ON THE SYMMETRIC SQUARE: TWISTED TRACE FORMULA

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Abstract. A trace formula – for a smooth compactly supported function f on the adele group $\mathrm{PGL}(3, \mathbb{A})$ – twisted by the outer automorphism σ – is computed. The resulting formula is then compared with trace formulae for $H = H_0 = \mathrm{SL}(2)$ and $H_1 = \mathrm{PGL}(2)$, and matching functions f_0 and f_1 thereof. We obtain a trace formula identity which plays a key role in the study of the symmetric square lifting from $H(\mathbb{A})$ to $G(\mathbb{A})$. The formulae are remarkably simple, due to the introduction of a new concept of a regular function. This eliminates the singular and weighted integrals in the trace formulae.

0. Introduction

The purpose of this part is to compute explicitly a trace formula for a test function $f = \otimes f_v$ on $G(\mathbb{A})$, where $G = \mathrm{PGL}(3)$ and $\mathbb{A}$ is the ring of adeles of a number field F. This formula is twisted with respect to the outer twisting

$$\sigma(g) = \mathcal{J}{}^t g^{-1}\mathcal{J}, \qquad \mathcal{J} = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$$

and plays a key role in the study of the symmetric square lifting. We also stabilize the formula and compare it with the stable trace formula for a matching test function $f_0 = \otimes f_{0v}$ on $H(\mathbb{A}), H = \mathrm{SL}(2)$, and the trace formula for a matching test function $f_1 = \otimes f_{1v}$ on $H_1(\mathbb{A}), H_1 = \mathrm{PGL}(2)$. The final result concerns a distribution J in f, f_0, f_1 of the form

$$J = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I_1' - [I_0 + \frac{1}{2}I_0' + \frac{1}{4}I_0'' + \frac{1}{2}I_1],$$

where each I is a sum of traces of convolution operators. The final result asserts:
(3.5(1))$J = 0$ if f has two discrete components;
(3.5(2))J is equal to a certain integral if f has (i) a discrete component and (ii) a component which is sufficiently regular with respect to all other components.

The result (3.5(1)) is used in the study of the local symmetric square lifting in [IV]. The result (3.5(2)) is used in [IV] to show that $J = 0$ and to establish the global symmetric square lifting for automorphic forms with an elliptic component.

The vanishing of J for general matching functions is proven in [VI].

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Our formulae here are essentially those of the unpublished manuscript [Fadj], where we suggested, in the context of the (first non-trivial) symmetric square case, a truncation with which the trace formula, twisted by an automorphism σ, can be developed. This formula was subsequently computed in [CLL] to which we refer for proofs of the general form of the twisted trace formula. Our formulae here are considerably simpler than those of [Fadj]. This is due to the fact that we introduce here a new notion, of a regular function, and compute only an asymptotic form of the formula for a test function with a component which is sufficiently regular with respect to all other components. For such a function f the truncation is trivial; in fact f vanishes on the $G(\mathbb{A})$-σ-orbits of the rational elements (in $G(F)$) which are not σ-elliptic regular, and no weighted orbital integrals appear in our formulae. In [IV] and [VI] we show that this simple, asymptotic form of the formula suffices to establish the symmetric square lifting, unconditionally. Similar ideas are used in [Fgl2] to give a simple proof of base change for GL(2), and in our work on base change for $U(3)$ (see [Fu]) and other lifting problems.

1. Geometric Side

1.1. Let F be a number field, $\mathbb{A}$ its ring of adeles, G a reductive group over F with an anisotropic center, and L the space of complex valued square integrable functions ψ on $G(F)\backslash G(\mathbb{A})$. The group $G(\mathbb{A})$ acts on L by right translation, thus $(r(g)\psi)(h) = \psi(hg)$, and each irreducible constituent of the $G(\mathbb{A})$-module L is called an automorphic G-module (or representation). Let σ be an automorphism of G of finite order, and $G' = G \times \langle\sigma\rangle$ the semi-direct product of G and the group $\langle\sigma\rangle$ generated by σ. Extend r to a representation of $G'(\ A)$ on L by putting $(r(\sigma)\psi)(h) = \psi(\sigma^{-1}(h))$. Fix a Haar measure $dg = \otimes dg_v$ on $G(\mathbb{A})$. Let f be any smooth complex valued compactly supported function on $G(\mathbb{A})$. Let $r(f)$ be the (convolution) operator on L which maps ψ to

$$(r(f)\psi)(h) = \int f(g)\psi(hg)dg \qquad (g \text{ in } G(\mathbb{A})).$$

Then $r(f)r(\sigma)$, which we also denote by $r(f \times \sigma)$, is the operator on L which maps ψ to

$$h \mapsto \int_{g\in G(\mathbb{A})} f(g)\psi(\sigma^{-1}(hg))dg = \int_{g\in G(F)\backslash G(\mathbb{A})} K(h,g)\psi(g)dg,$$

where

$$K(h,g) = K_f(h,g) = \sum_{\gamma\in G(F)} f(h^{-1}\gamma\sigma(g)). \tag{1.1.1}$$

The theory of Eisenstein series provides a direct sum decomposition of the $G(\mathbb{A})$-module L as $L_d \oplus L_c$, where L_d, the "discrete spectrum", is a direct sum with finite multiplicities of irreducibles, and L_c, the "continuous spectrum", is a direct integral of such. This theory also provides an alternative formula for the kernel. The Selberg trace formula is an identity obtained on (essentially) integrating the two expressions for the kernel over the diagonal

$g = h$. To get a useful formula one needs to change the order of summation and integration. This is possible if G is anisotropic or if f has some special properties (see, e.g., [FK]). In general one needs to truncate the two expressions for the kernel in order to be able to change the order of summation and integration.

When σ is trivial, the truncation introduced by Arthur involves a term for each standard parabolic subgroup P of G. For $\sigma \neq 1$ it was suggested in [Fadj] (in the context of the symmetric square) to truncate only with the terms associated with σ-invariant P, and to use a certain normalization of a vector which is used in the definition of truncation. The consequent (non-trivial) computation of the resulting twisted (by σ) trace formula is carried out in [CLL] for general G and σ. In (2.1) we record the expression, proven in [CLL], for the analytic side of the trace formula, which involves Eisenstein series. In (2.2) and (2.3) we write out the various terms in our case of the symmetric square.

