ON THE SYMMETRIC SQUARE: TOTAL GLOBAL COMPARISON

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Put $G = PGL(3)$ and $H = H_0 = SL(2)$. Let F be a local field, and μ, μ', μ'' complex-valued characters of $F^\times$. Denote by $I_0(\mu) = \mathrm{Ind}\,(\delta_0^{1/2}\mu; P_0, H(F))$ the $H(F)$-module normalizedly induced from the character $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \to \mu(a)$ of the upper triangular subgroup P_0 of $H(F)$; here $\delta_0(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = |a|^2$. Denote by $I(\mu', \mu'') = \mathrm{Ind}\,(\delta^{1/2}(\mu', \mu''); P, G(F))$ the $G(F)$-module normalizedly induced from the character $(\mu', \mu'') : \begin{pmatrix} a & & * \\ & b & \\ 0 & & c \end{pmatrix} \to \mu'(a/b)\mu''(b/c)$ of the upper triangular subgroup P of $G(F)$; here $\delta\ (\begin{pmatrix} a & & * \\ & b & \\ 0 & & c \end{pmatrix}) = |a/c|^2$. The Grothendieck group $K(H)$ is the free abelian group generated by the set of equivalence classes of the irreducible $H(F)$-modules. Let $[\pi_0]$ denote the image in $K(H)$ of an $H(F)$-module π_0. Put $\mathcal{J} = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{pmatrix}$, and $\sigma(g) = \mathcal{J}\,{}^t g^{-1} \mathcal{J}$ for g in $G(F)$.

An irreducible $G(F)$-module π is called σ-*invariant* if it is equivalent to the $G(F)$-module ${}^\sigma\pi$ defined by ${}^\sigma\pi(g) = \pi(\sigma g)$. The notion of σ-invariance extends to the Grothendieck group $K(G)$ of $G(F)$. The subgroup $K(G, \sigma)$ of $K(G)$ which is generated by the irreducible σ-invariant $G(F)$-modules is a direct summand of $K(G)$; its complement is generated by the irreducible non-σ-invariant $G(F)$-modules. Let $[\pi]$ denote the image in $K(G, \sigma)$ of a $G(F)$-module π. Denote $I(\mu, \mu)$ by $I(\mu)$. It is clear that $I(\mu)$ is σ-invariant and $[I(\mu)] \neq 0$. We say that $[I_0(\mu)]$ *lifts* to $[I(\mu', \mu'')]$, and $I_0(\mu)$ to $I(\mu', \mu'')$, if $[I(\mu', \mu'')] = [I(\mu)]$. This is a trivial case of a definition in terms of character relations (see [II]) of a surjective lifting monomorphism from $K(H)$ to $K(G, \sigma)$. The trivial $H(F)$-module lifts to the trivial $G(F)$-module, and consequently the Steinberg $H(F)$-module lifts to the Steinberg constituent in $[I(\nu)]$, where $\nu(x) = |x|$.

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If F is non-archimedean, let R denote its ring of integers , put $K=G(R)$ and $K_0=H(R)$. A $G(F)$-module π is called *unramified* if it has a non-zero K-fixed vector. An $H(F)$-module is called unramified if it has a non-zero K_0-fixed vector. If $I(\mu',\mu'')$ is σ-invariant and unramified then there is μ with $[I(\mu',\mu'')]=[I(\mu)]$. If $[I(\mu',\mu'')]\neq 0$ and it is not of the form $[I(\mu)]$, then μ',μ'' and $\mu'\mu''$ are characters of (strict) order two.

Let F be a global field. Let $\mathbb{A}$ be the ring of adeles of F. Denote the completion of F at its place v by F_v. An irreducible $G(\mathbb{A})$-module π is the restricted direct product $\otimes\pi_v$ over all places v of F of irreducible $G_v=G(F_v)$-modules π_v; for almost all v the component π_v is unramified and there are unramified characters μ_v and μ'_v of $F_v^\times$ such that π_v is the unique unramified subquotient of $I(\mu_v,\mu'_v)$. If π is σ-invariant then π_v is the unique unramified subquotient of $I(\mu_v)$ for some unramified μ_v. Similarly an irreducible $H(\mathbb{A})$-module π_0 is $\otimes\pi_{0v}$ with unramified π_{0v} in $I_0(\mu_v),\mu_v$ unramified, for almost all v.

We say that an irreducible $H(\mathbb{A})$-module $\pi_0=\otimes\pi_{0v}$ *quasi-lifts* to the irreducible $G(\mathbb{A})$-module $\pi=\otimes\pi_v$ if for almost all v the induced representations $[I(\mu'_v,\mu''_v)]$ determined by π_v is $[I(\mu_v)]$ where π_{0v} determines $[I_0(\mu_v)]$. Let π_{0v} be an irreducible $H_v=H(F_v)$-module, and π'_{0v} an irreducible $GL(2,F_v)$-module whose restriction to H_v contains π_{0v}. The *packet* of π_{0v} is the set $\{[\pi_{0v}]\}$ of the images $[\pi_{0v}]$ in $K(H_v)$ of the irreducible H_v-modules π_{0v} in the restriction of π'_{0v} to H_v; it is independent of the choice of π'_{0v}; it consists of at most four elements, and it contains at most one unramified element. To simplify the notations we write $\{\pi_{0v}\}$ for $\{[\pi_{0v}]\}$; it is a set of H_v-modules which are determined only up to equivalence. The *packet* $\{\pi_0\}$ (see [LL]) of the irreducible $H(\mathbb{A})$-module $\pi_0=\otimes\pi_{0v}$ is the set of all (equivalence classes of) irreducible $H(\mathbb{A})$-modules $\pi'_0=\otimes\pi'_{0v}$ with $\pi'_{0v}\simeq\pi_{0v}$ for almost all v and π'_{0v} in $\{\pi_{0v}\}$ for all v. If π_0 quasi-lifts to π then each member of $\{\pi_0\}$ quasi-lifts to π and we say that the packet of π_0 quasi-lifts to π.

Let $L(H)$ be the space of smooth complex-valued slowly increasing functions ψ on
$H(F)\backslash H(\mathbb{A})$, and $L_0(H)$ the subspace of ψ with $\int_{U(F)\backslash U(\mathbb{A})}\psi(ux)du=0$. Here U is the group of upper triangular unipotent matrices in H. The group $H(\mathbb{A})$ acts on $L(H)$, and on $L_0(H)$, by right translation. An irreducible constituent of $L(H)$ (resp. $L_0(H)$) is called an *automorphic* (resp. *cuspidal*) $H(\mathbb{A})$-module.

Let K be a quadratic galois extension of F. Let N denote the norm map from K to F. Let K^1 denote the kernel of the map $N:K^\times\to F^\times$, thus $K^1=\{x$ in $K^\times;x\overline{x}=1\}$. Let $\mathbb{A}_K^1$ be $\{x \text{ in } \mathbb{A}_K^\times;x\overline{x}=1\};\mathbb{A}_K^\times$ is the group of K-ideles, $\overline{x}$ is the image of x under $Gal(K/F)$. Let θ be a complex-valued character of $\mathbb{A}_K^1/K^1$. If v splits K/F, namely $K\otimes_F F_v=F_v\oplus F_v$, then $K_v^1=\{(x,y)$ in $F_v\oplus F_v;xy=1\}\simeq F_v^\times$, and the restriction of θ to K_v^1 (in $\mathbb{A}_K^1$) is a character θ_v of $F_v^\times$. Let χ denote the non-trivial character of $\mathbb{A}^\times$ with kernel $F^\times NA_K^\times$, and χ_v its restriction to $F_v^\times$. There exists a unique $H(\mathbb{A})$-packet $\{\pi_0(\theta)\}$ which contains an automorphic (cuspidal if $\theta\neq 1$) $H(\mathbb{A})$-module $\pi_0=\otimes\pi_{0v}$ such that for almost all places v of F we have that π_{0v} is in $I_0(\theta_v)$ if v splits in K, and

π_{0v} in $I_0(\chi_v)$ if v stays prime in K. Not every member of the packet $\{\pi_0(\theta)\}$ is automorphic but if a cuspidal $H(\mathbb{A})$-module lies in a packet $\{\pi_0(\theta)\}$ then its multiplicity in $L_0(H)$ is one; see [LL] or [IV, (1.3)]. A cuspidal $H(\mathbb{A})$-module π_0 is called *old* if there exists a field K and a character $\theta \neq 1$ such that π_0 lies in $\{\pi_0(\theta)\}$. A cuspidal $H(\mathbb{A})$-module is called *new* if it is not old. Each member of a packet of a new $H(\mathbb{A})$-module is cuspidal; see [LL] or [IV, (1.3)].

In this article we prove a trace identity (see Theorem below), which, using the techniques of [IV; §2] and the stable and unstable transfer of general test functions (see [I], §3), and unit elements of the Hecke algebras (see [V]), implies the following main theorem of the theory of the symmetric square. It is the following

Main Theorem. (1) *Each cuspidal $H(\mathbb{A})$-modules occurs in the cuspidal spectrum $L_0(H)$ with multiplicity one.* (2) *If the cuspidal $H(\mathbb{A})$-modules $\pi_0 = \otimes\pi_{0v}$ and $\pi'_0 = \otimes\pi'_{0v}$ have $\pi_{0v} \simeq \pi'_{0v}$ for almost all v then their packets $\{\pi_0\}$ and $\{\pi'_0\}$ are equal.* (3) *The relation of quasi-lifting defines a bijection from the set of packets of new $H(\mathbb{A})$-modules to the set of cuspidal σ-invariant $G(\mathbb{A})$-modules.*

As shown in [IV, §2], this follows (not without effort) from the Theorem presently to be stated. In the notations of [IV], it is clear that $\{\pi_0(\theta)\}$ quasi-lifts to $I(\pi(\theta/\overline{\theta}), \chi)$. It is also shown in [IV, §2] that the Theorem below implies: (4) the "twisted" character of a σ-invariant supercuspidal G_v-module is a σ-stable function (it is constant on each stable σ-conjugacy class of σ-regular elements in G_v), and (5) the σ-invariant automorphic $G(\mathbb{A})$-modules which are not quasi-lifts of automorphic $H(\mathbb{A})$-modules are of the form $I(\pi_1)$ (or $I(\pi_1, 1)$) where π_1 is a cuspidal or one-dimensional $PGL(2, \mathbb{A})$-module. Other applications of the theory of the symmetric square are described in [IV, §2].

