K \times k[X'']$-module $\tilde{E}_j$ is σ-invariant. Then for each $b_i \in B_j$ there are $f_{ik} = f'_{ik}/f''_{ik}$ with f'_{ik}, f''_{ik} in $k[X'']$ such that $\sigma b_i = \sum_k f_{ik} b_k$. Replacing X'' by its open subset which is defined by $f''_{ik} \neq 0$ for all i, k, we conclude that L''_j is σ-invariant for each j in J. The functions $\nu_{L''_j}$ and $\nu_{\sigma L''_j}$ are equal or disjoint. Hence, if $L''_{j,s} \simeq \sigma L''_{j,s}$ for some s in X'' then $L''_j \simeq \sigma L''_j$, and $\tilde{E}_j$ is σ-invariant (j lies in J). It follows that for $j \notin J$, the image of $L''_{j,s}$ in $\overline{R}^\sigma(G)$ is zero for every s in X''. This completes the proof of the lemma.

Corollary 1.1.2. *Let $\lambda, \nu_1, \cdots, \nu_n : X \to \overline{R}^\sigma(G)$ be regular functions, and λ irreducible. Denote by X_I the set of s in X such that $\lambda(s)$ lies in the subgroup of $\overline{R}^\sigma(G)$ generated by $\nu_1(s), \cdots, \nu_n(s)$. Then there is an étale dominant morphism $\phi : X' \to X$ such that $\phi^{-1}X_I$ is empty or is X'.*

Proof. There are irreducible disjoint regular functions $\lambda_1, \cdots, \lambda_n : X \to \overline{R}^\sigma(G)$ and positive integers a_{ij} such that $\nu_i = \sum_j a_{ij}\lambda_j$. We may assume that $\lambda = \lambda_1$. It remains to solve in integers $b_1, \cdots, b_n$ the equation $\sum_{i=1}^n b_i a_{ij} = \delta_{1,j}$.

Remark. A subset A of Θ is constructible if and only if it satisfies the condition:

(C) *For any locally closed subvariety X of Θ there exists a dominant étale morphism $\phi : X' \to X$ such that $\phi^{-1}(X_I)$, $X_I = X - X \cap A$, is either empty or X'.*

Proof of Proposition. To show that $\Theta^\sigma_{\text{disc}}$ is constructible we shall verify (C) for $A = \Theta^\sigma_{\text{disc}}$. Suppose that (N, ρ) is a cuspidal pair which defines Θ, and let $\nu_\rho : X(N) \to \Theta$ be the morphism defined by $\psi \mapsto (N, \psi\rho)$. For each standard Levi subgroup $M < G$ with $M = \sigma M > N$, denote by ν_M the regular function $X(N) \to \overline{R}^\sigma(M)$ defined by $\psi \mapsto i_{NM}(\psi\rho)$. Let X be a locally closed subvariety of Θ. Then by Lemma 1.1.1 there is a dominant étale morphism $\phi : X_1 \to X$ such that $\nu_M \circ \phi = \sum_j n_{M,j}\lambda_{M,j}$, $n_{M,j} > 0$ and $\lambda_{M,j} : X_1 \to \overline{R}^\sigma(M)$ are irreducible disjoint regular functions, for each such $M = \sigma M < G$. The set X_2 of points $s \in X_1$ where each $\lambda_{G,j}(s)$ lies in the subgroup of $\overline{R}^\sigma(G)$ generated by the regular functions $i_{GM}(\lambda_{M,k}(s))$, is $\phi^{-1}(X_I)$, $X_I = X - X \cap \Theta^\sigma_{\text{disc}}$. But then Corollary 1.1.2 implies that $\phi^{-1}(X_I)$ is empty or is X_1. Hence X satisfies (C) and the proposition follows.

The following Lemma will be used in the proof below of Theorem 1.

Lemma 1.2. *Given an irreducible σ-discrete G-module π there exists a tempered σ-discrete G-module π' and $\psi \in X^\sigma(G)$ with $\chi(\pi) = \chi(\psi\pi')$.*

Proof. Langlands' classification [BW; §XI] implies that any π in $\operatorname{Irr} G$ determines a unique triple (P, ρ, ψ_M) consisting of a standard parabolic subgroup $P = MU$ of G, a tempered (irreducible) M-module ρ, and $\psi_M \in X(M)$ which is positive with respect to U (see [BW]), such that π is the unique irreducible quotient of $i_{GM}(\psi_M\rho)$. The triple of ${}^\sigma\pi$ is $(\sigma P,\ {}^\sigma\rho,\ {}^\sigma\psi_M)$, and so if $\pi \simeq {}^\sigma\pi$ then $\sigma P = P,\ {}^\sigma\rho \simeq \rho,\ {}^\sigma\psi_M = \psi_M$. If the infinitesimal character of the M-module $\psi_M\rho$ is represented by the cuspidal pair $(N, \tau), N < M$, then each constituent of the G-module $i_{GM}(\psi_M\rho)$ is also a constituent of the G-module $i_{GN}(\tau)$, hence has the same infinitesimal character θ as π.

In $\overline{R}^\sigma(G)$ we have $\pi = i_{GM}(\psi_M\rho) - \sum_j \pi_j$, where π_j are the irreducible σ-invariant constituents of $i_{GM}(\psi_M\rho)$ other than π. Moreover, if (P_j, ρ_j, ψ_j) is the triple determined by π_j, then $\psi_j < \psi_M$ in the order $<$ introduced in [BW; XI, (2.13)]. Since the map $\chi : \operatorname{Irr} G \to \Theta(G)$ is finite to one, π_j lies in a fixed finite set determined by $\theta = \chi(\pi)$. By induction on the parameter ψ we may assume that each π_j is a $\mathbb{Z}$-linear combination of G-modules of the form $i_{GM'}(\psi'\rho')$, where $M' = \sigma M' < G,\ \psi' \in X^\sigma(M')$, and ${}^\sigma\rho' \simeq \rho'$ is tempered. Hence $\pi = \Sigma i_{GM'}(\psi'\rho')$ for some $M' = \sigma M' < G, \psi' \in X^\sigma(M')$, and tempered σ-invariant M'-modules ρ'. Since π is σ-discrete, at least one M' in the sum equals G, and the corresponding ρ' is σ-discrete. The lemma follows.

Proof of Theorem 1. The involution $+ : R(G) \to R(G)$ which assigns to each G-module π its Hermitian contragredient π^+, maps $\operatorname{Irr} G$ to $\operatorname{Irr} G$ and $\operatorname{Irr}^\sigma(G)$ to $\operatorname{Irr}^\sigma(G)$. It commutes with i_{GM} for each $M < G$, acts on $X(M)$ and on the set of cuspidal pairs (M, ρ), and consequently defines an involution $+$ on the complex algebraic variety $\Theta(G)$ which commutes with $\chi : \operatorname{Irr} G \to \Theta(G)$. It is clear that the action of $+$ on the algebraic varieties $X(M)$ and $\Theta(G)$ is anti-holomorphic and in particular anti-algebraic.

By Lemma 1.2 each $\theta \in \Theta^\sigma_{\text{disc}}$ is of the form $\chi(\psi\pi)$ where $\psi \in X^\sigma(G)$ and π is an irreducible tempered σ-invariant G-module. Since π is tempered it is unitary, and so $\pi^+ = \pi$. Hence $\theta^+ \in X^\sigma(G)\theta$. Consequently the subset $\overline{\Theta}^\sigma_{\text{disc}} = \Theta^\sigma_{\text{disc}}/X^\sigma(G)$ of the algebraic quotient variety $\overline{\Theta} = \Theta/X^\sigma(G)$, which is constructible by Proposition 1.1, is pointwise fixed by the anti-algebraic involution $+$. It follows that $\overline{\Theta}^\sigma_{\text{disc}}$ is finite, namely $\Theta^\sigma_{\text{disc}}$ consists of finitely many $X^\sigma(G)$-orbits, as asserted.

D. Induction.

Let L be a field of characteristic zero. A *G-module over L* is a smooth representation $\pi : G \to \operatorname{Aut} V$ of the group G on a vector space V over L. Denote by $R(G; L)$ the Grothendieck group of G-modules over L of finite length, and by $R^\sigma(G; L)$ the free abelian group generated by the pairs (π, S), where π is a G-module over L of finite length and $S \in \operatorname{Aut}^\sigma_G \pi$, subject to the relations (R_i) in §**A**. Note that $R^\sigma(G; \mathbb{C}) = R^\sigma(G)$.

Let $\mathbf{c} = (c_M; M = \sigma M \lneqq G)$ be a sequence of rational numbers. Then the operator $A_\sigma^{\mathbf{c}} = 1 + \sum_{M=\sigma M \lneqq G} c_M i_{GM} r_{MG}$ maps $R^\sigma(G;L)_Q$ to itself, and it is clear that for any π in $R^\sigma(G;L)_Q$ we have $A_\sigma^{\mathbf{c}}\pi \equiv \pi \bmod R_I^\sigma(G;L)_Q$, where $R_I^\sigma(G;L) = \sum_{M=\sigma M \lneqq G} i_{GM}(R^\sigma(M;L))$. We shall now show that the sequence $\mathbf{c}$ can be chosen so that $A_\sigma^{\mathbf{c}}$ distinguishes between induced and non-induced modules, in the following sense.

Theorem 2. *There exists a sequence* $\mathbf{c} = (c_M \in Q; M = \sigma M \lneqq G)$ *such that the endomorphism* $A_\sigma^{\mathbf{c}}$ *of* $R^\sigma(G;L)_Q$ *has the following property. Given* π *in* $R^\sigma(G;L)_Q$ *we have* $A_\sigma^{\mathbf{c}}\pi = 0$ *if and only if* π *lies in* $R_I^\sigma(G;L)_Q$.