In this section we compute and stabilize the "elliptic part" of the geometric side of the twisted formula in our case. Namely we take $G = \mathrm{PGL}(3)$ and $\sigma(g) = \mathcal{J}{}^t g^{-1} \mathcal{J}$, and consider

$$\int_{G(F)\backslash G(\mathbb{A})} \left[\sum_{\delta \in G(F)} f(g^{-1}\delta\sigma(g)) \right] dg, \tag{1.1.2}$$

where the sum ranges over the δ in $G(F)$ whose norm $\gamma = N\delta$ in $H(F), H = \mathrm{SL}(2)$, is elliptic. Here we use freely the norm map N of [I], §1, and its properties.

In [Fadj] the integral of the truncated $\sum_{\delta \in G(F)} f(g^{-1}\delta\sigma(g))$ was explicitly computed, and the correction argument of [Fgl3] was applied to the hyperbolic weighted orbital integrals, to show that their limits on the singular set equal the integrals obtained from the δ with unipotent $N\delta$. These computations are not recorded here for the following reasons. We need the trace formula only for a function f which has a regular component or two discrete components (the definitions are given below). In the first case $f(g^{-1}\delta\sigma(g)) = 0$ for every g in $G(\mathbb{A})$ and δ in $G(F)$ such that $N\delta$ is not elliptic regular in $H(F)$; hence the geometric side of the trace formula (twisted by σ) is (1.1.2). In the second case the computations of [CLL], which generalize those of [Fadj], suffice to show the vanishing of all terms in the geometric side, other than those obtained from (1.1.2).

1.2. To compute and stabilize (1.1.2) let $G_\delta^\sigma = \{g \in G; g^{-1}\delta\sigma(g) = \delta\}$ be the σ-centralizer of δ, and

$$\Phi_f^\sigma(\delta) = \int_{G_\delta^\sigma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\delta\sigma(g))dg$$

the σ-orbital integral of f at δ. Implicit is a choice of a Haar measure on $G_\delta^\sigma(\mathbb{A})$, which is chosen to be compatible with isomorphisms (of G_δ^σ with $G_{\delta'}^\sigma$, or $H_{N\delta}$, etc., cf. [L1, p. 82]). Let $\{\delta\}$ denote the set of σ-conjugacy classes in $G(F)$ of elements δ such that $N\delta$ is elliptic in $H(F)$. Then (1.1.2) is equal to

$$\sum_{\{\delta\}} \int_{G_\delta^\sigma(F)\backslash G(\mathbb{A})} f(g^{-1}\delta\sigma(g))dg = \sum_{\{\delta\}} c(\delta)\Phi_f^\sigma(\delta), \tag{1.2.1}$$

where the volume

$$c(\delta)=|G_\delta^\sigma(F)\backslash G_\delta^\sigma(\mathbb{A})|$$

is finite since $N\delta$ is elliptic in $H(F)$. It is equal to $|H_\gamma(F)\backslash H_\gamma(\mathbb{A})|$ if $\gamma=N\delta$ is elliptic regular (in $H(F)$), to $|H(F)\backslash H(\mathbb{A})|$ if $\gamma=-I$, and to $|H_1(F)\backslash H_1(\mathbb{A})|$ if $\gamma=I$, where $H_1=\mathrm{PGL}(2)$.

Recall from [I], §1, that $D(\delta/F)$ denotes the set of σ-conjugacy classes within the stable σ-conjugacy class of δ in $G(F)$, and $D(\delta/F_v)$ denotes the local analogue for any place v of F. For any local or global field, $D(\delta/F)$ is a pointed set, isomorphic to $H^1(F,G_\delta^\sigma)$, and we put $D(\delta/\mathbb{A})=\oplus D(\delta/F_v)$ and $H^1(\mathbb{A},G_\delta^\sigma)=\oplus H^1(F_v,G_\delta^\sigma)$ (pointed direct sums). If $\gamma=N\delta$ is -1, we have $G_\delta^\sigma=H=\mathrm{SL}(2)$ and $H^1(F,G_\delta^\sigma)=H^1(\mathbb{A},G_\delta^\sigma)$ is trivial. If $\gamma=N\delta$ is I or elliptic regular then $H^1(F,G_\delta^\sigma)$ embeds in $H^1(\mathbb{A},G_\delta^\sigma)$ and the quotient is a group of order two. Denote by k the non-trivial character of this group.

Denote by $\Phi_{f_v}^\sigma(\delta)$ the σ-orbital integral at δ in $G(F_v)$ of a smooth compactly supported complex valued function f_v on $G(F_v)$. If F_v is nonarchimedean, denote its ring of integers by R_v. Let f_v^0 be the unit element in the Hecke algebra $\mathbb{H}_v$ of compactly supported $K_v=G(R_v)$-biinvariant functions on $G(F_v)$. Consider $f=\otimes f_v$, product over all places v of F, where $f_v=f_v^0$ for almost all v. Then, for every δ in $G(F)$ we have $\Phi_f^\sigma(\delta)=\prod_v\Phi_{f_v}^\sigma(\delta)$, where the product is absolutely convergent. It is easy to see that the sum

$$\sum_{\delta'\in D(\delta/F)}\Phi_f^\sigma(\delta')=\sum_{\delta'\in\Im[D(\delta/F)\to D(\delta/\mathbb{A})]}\prod_v\Phi_{f_v}^\sigma(\delta')$$

is finite for each f and δ. If $\gamma=N\delta$ is elliptic regular or the identity and k_v is the component at v of the associated quadratic character k on $D(\delta/\mathbb{A})/D(\delta/F)$, then the sum can be written in the form

$$\frac{1}{2}\prod_v\left[\sum_{\delta'\in D(\delta/F_v)}\Phi_{f_v}^\sigma(\delta')\right]+\frac{1}{2}\prod_v\left[\sum_{\delta'\in D(\delta/F_v)}\kappa_v(\delta')\Phi_{f_v}^\sigma(\delta')\right].\tag{1.2.2}$$

Note that for a given f and δ, for almost all v, the integral $\Phi_{f_v}^\sigma(\delta')$ vanishes unless δ' and δ are equal σ-conjugacy classes in $G(F_v)$.

Denote by f_{0v}^0 the unit element of the Hecke algebra $\mathbb{H}_{0v}$ of $H(F_v)$ with respect to $K_{0v}=H(R_v)$. Similarly introduce K_{1v}, $\mathbb{H}_{1v}$, and f_{1v}^0. Recall that the norm maps N, N_1 from the set of σ-stable conjugacy classes in $G(F)$ to the set of stable conjugacy classes in $H(F)$, $H_1(F)$ are defined in [I], §1.