Put $H_1 = PGL(2)$. Let f_v (resp. f_{0v}, f_{1v}) denote a complex-valued, smooth (that is, locally-constant if F_v is non-archimedean), compactly-supported function on F_v (resp. H_v, H_{1v}). If F_v is non-archimedean put $K_{1v} = H_1(R_v)$, and let f^0_v (resp. f^0_{0v}, f^0_{1v}) be the measure of volume one which is supported on K_v (resp. K_{0v}, K_{1v}) and is constant on this group. Here we used the uniqueness of the Haar measure (up to a constant) to identify the space of locally-constant compactly-supported measures with the space of locally-constant compactly-supported functions on G_v (resp. H_v, H_{1v}) once a Haar measure is chosen.

At any place v, the functions f_v and f_{0v} (resp. f_v and f_{1v}) are called *matching* if they have matching orbital integrals (for a definition see [I, §3]; briefly, they satisfy $\Delta(\delta)\Phi^{\mathrm{st}}(\delta, f_v) = \Delta_0(\gamma)\Phi^{\mathrm{st}}(\gamma, f_{0v})$ for every δ in G_v with regular norm $\gamma = N\delta$, respectively, $\Delta(\delta)\Phi^{\mathrm{us}}(\delta, f_v) = \Delta_1(\gamma_1)\Phi_1(\gamma_1, f_{1v})$ for every δ in G_v with regular norm $\gamma_1 = N_1\delta; \Phi^{\mathrm{st}}(\delta, f_v)$ means "stable σ-orbital integral of f_v at δ", and $\Phi^{\mathrm{us}}(\delta, f_v)$ is the "unstable σ-orbital integral of f_v at δ". These are defined and studied in [I, §3].

The Theorem of [V] asserts that f^0_v and f^0_{0v} are matching, and that f^0_v and f^0_{1v} are matching. This local proof relies on recent work of Waldspurger. There are other proofs of these assertions (see, e.g., [I, §4], for a proof of the first assertion), but they are more complicated. Let $f = \otimes f_v$ (resp. $f_0 = \otimes f_{0v}, f_1 = \otimes f_{1v}$) be functions on $G(\mathbb{A})$ (resp. $H(\mathbb{A}), H_1(\mathbb{A})$) such that (1) $f_v = f_v^0, f_{0v} = f_{0v}^0, f_{1v} = f_{1v}^0$ for almost all v, and such that (2) f_v and f_{0v}, and f_v and f_{1v}, are matching for all v. The functions f, f_0, f_1 exist since the conditions (1) and (2) are compatible, namely f_v^0 and f_{0v}^0 as well as f_v^0 and f_{1v}^0 are matching.

In [III, §3] we defined various sums, denoted by I_i^*, of traces (such as $\operatorname{tr}\{\pi_0\}(f_0)$, $\operatorname{tr}\pi_1(f_1)$, $\operatorname{tr}\pi(f \times \sigma)$) of convolution operators ($\{\pi_0\}(f_0)$, $\pi_1(f_1)$ and $\pi(f \times \sigma)$); the sums I, I', I'', I_1' depend on f; the sums I_0, I_0', I_0'' depend on f_0, and I_1 on f_1. Put

$$T = I + \frac{1}{2}I' + \frac{1}{4}I'' + \frac{1}{2}I_1' - I_0 - \frac{1}{2}I_0' - \frac{1}{4}I_0'' - \frac{1}{2}I_1.$$

It follows from the proofs of [IV, §2] that the Main Theorem is a consequence of the following

Theorem. *We have $T = 0$ for any matching f, f_0, f_1 as above.*

It is also shown in [IV, §2] that when $T = 0$ then $I = I_0, I' = I_0', I'' = I_0'', I_1 = I_1'$. In [IV, (1.6.3)], the theorem is proven in the case that there exists a place u of F such that $f_{1u} = 0$. Then $f_1 = 0$ and $I_1 = 0$, and T depends only on the f and f_0 with matching f_v and f_{0v}, such that f_u and f_{0u} satisfy the conditions implied by $f_{1u} = 0$. In particular, [IV, §2] uses the fact that f_v^0 and f_{0v}^0 are matching, but not the statement that f_v^0 and f_{1v}^0 are matching. In [IV, §2] the special case of the Main Theorem which concerns cuspidal $H(\mathbb{A})$-modules with a square-integrable component (at u) is deduce from the special case $T = 0$ if $f_{1u} = 0$ of the Theorem. The assertion that f_v^0 and f_{1v}^0 are matching is proven by local means in [V]. Assuming this result, our Theorem becomes accessible to the method of proof of [IV, §2], and the purpose of this article is to prove it. The proof is a natural extension of the proof given in [IV, (1.6.3)] under the assumption that $f_{1u} = 0$. It is based on the usage of regular, or Iwahori type, functions.

It is clear from the proof given below that it applies to establish relatively effortlessly, and conceptually, the analytic part of the comparison of trace formulae for general test functions in any lifting situation where all groups involved have (split) rank bounded by one. In our case the ("twisted") rank of $G = PGL(3)$ is one. In particular our technique establishes the comparison of trace formulae for any test functions in the cases of (1) base-change from $U(3)$ to $GL(3, E)$ which is studied in [F3], [F4], [F7; IV], and [F5] ([F6] contains another proof of the trace formulae comparison for a general test function in the case of base-change from $U(3)$ to $GL(3, E)$; it relies on properties of quasi-spherical functions, but does not generalize to establish our Theorem), (2) cyclic base-change lifting for $GL(2)$ (see [F8] where our present technique is used to give a simple proof of this problem); (3) base-change form $U(2)$ to $GL(2, E)$ (see [F2]); (4) metaplectic correspondence for $GL(2)$ (see [F1]).

The proof of the Theorem is based on the usage of regular functions in the sense of [IV], [FK], and [F7; III, IV]. That such functions would be useful in this context was discovered by us while working on the joint paper [FK] with D. Kazhdan, being inspired by the proof – see [FK], Sections 16, 17 – of the metaplectic correspondence for representations of $GL(n)$ with a vector fixed by an Iwahori subgroup. Although these functions can be introduced for any quasi-split group, to simplify the notations we discuss these functions here only in the case of the group $GL(n)$ (and $SL(n), PGL(n)$).

Let F be a local non-archimedean field, R its ring of integers, $\boldsymbol{\pi}$ a local uniformizer in $R, \mathbf{q} = \boldsymbol{\pi}^{-1}, q$ the cardinality of the residue field $R/(\boldsymbol{\pi}), |\cdot|$ the valuation on F normalized to have $|\boldsymbol{\pi}| = q^{-1}$ (thus $|\mathbf{q}| = q$), G the group $GL(n,F), K = GL(n,R)$ a maximal compact subgroup in G, B the Iwahori subgroup of G which consists of matrices in K which are upper triangular modulo $\boldsymbol{\pi}, A$ the diagonal subgroup of G, $A(R) = A \cap K = A \cap B$, and U the upper triangular unipotent subgroup; AU is a minimal parabolic subgroup.

The vector $\mathbf{m} = (m_1, \cdots, m_n)$ in $\mathbb{Z}^n$ is called *regular* if $m_i > m_{i+1}$ for all i $(1 \le i < n)$. Let $\mathbf{q}^{\mathbf{m}}$ be the matrix $\operatorname{diag}(\mathbf{q}^{m_1}, \cdots, \mathbf{q}^{m_n})$ in A. The matrix $\mathbf{a} = \operatorname{diag}(a_1, \cdots, a_n)$ in A is called *strongly regular* if $|a_i| > |a_{i+1}|$ for all i, and $\mathbf{m}$*-regular* if $\mathbf{a} = u\mathbf{q}^{\mathbf{m}}$ for a regular $\mathbf{m}$ and u in $A(R)$. A conjugacy class in G is called *strongly* (resp. $\mathbf{m}$-)*regular* if it contains a strongly (resp. $\mathbf{m}$-) regular element. An element of G is called *regular* if its eigenvalues are distinct.

Denote by $\mathcal{J}$ the matrix whose (i,j) entry is $\delta_{i,n-j}$. Put $\sigma(g) = \mathcal{J}^t g^{-1} \mathcal{J}$. The elements g and g' of G are called σ*-conjugate* if there is x in G with $g' = xg\sigma(x)^{-1}$. For $\mathbf{m} = (m_1, \cdots, m_n)$ in $\mathbb{Z}^n$ put $\sigma\mathbf{m} = (-m_n, \cdots, -m_2, -m_1)$, and say that $\mathbf{m}$ is σ*-regular* if $\mathbf{m} + \sigma\mathbf{m}$ is regular. The element $\mathbf{a}$ of A is called $\mathbf{m}$-σ*-regular* if $\mathbf{m}$ is σ-regular and $\mathbf{a}\sigma(\mathbf{a})$ is $(\mathbf{m} + \sigma\mathbf{m})$-regular; $\mathbf{a}$ is called *strongly* σ*-regular* if it is $\mathbf{m}$-σ-regular for some $\mathbf{m}$. A σ-conjugacy class in G is called *strongly* (or $\mathbf{m}$-) σ*-regular* if it contains a strongly (or $\mathbf{m}$) σ-regular element in A. Note that if $\mathbf{a}$ is $\mathbf{m}$*-regular* then $\mathbf{a}$ is $\mathbf{m}$-σ-regular since $\mathbf{a}\sigma(\mathbf{a})$ is $(\mathbf{m} + \sigma(\mathbf{m}))$-regular. We have

Proposition 1. *If* $\mathbf{a}$ *is* $\mathbf{m}$*-regular then (1) each conjugacy class in* G *which intersects* $B\mathbf{a}B$ *is* $\mathbf{m}$*-regular; (2) each* σ*-conjugacy class in* G *which intersects* $B\mathbf{a}B$ *contains an* $\mathbf{m}$*-regular element in* A*; in particular it is* $\mathbf{m}$-σ*-regular.*

Proof. We shall prove (2); (1) follows by the same method on erasing σ throughout. Write $g' \sim g$ if g is σ-conjugate to g' in G. We have to show that any $b'\mathbf{a}b$ $(b', b$ in $B)$ is σ-conjugate to an $\mathbf{m}$-regular element. Since $\sigma B = B$, up to σ-conjugacy we may assume that $b' = 1$. Each element b in B can be written in a unique way as a product $b_0 b_- b_+$ with b_0 in $A(R)$, $b_- = 1 + n_-$, $b_+ = 1 + n_+$, where n_- (resp. n_+) is a lower (resp. upper) triangular nilpotent matrix. Put $\tilde{\mathbf{a}} = \mathbf{a}b_0$. Then

$$\begin{aligned}\mathbf{a}\mathbf{b} = \tilde{\mathbf{a}}b_-b_+ \sim \sigma(b_+)\tilde{\mathbf{a}}b_- &= (\tilde{\mathbf{a}}b_-\tilde{\mathbf{a}}^{-1})\tilde{\mathbf{a}}(b_-^{-1}\tilde{\mathbf{a}}^{-1}\sigma(b_+)\tilde{\mathbf{a}}b_-) \\ &\sim \tilde{\mathbf{a}}(b_-^{-1}\tilde{\mathbf{a}}^{-1}\sigma(b_+)\tilde{\mathbf{a}}b_-)\sigma(\tilde{\mathbf{a}}b_-\tilde{\mathbf{a}}^{-1}).\end{aligned}$$

Denote by $\|x\|$ the maximum of the valuations of the entries of a matrix x in G. Put $b'_+ = \tilde{\mathbf{a}}^{-1}\sigma(b_+)\tilde{\mathbf{a}}$ and $b'_- = \sigma(\tilde{\mathbf{a}}b_-\tilde{\mathbf{a}}^{-1})$, and also $n'_+ = b'_+ - 1$ and

$n'_- = b'_- - 1$. Since σ stabilizes every congruence subgroup of G, and $\tilde{\mathbf{a}}$ is $\mathbf{m}$-regular, we have $\|n'_+\| < \|n_+\|$ and $\|n'_-\| < \|n_-\|$. Moreover, it is clear that $b_-^{-1}b'_+b_-b'_- = b''_0b''_-b''_+$ with $\max(\|n''_-\|, \|n''_+\|) \leq \max(\|n'_-\|, \|n'_+\|)$. Repeating this process we obtain a matrix of the form $\mathbf{a}'(1+\varepsilon)$ with $\mathbf{m}$-regular $\mathbf{a}'$ and ε with $\|\varepsilon\|$ smaller than any given positive number. The proposition follows.