Thus we need to find $\mathbf{c} = (c_M)$ such that $A_\sigma^{\mathbf{c}}(R_I^\sigma(G;L)_Q) = 0$.

Recall that the Weyl group W_G of G is $\mathrm{Norm}(M_0, G)/M_0$. For $M < G$ consider W_M as a subgroup of W_G. The standard Levi subgroups $M, N < G$ are called *associate* if there is w in W_G with $N = wMw^{-1}$. Each such w defines an isomorphism $w : R(M;L) \to R(N;L)$ which depends only on the double class of w in $W_M \backslash W_G / W_N$. If $w' : R(N';L) \to R(M;L)$ is defined, denote by $w \circ w'$ the composition $R(N';L) \to R(N;L)$.

Lemma 2.1. (i) *For* $N' < N < M < G$ *we have* $i_{MN'} = i_{MN} \circ i_{NN'}, r_{N'M} = r_{N'N} \circ r_{NM}$.

(ii) *If* $N = wMw^{-1}$ *then* $i_{GN} \circ w(\rho) = i_{GM}(\rho)$ *for all* ρ *in* $R^\sigma(M;L)$.

(iii) *For* $M, N < G$, *let* W_G^{NM} *be the set of representatives of* $W_N \backslash W_G / W_M$ *of minimal length. Then we have the following equality of functors from* $\mathbb{M}(M;L)$ *to* $\mathbb{M}(N;L)$ *:*

$$r_{NG} \circ i_{GM} = \sum_{w \in W_G^{NM}} i_{NN_w} \circ w \circ r_{M_w M},$$

where

$$M_w = w^{-1}Nw \cap M, \ N_w = wM_w w^{-1} = N \cap wMw^{-1}.$$

Proof. (i) follows from the definitions, (ii) is proven in [BDK], p. 189, and (iii) is [BZ2], (2.12).

Suppose that $M = \sigma M < G$. Then σ acts on W_M (and W_G). Since P_0 is σ-invariant we have $\ell(\sigma w) = \ell(w)$ where ℓ is the length function on W_G. If $N = \sigma N < G$ then σ acts on W_G^{NM}. Denote by $W_G^{NM}(\sigma)$ the subset of σ-fixed elements in W_G^{NM}.

Lemma 2.1. (iv). *For* $M = \sigma M, N = \sigma N < G$, *the homomorphism* $r_{NG} \circ i_{GM} : R^\sigma(M;L) \to R^\sigma(N;L)$ *is equal to*

$$\sum_{w \in W_G^{NM}(\sigma)} i_{NN_w} \circ w \circ r_{M_w M}.$$

Proof. The case of $\sigma = id$ follows at once from (iii). Denote by $\Sigma_1, \Sigma_2, \cdots$, the σ-orbits in W_G^{NM}. The length function is constant on each orbit Σ_i, and we index the Σ_i

to satisfy $\ell(\Sigma_i) \geq \ell(\Sigma_{i+1})$. Then $\ell(\Sigma_i) = 1$ if and only if $\Sigma_i \subset W_G^{NM}(\sigma)$. Index the elements w of W_G^{NM} as $w_1, w_2, \cdots, w_t$ such that if $s_i = |\Sigma_i|$, and $t_i = s_1 + \cdots + s_i$, then $\Sigma_i = \{w_{t_{i-1}+1}, \cdots, w_{t_i}\}$. Put $P = MP_0 = MU_M$, $Q = NP_0 = NU_N$ (U_M, U_N are the unipotent radicals of the standards parabolic subgroups $P, Q < G$ with Levi components M, N).

Given an M-module (ρ, E), the space of $i_{GM}\rho$ consists of the functions $f : G \to E$ with $f(mug) = \delta_P(m)^{\frac{1}{2}}\rho(m)f(g)$ $(m \in M, u \in U_M)$. Let E_k be the subspace of the f which are supported on $\bigcup_{1 \leq i \leq k} Pw_iQ$. Then E_k is Q-invariant, and [BZ2] define $F'_k(\rho)$ to be the image of E_k under r_{NG}. Moreover, [BZ2] show that $F'_1 \subset F'_2 \subset \cdots \subset F'_t$ is a functorial filtration of the functor $F'_t = F = r_{NG} \circ i_{GM} : \mathbb{M}(M; L) \to \mathbb{M}(N; L)$, such that $F'_i/F'_{i+1} = i_{NN_{w_i}} \circ w_i \circ r_{M_{w_i}M}$. Put $F_i = F'_{t_i}$. For any $\rho \in \operatorname{Irr}^\sigma(M) \cap R^\sigma(M; L)$, the N-module $F_i(\rho)/F_{i-1}(\rho)$ is the direct sum of s_i N-modules over L which are permuted by the action of σ. If $s_i > 1$, the image of $F_i(\rho)/F_{i-1}(\rho)$ in $R^\sigma(N; L)$ is then zero. Since $s_i = 1$ precisely for the elements of $W_G^{NM}(\sigma)$, the lemma follows.

Corollary 2.2. *For each $M = \sigma M < G$, the operator $T_M = i_{GM} \circ r_{MG} : R^\sigma(G; L) \to R^\sigma(G; L)$ satisfies*

$$(a) \qquad T_N \circ i_{GM} = \sum_{w \in W_G^{NM}(\sigma)} i_{GM_w} \circ r_{M_wM}, \text{ where } M_w = M \cap w^{-1}Nw;$$

$$(b) \qquad T_N \circ T_M = \sum_{w \in W_G^{NM}(\sigma)} T_{M_w}.$$

Proof. (a) $T_N \circ i_{GM} = i_{GN} \circ r_{NG} \circ i_{GM} \overset{(iv)}{=} \sum_w i_{GN} \circ i_{NN_w} \circ w \circ r_{M_wM}$
$\overset{(i)}{=} \sum_w i_{GN_w} \circ w \circ r_{M_wM} \overset{(ii)}{=} \sum_w i_{GM_w} \circ r_{M_wM}$.

(b) $T_N \circ T_M = T_N \circ i_{GM} \circ r_{MG} = \sum_w i_{GM_w} \circ r_{M_wM} \circ r_{MG} = \sum_w i_{GM_w} \circ r_{M_wG} = \sum_w T_{M_w}$.

Proof of Theorem 2. For $M = \sigma M < G$ put $d(M) = \dim X(M)$, and define a decreasing filtration R_σ^i on $R^\sigma(G; L)$ by $R_\sigma^i = \sum_{\{M = \sigma M < G; d(M) \geq i\}} i_{GM}(R^\sigma(M; L))$. Then $R_\sigma^i = R^\sigma(G; L)$ for $i \leq d(G), R_\sigma^{d(G)+1} = R_I^\sigma(G; L)$, and $R_\sigma^i = 0$ for $i > d(M_0)$. Corollary 2.2 (a) implies that the operator T_N for $N = \sigma N < G$ preserves the filtration $\{R_\sigma^i\}$. Put $[W_N^\sigma]$ for the cardinality of the set W_N^σ of σ-invariant elements in W_N. Put $d = d(N)$. The action of T_N on $R_\sigma^d/R_\sigma^{d+1}$ is given by

$$T_N(i_{GM}\rho) = \begin{cases} [W_N^\sigma] i_{GM}(\rho), & \text{if } M = \sigma M \text{ is conjugate to } N, \rho \in R^\sigma(M; L), \\ 0, & \text{if } M = \sigma M \text{ is not conjugate to } N, \text{ and } d(N) = d, \rho \in R^\sigma(M; L). \end{cases}$$

It follows that the operator $A_d = \prod_{\{N = \sigma N; d(N) = d\}} (T_N - [W_N^\sigma])$ preserves the filtration $\{R_\sigma^i\}$ and annihilates $R_\sigma^d/R_\sigma^{d+1}$. Put $A'_\sigma = A_{d(M_0)} \circ A_{d(M_0)-1} \circ \cdots \circ A_{d(G)+1}$. Then

$A'_\sigma(R^\sigma_I(G;L)) = 0$, and by Corollary 2.2 (b) the operator A'_σ takes the form $A'_\sigma = a(1 + \sum_{M=\sigma M \lneqq G} c_M T_M)$ with $c_M \in Q, a \in \mathbb{Z}, a \neq 0$, and $ac_M \in \mathbb{Z}$. The operator $A^{\mathbf{c}}_\sigma = a^{-1}A'_\sigma$, where $\mathbf{c} = (c_M)$, has the properties asserted in the theorem.

For $M = \sigma M \leq G$, denote by $i^*_{GM} : R^*_\sigma(G;L) \to R^*_\sigma(M;L)$ and $r^*_{MG} : R^*_\sigma(M;L) \to R^*_\sigma(G;L)$ the homomorphisms adjoint to i_{GM} and r_{MG}. A form F in $R^*_\sigma(G;L)$ is called σ*-discrete* if $F(R^\sigma_I(G;L)) = 0$. Denote by $R^*_\sigma(G;L)^{\text{disc}}$ the space of σ-discrete forms. Note that $R^*_\sigma(G;L) = \text{Hom}_{\mathbb{Z},\zeta}(R^\sigma(G;L),\mathbb{C})$ is denoted by $R^*_\sigma(G)$ when $L = \mathbb{C}$.

Corollary 2.3. *Given F in $R^*_\sigma(G;L)$, the form $F^d = F + \sum_{M=\sigma M \lneqq G} c_M r^*_{MG} i^*_{GM} F$ is σ-discrete.*

Proof. For π in $R^\sigma(G;L), F^d(\pi) = a^{-1}F(A'_\sigma\pi)$ vanishes if $\pi \in R^\sigma_I(G;L)$.