To rewrite (1.2.2) we recall the following

1.3. Proposition. *(1) For each smooth compactly supported f_v on $G(F_v)$ there exist smooth compactly supported f_{0v} on $H_0(F_v)$ and f_{1v} on $H_1(F_v)$ such that*

$$\Phi_{f_{0v}}^{st}(\gamma)=\sum_{\delta'\in D(\delta/F_v)}\Phi_{f_v}^\sigma(\delta')\qquad(\gamma=N\delta)\tag{1.3.1}$$

and

$$\Phi_{f_{1v}}(\gamma) = |(1+a)(1+b)|_v^{1/2} \sum_{\delta' \in D(\delta/F_v)} \kappa_v(\delta')\Phi^{\sigma}_{f_v}(\delta') \qquad (\gamma = N_1\delta) \tag{1.3.2}$$

for all δ *with regular* $\gamma = N\delta$. *Here* a, b *denote the eigenvalues of* $N\delta$.

(2) Moreover, if $\delta = I$ *then*

$$f_{0v}(I) = \sum \kappa_v(\delta')\Phi^{\sigma}_{f_v}(\delta') \quad \text{and} \quad f_{1v}(I) = \sum \Phi^{\sigma}_{f_v}(\delta'),$$

where the sums are taken over δ' *in* $D(\delta/F_v)$. *If* $N\delta = -I$ *then* $f_{0v}(-I) = \Phi^{\sigma}_{f_v}(\delta)$.

(3) Finally, if $f_v = f_v^0$ *and* F_v *has odd residual characteristic, then* $f_{0v} = f_{0v}^0$ *has the properties asserted by (1).*

Proof. (1) and (2) are proven in [I], §3, and (3) in [V]. □

Definition. The functions f_v, f_{0v} (resp. f_v, f_{1v}) are called *matching* if they satisfy (1.3.1) (resp. (1.3.2)) for all δ such that $\gamma = N\delta$ is regular.

From now on we work with functions $f = \otimes f_v$ which satisfy the following

1.3A. Assumption. There exists a component f_v of f such that $f_{1v} = 0$ matches f_v, or f_{1v}^0 matches f_v^0 for almost all v.

Remark. If $f_{1v} = 0$ then the second term in (1.2.2) is zero for every δ in $G(F)$. We work with f which has a component f_v which matches $f_{1v} = 0$ in [IV]. In [V] we show that f_v^0, f_{1v}^0 are matching for every local F_v with odd residual characteristic, namely that the assumption always holds. In [VI] we work with a general f to deduce the unrestricted symmetric square lifting.

Corollary. *Suppose that* $f = \otimes f_v$ *satisfies Assumption 1.3A. Put* $f_0 = \otimes f_{0v}$ *and* $f_1 = \otimes f_{1v}$, *where* (f_v, f_{0v}) *and* (f_v, f_{1v}) *are matching for all* v, *and* $f_{0v} = f_{0v}^0$ *and* $f_{1v} = f_{1v}^0$ *for almost all* v *(or* $f_1 = 0$*). Then (1.1.2)=(1.2.1) is the sum of*

$$\begin{aligned}\tilde{I}_0 =& |H(F)\backslash H(\mathbb{A})|[f_0(I) + f_0(-I)] \\ &+ \frac{1}{2}\sum_{\{T\}_{st}} \frac{1}{2}|T(F)\backslash T(\mathbb{A})| \sum_{\gamma \in T(F)} \Phi^{st}_{f_0}(\gamma)\end{aligned} \tag{1.3.3}$$

and $\frac{1}{2}$ *times*

$$\tilde{I}_1 = |H_1(F)\backslash H_1(\mathbb{A})| f_1(I) + \frac{1}{2}\sum_{\{T\}} |T(F)\backslash T(\mathbb{A})| {\sum}'_{\gamma \in T(F)} \Phi_{f_1}(\gamma). \tag{1.3.4}$$

$\{T\}_{st}$ *in (1.3.3) is the set of stable conjugacy classes of elliptic* F*-tori* T *in* H. $\{T\}$ *in (1.3.4) is the set of conjugacy classes of elliptic* F*-tori* T *in* $H_1 = \mathrm{SO}(3)$. *The sum* $\sum'$ *in*

(1.3.4) ranges over the γ in $T(F) \subset \mathrm{SO}(3,F)$ whose eigenvalues are distinct (not -1). The sums are absolutely convergent.

Proof. (1.2.1) is a sum over σ-stable conjugacy classes δ which are equal to $c(\delta)$ times (1.2.2) if $N\delta$ is I or elliptic regular. If $N\delta$ is elliptic regular then the first term in (1.2.2) makes a contribution in the sum of (1.3.3) by (1.3.1), and the second term in (1.2.2) contributes to (1.3.4) by (1.3.2). If $N\delta = I$ then the order is reversed, by (2) in the proposition. The single σ-conjugacy class δ in $G(F)$ with $N\delta = -I$ makes the term of $f_0(-I)$ in (1.3.3). The coefficient of $f_0(I)$ in (1.3.3) is $|H(F)\backslash H(\mathbb{A})|$ since the Tamagawa number of $\mathrm{SO}(3) = \mathrm{PGL}(2)$ is twice that of $\mathrm{SL}(2)$. The first one-half which appears in (1.3.3) and (1.3.4) exists since the number of regular γ in $T(F)$ which share the same set of eigenvalues is two. The sums in (1.3.3) and (1.3.4) are absolutely convergent since they are parts of the trace formula for f_0 on H and f_1 on H_1. $\square$

Analytic Side

2.1. As suggested in (1.1) we shall now record the expression of [CLL] for the analytic side, which involves traces of representations, in the twisted trace formula. Let P_0 be a minimal σ-invariant F-parabolic subgroup of G, with Levi subgroup M_0. Let P be any standard (containing P_0) F-parabolic subgroup of G; denote by M the Levi subgroup which contains M_0 and by A the split component of the center of M. Then $A \subset A_0 = A(M_0)$. Let $X^*(A)$ be the lattice of rational characters of A, $\mathcal{A}_M = \mathcal{A}_P$ the vector space $W_*(A) \otimes \mathbb{R} = \mathrm{Hom}(X^*(A), \mathbb{R})$, and $\mathcal{A}^*$ the space dual to $\mathcal{A}$. Let $W_0 = W(A_0, G)$ be the Weyl group of A_0 in G. Both σ and every s in W_0 act on $\mathcal{A}_0$. The truncation and the general expression to be recorded depend on a vector T in $\mathcal{A}_0 = \mathcal{A}_{M_0}$. In the case of (2.2) below this T becomes a real number, the expression is linear in T, and we record in (2.2) only the value at $T = 0$.