Let f be a locally-constant compactly-supported complex-valued function on G, dx a Haar measure on G, and $\Phi^\sigma(\gamma, f) = \int f(x^{-1}\gamma\sigma(x))dx/d_\gamma$ the (*twisted* or) σ-*orbital integral* of f at the element γ of G (the integration is taken over $G_\gamma^\sigma\backslash G$, where G_γ^σ is the σ-centralizer of γ in G, and d_γ is a Haar measure on G_γ^σ). Denote by $L(G)$ the Lie algebra of G; if $G = GL(n)$ then $L(G) = M_n$ (the algebra of $n \times n$ matrices). Put $\mathrm{Ad}\,(\sigma)X = -J^tXJ$ for X in $L(G)$. Denote by $\mathrm{Ad}\,(\gamma)$ the adjoint action of γ on $L(G)$. We say that γ is σ-*regular* if $\gamma\sigma(\gamma)$ is regular (has distinct eigenvalues) in G. If γ is σ-regular, its σ-orbit is closed, and the convergence of $\Phi^\sigma(\gamma, f)$ is clear; this is the only case to be used in this paper, but the convergence of $\Phi^\sigma(\gamma, f)$ is known in general. Put

$$\Delta^\sigma(\gamma) = |\det\{(1 - \mathrm{Ad}\,(\gamma\sigma))|L(G)/L(G_\gamma^\sigma)\}|^{1/2}.$$

This is well-defined since $\mathrm{Ad}\,(\gamma\sigma)$ acts trivially on G_γ^σ and therefore trivially also on $L(G_\gamma^\sigma)$. Put $F^\sigma(\gamma, f) = \Delta^\sigma(\gamma)\Phi^\sigma(\gamma, f)$. Let U be the unipotent upper triangular subgroup in G, A the diagonal subgroup, and K the maximal compact subgroup $G(R)$. Each of A, U, K is σ-invariant, and A normalizes U. Put $A^\sigma = \{a \text{ in } A; \sigma a = a\}$. For γ in A put $\delta^\sigma(\gamma) = |\det \mathrm{Ad}\,(\gamma\sigma)|L(U)|$ and

$$f_U^\sigma(\gamma) = \delta^\sigma(\gamma)^{1/2} \int_{A^\sigma\backslash A} \int_U \int_K f(\sigma(k)^{-1}\sigma(a)^{-1}\gamma\, auk)\, dk\, du\, da.$$

A standard formula of change of variables (see, e.g., [FK], §7) asserts that for any σ-regular γ in A we have $F^\sigma(\gamma, f) = f_U^\sigma(\gamma)$. Consequently it is clear from Proposition 1(2) that if f is (a multiple of) the characteristic function of $B\mathbf{a}B$, where $\mathbf{a}$ is an $\mathbf{m}$-regular element, then $F^\sigma(\gamma, f)$ is a scalar multiple of the characteristic function of the union of the σ-conjugacy classes in G which contain an $\mathbf{m}$-regular element, namely of the set of the $\mathbf{m}$-σ-regular σ-conjugacy classes in G. Consequently we can introduce the following

Definition. For any regular $\mathbf{m}$ in Z^n let $\phi_{\mathbf{m},\sigma}$ denote the multiple of the characteristic function of $B\mathbf{q}^{\mathbf{m}}B$ such that $F^\sigma(\gamma, \phi_{\mathbf{m},\sigma})$ is zero unless γ lies in an $\mathbf{m}$-σ-regular σ-conjugacy class in G, where $F^\sigma(\gamma, \phi_{\mathbf{m},\sigma}) = 1$.

Analogous definitions will now be introduced in the non-twisted case. We simply have to erase σ everywhere. Thus the orbital integral of a locally-constant compactly-supported complex-valued function f on G at γ in G is denoted by $\Phi(\gamma, f) = \int f(x^{-1}\gamma x)dx/d_\gamma; x$ ranges over $G_\gamma\backslash G$, where G_γ is the centralizer of γ in G. If γ is regular, namely it has distinct eigenvalues $\gamma_1, \dots, \gamma_n$, the orbit of γ is closed and $\Phi(\gamma, f)$ is clearly convergent. Put

$$\Delta(\gamma) = |\det\{(1 - \mathrm{ad}\,(\gamma))|L(G)/L(G_\gamma)\}|^{1/2};$$

it is equal to

$$|\Pi_{i<j}(\gamma_i - \gamma_j)^2|^{1/2}/|\det \gamma|^{(n-1)/2}.$$

Put $F(\gamma, f) = \Delta(\gamma)\Phi(\gamma, f)$. If γ lies in A put $\delta(\gamma) = |\det \mathrm{Ad}(\gamma)|L(U)|$; it is equal to $\Pi_{i<j}|\gamma_i/\gamma_j|$. Put $f_U(\gamma) = \delta(\gamma)^{1/2}\int_U\int_K f(k^{-1}\gamma nk)\,dk\,dn$. Since $F(\gamma, f) = f_U(\gamma)$ for all regular γ in A it is clear from Proposition 1(1) that if f is (a multiple of) the characteristic function of $B\mathbf{a}B$, where $\mathbf{a}$ is an $\mathbf{m}$-regular element, than $F(\gamma, f)$ is a scalar multiple of the characteristic function of the union of the $\mathbf{m}$-regular conjugacy classes in G. Consequently we can introduce the following

Definition. Denote by $\phi_{\mathbf{m}}$ the multiple of the characteristic function of $B\mathbf{q}^{\mathbf{m}}B$ such that $F(\gamma, \phi_{\mathbf{m}})$ is 0 unless γ lies in an $\mathbf{m}$-regular conjugacy class, where $F(\gamma, \phi_{\mathbf{m}}) = 1$.

Let π be an admissible G-module. Let $\pi(f)$ be the convolution operator $\int f(x)\pi(x)dx$; it is of finite rank, hence has a trace, denoted by $\mathrm{tr}\,\pi(f)$. It is easy to see that there exists a conjugacy invariant locally-constant complex-valued function χ on the regular set (distinct eigenvalues) of G, with $\mathrm{tr}\,\pi(f) = \int_G \chi(x)f(x)dx$ for any f supported on the regular set of G. The function $\chi = \chi_\pi$ is called the *character* of π; it is clearly independent of the choice of the measure dx.

If V is the space of π, then $V_U = \{\pi(u)v - v; v \text{ in } V,\ u \text{ in } U\}$ is stabilized by A since A normalizes U, and V/V_U is an admissible (namely it has finite length) A-module denoted by π'_U. The A-module $\pi_U = \delta^{-1/2}\pi'_U$ is called the *A-module of U-coinvariants of* π. The composition series of the admissible A-module π_U consists of finitely many irreducible A-modules, namely characters on A (since A is abelian), which we call here the *exponents* of π. The character $\chi(\pi_U)$ of π_U is the sum of the exponents of π.

If $\pi_U \neq \{0\}$ then by Frobenius reciprocity π is a subquotient of the G-module $I(\mu) = \mathrm{Ind}(\delta^{1/2}\mu; AU, G)$ normalizedly induced from the character μ of A extended to AU by one on U; here μ is any exponent of π. Let $W = N(A)/A$ be the Weyl group of A in G; $N(A)$ is the normalizer of A in G. Put $w\mu$ for the character $a \mapsto \mu(w(a))$ of A. Define $\mathcal{J} = (\delta_{i,n+1-i})$. The Theorem of [C] asserts that $(\Delta\chi_\pi)(\mathbf{a}) = (\chi(\pi_U))(\mathcal{J}\mathbf{a}\mathcal{J})$ for every strongly regular $\mathbf{a}$ in A. Hence $\chi(I(\mu)_U) = \Sigma\, w\mu$ (sum over w in W), and each exponent of π is of the form w in W. Since $\phi_{\mathbf{m}}$ is supported on the $\mathbf{m}$-regular set, the Weyl integration formula implies that

$$\mathrm{tr}\ \pi(\phi_{\mathbf{m}}) = [W]^{-1}\int_A (\Delta\chi_\pi)(\mathbf{a})F(\mathbf{a}, \phi_{\mathbf{m}})d\mathbf{a} = (\chi(\pi_U))(\mathbf{q}^{\mathbf{m}})\int_{A(R)} \mu(\mathbf{a})d\mathbf{a}.$$

Namely the trace $\mathrm{tr}\ \pi(\phi_{\mathbf{m}})$ is zero unless the composition series of π_U consists of unramified characters, in which case (for a suitable choice of measures) $\mathrm{tr}\ \pi(\phi_{\mathbf{m}})$ is the sum of $\mu(\mathbf{q}^{\mathbf{m}})$ over the exponents (with multiplicities) of π. We conclude:

Proposition 2. *If μ is an unramified character of A then*

$$tr(I(\mu))(\phi_{\mathbf{m}}) = \Sigma_w (w\mu)(\mathbf{q}^{\mathbf{m}}) \qquad (w \text{ in } W).$$

Let V denote the space of π, $V_B(\pi)$ the subspace of B-fixed vectors in V, and $V_B(\mu)$ the space $V_B(\pi)$ when $\pi = I(\mu)$. Then $\pi(\phi_{\mathbf{m}})$ acts on $V_B(\pi)$, and we have

Proposition 3. *If μ in an unramified character of A then the dimension of $V_B(\mu)$ is the cardinality* [W] *of W. The set $\{\psi_w; w \text{ in } W\}$ of functions on G such that ψ_w is supported on $AUwB$ and satisfies $\psi_w(auwb) = (\mu\delta^{1/2})(a)$ (a in A, u in U, b in B), is a basis of the space $V_B(\mu)$.*

Proof. This is clear from the decomposition $AU\backslash G = (AU) \cap K \backslash (AU) \cap K \cdot W \cdot B$.