E. Dévissage.

Given a G-module π there is a special compact open σ-invariant subgroup K of G such that π^K generates π. Each subquotient π' of π is generated by π'^K. The map $\pi \to \pi^K$ is an equivalence from the category $\mathbb{M}_K(G)$ of G-modules π generated by π^K, to the category $\mathbb{M}(\mathbb{H}_K)$ of (nondegenerate) $\mathbb{H}_K$-modules. Since $\mathbb{M}(\mathbb{H}_K)$ has finite cohomological dimension ([B], see Appendix), the Grothendieck group $K(\mathbb{H}_K)$ of finitely generated $\mathbb{H}_K$-modules coincides with the Grothendieck group of finitely generated projective (and even free) $\mathbb{H}_K$-modules. The center $\mathcal{Z}_K = \mathcal{Z}(\mathbb{H}_K)$ of the algebra $\mathbb{H}_K$ is (equal to the center $\mathcal{Z}(\mathbb{M}(\mathbb{H}_K))$ of the category $\mathbb{M}(\mathbb{H}_K)$ and to) the ring $k[\Theta_K]$ of regular functions on the variety $\Theta_K; \Theta_K$ is a finite union of connected components Θ of $\Theta(G)$ with $\chi^{-1}(\Theta) \subset \text{Irr}^K(G)$.

Denote by $\text{Ann}\,(\pi, \mathcal{Z}_K)$ the annihilator of the $\mathbb{H}_K$-module π in the ring $\mathcal{Z}_K$. This is an ideal in $\mathcal{Z}_K$. The corresponding subvariety $\text{supp}\,\pi$ of $\Theta_K \subset \Theta(G)$ is called the *support* of π. If the distinct irreducible components of $\text{supp}\,\pi$ are denoted by Y then $\text{supp}\,\pi = \cup Y$.

Let A be a $\mathbb{C}$-algebra and denote by σ an automorphism of A of finite order ℓ.

Definition. The σ*-cocenter* $\tau_\sigma(\mathbb{M}(A))$ of the category $\mathbb{M}(A)$ of (non-degenerate) A-modules is defined to be the quotient of the free abelian group generated over $\mathbb{C}$ by the triples (P,S,α), where P is a projective finitely generated A-module, $S \in \text{Aut}^\sigma_A P$ (thus $S : P \tilde{\to} P$ is a vector space automorphism with $S(hp) = \sigma(h)^{-1}S(p)$ for $p \in P, h \in A$, and $S^\ell = 1$), and $\alpha \in \text{End}_A P$, subject to the following relations:

(1) $(P,S,\alpha) \sim (P',S',\alpha') + (P'',S'',\alpha'')$ if $0 \to (P',S',\alpha') \to (P,S,\alpha) \to (P'',S'',\alpha'') \to 0$ is exact;

(2) $(P,S,\alpha+\beta) \sim (P,S,\alpha) + (P,S,\beta)$, $(P,S,\alpha\sigma(\beta) - \beta\alpha) \sim 0$, $(P,\zeta S, t\alpha) \sim \zeta t(P,S,\alpha)$, $(\alpha,\beta \in \text{End}_A P,\ \zeta^\ell = 1,\ t \in \mathbb{C})$;

(3) If $P = \oplus_i P_i, \alpha(P_i) \subset P_i$ and for each i there is j such that $SP_i = P_j$, then $(P, S, \alpha) \sim \Sigma_i(P_i, S_i, \alpha_i)$, where the sum ranges over the i with $j(i) = i$, and $\alpha_i = \alpha|P_i, S_i = S|P_i$.

Write $\tau_\sigma(G)$ for $\tau_\sigma(\mathbb{M}(\mathbb{H}_G))$. Write $\tau_\sigma(\Theta)$ for $\tau_\sigma(\mathbb{M}(\mathbb{H}_\Theta))$; it is a direct summand of $\tau_\sigma(G)$. When K is σ-invariant and special, $\tau_\sigma(\Theta_K) = \tau_\sigma(\mathbb{M}(\mathbb{H}_K))$ is also a direct summand of $\tau_\sigma(G)$, being the direct sum of $\tau_\sigma(\Theta)$ over the $\Theta \subset \Theta_K$. Put $\tau_{\sigma,I}(G) = \Sigma_{M=\sigma M \lneqq G} i_{GM}(\tau_\sigma(M))$.

Define $\tau_{\sigma,i}(\Theta)$ to be the quotient by the relations (1), (2), (3) of the free abelian group generated over $\mathbb{C}$ by the triples (P, S, α) (P: projective finitely generated $\mathbb{H}_\Theta$-module, $S \in \operatorname{Aut}^\sigma_{\mathbb{H}_G} P$, $\alpha \in \operatorname{End}_{\mathbb{H}_G} P$) such that P is supported on a subvariety Y of Θ whose image $\overline{Y}$ in the quotient variety $\overline{\Theta} = \Theta / X^\sigma(G)$ is of dimension at most i. Recall that the *dimension* of a subvariety Y of Θ, corresponding to a prime ideal I in the ring $k[\Theta]$, is defined to be the supremum of the lengths n of all finite strictly increasing chains $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals P_i in $k[\Theta]$, with $P_n = I$. The identity induces a natural map $\tau_{\sigma,i}(\Theta) \to \tau_{\sigma,i+1}(\Theta)$, and for all sufficiently large i we have $\tau_{\sigma,i}(\Theta) = \tau_{\sigma,i+1}(\Theta)$. Define $\tau_{\sigma,i}(G)$ similarly, and note that $\tau_{\sigma,i}(G) = \oplus_\Theta \tau_{\sigma,i}(\Theta)$. Note that $\tau_{\sigma,0}(\Theta) \subset \widetilde{R}^\sigma(\Theta)_\mathbb{C}$ and $\tau_{\sigma,0}(G) \subset \widetilde{R}^\sigma(G)_\mathbb{C}$, where $R^\sigma(\Theta)$ is the subgroup of $R^\sigma(G)$ generated by the pairs (π, S) with $\operatorname{supp} \pi \subset \Theta$. As usual, $R_T = R \otimes_\mathbb{Z} T$ for any $\mathbb{Z}$-modules R and T, and $\widetilde{R}^\sigma$ indicates the quotient of R^σ by the relations $(P, \zeta S) \sim \zeta(P, S), \zeta \in \mathbb{C}, \zeta^\ell = 1$. Note that $\tau_{\sigma,0}(G)$ is generated by the (P, S, α) where P is *projective* of finite length.

The triple (π, S, α) represents an element of $\tau_{\sigma,i}(\Theta)$ if $\operatorname{supp} \pi = \cup Y, Y \subset \Theta$, $\dim \overline{Y} \le i$ for all i, $\alpha \in \operatorname{End}_{\mathbb{H}_G} \pi$ and $S \in \operatorname{Aut}^\sigma_{\mathbb{H}_G} \pi$. The automorphism S satisfies $S(hp) = \sigma(h)^{-1} S(p) (h \in \mathbb{H}_\Theta, p \in \pi)$. In particular $S(zp) = \sigma(z)^{-1} S(p)$ for all $z \in \mathcal{Z}_\Theta = \mathcal{Z}(\mathbb{H}_\Theta) \subset \mathbb{H}_\Theta$, and so $\operatorname{Ann}({}^\sigma\pi, \mathcal{Z}_\Theta)$ is an ideal in $\mathcal{Z}_\Theta$ which corresponds to $\cup_Y \sigma Y$.

For any subvariety Y of Θ_K (or Θ) put $J_Y = \operatorname{Ann}(Y, \mathcal{Z}_K)$. It is an ideal in the ring $k[\Theta_K]$, which is prime if and only if Y is irreducible.

For any subfield L of $\mathbb{C}$ and algebra homomorphism $\theta : \mathcal{Z}_K \to L$, denote by $R_\theta(L)$ the Grothendieck group of (non-degenerate) $\mathbb{H}_K$-modules of finite length over L on which $\mathcal{Z}_K$ acts via θ. Let $R^\sigma_\theta(L)$ be the quotient of the free abelian group generated by the pairs (π, S) where π is an $\mathbb{H}_K$-module over L on which $\mathcal{Z}_K$ acts via θ, and $S \in \operatorname{Aut}^\sigma_{\mathbb{H}_K} \pi$, by the relations (R_i) in §**A**.

Theorem 3. *For every connected component Θ of $\Theta(G)$, and $i \ge 0$, the map*

$$(\pi, S, \alpha) \mapsto \sum_Y \sum_{j \ge 0} ((J^j_Y \pi / J^{j+1}_Y \pi) \otimes_{k[Y]} k(Y), S, \alpha),$$

where Y ranges over all irreducible subvarieties of $(\operatorname{supp} \pi \subset)\Theta$ with $\dim \overline{Y} = i$ and $\sigma Y = Y$, yields an isomorphism

$$\tau_{\sigma,i}(\Theta) / \operatorname{Im} \tau_{\sigma,i-1}(\Theta) \tilde{\to} \oplus_Y \widetilde{R}^\sigma_\theta(k(Y))_\mathbb{C}.$$

Here $k(Y)$ *is the field of rational functions on the variety* Y *, and* $\theta=\theta_Y$ *is the generic point* $\theta:(\mathcal{Z}_K\to)k[\Theta]\to k[Y]$ *of* Y *(corresponding to* $Y\hookrightarrow\Theta$ *). We fix an embedding of* $k(Y)$ *in* $\mathbb{C}$ *.*

Remark. For each z in $\mathcal{Z}_K$ we have $z\alpha=\alpha z$ and $Sz=\sigma^{-1}(z)S$. If $Y\neq\sigma Y$ then $\theta_Y\neq\theta_{\sigma Y}={}^{\sigma}\theta_Y$, and $R_\theta^\sigma(k(Y))=\{0\}$. If $Y=\sigma Y$ then S induces an automorphism of $J_Y^j\pi/J_Y^{j+1}\pi$, and so does α.