Proposition [CLL]. *The analytic side of the trace formula is equal to a sum over*
(1) the set of Levi subgroups M which contain M_0 of F-parabolic subgroups of G;
(2) the set of subspaces $\mathcal{A}$ of $\mathcal{A}_0$ such that for some s in W_0 we have $s\mathcal{A} = \mathcal{A}_M^\sigma$, where $\mathcal{A}_M^\sigma$ is the space of σ-invariant elements in the space $\mathcal{A}_M$ associated with a σ-invariant F-parabolic subgroup P of G;
(3) the set $W^{\mathcal{A}}(\mathcal{A}_M)$ of distinct maps on $\mathcal{A}_M$ obtained as restrictions of the maps $s \times \sigma$ (s in W_0) on $\mathcal{A}_0$ whose space of fixed vectors is precisely $\mathcal{A}$; and
(4) the set of discrete series representations τ of $M(\mathbb{A})$ with $(s \times \sigma)\tau \simeq \tau$. The terms in the sum are equal to the product of

$$\frac{[W_0^M]}{[W_0]}(\det(1 - s \times \sigma)|_{\mathcal{A}_M/\mathcal{A}})^{-1} \tag{2.1.1}$$

and

$$\int_{i\mathcal{A}^*} \mathrm{tr}[\mathcal{M}_{\mathcal{A}}^T(P,\lambda) M_{P|\sigma(P)}(s,0) I_{P,\tau}(\lambda, f \times \sigma)]|d\lambda|.$$

Here $[W_0^M]$ is the cardinality of the Weyl group $W_0^M = W(A_0, M)$ of A_0 in M; P is an F-parabolic subgroup of G with Levi component M; $M_{p|\sigma(P)}$ is an intertwining operator; $\mathcal{M}_{\mathcal{A}}^T(P,\lambda)$ is a logarithmic derivative of intertwining operators, and $I_{P,\tau}(\lambda)$ is the $G(\mathbb{A})$-module normalizedly induced from the $M(\mathbb{A})$-module $m \mapsto \tau(m)e^{\langle ,\lambda H(m)\rangle}$ (in standard notations).

Remark. The sum of the terms corresponding to $M = G$ in (1) is equal to the sum $I = \sum \operatorname{tr} \pi(f \times \sigma)$ over all discrete series representations π of $G(\mathbb{A})$.

2.2. We shall now describe, in our case of $G = \mathrm{PGL}(3)$ and $\sigma(g) = \mathcal{J}{}^t g^{-1}\mathcal{J}$, the terms corresponding to $M \neq G$ in (1) of Proposition 2.1. There are three such terms. Let $M_0 = A_0$ be the diagonal subgroup of G.

(a) For the three Levi subgroups $M \supset A_0$ of maximal parabolic subgroups P of G we have $\mathcal{A} = \{0\}$. The corresponding contribution is

$$\begin{aligned}\sum_M \sum_\tau \frac{2}{6}\cdot\frac{1}{2} \operatorname{tr} M(s,0) I_{P,\tau}(0, f\times\sigma) \\ = \frac{1}{2}\sum_\tau \operatorname{tr} M(\alpha_2\alpha_1, 0) I_{P1}(\tau, f\times\sigma).\end{aligned} \tag{2.2.1}$$

Here P_1 denotes the upper triangular parabolic subgroup of G of type (2,1). We write $\alpha_1 = (12), \alpha_2 = (23), \mathcal{J} = (13)$ for the transpositions in the Weyl group W_0.

(b) The contribution corresponding to $M = M_0$ and $\mathcal{A} = \{0\}$ is

$$\begin{aligned}\frac{1}{6}\cdot\frac{1}{4}\sum_\tau \operatorname{tr} M(\mathcal{J},0) I_{P_0}(\tau, f\times\sigma) \\ + \frac{1}{6}\sum_\tau \operatorname{tr} M(\alpha_1,0) I_{P_0}(\tau, f\times\sigma) + \frac{1}{6}\sum_\tau \operatorname{tr} M(\alpha_2,0) I_{P_0}(\tau, f\times\sigma).\end{aligned} \tag{2.2.2}$$

(c) Corresponding to $M = M_0$ and $\mathcal{A} \neq \{0\}$ we obtain three terms, with $\mathcal{A} = \{(\lambda, 0, -\lambda\}$ and $s = 1$, with $\mathcal{A} = \{(\lambda, -\lambda, 0)\}$ and $s = \alpha_2\alpha_1$, and with $\mathcal{A} = \{(0, \lambda, -\lambda)\}$ and $s = \alpha_1\alpha_2$. The value of (2.1.1) is $\frac{1}{12}$. It is easy to see that the three terms are equal and that their sum is

$$\frac{1}{4}\sum_\tau \int_{i\mathbb{R}} \operatorname{tr}[\mathcal{M}(\lambda,0,-\lambda) I_{P_0,\tau}((\lambda,0,-\lambda); f\times\sigma)]|d\lambda|. \tag{2.2.3}$$

The operator $\mathcal{M}$ is a logarithmic derivative of an operator $M = m \otimes_v R_v$. Here R_v denotes a normalized local intertwining operator. It is normalized as follows. If $I(\tau_v)$ is unramified, its space of K_v-fixed vectors is one-dimensional, and R_v acts trivially on this space. In particular $R'_{\tau_v}(\lambda) I_{\tau_v}(\lambda, f_v \times \sigma)$ is zero if f_v is spherical, where $R'_{\tau_v}(\lambda)$ is the derivative of $R_{\tau_v}(\lambda)$ with respect to λ.