For each i $(1 \leq i \leq n)$ let $\mathbf{e}_i$ be the vector $(0, \dots, 0, 1, 0, \dots, 0)$ in $\mathbb{Z}^n$; the non-zero entry is at the i-th place. A vector $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j$ $(i \neq j)$ is called here a *root* of A. It is called *positive* if $i < j$, negative if $i > j$, and *simple* if $j = i + 1$ $(1 < i < n)$. Put $\rho = \Sigma_{\alpha > 0} \alpha$ $(= (n-1, n-3, \dots, 1-n))$. Then $\delta(\mathbf{q}^{\mathbf{m}}) = q^{\langle \rho, \mathbf{m} \rangle}$. Denote by $\overline{U}$ the unipotent lower triangular subgroup. We have

Proposition 4. *(1) If $\mathbf{m} = (m_1, \dots, m_n) = \Sigma_{i=1}^n m_i \mathbf{e}_i$ satisfies $m_1 \geq \dots \geq m_n$, and $h = \mathbf{q}^{\mathbf{m}}$, then the cardinality of the set BhB/B is $\delta(h)$. (2) Put $B_- = B \cap \overline{U}$. Then for every w in W, the cardinality of the set*

$$w[h^{-1}B_-h/B_- \cap h^{-1}B_-h]w^{-1}/\overline{U} \cap wh^{-1}B_-hw^{-1}$$

is $\delta^{1/2}(h)/\delta^{1/2}(whw^{-1})$.

Proof. If $B_+ = B \cap U, B_0 = B \cap A$, then $B = B_- B_0 B_+, h^{-1}B_-h \supset B_-, h^{-1}B_+h \subset B_+$ and

$$BhB/B \simeq h^{-1}Bh \cdot B/B = h^{-1}B_-h \cdot B/B \simeq h^{-1}B_-h/h^{-1}B_-h \cap B_-;$$

(1) follows; the proof of (2) is similar.

The Weyl group W is isomorphic to the symmetric group S_n on n letters. It is generated by the simple transpositions $s_i = (i, i+1)(1 \leq i \leq n)$. The length function ℓ on W associates to each w in W the least non-negative integer $\ell(w)$ such that w can be expressed as a product of $\ell(w)$ simple transpositions. It is easy to verify that $(\pi(\phi_{\mathbf{m}})\psi_w)(u)$ is zero for every $u \neq w$ in W with $\ell(u) \geq \ell(w)$.

Proposition 5. *For every w in W we have $(\pi(\phi_{\mathbf{m}})\psi_w)(w) = \mu(whw^{-1})$ (where $h = \mathbf{q}^{\mathbf{m}}$), and $\phi_{\mathbf{m}}$ is equal to $|BhB|^{-1}\delta^{1/2}(h)\, ch(BhB)$.*

Proof. Compute:

$$(\pi(\operatorname{ch}(BhB))\psi_w)(w) = \int_{BhB} \psi_w(wx)dx = |B|\Sigma_{x\in BhB/B}\psi_w(wh\cdot h^{-1}x)$$
$$= |B|(\mu\delta^{1/2})(whw^{-1})\Sigma_{x\in h^{-1}B_-h/B_-\cap h^{-1}B_-h}\psi_w(wxw^{-1}\cdot w)$$
$$= |B|(w\mu)(h)\cdot\delta^{1/2}(whw^{-1})\cdot(\delta^{1/2}(h)/\delta^{1/2}(whw^{-1}))\psi_w(w)$$
$$= |B|(w\mu)(h)\delta^{1/2}(h)\psi_w(w) = |BhB|\cdot\delta^{-1/2}(h)\cdot(w\mu)(h).$$

Conclude:

$$\operatorname{tr}\ \pi[|BhB|^{-1}\delta^{1/2}(h)\ \operatorname{ch}(BhB)] = \Sigma_w(w\mu)(h) = \operatorname{tr}\ \pi(\phi_{\mathbf{m}}).$$

Since $\phi_{\mathbf{m}}$ is by definition a multiple of $\operatorname{ch}(BhB)$, the proposition follows.

We conclude that the matrix of $\pi(\phi_{\mathbf{m}})$ with respect to the basis $\{\psi_w; w \text{ in } W\}$ of $V_B(\mu)$ (this basis is partially ordered by the length function ℓ on W) is of the form $Z+N$, where Z is a diagonal matrix with diagonal entries $\mu(whw^{-1})$ (w in W), and N is a strictly upper triangular nilpotent matrix of size $[W]\times[W]$. Thus we have $N^{[W]}=0$.

Proposition 6. *If* $\mathbf{m}=(m_i)$ *and* $\mathbf{m}'=(m_i')$ *satisfy* $m_i\ge m_{i+1}, m_i'\ge m_{i+1}'(1\le i<n)$ *then* $\pi(\phi_{\mathbf{m}})\pi(\phi_{\mathbf{m}'})=\pi(\phi_{\mathbf{m}+\mathbf{m}'})$.

Proof. Since $hB_-h^{-1}\subset B_-$ and $h^{-1}B_+h\subset B_+$, we have $B\mathbf{q}^{\mathbf{m}}B\mathbf{q}^{\mathbf{m}'}B = B\mathbf{q}^{\mathbf{m}}\mathbf{q}^{\mathbf{m}'}B = B\mathbf{q}^{\mathbf{m}+\mathbf{m}'}B$.

We shall consider only operators $\pi(\phi_{\mathbf{m}})$ with regular $\mathbf{m}$. Since the semi-group of $\mathbf{m}$ in $\mathbf{Z}^n$ with $m_i\ge m_{i+1}\ge 0$ $(1\le i<n)$ is generated by $\Sigma_{i=1}^j\mathbf{e}_i = (1,\dots,1,0,\dots,0)(1\le j<n)$, we need only consider (products of finitely many commuting) matrices of the form $(Z+N)^m$, $m\ge 0$.

Proposition 7. *Let* Z *be a diagonal matrix with entries* z_α *along the diagonal. Let* $N=(n_{\alpha,\beta})$ *be a strictly upper triangular matrix with* $N^s=0$. *Then* $(Z+N)^m$ *is the matrix whose* (α_1,α_r) *entry is*

$$\Sigma_{r=1}^s[\Sigma_{\{\alpha_1<\alpha_2<\dots<\alpha_r\}}n_{\alpha_1,\alpha_2}\cdots n_{\alpha_{r-1},\alpha_r}\Sigma_{1\le k\le r}(-1)^{k-1}z_{\alpha_k}^m\prod_{\substack{1\le i<j<r\\ i,j\ne k}}(z_{\alpha_i}-z_{\alpha_j})/\prod_{1\le i<j\le r}(z_{\alpha_i}-z_{\alpha_j})].$$

Proof. This is easily proven by induction. To obtain this formula, we argue as follows. The non-commutative binomial expansion, easily verified by induction, asserts

$$(Z+N)^m = \Sigma_{r=1}^s(\Sigma_{\{(i_j);\Sigma_{j=1}^r i_j=m+1-r\}}Z^{i_1}NZ^{i_2}\dots NZ^{i_r}).$$

Here

$$Z^{i_1}N\dots NZ^{i_r} = (z_{\alpha_1}^{i_1})(n_{\alpha_1,\alpha_2})(z_{\alpha_2}^{i_2})\dots(n_{\alpha_{r-1},\alpha_r})(z_{\alpha_r}^{i_r})$$
$$= (\Sigma_{\alpha_2,\alpha_3,\dots,\alpha_{r-1}} n_{\alpha_1,\alpha_2} n_{\alpha_2,\alpha_3}\dots n_{\alpha_{r-1},\alpha_r}\cdot z_{\alpha_1}^{i_1}\cdots z_{\alpha_r}^{i_r}).$$

To take the sum over (i_j) we note that by induction we have

$$\Sigma_{\Sigma_{j=1}^r i_j = m+1-r} z_1^{i_1}\cdots z_r^{i_r} = \Sigma_{k=1}^r(-1)^{k+1} z_k^m \prod_{\substack{1\le i<j<r\\ i,j\ne k}} (z_i - z_j) / \prod_{1\le i<j\le r}(z_i - z_j).$$

The proposition follows.

As usual, let μ be an unramified character on A. Let $\psi_{K,\mu}$ be the function on G defined by

$$\psi_{K,\mu}(pk) = (\mu\delta^{1/2})(p) \qquad (p \text{ in } P = AN, \quad k \text{ in } K).$$

It lies in the space of $I(\mu)$. Put $\mu_i = \mu(\mathbf{q}^{\mathbf{e}_i})$. Suppose that $\mu_i \ne q\mu_j$ for all $i \ne j$. Put

$$c_\alpha(\mu) = \frac{1-\mu_i/\mu_j}{1-\mu_i/q\mu_j} \quad \text{if } \alpha = \alpha_{ij}, \tag{7.1}$$

and

$$c_w(\mu) = \Pi_\alpha c_\alpha(\mu) \quad (\alpha > 0, w\alpha < 0).$$

The Weyl group W acts on the set of roots. Suppose that $\mu_i \ne \mu_j$ for all $i \ne j$. Then for each w in W there exists a unique G-morphism $R_{w,\mu}$ from $I(\mu)$ to $I(w\mu)$ which maps $\psi_{K,\mu}$ to $\psi_{K,w\mu}$; this is the content of $[C2]$, Theorem 3.1, where our μ is denoted by χ, our $c_w(\mu)$ is denoted by $c_w(\chi)^{-1}$ in $[C2]$, and it is shown in $[C2]$, (3.1), that our $R_{w,\mu}$ has the form $c_w(\chi)^{-1}T_w$ (in the notations of $[C2]$). The uniqueness of $R_{w,\mu}$ implies that if $w = w_t\dots w_2w_1$ in W, then

$$R_{w,\mu} = R_{w_t, w_{t-1}\dots w_2w_1\mu}\cdots R_{w_2,w_1\mu}R_{w_1,\mu}. \tag{7.2}$$

Put $c_i(\mu)$ for $c_{s_i}(\mu)$. The action of $R_{w,\mu}$ on $V_B(\mu)$ is described in $[C2]$. Theorem 3.4, which asserts the following

