ELEMENTARY PROOF OF THE FUNDAMENTAL LEMMA FOR A UNITARY GROUP

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ABSTRACT. The fundamental lemma in the theory of automorphic forms is proven for the (quasi-split) unitary group $U(3)$ in three variables associated with a quadratic extension of p-adic fields, and its endoscopic group $U(2)$, by means of a new, elementary technique. This lemma is a prerequisite for an application of the trace formula to classify the automorphic and admissible representations of $U(3)$ in terms of those of $U(2)$ and base change to $GL(3)$. It compares the (unstable) orbital integral of the characteristic function of the standard maximal compact subgroup K of $U(3)$ at a regular element (whose centralizer T is a torus), with an analogous (stable) orbital integral on the endoscopic group $U(2)$. The technique is based on computing the sum over the double coset space $T\backslash G/K$ which describes the integral, by means of an intermediate double coset space $H\backslash G/K$ for a subgroup H of $G = U(3)$ containing T. Such an argument originates from Weissauer's work on the symplectic group. The lemma is proven for both ramified and unramified regular elements, for which endoscopy occurs (the stable conjugacy class is not a single orbit).

A. Introduction.

Let E/F be an unramified quadratic extension of p-adic fields, $p > 2$, $\mathbf{G} = U(2,1;E/F)$ the unitary group in 3 variables associated with E/F, $\mathbf{H} = U(1,1) \times U(1)$ a subgroup of $\mathbf{G}$, where $U(1,1) = U(1,1;E/F)$ is a quasi-split unitary group in 2 variables and $U(1) = U(1;E/F)$ is an anisotropic torus, and $\mathbf{T}$ an anisotropic F-torus in $\mathbf{H}$ (and $\mathbf{G}$) which splits over E; then $\mathbf{T} = U(1) \times U(1) \times U(1)$. Put $T = \mathbf{T}(F)$, $H = \mathbf{H}(F)$, $G = \mathbf{G}(F)$ for the group of F-points of the F-groups $\mathbf{T}$, $\mathbf{H}$, $\mathbf{G}$. Denote the group of F-points of $U(1)$ by $E^1 = \{x \in E^\times; Nx = 1\}$, $N = N_{E/F}$ signifies the norm map from E to F. Let K be the standard hyperspecial maximal compact subgroup of G, and 1_K the unit element in the Hecke algebra of K-biinvariant compactly supported functions on G.

For a suitable character $\kappa \neq 1$ on the set (with a group structure) of conjugacy classes within the stable conjugacy class of $t = (a,b,c)$, a regular ($a \neq b \neq c \neq a$) element in $T = (E^1)^3$, the κ-orbital integral $\Phi^\kappa_{1_K}(t)$ is defined to be the sum – weighted by the values of κ – of the orbital integrals of 1_K over the conjugacy classes within the stable conjugacy class of t.

Analogously one has the standard maximal compact subgroup K_H in H, the measure 1_{K_H}, and the stable orbital integral $\Phi^{st}_{1_{K_H}}(t)$ on H, where "st" indicates $\kappa = 1$.

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The "endoscopic fundamental lemma" asserts that $\Delta_{G/H}(t)\Phi^{\kappa}_{1_K}(t) = \Phi^{st}_{1_{K_H}}(t)$, where in this case the transfer factor $\Delta_{G/H}(t)$ (defined by Langlands [L], p. 51, and in general by Langlands and Shelstad [LS]) is $(-q)^{-N_1-N_2}$. Here $q = \#(R/\boldsymbol{\pi}R)$ is the residual cardinality of F (R : ring of integers in F, $\boldsymbol{\pi}$: generator of the maximal ideal in R), and $a - b \in \boldsymbol{\pi}^{N_1}R_E^{\times}$, $c - b \in \boldsymbol{\pi}^{N_2}R_E^{\times}$, define the non-negative integers N_1, N_2 (R_E: ring of integers in E).

The other "endoscopic fundamental lemma" concerns the anisotropic F-torus $\mathbf{T}_L$ in $\mathbf{H}$ and $\mathbf{G}$ whose splitting field is a biquadratic extension EL of F. Thus L is a ramified quadratic extension of F. Then $T_L \simeq (EL)^1 \times E^1$ consists of scalar multiples (in E^1) of $t = (t_1, 1)$, and t is regular if t_1 ($\in (EL)^1 = \{x \in (EL)^{\times}; Nx = 1\}$, N=norm from EL to the quadratic extension of F other than E, L) does not lie in E^1. Define n by $t_1 - 1 \in \boldsymbol{\pi}^n_{EL}R^{\times}_{EL}$. The transfer factor $\Delta_{G/H}(t)$ is $(-q)^{-n}$. Once again the "lemma" asserts $\Delta_{G/H}(t)\Phi^{\kappa}_{1_K}(t) = \Phi^{st}_{1_{K_H}}(t)$ for a regular t.

Langlands – who stated the fundamental lemma and explained its importance to the study of automorphic forms by means of the trace formula – suggested a proof based on counting vertices of the Bruhat-Tits building of G. Such a proof ([LR], p. 360 [by Kottwitz, in the EL – or ramified – case], and p. 388 [by Blasius-Rogawski, in the E – or unramified – case]; both cases are attributed by [L], p. 52 to the last author [who claimed them in the last page of his thesis]) presumes building expertise, which I do not have. This technique has not yet been applied in rank > 1 unstable cases.