The τ in (2.2.3) are unitary characters (μ_1, μ_2, μ_3) of $M_0(\mathbb{A})/M_0(F)$, which are σ-invariant; thus $\mu_2 = 1$ and $\mu_1, \mu_3 = 1$. According to $[Shah]$, where the R_v are studied, the normalizing factor $m = m(\lambda)$ is the quotient

$$L(1-2\lambda, \mu_3/\mu_1)/L(1+2\lambda, \mu_1/\mu_3)$$

of L-functions. In this case the logarithmic derivative $\mathcal{M}$ has the form

$$m'(\lambda)/m(\lambda) + (\otimes R_v^{-1})\frac{d}{d\lambda}(\otimes R_v).$$

Hence (2.2.3) is equal to $\frac{1}{4}(S+S')$, where

$$S = \sum_\tau \int_{i\mathbb{R}} \frac{m'(\lambda)}{m(\lambda)} \left[\, \textstyle\prod_v \operatorname{tr} I_{\tau_v}(\lambda, f_v \times \sigma) \,\right] |d\lambda| \tag{2.2.4}$$

and

$$\begin{aligned} S' = \sum_\tau \sum_v \int_{i\mathbb{R}} & [\operatorname{tr} R_{\tau_v}(\lambda)^{-1} R_{\tau_v}(\lambda)' I_{\tau_v}(\lambda, f_v \times \sigma)] \\ & \cdot \prod_{w \neq v} \operatorname{tr} I_{\tau_w}(\lambda, f_w \times \sigma) \cdot |d\lambda|. \end{aligned} \tag{2.2.5}$$

In view of the normalization of the $R_v = R_{\tau_v}(\lambda)$, the inner sum in S' extends only over the places v where f_v is not spherical.

The terms (2.2.1) and (2.2.2) contain arithmetic information which is crucial for the study of the symmetric square. They are analyzed in (2.3) and (2.4) below.

2.3. We shall now study the representations τ which occur in (2.2.1). Such a τ is a discrete series representation of the Levi component $M(\mathbb{A})$ of a maximal parabolic subgroup of $G(\mathbb{A})$. Hence it has the form $(\tilde{\pi}, \chi)$, where $\tilde{\pi}$ is a discrete series representation of $\mathrm{GL}(2, \mathbb{A})$ and χ is a (unitary) character of $\mathbb{A}^\times/F^\times$. The central character of $\tilde{\pi}$ is χ^{-1} since G is the projective group PGL(3). The representation $(\tilde{\pi}, \chi)$ is σ-invariant. Hence $\chi = \chi^{-1}$, and $\tilde{\pi}$ is equivalent to its contragredient $\tilde{\pi}^\vee$ which is $\tilde{\pi} \otimes \chi^{-1}$. If $\chi = 1$, then $\tilde{\pi}$ is a representation π_1 of $\mathrm{PGL}(2, \mathbb{A})$. If $\chi \neq 1$ then χ is quadratic, and it determines a quadratic extension K of F by class field theory. Since $\tilde{\pi} \simeq \tilde{\pi} \otimes \chi$, we have (by [LL]) that $\tilde{\pi}$ is of the form $\tilde{\pi}(\theta')$, associated to the two-dimensional representation $\mathrm{Ind}(\theta'; W_K, W_F)$ of the Weil group W_F of F induced from the character θ' of W_K; θ' can also be viewed as a character of the idèle class group $\mathbb{A}_K^\times/K^\times$. The central character of $\tilde{\pi}(\theta')$ is $\chi \cdot \theta'|_{\mathbb{A}^\times}$; this has to be equal to χ. Hence $\theta' = 1$ on $\mathbb{A}^\times$, and so θ' factors through the map $z \mapsto z/\overline{z}$ of $\mathbb{A}_K^\times$, namely $\theta'(z) = \theta(z/\overline{z})$ for some character θ of $\mathbb{A}_K^\times$. Here the bar indicated the non-trivial automorphism of K over F. Hence $\tilde{\pi} = \tilde{\pi}(\theta/\overline{\theta})$ if $\chi \neq 1$, and (2.2.1) is equal to $\frac{1}{2}(I_1' + I')$, where

$$I_1' = \sum_{\pi_1} \operatorname{tr} I_{P_1}((\pi_1, 1); f \times \sigma) \tag{2.3.1}$$

and

$$I' = \sum_{\{\chi,\theta;\chi\neq 1,(\theta/\overline{\theta})^2\neq 1\}} \operatorname{tr} I_{P_1}((\pi(\theta/\overline{\theta}),\chi); f\times\sigma). \tag{2.3.2}$$

The sum of I_1' ranges over the discrete spectrum of H_1. The sum of I' ranges over all quadratic characters χ of $\mathbb{A}^\times/F^\times$, and characters $z \mapsto \theta(z/\overline{z})$ of $\mathbb{A}_K^\times/K^\times$, where K is determined by χ. We require that $\theta(z/\overline{z}) \neq \theta(\overline{z}/z)$ to have a discrete series $\pi(\theta/\overline{\theta})$, which is necessarily cuspidal. The intertwining operator $M(s,\pi)$ of (2.1.1) is equal to $m(\pi)\otimes_v R(s,\pi_v)$. Here $m(\pi)=L(1,\check{\pi})/L(1,\pi)$, and each $R(s,\pi_v)$ is equal to the identity, by [Shah]. Note that $\pi=\pi_1$ or $\pi(\theta/\overline{\theta}$ is self-contragredient, and $L(s,\pi)$ is analytic at $s=1$ since π is cuspidal or one-dimensional. Hence $m(\pi)=1$, and $M(s,0)$ is the identity everywhere in (2.2.1).

2.4. The representations τ which appear in (2.2.2) are (unitary) characters $\eta=(\mu_1,\mu_2,\mu_3)$, μ_i being a character of $\mathbb{A}^\times/F^\times$, and $\mu_1\mu_2\mu_3=1$. In the first sum appear all η with $\mu_i^2=1$, but in the other two sums appear only the η with $(s\times\sigma)\eta=\eta$, namely $\eta=(1,1,1)$. Since all representations which appear here are irreducible, the intertwining operators $M(s,\eta)$ are scalars. They can be seen to be equal to -1, as in the case of $\mathrm{GL}(2)$, unless μ_i are all distinct, where they are equal to 1. It remains to note that in the first sum each representation $I(\eta)$ with $\mu_i\neq 1 (i=1,2,3)$ occurs six times, three times if $\mu_i=1$ for a single i, and once if $\mu_i=1$ for all i. Then (2.2.2) takes the form $\frac{1}{4}I''-\frac{3}{8}I^*-\frac{1}{8}I^{**}$, where

$$I'' = \sum_{\eta=\{\chi,\mu\chi,\mu\}} \operatorname{tr} I(\eta, f\times\sigma) \tag{2.4.1}$$

and

$$I^* = \operatorname{tr} I(1, f\times\sigma), \qquad I^{**} = \sum_{\eta=(\mu,1\mu)} \operatorname{tr} I(\eta, f\times\sigma) \tag{2.4.2}$$

The χ and μ are characters of $\mathbb{A}^\times/F^\times$ of order exactly two. The symbol $\{\chi,\mu\chi,\mu\}$ means an unordered triple of distinct characters.