Proposition 8. *For each* $i(1 \le i < n)$, *put* $R_i = R_{s_i,\mu}$. *If* $\ell(s_iw) > \ell(w)$, *then*

$$R_i(\psi_w) = (1 - c_i(\mu))\psi_w + q^{-1}c_i(\mu)\psi_{s_iw}$$

and

$$R_i(\psi_{s_iw}) = c_i(\mu)\psi_w + (1 - q^{-1}c_i(\mu))\psi_{s_iw}.$$

Next we analyze in greater detail the case when G is $H = SL(2)$. Here we put $\mathbf{m} = (m, -m)$ where m is a positive integer, $h = \mathbf{q}^{\mathbf{m}} = \begin{pmatrix} \mathbf{q}^m & 0 \\ 0 & \mathbf{q}^{-m}\end{pmatrix}$. Note that $\delta(h) = q^{2m}$. Let z be a non-zero complex number, and μ the unramified character of $A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\}$ with $\mu(\begin{pmatrix} \mathbf{q} & 0 \\ 0 & 1/\mathbf{q} \end{pmatrix}) = z$. Thus, if $\tilde{\mu}$ is an extension of μ to the diagonal subgroup in $GL(2)$, then $z = \tilde{\mu}_1/\tilde{\mu}_2$ in our previous notations. The Weyl group W consists of two elements. If s denotes the non-trivial one, put c for $c_s(\mu)$; then $c = (1-z)/(1-z/q)$. With respect to the basis $\{\psi_1, \psi_s\}$, the matrix of $R = R_{s,\mu}$ is $\begin{pmatrix} 1-c & c \\ c/q & 1-c/q \end{pmatrix}$. Then $\frac{dc}{dz} = q(1-q)/(q-z)^2$ and $\det R = (1-qz)/(z-q)$. Hence

$$R^{-1} = \frac{z-q}{1-qz}\begin{pmatrix} 1-c/q & -c \\ -c/q & 1-c \end{pmatrix}, \quad R' = \frac{d}{dz}R = \frac{1-q}{(z-q)^2}\begin{pmatrix} -q & q \\ 1 & -1 \end{pmatrix},$$

and

$$R'R^{-1} = \frac{q-1}{(z-q)(qz-1)}\begin{pmatrix} -q & q \\ 1 & -1 \end{pmatrix}.$$

Proposition 9. *The matrix of the operator* $\pi(\phi_{\mathbf{m}})$, *where* $\pi = I(\mu)$ *and* $\phi_{\mathbf{m}} = |BhB|^{-1}\delta^{1/2}(h)\, ch\,(BhB)$, *with respect to the basis* $\{\psi_1, \psi_s\}$, *is*

$$\begin{pmatrix} z^m & (q-1)z(1-z)^{-1}(z^{-m}-z^m) \\ 0 & z^{-m} \end{pmatrix}.$$

Proof. For w, u in $W = \{1, s\}$, we are to compute

$$|B|^{-1}(\pi(\mathrm{ch}\,(BhB))\psi_w)(u) = \Sigma_{x \in h^{-1}B_-h/h^{-1}B_-h\cap B_-}\psi_w(uhx).$$

If $u = s$ we obtain $|BhB|\psi_w(sh)$, which is zero if $w=1$ and $|BhB|(\mu\delta^{1/2})(shs^{-1})$ if $w = s$. If $u = 1$ we obtain

$$(\mu\delta^{1/2})(h)\sum_x \psi_w(\begin{pmatrix} 1 & 0 \\ \mathbf{q}^{2m-1}x & 1 \end{pmatrix}), \quad (x \text{ in } R/\boldsymbol{\pi}^{2m}R).$$

Using the relation

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1/t \\ 0 & 1 \end{pmatrix}$$

it is clear that when $w=1$ only the term of $x=0$ in $R/\boldsymbol{\pi}^{2m}R$ is non-zero, and we obtain $(\mu\delta^{1/2})(h)$. When $w = s$ only the terms of $x \neq 0$ are non-zero; there are $(q-1)q^{2m-i-1}$ elements x in $R/\boldsymbol{\pi}^{2m}R$ with absolute value $q^{-i}(0 \leq i < 2m)$, and our sum becomes

$$(q-1)\sum_{i=0}^{2m-1} q^{2m-i-1}(\mu\delta^{1/2})\left(\begin{pmatrix} \mathbf{q}^{1-2m}\mathbf{q}^i & 0 \\ 0 & \mathbf{q}^{-i}\mathbf{q}^{2m-1} \end{pmatrix}\right) = (q-1)\sum_{i=0}^{2m-1} q^{2m-i-1}(qz)^{i+1-2m}$$
$$= (q-1)z^{1-m}(1-z)^{-1}(z^{-m}-z^m).$$

Since $(\mu\delta^{1/2})(h) = (qz)^m$ and $|BhB|^{-1}\delta^{1/2}(h) = q^{-m}$, the proposition follows.

Corollary 10. *For any* $m \geq 0$ *we have*

$$\begin{aligned} & tr[R' \cdot R^{-1} \cdot I(\mu, \phi_{\mathbf{m}})] \\ (10.1) \qquad & = \frac{(q-1)/z}{(z-q)(z^{-1}-q)}[z^{-m} + qz^m - (q-1)z(z-1)^{-1}(z^m - z^{-m})]. \end{aligned}$$

We shall now use these computations to express the trace fomula for $H = SL(2)$ in a convenient form. Thus let F be a global field, fix a non-archimedean place u of F, fix a function f_{0v} for all $v \neq u$ such that $f_{0v} = f_{0v}^0$ for almost all v.

Proposition 11. *There exists a positive integer* m_0, *depending on* $\{f_{0v}; v \neq u\}$, *with the following property. Suppose that* $m \geq m_0$; f_{0u} *is the function* $\phi_{\mathbf{m}}$ *on* H_u; f_0 *is* f_{0v}; *and* x *is an element of* $H(F)$ *with eigenvalues in* $F^\times$. *Then* $f_0(x) = 0$.

Proof. Denote the eigenvalues of x by a and a^{-1}. If $f_0(x) \neq 0$ then $f_{0v}(x) \neq 0$ for all v, and there are $C_{0v} \geq 1$ with $C_{0v} = 1$ for almost all v such that

$$(*)_v \qquad C_{0v}^{-1} \leq |a|_v \leq C_{0v}$$

holds for all $v \neq u$. Since a lies in $F^\times$ we have $\Pi_v |a|_v = 1$. Hence $(*)_u$ holds with $C_{0u} = \Pi_{v \neq u} C_{0v}$. But if $f_{0u} = \phi_{\mathbf{m}}$ and $f_{0u}(x) \neq 0$ then $|a|_u = q_u^m$ or q_u^{-m}. The choice of m_0 with $q_u^{m_0} > C_{0u}$ establishes the proposition.

We conclude that for $f_0 = f_{0v}$ as in Proposition 11, the group theroetic side of the trace formula consists only of orbital integrals of elliptic regular elements; weighted orbital integrals and orbital integrals of singular classes do not appear.

In the representation theoretic side of the trace formula there appears a sum of traces $\operatorname{tr} \pi_0(f_0)$, described as I_0, I_0', I_0^* in [III, (3.3)], Proposition (1), p. 207, and [IV, (1.3)]. There are two additional terms, denoted by S_0, S_0' in [III], p. 207, $\ell.\ -5$. They involve integrals over the analytic manifold of unitary characters $\mu(a) = \mu_0(a)|a|^s$ (s in $i\mathbb{R}$) of $\mathbb{A}^\times / F^\times$; each connected component of this manifold is isomorphic to $\mathbb{R}$. The first term, denoted by $S_0/2$ in [III], p. 208, is

$$(11.1) \qquad \frac{1}{2}\Sigma_{\mu_0} \int_{i\mathbb{R}} \frac{m'(\mu)}{m(\mu)} \Pi_v \operatorname{tr}(I_0(\mu_v))(f_{0v})|ds|.$$

The sum ranges over a set of representatives for the connected components, $m(\mu)$ is the quotient $L(1, \mu)/L(1, \mu^{-1})$ of values of L-functions (see [III, §3]). Since all sums and products in the trace formula are absolutely convergent we obtain

$$(11.1)' \qquad \int_{|z|=1} d(z)(z^m + z^{-m})|d^\times z|.$$

Here $d(z)$ is an integrable functions on the unit circle $|z| = 1$ in $\mathbb{C}$. We used the fact that $\operatorname{tr}(I_0(\mu_u))(\phi_{\mathbf{m}}) = z^m + z^{-m}$, where $z = \mu_u(\begin{pmatrix} \mathbf{q} & 0 \\ 0 & \mathbf{q}^{-1} \end{pmatrix})$.

The second term, denoted by $S_0'/2$ in [III], p. 208, ℓ. 6, is the sum over all places w of the terms

$$(11.2)_w \qquad \frac{1}{2}\Sigma_{\mu_0}\int_{i\mathbb{R}} \mathrm{tr}\,[R_w^{-1}R_w' I_0(\mu_w)](f_{0w})\cdot\Pi_{v\neq w}\,\mathrm{tr}\,[I_0(\mu_v)](f_{0v})|ds|.$$

The summands $(11.2)_w$ which are indexed by $w\neq u$ depend on f_{0u} via $\mathrm{tr}\,[I_0(\mu_u)](f_{0u}) = z^m + z^{-m}$; they can be included in the expression $(11.1)'$ on changing $d(z)$ to another function with the same properties. Left is only $(11.2)_u$, in which $\mathrm{tr}\,[R_w^{-1}R_w' I_0(\mu_w, f_{0w})]$ is given by Corollary 10.