Proof. (i) It suffices to show that the map of the theorem defines an isomorphism $\tau_{\sigma,i}(\Theta)_Q/\operatorname{Im}\tau_{\sigma,i-1}(\Theta)_Q\tilde{\to}\oplus_Y R_\theta^\sigma(k(Y))_Q$, where $\tau_{\sigma,i}(\Theta)_Q$ is defined to be the quotient of the free abelian group generated by the (P,S,α) over Q, rather than $\mathbb{C}$, by the relations (1) - (3), where in (2) we take $t\in Q$ and $\zeta=1$. In the course of this proof *only* we denote $\tau_{\sigma,i}(\Theta)_Q$ by $\tau_{\sigma,i}(\Theta)$.

(ii) The map is well-defined. Indeed, $X=\operatorname{supp}\pi$ is a subvariety of Θ corresponding to the ideal $I=\operatorname{Ann}\pi$ in the noetherian ring $A=k[\Theta]$. Let $I=\cap_k I_k$ be a minimal primary decomposition of I. The radical $J_k=r(I_k)$ is a prime ideal. It is finitely generated since A is noetherian. Hence there is $h_k\geq 1$ such that $J_k^{h_k}\subset I_k$, for each k. Let Y_k be the subvariety of Θ corresponding to J_k. Then $J_{Y_k}=\operatorname{Ann}Y_k$ is J_k. Now $X=\cup_k Y_k$ has only finitely many connected components Y (in particular with $\dim\overline{Y}=i$, and $\sigma Y=Y$). Each $J_Y^j\pi/J_Y^{j+1}\pi$ is annihilated by J_Y, hence is supported on $Y\subset\operatorname{supp}\pi$. Put $h(Y)$ for h_k if Y is Y_k.

To show that for each (π,S,α) the sum over j is finite, note that for each variety Y in the first sum, the module $J_Y^h\pi$ is annihilated by $\prod_{Y'\neq Y}J_{Y'}^{h(Y')}$; here we put $h=h(Y)$, and Y' ranges over the connected components of $\operatorname{supp}\pi$ other than Y. Hence $J_Y^h\pi$ is supported on $\bigcup_{Y'\neq Y}Y'$, and $J_Y^h\pi/J_Y^{h+1}\pi$ on $Y\cap\bigcup_{Y'\neq Y}Y'$, a proper subvariety of Y (in particular, of lower dimension). Hence

$$(J_Y^j\pi/J_Y^{j+1}\pi)\otimes_{k[Y]}k(Y)=0\quad\text{for } j\geq h.$$

(iii) The map is surjective. Let $\theta:\mathcal{Z}_K\to k(Y)$ (i.e. $Y\hookrightarrow\Theta\hookrightarrow\Theta_K$) be a generic point of an irreducible subvariety $Y=\sigma Y$ of Θ with $\dim\overline{Y}=i$. An irreducible π_1 in a pair (π_1,S_1) in $R_\theta^\sigma(k(Y))$ is a finite dimensional vector space over the field $k(Y)$, σ-invariant and irreducible as an $\mathbb{H}_K$-module, on which $\mathcal{Z}_K$ acts by multiplication by θ. Let B be a σ-invariant finite set which spans π_1 over $k(Y)$. Then $\pi=\mathbb{H}_K B$ is a finitely generated σ-invariant $\mathbb{H}_K$-module on which $\mathcal{Z}_K$ acts by multiplication via θ. It is therefore supported on $Y(\subset\Theta,\dim\overline{Y}=i,Y=\sigma Y)$, and so (π,S,id) defines an element in $\tau_{\sigma,i}(\Theta)$, where $S\in\operatorname{Aut}_{\mathbb{H}_K}^{\sigma}\pi$ exists since π is σ-invariant. Note that S is unique up to an ℓth root of unity, since π is irreducible. Choose S to coincide with S_1 on π_1. Note that since π is irreducible, any $\alpha\in\operatorname{End}_{\mathbb{H}_K}\pi$ is a scalar by Schur's lemma. Then $J_Y\pi=0$, and since $\pi\otimes_{\mathcal{Z}_{K\vec{\theta}}}k(Y)=\pi_1$, our (π_1,S_1) is the image of (π,S,id).

(iv) The map is injective. To show this, note that any element of $\tau_{\sigma,i}(\Theta)$ can be represented as a difference $n_1(\pi_1,S_1,\alpha_1)-n_2(\pi_2,S_2,\alpha_2)$, where $n_k\geq 0$ are rational, π_k are projective finitely generated $\mathbb{H}_\Theta$ -modules, $S_k \in \mathrm{Aut}^{\sigma}_{\mathbb{H}_G}\pi_k$ and $\alpha_k \in \mathrm{End}_{\mathbb{H}_G}\pi_k$. Suppose this difference maps to zero by the map of the theorem. Multiplying by the denominators of n_k we may assume that the n_k are non-negative integers. Moreover, replacing α_k by $n_k\alpha_k$, or π_k by 0, we may assume that $n_k = 1$.

To simplify the notations, fix $k(= 1$ or $2)$, and delete it from the notations. In the notations of (ii), we may replace (π, S, α) by $\Sigma_Y \Sigma_{0\le j\le h}(J^j_Y\pi/J^{j+1}_Y\pi, S, \alpha)$ in $\tau_{\sigma,i}(\Theta)/\ \mathrm{Im}\ \tau_{\sigma,i-1}(\Theta)$. To prove injectivity it suffices to assume that the sum ranges over a single Y. Namely we may assume that π is a sum of finitely many modules, denoted again by π to simplify the notations, and these are supported on $Y = \sigma Y$ with $\dim \overline{Y} = i$, and $J_Y\pi = 0$.

As in the proof of Lemma 1.1.1, we fix a special open compact σ-invariant subgroup K of G such that π is generated by π^K as a G-module, and we work with the $\mathbb{H}_K$-module π^K. As there, there is a finite étale dominant morphism $Y' \to Y$ such that the $\mathbb{H}_K \times k(Y')$-module $k(Y') \otimes_{k[Y]} \pi^K$ is completely reducible. Let $0 = E'_0 \subset E'_1 \subset \cdots \subset E'_r$ be a composition series; the quotients $E_\ell = E'_\ell/E'_{\ell-1}$ are irreducible $\mathbb{H}_K$-modules over $k(Y')$. In each E_ℓ we can find a lattice L_ℓ (finitely generated projective $\mathbb{H}_K \times k[Y]$-module with $k(Y')\otimes_{k[Y]} L_\ell = E_\ell$) which is generically irreducible. Since the endomorphism α commutes with $\mathbb{H}_K$, it maps each E'_ℓ to itself, and induces an endomorphism, denoted α_ℓ, on E_ℓ. The lattice L_ℓ can, and will, be chosen to satisfy $\alpha_\ell L_\ell \subset L_\ell$. By induction we may (and will) choose the E'_ℓ to have the property that there are $1 \le \ell_1 < \ell_2 < \cdots < \ell_t = r$ such that $SE'_{\ell_s} = E'_{\ell_s}$ and $E_{\ell_{s+1}} = E'_{\ell_{s+1}}/E'_{\ell_s}$ is the direct sum of the orbit of $E_{\ell_s+1} = E'_{\ell_s+1}/E'_{\ell_s}$ under the action of S, and $\ell_{s+1} - \ell_s$ is the length of the orbit. Denote by $S_{\ell_{s+1}}$ the restriction of S to $E_{\ell_{s+1}}$ when $\ell_{s+1} = \ell_s + 1$ (i.e. E'_{ℓ_s+1} is invariant under S). We may and will choose the lattice L_ℓ to be invariant under S_ℓ if S_ℓ is defined $(\ell = \ell_{s+1} = \ell_s+1)$.

Returning to the original notations (undeleting k), we conclude that there are finitely many generically irreducible $\mathbb{H}_K$-modules $L_{k\ell}$, suppported on $Y(= \sigma Y, \dim \overline{Y} = i)$ with $J_Y = \mathrm{Ann}\,(L_{k\ell}, k[\Theta])$, and $S_{k\ell} \in \mathrm{Aut}^{\sigma}_{\mathbb{H}_K} L_{k\ell}$, and $\alpha_{k\ell} \in \mathrm{End}_{\mathbb{H}_K} L_{k\ell}$, such that

$$(\pi_k, S_k, \alpha_k) \equiv \Sigma_\ell(L_{k\ell}, S_{k\ell}, \alpha_{k\ell}) \text{ in } \tau_{\sigma,i}(\Theta)/\ \mathrm{Im}\ \tau_{\sigma,i+1}(\Theta)\ (k = 1, 2).$$

This is a "pre-semi-simplification" of π_k. The "semi-simplification" of $k(Y) \otimes_{k[Y]} \pi_k$ is $\oplus_\ell(k(Y) \otimes_{k[Y]} L_{k\ell})$.

To prove injectivity we assume that $\Sigma_\ell(E_{1\ell}, S_{1\ell}, \alpha_{1\ell}) = \Sigma_\ell(E_{2\ell}, S_{2\ell}, \alpha_{2\ell})$, where $E_{k\ell} = k(Y) \otimes_{k[Y]} L_{k\ell}$. Since the $E_{k\ell}$ are all irreducible, the existence and uniqueness of the Jordan-Holder composition series implies that up to reordering indices we have $(E_{1\ell}, S_{1\ell}, \alpha_{1\ell}) = (E_{2\ell}, S_{2\ell}, \alpha_{2\ell})$ for all ℓ. But $L_{1\ell}$ and $L_{2\ell}$ are both lattices in the same vector space $E_{k\ell}$. Their intersection $L_{1\ell} \cap L_{2\ell}$ is a lattice, and the quotient $L_{k\ell}/L_{1\ell} \cap L_{2\ell}$ is supported on a lower dimensional variety. Hence in $\tau_{\sigma,i}(\Theta)/\ \mathrm{Im}\ \tau_{\sigma,i-1}(\Theta)$ we have $(\pi_k, S_k, \alpha_k) = \Sigma_\ell(L_{1\ell} \cap L_{2\ell}, S_{1\ell}, \alpha_{1\ell})$ for both $k = 1$ and $k = 2$, as required.