3. Formulae

3.1. We shall next state the twisted trace formula. This can be done for a general test function f on using the computations of [Fadj] (or [CLL]) of the weighted orbital integrals on the non-elliptic σ-orbits. However, we shall use the formula only for f with a regular component or two discrete components (definitions soon to follow). For such f the formula simplifies considerably, and we consequently state the formula only in this case.

Definition. The function $f=\otimes f_v$ on $G(\mathbb{A})$ is of *type E* if for every δ in $G(F)$ and g in $G(\mathbb{A})$ we have $f(g^{-1}\delta\sigma(g))=0$ unless $N\delta$ is elliptic regular in $H(F)$.

Example. If f has a component f_v which is supported on the set of g in $G(F_v)$ such that Ng is elliptic regular in $H(F_v)$, then f is of type E.

If f is of type E then $K(g,g)$ of (1.1.1) is equal to the integrand of (1.1.2), and the truncation which is applied to $K(g,g)$ in [CLL] (or [Fadj]) is trivial (it does not change $K(g,g)$). Hence the computations of Sections 1 and 2 imply the following form of the twisted trace formula. Put

$$I = \sum_{\pi} \operatorname{tr}(f \times \sigma), \tag{3.1.1}$$

where π ranges over all discrete series (cuspidal or one-dimensional) G-modules which are σ-invariant: π is called σ-invariant if $\pi \simeq {}^{\sigma}\pi$, where ${}^{\sigma}\pi(g) = \pi(\sigma(g))$.

Proposition. *Suppose that f is a function of type E which satisfies Assumption (1.3A). Then we have*

$$\widetilde{I}_0 + \frac{1}{2}\widetilde{I}_1 = I + \frac{1}{2}I_1' + \frac{1}{2}I' + \frac{1}{4}I'' - \frac{3}{8}I^* - \frac{1}{8}I^{**} + \frac{1}{4}S + \frac{1}{4}S'.$$

$\widetilde{I}_0$ is defined in (1.3.3), $\widetilde{I}_1$ in (1.3.4), I in (3.1.1), I_1' in (2.3.1), I' in (2.3.2), I'' in (2.4.1), I^* and I^{**} in (2.4.2), S in (2.2.4), and S' in (2.2.5). These are distributions in f.

3.2. We shall next introduce a class of functions f of type E which suffices to establish in [IV] and [VI] the symmetric square lifting. Fix a non-archimedean place u of F. Denote by ord_u the normalized additive valuation on F_u; thus $\operatorname{ord}_u(\pi_u) = 1$ for a uniformizer π_u in R_u. Put q_u for the cardinality of the residue field $R_u/(\pi_u)$. Given an element δ of $G(F_u)$, denote by a, a^{-1} the eigenvalues of $N\delta$ and put

$$F(\delta, f_u) = |a - a^{-1}|_u^{1/2} \;\; \Phi_{f_u}^{\sigma}(\delta);$$

here $|\cdot|_u$ is the normalized valuation on F_u.

Definition. Let n be a positive integer. The function f_u on $G(F_u)$ is called *n-regular* if it is (compactly) supported on the set of δ with $|\operatorname{ord}_u(a)| = n$, and $F(\delta, f_u) = 1$ for such δ.

3.2.1. Proposition. *For every $f^u = \otimes f_v$ (product over $v \neq u$) there exists $n' > 0$, such that $f = f_u \otimes f^u$ is of type E if f_u is n-regular with $n \geqslant n'$.*

Proof. Given f^u there exists $C_v \geqslant 1$ for each $v \neq u$, with $C_v = 1$ for almost all v (C_v depends only on the support of f_v) with the following property. Let $\mathbb{A}^u$ be the ring of adeles of F without component at u. If δ is an element of $G(F)$ such that the eigenvalues a, a^{-1} of $N\delta$ lie in $F^{\times}$, then $C_v^{-1} \leqslant |a|_v \leqslant C_v (v \neq u)$. Put $C_u = \prod_{v \neq u} C_v$. The product formula $\prod_v |a|_v = 1$ on $F^{\times}$ implies that $C_u^{-1} \leq |a|_u \leqslant C_u$. The least integer n' with $q_u^{n'} > C_u$ clearly has the property asserted by the proposition. □

Let μ_u be a σ-invariant character of the diagonal subgroup $A(F_u)$. Then there is a character μ_{0u} of F_u with $\mu_u((a,b,c)) = \mu_{0u}(a/c)$. Denote by $I(\mu_u)$, as in [II], the $G(F_u)$-module normalizedly induced from the associated character μ_u of the upper triangular subgroup, and by $I_0(\mu_{0u})$ the $H(F_u)$-module normalizedly induced from $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \mu_{0u}(a)$. A standard computation [II], §1, implies that if f_u, f_{0u} are matching then

$$\operatorname{tr} I(\mu_u, f_u \times \sigma) = \operatorname{tr} I_0(\mu_{0u}, f_{0u}). \tag{3.2.2}$$

If f_u is n-regular, then f_{0u} is n-regular: it is supported on the orbits of $\gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $|\operatorname{ord}_u(a)| = n$, and $F(\gamma, f_{0u}) = 1$ there. If now (3.2.1) is non-zero, then μ_{0u} and μ_u are unramified. Put $z = \mu_{0u}(\pi_u)$. We conclude

3.2.3. Lemma. *If f_u is n-regular then (3.2.2) is zero unless μ_u is unramified, in which case we have* $\operatorname{tr} I(\mu_u, f_u \times \sigma) = z^n + z^{-n}$.

Definition. The function f_v on $G(F_v)$ is called *discrete* if $\Phi^\sigma_{f_v}(\delta)$ is zero for every δ such that the eigenvalues a, a^{-1} of $N\delta$ are distinct and lie in $F_v^\times$.

Example. If f_v is supported on the σ-elliptic regular set then it is discrete.