This completes our discussion of the trace formula for $H = SL(2)$. Clearly this discussion applies also in the case of $H_1 = PGL(2)$. Again we take a global function $f_1 = \otimes f_{1v}$ (matching, as in the statement of the Theorem), whose component f_{1u} at u is sufficiently regular with respect to the other components, so that the analogue of Proposition 11 holds. The group theoretic part of the trace formula for H_1 then consists of orbital integrals of elliptic regular elements. There appears a sum of traces $\mathrm{tr}\,\pi_1(f_1)$, described as I_1' in [III], p. 209, ℓ. 5 (where the left side should be $\widetilde{I}_1$, not I_1' as misprinted there), and [IV, (1.3)], and a term analogous to (11.1) (or $(11.1')$), denoted by $S_1/2$ in [III], p. 209, ℓ. 5, and a sum of terms of the form $(11.2)_w$ over all places w of F, which comes from the term $S_1'/2$ of [III], p. 209, ℓ. 5. Note that the contribution of $\widetilde{I}_1$ to J is multiplied by $1/2$. We need consider only the analogue for H_1 of $(11.2)_u$, since $(11.2)_w$ for $w\neq u$ can be included in $(11.1)'$. Here write z for $\mu(\mathbf{q})$, when the induced representation $I_1(\mu)$ of $H_1(F_u)$ from the character $\begin{pmatrix} a & * \\ 0 & b\end{pmatrix}\to\mu(a/b)$ is considered. Then $\mu_1 = z, \mu_2 = z^{-1}$ and $c = (1-z^2)/(1-z^2/q)$ in the notations of (6.1). Hence $\frac{dc}{dz} = 2zq(1-q)/(q-z^2)^2$, $\det R = (1-qz^2)/(z^2-q)$,

$$R = \begin{pmatrix} 1-c & c \\ c/q & 1-c/q\end{pmatrix}, \qquad R^{-1}R' = \frac{2z(q-1)}{(z^2-q)(1-qz^2)}\begin{pmatrix} q & -q \\ -1 & 1\end{pmatrix}$$

and

$$I_1(\phi_{1,\mathbf{m}}) = \begin{pmatrix} z^m & (q-1)z(z^m - z^{-m})/(z-z^{-1}) \\ 0 & z^{-m}\end{pmatrix}$$

where $I_1 = I_1(\mu)(= I_1(z))$ and $\phi_{1,\mathbf{m}}$ is the function $|BhB|^{-1}\delta^{1/2}(h)\,\mathrm{ch}\,(BhB)$ associated with $h = \begin{pmatrix}\mathbf{q}^m & 0 \\ 0 & 1\end{pmatrix}$ in $H_1(F_u)$. Namely we have

Proposition 12. *For every $m\geq 0$ we have*

$$(12.1)\qquad \begin{aligned} &tr\,[R^{-1}R' I_1(\mu,\phi_{1,\mathbf{m}})] \\ &= \frac{2(q-1)/z}{(z^2-q)(z^{-2}-q)}[qz^m + z^{-m} - (q-1)z(z^m - z^{-m})/(z-z^{-1})].\end{aligned}$$

This completes our discussion of the trace formula for $H_1 = PGL(2)$.

Remark. The above discussion applies for any group of rank one. For example it applies also in the case of the unitary group $U(3)$ in three variables, defined by means of a quadratic extension E/F (see [F3], [F4] and [F5]). Here we take a place u which stays prime in E, and note that the definition of $c_w(\mu)$ in the quasi-split case is different from the split case discussed here; see [C2], p. 397.

It remains to carry out analogous discussion of the twisted trace formula of $G = PGL(3)$ for a function $f = \otimes f_v$ as in the Theorem whose component f_u at u is sufficiently regular with respect to the other components. Again the trace formula consists of (1) twisted orbital integrals of σ-elliptic regular elements only, by virtue of the immediate twisted analogue of Proposition 11; (2) discrete sum described as I, I', I'' in [III], p. 201 and p. 203, and [IV, (1.3)]; (3) an integral as in $(11.1)'$, see S of [III], (2.2.4) on p. 202; (4) a sum over w of terms analogous to $(11.2)_w$, see S' of [III], (2.2.5), p. 202. Note that the contribution to our formulae is $(S+S')/4$, see the line prior to (2.2.4), [III], p. 202. Only the term at $w = u$ has to be explicitly evaluated, and we proceed to establish the suitable analogue of Corollary 10 and Proposition 12 for PGL(3), twisted by σ.

Recall that if π is a G-module we define ${}^\sigma\pi$ to be the G-module ${}^\sigma\pi(g) = \pi(\sigma g)$. The notion of a σ-invariant G-module is defined in the introduction. If μ' is a character of A, put $\sigma\mu'$ for the character $\mu' \circ \sigma$ of A. Then ${}^\sigma I(\mu')$ is $I(\sigma\mu')$. We denote by $\pi(\sigma)$ the operator from $I(\mu')$ to $I(\sigma\mu')$ which maps ψ in the space of $I(\mu')$ to $\psi \circ \sigma$. In particular, when μ' is unramified, $\pi(\sigma)$ maps $\psi_{w,\mu'}$ in $V_B(\mu')$ to $\psi_{\sigma w,\sigma\mu'}$ in $V_B(\sigma\mu')$. If $I(\mu')$ is σ-invariant then $[I(\mu')]$ and $[I(\sigma\mu')]$ are equal as elements of the Grothendieck group $K(G,\sigma)$, and there exists w in W with $\sigma\mu' = w\mu'$. If $G = PGL(3)$ and $\mu' = \sigma\mu'$ then there is a character μ of $F^\times$ such that $\mu'(\mathrm{diag}(a,b,c)) = \mu(a/c)$. Suppose in addition that μ' is unramified, and fix as a basis of $V_B(\mu') = V_B(\sigma\mu')$ the set $\psi_1 = \psi_{id}, \psi_2 = \psi_{(12)}, \psi_3 = \psi_{(23)}, \psi_4 = \psi_{(23)(12)}, \psi_5 = \psi_{(12)(23)}, \psi_6 = \psi_{(13)}$, where $W = \{id, (12), (23), (12)(23), (23)(12), (13)\}$. Then the matrix of $\pi(\sigma)$ with respect to this basis is the 6×6 matrix whose non-zero entries are equal to one and located at $(1,1),(2,3),(3,2),(4,5),(5,4),(6,6)$. Here $\pi = I(\mu')$. Denote by A the matrix of $\pi(\phi_{\mathbf{m}})$, with $\mathbf{m} = (1,0,0)$, with respect to our basis, and by B the matrix of $\pi(\phi_{\mathbf{m}})$ with $\mathbf{m} = (1,1,0)$. Then A^n (resp. B^m) is the matrix of $\pi(\phi_{\mathbf{m}})$ with $\mathbf{m} = (n,0,0)$ (resp. $\mathbf{m} = (m,m,0)$), and $A^nB^m = B^mA^n$ by Proposition 6. A direct computation, as in Proposition 9, shows that

$$A = \begin{pmatrix} z & (q-1)z & 0 & 0 & 0 & q(q-1)z \\ 0 & 1 & 0 & q-1 & 0 & 0 \\ 0 & 0 & z & (q-1)z & (q-1)z & (q-1)^2z \\ 0 & 0 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & q-1 \\ 0 & 0 & 0 & 0 & 0 & z^{-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} z & 0 & (q-1)z & 0 & 0 & q(q-1)z \\ 0 & z & 0 & (q-1)z & (q-1)z & (q-1)^2 z \\ 0 & 0 & 1 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 1 & 0 & q-1 \\ 0 & 0 & 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & z^{-1} \end{pmatrix}$$

Here $z=\mu(\mathbf{q})$. Proposition 7 implies that

$$A^n = \begin{pmatrix} z^n & (q-1)z\alpha(n) & 0 & (q-1)^2 z\beta(n) & 0 & q(q-1)z\gamma(n) \\ 0 & 1 & 0 & (q-1)\delta(n) & 0 & 0 \\ 0 & 0 & z^n & (q-1)z\gamma(n) & (q-1)z\alpha(n) & (q-1)^2 z(\gamma(n)+\beta(n)) \\ 0 & 0 & 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & (q-1)\delta(n) \\ 0 & 0 & 0 & 0 & 0 & z^{-n} \end{pmatrix},$$

where $\alpha(n)=(z^n-1)/(z-1)$;

$$\beta(n)=[z^n(1-z^{-1})-(z-z^{-1})+z^{-n}(z-1)]/(z-1)(1-z^{-1})(z-z^{-1});$$
$$\gamma(n)=(z^n-z^{-n})/(z-z^{-1}); \qquad \delta(n)=(1-z^{-n})/(1-z^{-1});$$

and

$$B^m = \begin{pmatrix} z^m & 0 & (q-1)z\alpha(m) & 0 & (q-1)^2 z\beta(m) & q(q-1)z\gamma(m) \\ 0 & z^m & 0 & (q-1)z\alpha(m) & (q-1)z\gamma(m) & (q-1)^2 z(\beta(m)+\gamma(m)) \\ 0 & 0 & 1 & 0 & (q-1)\delta(m) & 0 \\ 0 & 0 & 0 & 1 & 0 & (q-1)\delta(m) \\ 0 & 0 & 0 & 0 & z^{-m} & 0 \\ 0 & 0 & 0 & 0 & 0 & z^{-m} \end{pmatrix}.$$

In particular we conclude the following

Proposition 13. *For any* $\mathbf{m}=(m_1,m_2,m_3)$ *with* $m_1\ge m_2\ge m_3$ *we have*

$$tr[\pi(\phi_{\mathbf{m}})\pi(\sigma)]=\mu'(h_{\mathbf{m}})+\mu'(\mathcal{J}h_{\mathbf{m}}\mathcal{J})=\mu(h_{\mathbf{m}}\sigma(h_{\mathbf{m}}))+\mu(\mathcal{J}h_{\mathbf{m}}\sigma(h_{\mathbf{m}})\mathcal{J}),$$

where $h_{\mathbf{m}}=\mathbf{q}^{\mathbf{m}}$ *, that is,* $=z^{m_1-m_3}+z^{m_3-m_1}$.

On the other hand it is easy to compute the twisted character $\chi=\chi_\pi$ of $\pi=I(\mu')$; see [II, (1.4)]. Recall that χ is a locally constant function on the σ-regular set of G with $\operatorname{tr}\pi(f\times\sigma)=\int f(g)\chi(g)dg$ for every locally-constant function on the σ-regular set of G. Now the twisted character χ of $\pi=I(\mu')$ is supported on the set of g in G such that $g\sigma(g)$ is conjugate to a diagonal element, where $\Delta(h)\chi(h)=z^{m_1-m_3}+z^{m_3-m_1}$ at $h=h_{\mathbf{m}}$. Using the Weyl integration formula we conclude that

$$\operatorname{tr}[\pi(\phi_{\mathbf{m},\sigma})\pi(\sigma)]=z^{m_1-m_3}+z^{m_3-m_1},$$

where $\phi_{\mathbf{m},\sigma}$ is the unique multiple of $\operatorname{ch}(Bh_{\mathbf{m}}B)$ with $F^\sigma(h_{\mathbf{m}},\phi_{\mathbf{m},\sigma})=1$. It follows from Proposition 13 that we have

Proposition 14. *We have* $\phi_{\mathbf{m},\sigma} = \phi_{\mathbf{m}} (= \delta^{1/2}(h_{\mathbf{m}}) |Bh_{\mathbf{m}}B|^{-1}\, ch(Bh_{\mathbf{m}}B))$.