Corollary 3.1. *The map* $\widetilde{R}^\sigma(G)_{\mathbb{C}} \to \tau_\sigma(G)/\ \mathrm{Im}\ \tau_{\sigma,I}(G)$, *induced by the natural map* $R^\sigma(G) \to K^\sigma(G)$ *and* $K^\sigma(G) \to \tau_\sigma(G)$ *by* $(P, S) \mapsto (P, S, id)$, *is surjective.*

Proof. Let Y be an irreducible subvariety of $\Theta \subset \Theta_K$ as in Theorem 3, and $\theta : \mathcal{Z}_K \to k[Y]$ its generic point, corresponding to $Y \hookrightarrow \Theta \hookrightarrow \Theta_K$. Denote by $\overline{k(Y)}$ an algebraic closure of the field $k(Y)$ of rational functions on Y, and fix an embedding $\overline{k(Y)} \hookrightarrow \mathbb{C}$; $k[Y]$ is naturally embedded in its fraction field $k(Y)$, and so in $\overline{k(Y)}$. Then θ defines also maps $\mathcal{Z}_K \to \overline{k(Y)}$ and $\mathcal{Z}_K \to \mathbb{C}$, denoted again by θ.

If L'/L is a finite field extension, $\theta : \mathcal{Z}_K \to L$ a homomorphism, and θ' is its composition with the embedding $L \hookrightarrow L'$, then $R^\sigma_\theta(L)$ embeds in $R^\sigma_{\theta'}(L')$ via $j' = j/[L' : L']$. Here j maps $V_L \in R^\sigma_\theta(L)$ to $V_{L'} = V_L \otimes_L L' \in R^\sigma_{\theta'}(L')$. Indeed, the restriction of the L'-module $j'(V_L)$ to L is V_L. Let $\overline{L}$ denote an algebraic closure of L, and $\overline{\theta} : \mathcal{Z}_K \to \overline{L}$ the composition of θ with $L \hookrightarrow \overline{L}$. We conclude that $R^\sigma_\theta(L)$ embeds in $R^\sigma_{\overline{\theta}}(\overline{L}) = \lim_{\rightarrow} R^\sigma_{\theta'}(L')$ (limit over $L', L \subset L' \subset \overline{L}$).

If $\overline{L} \subset \overline{E}$ are algebraically closed, and $\theta : \mathcal{Z}_K \to \overline{E}$ is the composition of $\theta : \mathcal{Z}_K \to \overline{L}$ and $\overline{L} \hookrightarrow \overline{E}$, then $R^\sigma_\theta(\overline{L}) \tilde{\to} R^\sigma_\theta(\overline{E})$. Indeed, any irreducible $\mathbb{H}_K$-module over $\overline{L}$ is absolutely irreducible, namely it stays irreducible after tensorring with $\overline{E}$ over $\overline{L}$. On the other hand, given an irreducible in $R^\sigma_\theta(\overline{E})$ with a basis B as a vector space over $\overline{E}$, it is obtained from $\mathbb{H}_K B \otimes_Q \overline{L}$ in $R^\sigma_\theta(\overline{L})$ on tensorring with $\overline{E}$ over $\overline{L}$. Here $\mathbb{H}_K$ is the Hecke algebra over Q associated with K. Note that any element of $R^\sigma_{\overline{\theta}}(\overline{L})$ lies in $R^\sigma_{\theta'}(L')$ for some finite extension L' of L in $\overline{L}$.

In view of these commments we have the natural inclusions

$$R^\sigma_\theta(k(Y)) \hookrightarrow R^\sigma_\theta(\overline{k(Y)}) \hookrightarrow R^\sigma_\theta(\mathbb{C}).$$

Theorem 1 implies that if θ is σ-discrete, namely $\theta \in \Theta^\sigma_{\rm disc}(G)$, then $\dim \overline{\theta}(= \dim \overline{Y}) = 0$. In particular $R^\sigma_\theta(\mathbb{C}) \subset R^\sigma_I(G)_\mathbb{C}$ if $\dim \overline{\theta} > 0$. Theorem 2 asserts the existence of an operator $A_\sigma = A^{\mathbf{c}}_\sigma$ on $\widetilde{R}^\sigma(G)_\mathbb{C}$ such that for any field L of characteristic zero and $\pi \in \widetilde{R}^\sigma(G;L)_\mathbb{C}$ we have $A_\sigma \pi = 0$ iff $\pi \in \widetilde{R}^\sigma_I(G;L)_\mathbb{C}$. Consequently $\pi \in \widetilde{R}^\sigma_\theta(k(Y))_\mathbb{C} \subset \widetilde{R}^\sigma_\theta(\mathbb{C})_\mathbb{C}$ lies in $\widetilde{R}^\sigma_{\theta,I}(k(Y))_\mathbb{C}$ iff $A_\sigma \pi = 0$, namely iff $\pi \in \widetilde{R}^\sigma_{\theta,I}(\mathbb{C})_\mathbb{C}$ (by a double application of Theorem 2), and if $\dim \overline{\theta} > 0$, by Theorem 1.

Theorem 3 provides an isomorphism

$$\tau_{\sigma,i}(\Theta)/ \text{ Im } \tau_{\sigma,i-1}(\Theta) \simeq \oplus_{Y \subset \Theta} \widetilde{R}^\sigma_\theta(k(Y))_\mathbb{C} \text{ (irreducible } Y = \sigma Y, \ \dim \overline{Y} = i).$$

If $i > 0$ then $\widetilde{R}^\sigma_\theta(k(Y))_\mathbb{C} \subset \widetilde{R}^\sigma_I(G)_\mathbb{C}$ as was just observed. Hence by Theorem 2 we have that $A_\sigma[\tau_{\sigma,i}(\Theta)/ \text{ Im } \tau_{\sigma,i-1}(\Theta)] = 0$. It follow that for some $j \geq 0$ we have $A^j_\sigma(\tau_\sigma(G)) = \tau_{\sigma,0}(G) \subset \widetilde{R}^\sigma(G)_\mathbb{C}$. In other words, given (π, S, α) in $\tau_\sigma(G)$, it is equal to $A^j_\sigma(\pi, S, \alpha) \in \widetilde{R}^\sigma(G)_\mathbb{C}$ up to $(\pi, S, \alpha) - A^j_\sigma(\pi, S, \alpha) \in \tau_{\sigma,I}(G)$. Hence the map $\widetilde{R}^\sigma(G)_\mathbb{C} \to \tau_\sigma(G)/ \text{ Im } \tau_{\sigma,I}(G)$ is surjective.

F. Categorical cocenter.

We need to relate the categorical σ-cocenter $\tau_\sigma(\mathbb{M}(\mathbb{H}_K))$ of §**E** with the algebra σ-cocenter $\tau_\sigma(\mathbb{H}_K)$ which occurs in the statement of the Main Theorem. Instead of $\mathbb{H}_K$ we shall work with a $\mathbb{C}$-algebra A with identity, and denote by σ an automorphism of A of finite order ℓ. The semi-direct product $A^{\#} = A \rtimes < \sigma >$ contains the coset $A\sigma$. Put

$$\tau A = A/[A,A],\ \tau A^{\#} = A^{\#}/[A^{\#}, A^{\#}],$$

and

$$\tau_\sigma A = A/ < a\sigma(b) - ba; a, b \in A > \simeq A\sigma/[A\sigma, A] = A\sigma/A\sigma \cap [A^{\#}, A^{\#}].$$

Let $\mathbb{M}(A)$ (resp. $\mathbb{M}(A^{\#})$) be the category of (non-degenerate) A-modules (resp. $A^{\#}$-modules). An $A^{\#}$-module is a pair (P,S) consisting of an A-module P and an element S in the set $\mathrm{Aut}_A^\sigma P$ of vector space automorphisms $S : P \tilde{\to} P$ of order ℓ which satisfy $S(ap) = \sigma^{-1}(a)S(p)(a \in A, p \in P)$; σ acts on P via S.

The cocenter $\tau(\mathbb{M}(A))$ of the category $\mathbb{M}(A)$ is the quotient of the free abelian group generated over $\mathbb{C}$ by the pairs (P,α) consisting of a projective finitely generated A-module P and $\alpha \in \mathrm{End}_A P$, by the relations

(1) $(P,\alpha) \sim (P',\alpha') + (P'',\alpha'')$ if $0 \to (P',\alpha') \to (P,\alpha) \to (P'',\alpha'') \to 0$ is exact;

(2) $(P,\alpha+\beta) \sim (P,\alpha) + (P,\beta)$, $(P,\alpha\beta) \sim (P,\beta\alpha)$, $(P,t\alpha) \sim t(P,\alpha)$ $(t \in \mathbb{C}; \alpha,\beta \in \mathrm{End}_A P)$.

Similarly $\tau(\mathbb{M}(A^{\#}))$ is the quotient of the Grothendieck group of pairs (P,α) of a projective finitely generated $A^{\#}$-module P and $\alpha \in \mathrm{End}_{A^{\#}} P$, by the analogous relations. The σ-cocenter $\tau_\sigma(\mathbb{M}(A))$ has already been defined in §**E**; it coincides with $\tau(\mathbb{M}(A))$ when $\sigma =$ identity.