3.2.4. Corollary. *Fix a finite place u of F. For every $f^u = \otimes_{v \neq u} f_v$ which has a discrete component (at $u' \neq u$) and satisfies Assumption (1.3A), there exists a bounded integrable function $d(z)$ on the unit circle in the complex plane with the following property. For every $n \geqslant n'(f^u)$ and n-regular f_u, we have*

$$\widetilde{I}_0 + \frac{1}{2}\widetilde{I}_1 = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I'_1 + \int_{|z|=1} d(z)(z^n + z^{-n})|d^\times z|.$$

Proof. Recall that the I are linear functionals in $f = f_u \otimes f^u$. Since f^u, hence also f, has a discrete component, it is clear (from (3.2.2)) that $I^* = I^{**} = S = 0$, and that the sum over v in (2.2.5) (where S' is defined) ranges over $v = u'$ only. The sum over τ in (2.2.5) ranges over a set of representatives for the connected components of the one-dimensional complex manifold of σ-invariant characters of $A(\mathbb{A})/A(F)$ whose component τ_u at u is unramified. We may choose τ with $\tau_u = 1$. Put $z = q_u^\lambda$ for λ in $i\mathbb{R}$. Then $\operatorname{tr} I_{\tau_u}(\lambda, f_u \times \sigma) = z^n + z^{-n}$ by Lemma (3.2.3). Of course, z depends on λ only modulo $2\pi i\mathbb{Z}/\log q_u$. Since the sum over τ, the integral over $i\mathbb{R}$, and product over $w \neq u, u'$ in (2.2.5) are absolutely convergent, the function

$$\begin{aligned} d(z) = \sum_\tau \sum_{k \in \mathbb{A}} & [\operatorname{tr} R_{\tau'_u}(\lambda + k')^{-1} R_{\tau'_u}(\lambda + k', f_{u'} \times \sigma)] \\ & \cdot \prod_{w \neq u, u'} \operatorname{tr} I_{\tau_W}(\lambda + k', f_w \times \sigma), \end{aligned}$$

where $k' = k2\pi i/\log q_u$, has the required properties. □

This corollary plays a key role in the proof of [IV] of the symmetric square lifting for automorphic forms with an elliptic component. For the local work in [IV] we use also a simpler form of the formula, as follows.

3.2.5. Proposition. *If $f = \otimes f_v$ has two discrete components and it satisfies Assumption(1.3A), then*

$$\widetilde{I}_0 + \frac{1}{2}\widetilde{I}_1 = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I_1'.$$

Proof. The terms in the geometric side of the twisted trace formula which are associated with non-elliptic σ-conjugacy classes are computed explicitly in [Fadj] and also in [CLL]. They are similar to those obtained in the trace formulae of groups of rank one. In particular, they vanish if f has two discrete components. As noted in (3.2.4) we have $I^* = I^{**} = S = 0$ if f has a single discrete component. It is clear that $S' = 0$ if f has two discrete components, and the proposition follows. □

Remark. If f has a discrete component and a component as in Example (3.2.3) then f is of type E and Proposition 3.2.5 follows at once from Proposition 3.1.

3.3. The twisted trace formulae for f on G of (3.2.4) and (3.2.5) will be compared with the stable trace formula for $f_0 = \otimes f_{0v}$ on $H = \mathrm{SL}(2)$ from [LL] and the trace formula for $f_1 = \otimes f_{1v}$ on $H_1 = \mathrm{PGL}(2)$. To recall the formula for H, we introduce:

Definition. The *packet* $\{\pi_{0v}\}$ of the irreducible $H(F_v)$-module π_{0v} is the set of irreducible $H(F_v)$-modules π_{0v}^g (g in $\mathrm{GL}(2, F_v)$). Here $\pi_{0v}^g(h) = \pi_{0v}(g^{-1}hg)$.

Remark. (1) An alternative definition of $\{\pi_{0v}\}$ is given as follows. Let $\widetilde{\pi}_{0v}$ be an irreducible $\mathrm{GL}(2, F_v)$-module whose restriction to $H(F_v)$ contains π_{0v}. Then $\{\pi_{0v}\}$ is the set of irreducibles in the restriction of $\widetilde{\pi}_{0v}$ to $H(F_v)$. It is independent of the choice of $\widetilde{\pi}_{0v}$.

(2) Denote by $\pi_0(\theta_v)$ the packet associated with the two-dimensional representation $\mathrm{Ind}(\theta_v; W_{F_v}, W_{K_v})$ of the (local) Weil group W_{F_v} of F_v induced from the character θ_v of W_{K_v}, equivalently of $K_v^\times$, where K_v is a quadratic extension of F_v. Denote by a bar the non-trivial element in $\mathrm{Gal}(K_v/F_v)$, and put $\overline{\theta}_v(z) = \theta_v(\overline{z})$. Then $\{\pi_{0v}\}$ consists of one element unless $\{\pi_{0v}\} = \pi_0(\theta_v)$. Then it consists of two elements if $\overline{\theta}_v^2 \neq \theta_v^2$, and of four if $\overline{\theta}_v^2 = \theta_v^2$ but $\overline{\theta}_v \neq 0_v$.

Definition. Let P_{0v} be a packet for each v, such that P_{0v} contains an unramified $H(F_v)$-module π_{0v}^0 for almost all v. The associated global *packet* P is the set of all $H(\mathbb{A})$-modules $\otimes_v \pi_{0v}$ with π_{0v} in P_{0v} for all v and $\pi_{0v} \simeq \pi_{0v}^0$ for almost all v.

Proposition. *(1) For every $f_0^u = \otimes f_{0v} (v \neq u)$ there is $n' > 0$ such that for every n-regular f_{0u} with $n \geqslant n'$ we have*

$$\widetilde{I}_0 = I_0 + \frac{1}{2}I_0' + \frac{1}{4}I_0'' - \frac{1}{4}I^* + \frac{1}{2}S_0 + \frac{1}{2}S_0'.$$

(2) If in addition f_0^u has a discrete component $f_{0u'}$ then there is a function $d_0(z)$, bounded and integrable on $|z| = 1$, depending only on f_0^u, such that

$$\widetilde{I}_0 = I_0 + \frac{1}{2}I_0' + \frac{1}{4}I_0'' + \int_{|z|=1} d_0(z)(z^n + z^{-n})|d^\times z|$$

for every n-regular f_{0u} with $n \geqslant n'$.

(3) If $f_0 = \otimes f_{0v}$ has two elliptic components then $\widetilde{I}_0 = I_0 + \frac{1}{2} I_0' + \frac{1}{4} I_0''$.