The operator $R = R((13))$ from $V_B(\mu')$ to $V_B(\mathcal{J}\mu')$ is the product of three operators, according to (7.2). Write $V_B(\mu_1, \mu_2, \mu_3)$ for $V_B(\mu')$ if $\mu_i (i = 1, 2, 3)$ are the parameters associated to μ' in (7.1). Then R is the product of $R_1 = R((12))$ form $V_B(z, 1, z^{-1})$ to $V_B(1, z, z^{-1})$, then $R_2 = R((23))$ to $V_B(1, z^{-1}, z)$, and then $R_3 = R((12))$ to $V_B(z^{-1}, 1, z)$. Put $c_1 = (1 - z)/(1 - z/q)$, $c_2 = (1 - z^2)/(1 - z^2/q)$. Put

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1/q & -1/q & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1/q & 0 & -1/q & 0 \\ 0 & 0 & 0 & 1/q & 0 & -1/q \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1/q & 0 & -1/q & 0 & 0 & 0 \\ 0 & 1/q & 0 & -1/q & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1/q & -1/q \end{pmatrix}.$$

Then $R_1 = R_3 = I + c_1 A_1$ and $R_2 = I + c_2 A_2$; further, $R = R_3 R_2 R_1$. Now denote (the right side of) (10.1) by $X(z; m)$, that of (12.1) by $Y(z; m)$, and $\mathrm{tr}\,[R^{-1} R' A^n B^m \pi(\sigma)]$ by $Z(z; n, m)$. Then we have

Proposition 15. *For every* $m, n \geq 0$ *we have*

$$2X(z; n + m) + Y(z; n + m) = Z(z; n, m).$$

Proof. We proved this using the symbolic manipulation language Mathematica. The difference of the two sides of the Proposition is denoted by DIFF in the file given below, and it took the OSU Sun computer three minutes to arrive at the conclusion that DIFF=0. The file is given as an appendix at the end of the paper. In this appendix we denote A_1 by A , A_2 by B , c_1 by c , c_2 by d , R_i by Ri , R^{-1} by S , $\pi(\sigma)$ by s , $\alpha(n)$, etc., by an , etc., A^n, B^m by An, Bm , $Z(z; n, m)$ by Z , $X(z; n + m)$ by X , $Y(z; n + m)$ by Y .

Remark. The fact that $Z(z; n, m)$ depends only on $n + m$ is remarkable.

Corollary 16. *The sum of twice* $(11.2)_u$ *for* $H = SL(2)$ *with* $(11.2)_u$ *for* $H_1 = PGL(2)$ *is equal to the term* $(11.2)_u$ *for* $G = PGL(3)$.

Proof. It follows from Proposition 14 that the function $\phi_{\mathbf{m},\sigma}$ with $\mathbf{m} = (m + n, n, 0)$ matches the function $\phi_{(m+n,-m-n)}$ on $H = SL(2)$ and $\phi_{(m+n,0)}$ on

$H_1 = PGL(2)$. Using [III], p. 204, ℓ. -4 , p. 207, ℓ. -5 , and p. 209, ℓ. 5 (where the left side is $\widetilde{I}_1$ and not I'_1 as misprinted there), we obtain that J of [III], p. 209, ℓ. -3 , is equal to

$$(S + S')/4 - (S_0 + S'_0)/2 - (S_1 + S'_1)/4$$

in the notations of [III]. The S'_i are those leading to the $(11.2)_u$ here. The corollary then follows from Proposition 15.

The Theorem now follows as in [IV, (1.6.3)]. On the one hand T of the Theorem is a discrete sum of the form

$$\sum_i c_i(z_i^m + z_i^{-m}) + \sum_j a_j z_j^m,$$

where z_j lies in the finite set $\{q, q^{-1}, q^{1/2}, q^{-1/2}, -q^{1/2}, -q^{-1/2}\}$, and z_i in $|z_i| = 1$ or $q^{-1/2} < z_i < q^{1/2}$ or $-q^{1/2} < z_i < -q^{-1/2}$. On the other hand T is equal to an integral of the form $(11.1)'$. Here m is a sufficiently large positive integer. The argument of [IV, (1.6.3)] implies that the coefficients c_i and a_j are zero. In particular $T = 0$, and the Theorem follows.

Correction to [F9]. As noted in [F10], p. 3, the sentence on p. 141, ℓ. 4-5, of [F9], does not suffice to pass from Lemma 6.4 to Lemma 6.5 of [F9], but Proposition 8 of [L] does. To complete [F9], this passage is carried out below. It relies on a property of a spherical representation which distinguishes it from other representations with an Iwahori fixed vector, which occur in [F9], (6.4). In an attempt to make this correction readable, we reproduce here some material from [F9]. We put this correction here as both this paper and [F9] use Iwahori-regular functions.

Let G be a quasi-split reductive group over a local non-archimedean field F, which splits over an unramified extension of F, and $B = AU$ a minimal parabolic subgroup over F (such that both the Levi subgroup A and unipotent radical U are invariant under the automorphism σ of [F9]). Denote by Δ a set of simple roots of A on U. It is a subset of $X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A)$. There is a canonical isomorphism $X_*(A) \otimes F^\times \widetilde{\longrightarrow} A(F)$; denote the image of $\alpha \otimes \boldsymbol{\pi}$, $\alpha \in \Delta$, $\boldsymbol{\pi} =$ uniformizer in F, by a_α. Any unramified character (η_u in [F9], Lemma 6.4) of $A(F)$, can be written as the product $\eta\nu$, where η is a unitary unramified character of $A(F)$, and ν is an unramified positive valued character of $A(F)$, with $\nu(a_\alpha) \geq 1$ for all $\alpha \in \Delta_0$. The set $\Delta_M = \{\alpha \in \Delta; \nu(a_\alpha) = 1\}$ is a basis for a set of roots of A in $U \cap M = N$, where M is the standard $(M \supset A)$ Levi subgroup of a standard parabolic subgroup $P = MU$ which is uniquely defined by this basis. Further we fix an element ω in the Weyl group $W_F = \mathrm{Norm}(A(F), G(F))/A(F)$, of minimal length, such that ${}^\omega\nu(t) = \nu^{\omega^{-1}}(t) = \nu(\omega^{-1}(t))$ satisfies ${}^\omega\nu(a_\alpha) \leq 1$ for all $\alpha \in \Delta$. The minimality implies that for $\alpha \in \Delta$ we have $\omega\alpha < 0$ if and only if $\alpha \in \Delta - \Delta_M$, and ${}^\omega\nu(a_\alpha) < 1$.

Consider the $G(F)$-module $i_A^G({}^\omega(\eta\nu))$ which is normalizedly induced from the character ${}^\omega(\eta\nu)$ of $B(F)$ (extended from $A(F)$ by 1 on $U(F)$). If M_ω is the Levi subgroup of the F-parabolic P_ω defined by $\Delta_{M_\omega} = \{\alpha \in \Delta; {}^\omega\nu(a_\alpha) = 1\}$, then ${}^\omega\nu$ extends to a character of $M_\omega(F)$, and $i_A^G({}^\omega(\eta\nu)) = i_{M_\omega}^G({}^\omega\nu \otimes i_A^{M_\omega}({}^\omega\eta))$ is induced in stages. The normalizedly induced $M_\omega(F)$-module $i_A^{M_\omega}({}^\omega\eta)$ is unitarizable, since ${}^\omega\eta$ is a unitary character (of $A(F)$, hence of $B(F) \cap M_\omega(F) = A(F)N_\omega(F)$). Hence it splits as a direct sum of tempered representations $\oplus\tau_{K_\omega}$, where – according to [K], Theorem, p. 400 – K_ω ranges over the set of good maximal compact subgroups of M_ω. In particular, by [T] the adjoint group $M_{\omega,ad}(F)$ of $M_\omega(F)$ acts transitively on the set $\{\tau_{K_\omega}\}$ of components of $i_A^{M_\omega}({}^\omega\eta)$.

By the Langlands' classification ([BW], Ch. XI, or [S]), each of $i_{M_\omega}^G({}^\omega\nu \otimes \tau_{K_\omega})$ has a unique quotient $L_{M_\omega}^G({}^\omega\nu \otimes \tau_{K_\omega})$, which is the image of the standard intertwining operator T_ω from $i_{M_\omega}^G({}^\omega\nu \otimes \tau_{K_\omega})$ to $i_M^G(\nu \otimes \tau_K)$. Here we write τ_K for the irreducible (tempered) constituents (in fact direct summands) of $i_A^M(\eta)$; K ranges over the good maximal compact subgroups of $M(F)$, and $M_{ad}(F)$ acts transitively on the set of τ_K. Then $L_{M_\omega}^G({}^\omega\nu \otimes \tau_{K_\omega})$ is a subrepresentation of $i_M^G(\nu \otimes \tau_K)$ for some K depending on K_ω, and the direct sum over K_ω, which we denote by $L_A^G({}^\omega(\eta\nu))$, is a subrepresentation of $i_A^G(\eta\nu) = \bigoplus_K i_M^G(\nu \otimes \tau_K)$, obtained as the image of the intertwining operator $T_\omega : i_A^G({}^\omega(\eta\nu)) \to i_A^G(\eta u)$.

Since the character $\eta\nu$ is unramified, the representation $i_A^G({}^\omega(\eta\nu))$ is spherical, its $K(F)$-fixed vector $\phi_{K(F),{}^\omega(\eta\nu)}$ is defined by the characteristic function of $K(F)$(= the fixed hyperspecial compact open subgroup of $G(F)$). According to [C2], Theorem 3.1, p. 397, the image $T_\omega(\phi_{K(F),{}^\omega(\eta\nu)})$ is the product of $\phi_{K(F),\eta\nu}$, and a product over $\alpha\in\Delta$ with $\omega\alpha<0$ of some numbers $c_\alpha({}^\omega(\eta\nu))$, which are non-zero since $|{}^\omega(\eta\nu)(a_\alpha)| = {}^\omega\nu(a_\alpha) < 1$ for all such α. Hence $L_A^G({}^\omega(\eta\nu))$ is spherical, containing a non-zero $K(F)$-fixed vector.