Theorem 4. *We have* $\tau_\sigma(\mathbb{M}(A)) \simeq \tau_\sigma A$ *; in particular* $\tau(\mathbb{M}(A)) \simeq \tau A$ *.*

Proof. Let P be a free finitely-generated A-module, and $e_1, \cdots, e_k$ a basis of P over A. Fix S in $\mathrm{Aut}_A^\sigma P$; then P extends to an $A^{\#}$-module by $\sigma(p) = S(p)$. Given $\alpha \in \mathrm{End}_A P$ we shall associate to (P,S,α) an element in $\tau_\sigma A$ as follows. Since $\alpha\sigma$ is an endomorphism of P there are $\overline{\alpha}_{ij}$ in A such that

$$\alpha\sigma e_i = \sum_j \overline{\alpha}_{ij} e_j. \text{ Define } \mathrm{tr}_P(\alpha\sigma) \text{ to be } \sum_i \overline{\alpha}_{ii} (\in A).$$

We claim that $\mathrm{tr}_P(\alpha\sigma)$ is a well-defined element of $A/ < a\sigma(b) - ba >$. We need to show that $\mathrm{tr}_P(\alpha\sigma)$ is independent of the choice of the basis $\{e_i\}$. If $f_1, \cdots, f_k$ is another basis of P over A then $\alpha\sigma f_i = \sum_j \overline{\beta}_{ij}\ f_j$ $(\overline{\beta}_{ij} \in A)$; moreover, there are $f_{ij},\ e_{ij} \in A$ with $f_i = \sum_j f_{ij} e_j,\ e_i = \sum_j e_{ij} f_j$. Consequently $\sum_j f_{ij} e_{jk} = \delta_{ik} = \sum_j e_{ij} f_{jk}$. Then

$$\sum_{jk} \overline{\beta}_{ij} f_{jk} e_k = \sum_j \overline{\beta}_{ij} f_j = \alpha\sigma f_i = \sum_j \sigma^{-1}(f_{ij}) \alpha\sigma(e_j) = \sum_{jk} \sigma^{-1}(f_{ij}) \overline{\alpha}_{jk} e_k,$$

and

$$\overline{\alpha}_{\ell k} = \sum_{ij} \sigma^{-1}(e_{\ell i})\sigma^{-1}(f_{ij})\overline{\alpha}_{jk} = \sum_{ij} \sigma^{-1}(e_{\ell i})\overline{\beta}_{ij} f_{jk} \equiv \sum_{ij} \overline{\beta}_{ij} f_{jk} e_{\ell i} (\bmod\ < \sigma^{-1}(b)a - ab >).$$

Hence

$$\operatorname{tr}_P(\alpha\sigma) = \sum_i \overline{\alpha}_{ii} \equiv \sum_i \overline{\beta}_{ii}\ (\bmod\ \ < \sigma^{-1}(b)a - ab >)$$

is well-defined, as claimed.

If P is projective then there is a finitely generated A-module Q such that $P \oplus Q$ is free. Define σQ to be the vector space Q on which $a \in A$ acts by $\sigma^{-1}(a)$. Put $Q_\sigma = Q \oplus \sigma Q \oplus \cdots \oplus \sigma^{\ell-1} Q$. Then $\operatorname{Aut}_A^\sigma Q_\sigma$ is non-empty, and we define $\operatorname{tr}_P(\alpha\sigma)$ to be $\operatorname{tr}_{P\oplus\cdots\oplus P \oplus Q_\sigma}(\alpha\sigma \oplus 0 \oplus \cdots \oplus 0)$; it is independent of the choice of Q.

A basis of the trivial A-module A is its identity, which is fixed by σ. If a denotes multiplication of A by $a \in A$, then $\operatorname{tr}_A(a\sigma) = \operatorname{tr}_A a = a$. It follows that the map $\operatorname{tr} : \tau_\sigma(\mathbb{M}(A)) \to \tau_\sigma A$ is surjective.

If $\alpha\sigma e_i = \sum_j \overline{\alpha}_{ij} e_j$ and $\beta e_i = \sum_j \beta_{ij} e_j$ $(\overline{\alpha}_{ij}, \beta_{ij} \in A)$, where $\beta \in \operatorname{End}_A P$, then

$$\beta \cdot \alpha\sigma e_i = \sum_{jk} \overline{\alpha}_{ij}\beta_{jk} e_k, \qquad \alpha\sigma \cdot \beta e_i = \sum_{jk} \sigma^{-1}(\beta_{ij})\overline{\alpha}_{jk} e_k,$$

and so

$$\operatorname{tr}_P(\alpha\sigma \cdot \beta - \beta \cdot \alpha\sigma) = \sum_{ij} [\sigma^{-1}(\beta_{ij})\overline{\alpha}_{ji} - \overline{\alpha}_{ji}\beta_{ij}] \in \langle \sigma^{-1}(b)a - ab \rangle.$$

To prove injectivity , given $\alpha \in \operatorname{End}_A P$ with $\operatorname{tr}_P(\alpha\sigma) \in \langle \sigma^{-1}(b)a - ab \rangle$ we need to exhibit $\beta, \gamma \in \operatorname{End}_A P$ with $\alpha\sigma = \gamma\sigma \cdot \beta - \beta \cdot \gamma\sigma$. If $\operatorname{tr}_P(\alpha\sigma) = \sum_i (\sigma^{-1}(b_i)a_i - a_i b_i)$, let P_1 be a free A-module with basis $\{e_i\}$, and $\beta, \gamma \in \operatorname{End}_A P_1$ endomorphisms with $\gamma\sigma e_i = a_i e_i, \beta e_i = b_i e_i$. Then $\operatorname{tr}_{P_1}(\beta \cdot \gamma\sigma - \gamma\sigma \cdot \beta) = \sum_i (a_i b_i - \sigma^{-1}(b_i)a_i)$, and $\operatorname{tr}_{P\oplus P_1}[\alpha\sigma \oplus (\beta \cdot \gamma\sigma - \gamma\sigma \cdot \beta)] = 0$. Consequently, we may assume that $\operatorname{tr}_P(\alpha\sigma) = \sum_i \overline{\alpha}_{ii}$ is zero (on replacing (P, α) by $(P \oplus P_1, \alpha \oplus 0)$). Again we need to present an A-module P_1 with σ-action and $\beta, \gamma \in \operatorname{End}_A P_1$ such that $(P, \alpha) \sim (P_1, \gamma\sigma \cdot \beta - \beta \cdot \gamma\sigma)$ in $\mathbb{M}(A)$. By (1) it suffices to take P_1 free on a basis e_1, e_2, and assume that (i) $\alpha\sigma e_1 = be_2, \alpha\sigma e_2 = ae_1$, or: (ii) $\alpha\sigma e_1 = ae_1, \alpha\sigma e_2 = -ae_2$. In the first case (i), take β with $\beta e_1 = e_1, \beta e_2 = 0$, and γ with $\gamma\sigma e_1 = be_2, \gamma\sigma e_2 = -ae_1$; then $(\gamma\sigma \cdot \beta - \beta \cdot \gamma\sigma)e_1 = be_2, (\gamma\sigma \cdot \beta - \beta \cdot \gamma\sigma)e_2 = ae_1$. In the second case (ii), take β, γ with $\beta e_1 = e_2, \beta e_2 = e_1, \gamma\sigma e_1 = e_2, \gamma\sigma e_2 = (a+1)e_1$. Then $(\gamma\sigma \cdot \beta - \beta \cdot \gamma\sigma)e_1 = (a+1)e_1 - e_1 = ae_1,\ (\gamma\sigma \cdot \beta - \beta \cdot \gamma\sigma)e_2 = e_2 - (a+1)e_2 = -ae_2$, as required.

Consider the map $\Psi : \tau_\sigma(\mathbb{H}_K) = \mathbb{H}_K / \langle h_1\sigma(h_2) - h_2 h_1 \rangle \to R_\sigma^*(G)$, given by $\Psi(h) = F_h$, where $F_h((\pi, \sigma)) = \operatorname{tr} \pi(h\sigma)$. Since $\tau_\sigma(\mathbb{H}_K) = \tau_\sigma(\mathbb{M}(\mathbb{H}_K))$ by Theorem 4, Ψ defines a map $\tau_\sigma(\mathbb{M}(\mathbb{H}_K)) \to R^*_\sigma(G)$, also denoted by Ψ. By definition $\Psi((P,S,\alpha)) = F_h$ where $h = \operatorname{tr}_P(\alpha S)$.

The functors r, i on $\mathbb{M}(G)$ define homomorphisms

$$i_M = i_{GM} : R^\sigma(M) \to R^\sigma(G) \text{ and } r_M = r_{MG} : R^\sigma(G) \to R^\sigma(M)$$

of the Grothendieck groups where $M = \sigma M < G$, and dual maps

$$i^*_M = i^*_{GM} : R^*_\sigma(G) \to R^*_\sigma(M) \text{ and } r^*_M = r^*_{MG} : R^*_\sigma(M) \to R^*_\sigma(G)$$

on the dual spaces. Recall that r_{MG} is defined using the standard parabolic subgroup $P = MP_0$, and $\overline{r}_{MG}$ using the opposite parabolic $\overline{P} = M\overline{P}_0$.

Corollary 4.1. *The homomorphism $\Psi : \tau_\sigma(\mathbb{M}(\mathbb{H}_G)) \to R^*_\sigma(G)$ intertwines the homomorphisms i_{GM}, r_{MG} with the homomorphisms $\overline{r}^*_{MG}, i^*_{GM}$. Namely*

$$\Psi(i_{GM}(P_M, S_M, \alpha_M)) = \overline{r}^*_{MG}(\Psi_M((P_M, S_M, \alpha_M)))$$

and

$$\Psi_M(r_{MG}(P,S,\alpha)) = i^*_{GM}(\Psi((P,S,\alpha)))$$

for all $(P_M, S_M, \alpha_M) \in \tau_\sigma(\mathbb{M}(M))$ and $(P,S,\alpha) \in \tau_\sigma(\mathbb{M}(G))$.