Proof. We have to explain the notations. $\widetilde{I}_0$ is defined in (1.3.3) and $I^* = \operatorname{tr} I(1, f_0)$ in (2.4.2). In standard notations,

$$S_0 = \int_{i\mathbb{R}} \sum_{\eta} \frac{m(\eta)'}{m(\eta)} \operatorname{tr} I_0(\eta, f_0) |d\lambda|$$

and

$$S_0' = \int_{i\mathbb{R}} \sum_{\eta} \sum_{v} \operatorname{tr}\{R_v(\eta)^{-1} R_v(\eta)' I_0(\eta, f_{0v})\} \cdot \prod_{w \neq v} \operatorname{tr} I(\eta_w, f_{0w}) \cdot |d\lambda|.$$

Put $\operatorname{tr}\{\pi_0\}(f_0)$ for $\prod_v \operatorname{tr}\{\pi_{0v}\}(f_{0v})$ if $\{\pi_0\}$ is the packet determined by the local packets $\{\pi_{0v}\}$. Then

$$I_0 = \sum_{\{\pi_0\}} m(\{\pi_0\}) \operatorname{tr}\{\pi_0\}(f_0).$$

The sum ranges over the set of packets $\{\pi_0\}$ of $H(\mathbb{A})$-modules which contain a cuspidal or one-dimensional H-module which is not of the form $\pi_0(\theta)$. Each element in $\{\pi_0\}$ is cuspidal or one-dimensional, and occurs in the discrete spectrum of $L^2(H(F)\backslash H(\mathbb{A}))$ with multiplicity $m(\{\pi_0\})$ which depends only on the packet. Next, we have

$$I_0' = \sum_{k} \sum_{\theta} \operatorname{tr}\{\pi_0(\theta)\}(f_0).$$

The K ranges over the set of quadratic extensions of F. The θ ranges over the set of characters of $\mathbb{A}_K^\times / K^\times$ such that $\overline{\theta}^2 \neq \theta^2$.

The last term I_0'' ranges over the set of packets $\pi_0(\theta)$ where $z \mapsto \theta(z/\overline{z})$ is a character of order precisely two of $\mathbb{A}_K^\times / K^\times$ for some quadratic extension K of F. Let χ be the non-trivial character of $\mathbb{A}^\times / K^\times N_{K/F} \mathbb{A}_K^\times$. Since $\theta/\overline{\theta} = \overline{\theta}/\theta \neq 1$ there is a character μ of $\mathbb{A}^\times / F^\times$ of order two with $\theta(z/\overline{z}) = \mu(z\overline{z})$ for all z in $\mathbb{A}_K^\times$, and $\mu \neq \chi$. Note that if E denotes the quadratic extension of F determined by μ and class field theory, and v is a character of $\mathbb{A}_E^\times / E^\times$ with $v(z/\overline{z}) = \chi(z\overline{z})$ (z in $\mathbb{A}_E^\times$), then $\{\pi_0(v)\} = \{\pi_0(\theta)\}$. In conclusion we have

$$I_0'' = \sum_{\{\mu, \chi, \mu\chi\}} \operatorname{tr}\{\pi_0(\theta)\}(f_0),$$

where the sum ranges over the unordered triples of distinct characters of $\mathbb{A}^\times / F^\times$ of order two. □

3.4. We also need the trace formula for a test function $f_1 = \otimes f_{1v}$ on $H_1 = \mathrm{PGL}(2)$. It suffices to consider f_1 analogous to the f_0 of (3.3). We first state the formula and then explain the notations.

Proposition. *(1) For every $f_1^u = \otimes_{v\neq u} f_{1v}$ there is $n' > 0$ such that for every n-regular f_{1u} with $n \geqslant n'$ we have*

$$\widetilde{I}_1 = I_1 - \frac{1}{4}I^* - \frac{1}{4}I^{**} + \frac{1}{2}S_1 + \frac{1}{2}S_1'.$$

(2) If in addition f_1^u has a discrete component $f_{1u'}$, then there is a function $d_1(z)$, bounded and integrable on $|z| = 1$, depending only on f_1^u, such that

$$\widetilde{I}_1 = I_1 + \int_{|z|=1} d_1(z)(z^n + z^{-n})|d^\times z|$$

for every n-regular f_{1u} with $n \geqslant n'$.

(3) If $f_1 = \otimes f_{1v}$ has two elliptic components then $\widetilde{I}_1 = I_1$.

Proof. Here $I_1 = \sum \operatorname{tr} \pi_1(f_1)$, where the sum ranges over all cuspidal and one-dimensional H_1-modules. I^* and I^{**} are defined in (2.4.2). Their sum is

$$I^* + I^{**} = \sum_{w\eta=\eta} \operatorname{tr} I_1(\eta, f_1);$$

for a character η of the diagonal subgroup of $H_1(\mathbb{A})$ we put $w\eta((a,b)) = \eta((b,a))$. As usual,

$$S_1 = \int_{i\mathbb{R}} \frac{m(\eta)'}{m(\eta)} \operatorname{tr} I_1(\eta, f_1)|d\lambda|$$

and

$$S_1' = \int_{i\mathbb{R}} \sum_\eta \sum_v \operatorname{tr}[R_v(\eta)^{-1} R_v(\eta)' I_1(\eta, f_{1v})] \cdot \prod_{w\neq v} \operatorname{tr} I_1(\eta_w, f_{1w}) \cdot |d\lambda|.$$

□

3.5. Finally we compare the formulae of (3.2), (3.3), (3.4) for functions $f = \otimes f_v$ on $G(\mathbb{A}), f_0 = \otimes f_{0v}$ on $H(\mathbb{A})$, and $f_1 = \otimes f_{1v}$ on $H_1(\mathbb{A})$, such that f_{0v} matches f_v for all v, and either f has a component f_v such that $f_{1v} = 0$ matches f_v and $f_1 = 0$, or f_{1v} matches f_v for all v. Define J to be the difference

$$J = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I_1' - [I_0 + \frac{1}{2}I_0' + \frac{1}{4}I_0'' + \frac{1}{2}I_1].$$

It is an invariant distribution in f, depending only on the orbital integrals of f.

Proposition. *(1) If f has two discrete components then $J = 0$.*

(2) Suppose that $f^u = \otimes_{v\neq u} f_v$ has a discrete component. Then there exists an integer $n' \geqslant 1$ and a bounded integrable function $d(z)$ on $|z| = 1$, depending only on f^u, f_0^u, f_1^u, such that for all n-regular functions f_u, f_{1u}, and f_{0u} we have

$$J = \int_{|z|=1} d(z)(z^n + z^{-n})|d^\times z|.$$

Proof. This follows at once from (3.2.4), (3.2.5), (3.3), and (3.4). □

Concluding Remarks. (1) is used in the local study of [IV]. (2) is used in [IV] to show that $J = 0$ for any f as in (2), and to establish the symmetric square lifting for automorphic forms with an elliptic component. Pursing the techniques of [IV] and studying the properties of regular functions, we show in [VI] that $J = 0$ for all matching f, f_0, f_1, and so reduce by virtue of the global work of [IV] the symmetric square lifting for all π to the local transfer of orbital integrals, which is proven in [V].

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