Since $L_A^G({}^\omega(\eta\nu))$ is a subrepresentation of the induced $i_A^G(\eta\nu)$, Frobenius reciprocity ([BZ])

$$\mathrm{Hom}_{G(F)}(L_A^G({}^\omega(\eta\nu)), i_A^G(\eta\nu)) = \mathrm{Hom}_{M(F)}(L_A^G({}^\omega(\eta\nu))_N, i_A^M(\eta\nu))$$

implies that there is a non-zero morphism $L_A^G({}^\omega(\eta\nu))_N \to i_A^M(\eta\nu)$. Since the group $M_{ad}(F)$ acts transitively on the set of irreducible constituents (direct summands) of $i_A^M(\eta\nu) = \nu\otimes i_A^M(\eta)$, and on $L_A^G({}^\omega(\eta v))_N$, this morphism is surjective. As a functor $\pi\to\pi_U$ of coinvariants is exact, the morphism $L_A^G({}^\omega(\eta\nu))_U \to i_A^M(\eta\nu)_U = \bigoplus_w {}^w\eta\nu$, $w\in W(A(F),M(F))$, is onto. The constituents of the module π_U of coinvariants will be called here *exponents* of π. We conclude that the exponents of the orbit under $G_{ad}(F)$ of the spherical subrepresentation π_0 of $i_A^G(\eta\nu)$ include the characters ${}^w\eta\nu$ $(w\in W_M = W(A(F),M(F)))$, but no other $G_{ad}(F)$-orbit of constituents of $i_A^G(\eta\nu)$ has these exponents. Denote by $W(\pi)$ a set of w in $W = W(A(F),G(F))$ such that the set of exponents of π is $\{{}^w(\eta\nu); w\in W(\pi)\}$.

Recall that Lemma 5 of [F9] asserts that for an irreducible $G(F)$-module π, and a *regular* function $f = f_t \in C_c^\infty(G(F))$ as defined in [F9], p. 133 (f is supported on the $G(F)$-orbits of $tA(F)$, R = ring of integers in F, t in $A(F)$ with $|\alpha(t)|\neq 1$ for all $\alpha\in\Delta$, and the normalized orbital integral $F(x,f)$ is the characteristic function of the $G(F)$-orbits of $tA(R)$ in $G(F)$), the trace $\mathrm{tr}\,\pi(f)$ is 0 unless π is a constituent of some $i_A^G(\eta\nu)$ as above ($\nu(a_\alpha)\geq 1$ for all $\alpha\in\Delta$), in which case $\mathrm{tr}\,\pi(f)$ is equal to $\sum_{w\in W(\pi)} {}^w(\eta\nu)(t)$, where $\{{}^w(\eta\nu); w\in W(\pi)\}$ is the set of exponents of π. If π_{ad} denotes the orbit of the irreducible π under $G_{ad}(F)$, the $W(\pi)$ are chosen to be pairwise disjoint, and $W(\pi_{ad}) = \bigcup_{\pi\in\pi_{ad}} W(\pi)$, then $\mathrm{tr}\,\pi_{ad}(f)$ equals $\sum_{\pi\in W(\pi_{ad})} {}^w(\eta\nu)(t)$. The set $W(\pi_{ad})$ contains W_M precisely when π_{ad} contains the spherical constituent of $i_A^G(\eta\nu)$.

Similar observations apply for a spherical representation π' of $G' = G(E)$ (where E/F is unramified cyclic extension as in [F9]) which is σ-invariant ($\mathrm{Gal}\,(E/F) = \langle\sigma\rangle$). Such π' is the subrepresentation of an induced $G(E)$-module $I'(\mu')$, where μ' is an unramified character of $A(E)$ with ${}^\sigma\mu' = \mu'$. Since $I'(\mu')$ is σ-invariant, it extends to a representation – denoted $I''(\mu')$ – of $G'' = G(E)\rtimes\mathrm{Gal}\,(E/F)$. Let ϕ be a regular function on $G(E)$ as in [F9], p. 131 (for some t' in $G(E)$ with $|\alpha(Nt')|\neq 1$ for all $\alpha\in\Delta$, ϕ is supported on the σ-conjugacy classes of $t'A(R_E)$, and the normalized σ-orbital integral $F(t'\times\sigma,\phi)$ is the characteristic function of this set in $G(E)$). A standard computation of the character of an induced representation implies that

$$\operatorname{tr} I'(\mu', \phi\sigma) = \int\limits_{A(E)} F(a \times \sigma, \phi)^{\sigma} \mu'(a) da = \sum_{w} {}^{w}({}^{\sigma}\mu')(t');$$

the sum ranges over the w in $W' = \operatorname{Norm}(A(E)\sigma, G(E))/A(E)$. Such w lies in $W_E = \operatorname{Norm}(A(E), G(E))/A(E)$, and it satisfies $\sigma(w) = w$, hence it lies in $W = W_F = W_E^{\langle\sigma\rangle}$. Since ${}^{\sigma}\mu' = \mu'$ there is an unramified character μ of $A(F)$ with $\mu'(a) = \mu(Na)$ $(a \in A(E))$. We write $\mu = \eta\nu$ as in the non-twisted case (η unitary unramified, $\nu > 0$, $\nu(a_\alpha) \geq 1$ for all $\alpha \in \Delta$), and $\mu' = \eta'\nu'$. The normalized module of $U(E)$-coinvariants of a G''-module π'' is denoted by π''_U; it is an $A'' = A(E) \rtimes \langle\sigma\rangle$-module. The value of the character $\chi(\pi'')$ at $t' \times \sigma$, multiplied by the factor $\Delta(t' \times \sigma)$, is equal – by the twisted analogue of [C] recorded in [F9], Lemma 1, p. 131 – to the value of $\chi(\pi''_U)$ at $t' \times \sigma$; contributions to this trace are obtained only from A''-modules whose restriction to $A(E)$ is a σ-invariant character. Namely the σ-exponents of any constituent of $I'(\eta'\nu')$ are among the ${}^{w}(\eta'\nu')(t') = {}^{w}(\eta\nu)(Nt)$. From the discussion in the non-twisted case, it follows that the exponents of the $G_{ad}(E)$-orbit of the spherical subrepresentation of $I'(\eta'\nu')$ include those parametrized by the w in W_M, but no other constituent of $I'(\eta'\nu')$ has these exponents. Since in the stable trace formula only orbits under the adjoint group appear, rather than individual representations, Lemma 6.5 now follows from Lemma 6.4 of [F9].

Correction to [F7; I, §7]. We have used the regular-Iwahori functions in many contexts, but the Theorem of [F9] was used only in [F7; IV]. We use this opportunity to make the following corrections to [F7; I, §7].

p. 159, ℓ. 4, add: For every $M < G$, put $av_M = \Sigma_w w$, where the sum ranges over a set of representatives w in W_G for the $\overline{w}$ in W_G/W_M with $\overline{w}(M) = \overline{w}M\overline{w}^{-1}$ equals M. Put $\tilde{r}_{GM} = av_M \circ r_{GM}$. Then $\tilde{r}^*_{GM} = r^*_{GM} \circ av^*_M$. Note that $w \circ i^*_{MG} = (i_{MG} \circ w)^*$ is equal to i^*_{MG} for any w in W_G with $wM = M$.

p. 160, ℓ. 4, before "This", insert: Hence $\tilde{r}^*_{GM} f^M = 0$ for $M = L$, and so for all M.

page	line	replace	by
157	$-11, -9$	BZ	BZ'
	-10	$\sum_w$	$= \sum_w$
159	1	R_{GM}	r_{GM}
	14	N	M
	21	$L \subset G$	$L \subset M$
	-3	$i_{M_w,M}$	$av_M \circ i_{M_w,M}$
	$6, 11, 15, 16, 20, 21, 23(\times 3), 24, 28(\times 2)$	r^*_{GM}(or r_{GM})	$\tilde{r}^*_{GM}$(or $\tilde{r}_{GM}$)
160	$2, 7, 8, 9(\times 2), 17(\times 2)$	ditto	ditto
	2	$s\rho$	$av_M \circ s\rho$
	2	where the	where
	$4(\times 2)$	f^L	$av^*_L \circ f^L$
	12	$i_{N_w,N}$	$av_N \circ i_{N_w,N}$

REFERENCES

[BZ] J. Bernstein, A. Zelevinski, Representations of the group $GL(n, F)$ where F is a non-archimedean local field, *Uspekhi Mat. Nauk* 31 (1976), 5–70.

[BW] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Ann. of Math. Study 94 (1980), Princeton Univ. Press.

[C] W. Casselman, Characters and Jacquet modules, *Math. Ann.* 230 (1977), 101-105.

[C2] W. Casselman, The unramified principal series of p-adic groups I. The spherical function, *Compos. Math.* 40 (1980), 387-406.

[F1] Y. Flicker, Automorphic forms on covering groups of $GL(2)$, *Invent. Math* 57 (1980),119-182.

[F2] Y. Flicker, Stable and labile base-change for $U(2)$, *Duke Math. J.* 49 (1982), 691-729.

[F3] Y. Flicker, Unitary quasi-lifting: preparations, *Contemporary Math.* 53 (1986), 119-139.

[F4] Y. Flicker, Unitary quasi-lifting: applications, *Trans. AMS* 294 (1986), 553-565.

[F5] Y. Flicker, Packets and liftings for $U(3)$, *J. Analyse Math.* 50 (1988), 19-63.

[F6] Y. Flicker, Base change trace identity for $U(3)$, *J. Analyse Math.* 52 (1989), 39-52.

[F7] Y. Flicker, Rigidity for automorphic forms, *J. Analyse Math.* 49 (1987), 135-202.

[F8] Y. Flicker, Regular trace formula and base change lifting, *Amer. J. Math.* 110 (1988), 739-764.

[F9] Y. Flicker, Stable base change for spherical function, *Nagoya Math. J.* 106 (1987), 121-142.

[F10] Y. Flicker, Regular trace formula and base change for $GL(n)$, *Ann. Inst. Fourier* 40 (1990), 1-30.

[I] Y. Flicker, On the symmetric square: Orbital integrals, *Math. Ann.* 279 (1987), 173-191.

[II] Y. Flicker, On the symmetric square: Definitions and lemmas, *Trans. AMS* 330 (1992), 111-124.

[III] Y. Flicker, On the symmetric square: Twisted trace formula, *J. Functional Anal.* 98 (1991), 194-210.

[IV] Y. Flicker, On the symmetric square: Applications of a trace formula, *Trans. AMS* 330 (1992), 125-152.

[V] Y. Flicker, On the symmetric square: Unit elements, (1993).

[FK] Y. Flicker, D. Kazhdan, Metaplectic correspondence, *Publ. Math. IHES* 64 (1987), 53-110.

[K] D. Keys, Reducibility of unramified unitary principal series representations of p-adic groups and class-1 representations, *Math. Ann.* 260 (1982), 397-402.

[L] J. P. Labesse, Fonctions élémentaires et lemme fondamental pour le changement de base stable, *Duke Math. J.* 61 (1990), 519-530.

[LL] J. P. Labesse, R. Langlands, L-indistinguishability for $SL(2)$, *Canad. J. Math.* 31 (1979), 726 -785.

[S] A. Silberger, The Langlands quotient theorem for p-adic groups, *Math. Ann.* 236 (1978), 95-104.

[T] J. Tits, Reductive groups over local fields, in *Proc. Sympos. Pure Math.* 33 I (1979), 29-69.