Proof. Denote by $\operatorname{Ext}^i(P,\pi) = \operatorname{Ext}^i_{\mathbb{H}^\#_G}(P,\pi)$ the ith Ext group of the $\mathbb{H}^\#_G$-modules P and π; it is an $\mathbb{H}^\#_G$-module. We first claim that the value of Ψ at $(P,S,\alpha) \in \tau_\sigma(\mathbb{M}(\mathbb{H}_K))$ is the homomorphism which takes $(\pi,\sigma) \in R^\sigma(G)$ to

$$\operatorname{tr}[\alpha\sigma; \operatorname{Ext}^*(P,\pi)] = \sum_i (-1)^i \operatorname{tr}[\alpha\sigma; \operatorname{Ext}^i(P,\pi)].$$

Since $\tau_\sigma(\mathbb{M}(\mathbb{H}_G))$ is generated by the (P,S,α), where P is a projective module, we may assume that P is projective. Then $\operatorname{Ext}^i(P,\pi) = \delta_{i,0} \operatorname{Hom}(P,\pi)$. Note that $P_K = C_c(G/K)$ is a projective generator of the category $\mathbb{M}(\mathbb{H}_K)$. Namely each projective module P in $\mathbb{M}(\mathbb{H}_K)$ is a direct summand of P_K. Extend α by 0 to P_K; then $\alpha \in \operatorname{End} P_K$, and

$$\begin{aligned}
\operatorname{tr} \pi(\alpha\sigma) &= \operatorname{tr} \pi^K(\alpha\sigma) \\
&= \operatorname{tr}[\alpha\sigma; \pi^K = \operatorname{Hom}_K(\mathbb{1}_K, \pi|K) = \operatorname{Hom}_G(i_{GK}\mathbb{1}_K, \pi) = \operatorname{Hom}(P_K, \pi)] \\
&= \operatorname{tr}[\alpha\sigma; \operatorname{Ext}^*(P_K, \pi)]
\end{aligned}$$

as claimed. To complete the proof of the corollary, we quote (from Bernstein [B]) the following

Second Adjointness Theorem ([B]). *The functor $\overline{r}_M$ is right adjoint to the functor i_M.*

Hence

$$[\Psi(i_M(E_M,S_M,\alpha_M))]((\pi,\sigma)) = F_{i_M(E_M,S_M,\alpha_M)}((\pi,\sigma)) = \operatorname{tr}[i_M\alpha_M\cdot\sigma;\ \operatorname{Ext}^*(i_ME_M,\pi)]$$
$$= \operatorname{tr}[\alpha_M\cdot\sigma;\ \operatorname{Ext}^*(E_M,\overline{r}_M\pi)] = [(\overline{r}^*_M\Psi)(E_M,S_M,\alpha_M)]((\pi,\sigma))$$

for all $(\pi,\sigma)\in R^\sigma(G)$ proving the first claim.

The other assertion of the corollary, that Ψ intertwines r_M on $\tau_\sigma(\mathbb{M}(\mathbb{H}_G))$ with i^*_M on $R^*_\sigma(G)$, follows from the Frobenius reciprocity (see [BZ1]), which says that r_M is left adjoint to i_M.

G. Trace Paley-Wiener theorem.

The following is (a twisted generalization of) the trace Paley-Wiener theorem of [BDK].

Theorem 5. *The map $\Psi:\tau_\sigma(\mathbb{H}_G)\to R^*_\sigma(G)_{\text{good}}$, by $h\mapsto F_h$, where $F_h((\pi,\sigma)) = \operatorname{tr}\,\pi(h\sigma)$, is surjective.*

For any subset X of $R^*(G)$ denote by $R^*_\sigma(X)_{\text{good}}$ and $R^*_\sigma(X)_{\text{trace}}$ the spaces of restrictions of elements of $R^*_\sigma(G)_{\text{good}}$ and of $R^*_\sigma(G)_{\text{trace}}$ to X. The corresponding forms will be called *good* or *trace* forms on X. Put $R^*_\sigma(G)^{\text{disc}}_{\text{good}}$ for $R^*_\sigma(\Theta^\sigma_{\text{disc}}(G))_{\text{good}}$.

Proposition 5.1. *The map $\Psi:\tau_\sigma(\mathbb{H}_G)\to R^*_\sigma(G)^{\text{disc}}_{\text{good}}$ is surjective.*

Proof. By Theorem 1, for every connected component Θ of $\Theta(G)$ the variety $\Theta^\sigma_{\text{disc}}$ is a finite union of $X^\sigma(G)$-orbits. Since an element of $R^*_\sigma(G)_{\text{good}}$ is supported only on finitely many groups $R^\sigma(\Theta)$, it suffices to show that for any finite union X of $X^\sigma(G)$-orbits in Θ the map $\Psi:\tau_\sigma(\mathbb{H}_G)\to R^*_\sigma(X)_{\text{good}}$ is onto.

If $X^\sigma(G)$ is finite then X is a finite set. Then the restriction to X of any linear form $F:R^\sigma(G)\to\mathbb{C}$ is a trace form, and in particular $R^*_\sigma(X)_{\text{trace}} = R^*_\sigma(X)_{\text{good}}$. Indeed, the twisted characters of irreducible σ-invariant G-modules are linearly independent functionals on $\mathbb{H}_G$.

In general X has the natural structure of an algebraic variety, as the union of finitely many $X^\sigma(G)$-orbits. By definition of good forms we have

$$R^*_\sigma(X)_{\text{trace}}\subset R^*_\sigma(X)_{\text{good}}\subset k[X],$$

where $k[X]$ is the algebra of regular functions on X.

Choose a σ-invariant cocompact lattice Λ in the center Z of G. Put $X(\Lambda) = \operatorname{Hom}(\Lambda,\mathbb{C}^\times)$, and $Y = X^\sigma(\Lambda)$ for the subgroup of σ-invariant characters. Then Y is an affine algebraic variety. The restriction map $X^\sigma(G)\to Y$ is a finite epimorphism of algebraic groups. Denote by ω_π the central character of $\pi\in\operatorname{Irr}G$; consider the map $\operatorname{Irr}^\sigma(G)\to Y, \pi\mapsto\omega_\pi|\Lambda$. Its restriction $X\to Y$ to X is a finite $X^\sigma(G)$-equivariant submersive morphism of algebraic varieties. Hence $k[X]$ is a finitely generated $k[Y]$-module. Note that $R_g = R^*_\sigma(X)_{\text{good}}$ is a $k[Y]$-submodule of $k[X]$, where $k[Y]$ acts by

$(fF)(\pi) = f(\omega_\pi|\Lambda)F(\pi)$ $(f \in k[Y], F \in R_g)$. Also $R_t = R^*_\sigma(X)_{\text{trace}}$ is a $k[Y]$-submodule via $f \cdot \text{tr } \pi(h\sigma) = \text{tr } z_f \cdot \pi(h\sigma)$; here $z_f \in k[X]$ is the image of $f \in k[Y]$ under the natural map $k[Y] \to k[X]$.

For any $y \in Y$ let $M_y \subset k[Y]$ be the maximal ideal consisting of those polynomial functions in $k[Y]$ which vanish at y. For each $k[Y]$-module E put $E_y = E/M_yE$ for the fiber of E at y. Since $u : X \to Y$ is finite and submersive, the set $X_y = u^{-1}(y)$ is finite, and the fiber $k[X]_y$ coincides with $k[X_y]$. Since X_y is a finite set, we have $R^*_\sigma(X_y)_{\text{trace}} = R^*_\sigma(X_y)_{\text{good}}$ as noted above; hence $R_g \subset R_t + M_y k[X]$. Put $E = k[X]/R_t, E' = R_g/R_t \subset E$. Then $E' \subset M_yE$ for each $y \in Y$. Since E is a finitely generated $k[Y]$-module it is locally free generically, namely at almost each $y \in Y$. Moreover, E is locally free at every $y \in Y$ since E is $X^\sigma(G)$-equivariant. Then $E' \subset M_yE$ for all $y \in Y$ implies that $E' = 0$, since a function which vanishes at each point of a variety is necessarily the zero function. Hence $R_t = R_g$ as required.

Proof of Theorem 5. We argue by induction on M; the case of M_0 follows from Proposition 5.1, since $R^*_\sigma(M_0)_{\text{good}} = R^*_\sigma(M_0)^{\text{disc}}_{\text{good}}$. By Corollary 2.3 there are $c_M \in Q$ such that for each $F \in R^*_\sigma(G)_{\text{good}}$ there is $F^d \in R^*_\sigma(G)^{\text{disc}}_{\text{good}}$ with $F = F^d + \sum_{M \subsetneqq G} c_M r^*_{MG}(i^*_{GM}F)$. Then $F_M = i^*_{GM}F$ lies in $R^*_\sigma(M)_{\text{good}}$, and by induction there is some $h_M \in \tau_\sigma(\mathbb{H}_M)$ which maps to F_M by the map Ψ_M of the theorem. Then $\Psi_M(h_M) = F_M$, and by Corollary 4.1 we have $\Psi_G(\bar{i}_{GM}h_M) = r^*_{MG}F_M = r^*_{MG}\Psi_M(h_M)$. Hence $r^*_{MG}i^*_{GM}F$ is in the image of Ψ_G, and so is F since F^d is in the image by Proposition 5.1.

H. Density theorem.

The following is (a twisted generalization of) the density theorem of [K1, Appendix].

Theorem 6. *The map* $\Psi : \tau_\sigma(\mathbb{H}_G) \to R^*_\sigma(G)_{\text{trace}} = R^*_\sigma(G)_{\text{good}}$ *of Theorem 5 is injective.*

This can be phrased as follows. Given $h \in \mathbb{H}_G$ with $\text{tr } \pi(h\sigma) = 0$ for all $(\pi, \sigma) \in \text{Irr}^{\,\sigma}(G)$, then h lies in the span $[\mathbb{H}_G\sigma, \mathbb{H}_G]\sigma^{-1}$ of $h_1\sigma(h_2) - h_2h_1$ $(h_1, h_2 \in \mathbb{H}_G)$. Here $\pi(h\sigma) = \int_G \pi(g\sigma)h(g)$ is a trace class operator.

We claim that it suffices to prove the theorem under the assumption that $X^\sigma(G)$ is finite. Indeed, let ω be a character of the center Z of G. By a standard reduction step we may work with the Hecke algebra of functions h which transform under Z by ω^{-1} and are compactly supported modulo Z, and forms on the Grothendieck group of G-modules π which transform under Z via ω. For $\pi \in \text{Irr}^{\,\sigma}(G)$ with central character ω, we have $\omega = {}^\sigma\omega$. Multiplying π by a σ-invariant unramified character we may assume that ω is trivial on a σ-stable lattice Λ of finite index in Z. Replacing G by G/Λ we may assume that $X^\sigma(G)$ is finite.

Suppose then that $X^\sigma(G)$ is finite. It suffices to show for each connected component Θ of $\Theta(G)$ that the map $\tau_\sigma(\mathbb{H}_\Theta) \to R^*_\sigma(\Theta)_{\text{good}}$ is injective. Put $\tau_\sigma(\mathbb{H}_\Theta)^d = \tau_\sigma(\mathbb{H}_\Theta)/\tau_{\sigma,I}(\mathbb{H}_\Theta)$. Corollary 3.1 and Theorem 4 assert that the map $\widetilde{R}^\sigma(\Theta)_\mathbb{C} \to \tau_\sigma(\mathbb{H}_\Theta)^d$ is surjective, and Proposition 5.1 implies the surjectivity of the map $\Psi : \tau_\sigma(\mathbb{H}_\Theta)^d \to$

$R_\sigma^*(\Theta)_{\text{good}}^{\text{disc}}$. Since $\widetilde{R}_I^\sigma(\Theta)_{\mathbb{C}}$ maps to zero in $\widetilde{R}^\sigma(\Theta)_{\mathbb{C}} \to \tau_\sigma(\mathbb{H}_\Theta)^d$, we obtain a surjective map

$$\widetilde{R}^\sigma(\Theta)_{\mathbb{C}}^d = \widetilde{R}^\sigma(\Theta)_{\mathbb{C}}/\widetilde{R}_I^\sigma(\Theta)_{\mathbb{C}} \to R_\sigma^*(\Theta)_{\text{good}}^{\text{disc}}.$$

Since $\Theta_{\text{disc}}^\sigma$ is a finite set for each Θ (by Theorem 1, under our assumption that $X^\sigma(G)$ is finite), the complex vector space $\widetilde{R}^\sigma(\Theta)_{\mathbb{C}}^d$ is finite dimensional, and has the same dimension as its dual $R_\sigma^*(\Theta)_{\text{good}}^{\text{disc}}$. In particular, the map above is an isomorphism, and if $h \in \widetilde{R}^\sigma(\Theta)_{\mathbb{C}}$ maps to zero in $R_\sigma^*(\Theta)_{\text{good}}$, then h lies in $\widetilde{R}_I^\sigma(\Theta)_{\mathbb{C}}$.

To prove the theorem consider h in $\tau_\sigma(\mathbb{H}_\Theta)$ which maps to zero in $R_\sigma^*(\Theta)_{\text{good}}$. For any $M = \sigma M < G$, the image of 0 by $i_{GM}^* : R_\sigma^*(\Theta)_{\text{good}} \to R_\sigma^*(\Theta_M)_{\text{good}}$, where $\Theta_M = i_{GM}^{-1}(\Theta)$, is 0. By induction on M, when $M = \sigma M \lneq G$ the inverse image of $0 \in R_\sigma^*(\Theta_M)_{\text{good}}$ by $\Psi_M : \tau_\sigma(\mathbb{H}_{\Theta_M}) \to R_\sigma^*(\Theta_M)_{\text{good}}$ is zero. Corollary 4.1 asserts that $\Psi_M(r_{MG}h) = i_{GM}^*(\Psi_G h)$. Hence $r_{MG}h = 0$. It follows that h lies in the intersection of $\ker\ r_{MG}$, $M = \sigma M \lneq G$. Consequently $h = A_\sigma h$ for $A_\sigma = A_\sigma^{\mathbf{c}}$ as in Theorem 2. As in the proof of Corollary 3.1, for a sufficiently large j we have that $A_\sigma^j h$ lies in $\widetilde{R}^\sigma(\Theta)_{\mathbb{C}}(\to \tau_\sigma(\mathbb{H}_\Theta))$. Hence h lies in $\widetilde{R}^\sigma(\Theta)_{\mathbb{C}}$, and it maps to 0 under the map $\widetilde{R}^\sigma(\Theta)_{\mathbb{C}}(\to \tau_\sigma(\mathbb{H}_\Theta)) \to R_\sigma^*(\Theta)_{\text{good}}$ mentioned above. Therefore it lies in $\widetilde{R}_I^\sigma(\Theta)_{\mathbb{C}}$, and $A_\sigma h = 0$ by Theorem 2. We conclude that $h = A_\sigma h$ is zero, as required.

Theorems 5 and 6 establish the surjectivity and injectivity of the map of the Main Theorem, whose proof is now complete.

Appendix. Cohomological dimension.

Theorem. *The category $\mathbb{M}(G)$ has finite cohomological dimension bounded by $d_0 = \dim X(M_0)$.*

Proof. We should show that each G-module X has a projective resolution of length $\leq d_0$.

(1) We proceed to construct the standard resolution of the trivial G-module $\mathbb{C}$ on using the theory of buildings (see Tits [T]). Recall that the building $B = B(G)$ associated with the group G is a CW-complex equipped with an action of G (on B). It has the following properties.

(i) All cells of B are polyhedra, and the action of G preserves cell decomposition.

(ii) For each cell τ of B, its stabilizer G_τ is an open compact subgroup of G which fixes all points in τ.

(iii) Modulo the action of G there are only finitely many cells. The dimension of any cell is bounded by d_0.

(iv) The building B is contractible as a topological space.

Consider the chain complex $C = \{0 \to C_{d_0} \to C_{d_0-1} \to \cdots \to C_0 \to 0\}$ of B with complex coefficients. This is a complex of G-modules. If $\tau_1, \cdots, \tau_k$ is a set of representatives of cells modulo the action of G, then $\oplus_j C_j = \bigoplus_{1\le i\le k} \mathrm{ind}\,(G, G_{\tau_i}, \mathbb{C})$ and (ii) implies that C_j are projective G-modules. Since B is contractible, we have $H^i(C) = 0$ for $i \neq 0$ and $H^0(C) = \mathbb{C}$; thus C is a projective resolution of $\mathbb{C}$ called the *standard resolution* of the trivial G-module $\mathbb{C}$.

(2) Let X be a G-module. Consider the complex $C_X = \{C_i \otimes_{\mathbb{C}} X\}$. Clearly this is a resolution of the G-module X of length d_0; we need to check that it is projective. Then let P be a projective G-module. We have to show that $P \otimes_{\mathbb{C}} X$ is also projective. For each G-module Y we have $\mathrm{Hom}_G(P \otimes X, Y) = \mathrm{Hom}_G(P, \mathrm{Hom}^0_{\mathbb{C}}(X, Y))$. Here $\mathrm{Hom}^0_{\mathbb{C}}(X, Y)$ is the smooth part of the G-module $\mathrm{Hom}_{\mathbb{C}}(X, Y)$. Hence it suffices to check that the functor $Y \mapsto \mathrm{Hom}^0_{\mathbb{C}}(X, Y)$ is exact. Fix an open compact subgroup K of G. As a vector space, $\mathrm{Hom}^0_{\mathbb{C}}(X, Y)$ depends only on the K-module structure of Y. Since the category $\mathbb{M}(K)$ of K-modules is completely reducible, each exact sequence in $\mathbb{M}(K)$ splits. Hence the functor $Y \mapsto \mathrm{Hom}^0_{\mathbb{C}}(X, Y)$ is exact, and $P \otimes_{\mathbb{C}} X$ is projective, as required.

Remark. The standard resolution C_X constructed above is not finitely generated in general, even when X is irreducible. If X is finitely generated then one can construct a resolution $0 \to P_{d_0} \to P_{d_0-1} \to \cdots \to P_0 \to X \to 0$ in which all P_i are finitely generated and $P_{d_0-1}, P_{d_0-2}, \cdots, P_0$ are projective. The Theorem implies that P_{d_0} is also projective.

References

[B] J. Bernstein, Notes for courses at Harvard and Tel-Aviv Universities, manuscript in preparation.

[BD] J. Bernstein, P. Deligne, Le "centre" de Bernstein, dans *Représentations des groupes reductifs sur un corps local*, Hermann, Paris 1985.

[BDK] J. Bernstein, P. Deligne, D. Kazhdan, Trace Paley-Wiener theorem for reductive p-adic groups, *J. Analyse Math.* 47 (1986), 180-192.

[BZ1] J. Bernstein, A. Zelevinski, Representations of the group $GL(n, F)$ where F is a non-archimedean local field, *Russian Math. Surveys* 31 (1976), 1-68.

[BZ2] J. Bernstein, A. Zelevinski, Induced representations of reductive p-adic groups I, *Ann. Sci. ENS* 10 (1977), 441-472.

[BW] A. Borel, N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Ann. of Math Studies 94 (1980).

[F] Y. Flicker, Rigidity for automorphic forms, *J. Analyse Math.* 49 (1987), 135-202.

[K1] D. Kazhdan, Cuspidal geometry of p-adic groups, *J. Analyse Math.* 47 (1986), 1-36.

[K2] D. Kazhdan, Representations of groups over close local fields, *J. Analyse Math.* 47 (1986), 175-179.

[T] J. Tits, Reductive groups over local fields, in *Proc. Sympos. Pure Math.* 33 I (1979), 29-69.