Since the orbital integrals are just integrals, our idea is simply to perform the integration in a naive fashion, using the fact that $T \subset H$, and using a double coset decomposition $H\backslash G/K$, which we easily establish here. We then obtain a direct and elementary proof, using no extraneous notions. The integrals which we compute are nevertheless non trivial, and this is reflected in our computations. We have used this direct approach to give a simple proof of the fundamental lemma for the symmetric square lifting [F1] from $SL(2)$ to $PGL(3)$ (in the stable and unstable cases), and a proof [F5] of this lemma for the lifting from $GSp(2)$ to $GL(4)$, a rank two case, by developing and combining twisted analogues of ideas of Kazhdan [K] and Weissauer [W], who had dealt with endoscopy for $GSp(2)$ (an alternative approach – using lattices – has recently been found by J. G. M. Mars). The importance of the fundamental lemma led us to test this technique in our case. Thus here we apply our direct approach to give an elementary and self contained proof in the unitary case.

The problem of studying the endoscopic lifting from $U(2)$ to $U(3)$ was raised by R. Langlands [L]. An attempt at this problem – based on stabilizing the trace formula for $U(3)$ alone – was made in reference [25] of [L] (= [Rogawski] in [GP]), but as explained in [F2], §4.6, p. 562/3, this attempt was conceptually insufficient for that purpose. The preprint "L-packets and liftings for $U(3)$" (reference [Flicker] in [GP], [2] of [A], and p. -2 in [L]) proposed studying the endoscopic lifting from $U(2)$ to $U(3)$ simultaneously with base-change from $U(3)$ to $GL(3, E)$ by means of the twisted trace formula. It introduced a definition of packets, and reduced a complete description of these packets – as well as the lifting from $U(2)$ to $U(3)$ and $U(3)$ to $GL(3, E)$ – to important technical assumptions, proven later (twisted trace formula, transfer of orbital integrals). Moreover, rigidity and multiplicity one theorem for $U(3)$ were reduced to the assertions of [GP], which was written later than our preprint. The papers [F2/3] contain a much improved exposition of the preliminary preprint. The paper [F4] contains a new technique, based on the usage of Iwahori-regular functions. It affords a proof of a trace formula identity for *all* test functions – thus extending the results of [F2/3] to all representations of $U(3)$ – by simple means. Later, an exposition of these techniques and results – but not of [F4] – was published by Rogawski (Ann. of Math. Studies (1990)), who subsequently ([LR], p. 395) corrected an error in the computation of the multiplicities of the non-tempered discrete series representations. Finally, we note that Waldspurger [Wa] has recently shown that the fundamental lemma implies the existence of smooth compactly supported functions with matching orbital integrals.

I lusted for an elementary proof as in this paper for a long time, but it was a conversation with T. Oda and A. Murase following my talk at the conference "Automorphic forms and algebraic groups" at RIMS, Kyoto 1995, organized by them, which helped me decompose $H\backslash G/K$ and initiated the present work. D. Zinoviev suggested treating $H''\backslash G/K$, H'' the anisotropic inner form of H, as in his thesis [Z]; this I need for the ramified case. They, the referees, and the support of the Humboldt Stiftung, are here warmly thanked.

B. Classes.

Let us review the structure of the set of (F-rational) conjugacy classes within the stable ($\overline{F}$-) conjugacy class of a regular element t in G. Being regular means that the centralizer $Z_{\mathbf{G}}(t)$ of t in $\mathbf{G}$ is a maximal F-torus $\mathbf{T}$. The elements t, t' of G are *conjugate* if there is g in G with $t' = g^{-1}tg$. They are *stably conjugate* if there is such a g in $\overline{G} = \mathbf{G}(\overline{F})$ ($\overline{F}$ is a separable closure of F). In this case $g_\sigma = g\sigma(g^{-1})$ lies in $\overline{T} = \mathbf{T}(\overline{F})$ for every σ in the Galois group , $= \mathrm{Gal}(\overline{F}/F)$, and $g \mapsto \{\sigma \mapsto g_\sigma\}$ defines an isomorphism from the set of conjugacy classes within the stable conjugacy class of the regular element t of G, to the pointed set $D(T/F) = \ker[H^1(F,\mathbf{T}) \to H^1(F,\mathbf{G})]$. This set is contained in the image $E(T/F) = \mathrm{Im}[H^1(F,\mathbf{T}^{sc}) \to H^1(F,\mathbf{T})]$, where $\mathbf{G}^{sc}$ denotes the simply connected covering group of the derived group of $\mathbf{G}$, and $\mathbf{T}^{sc}$ is the preimage in $\mathbf{G}^{sc}$ of the image of $\mathbf{T}$ in the derived group. When F is local and nonarchimedean, $H^1(F,\mathbf{G}^{sc})$ is trivial. When $H^1(F,\mathbf{G}^{sc}) = \{0\}$, $D(T/F) = E(T/F)$. In this case $D(T/F)$ is a group. Fix an F-torus $\mathbf{T}^*$ in $\mathbf{G}$. Put $\mathbf{N} = \mathrm{Norm}(\mathbf{T}^*, \mathbf{G})$, the normalizer of $\mathbf{T}^*$ in $\mathbf{G}$, and $W = \mathbf{N}/\mathbf{T}^*$ for the Weyl group of $\mathbf{T}^*$ in $\mathbf{G}$. The stable conjugacy classes are determined by means of the following.

1. Proposition. *The set of stable conjugacy classes of F-tori of $\mathbf{G}$ injects naturally in the image in $H^1(F,W)$ of* $\ker[H^1(F,\mathbf{N}) \to H^1(F,\mathbf{G})]$. *The map is bijective when $\mathbf{G}$ is quasi-split.*

Proof. The tori $\overline{T}$ and $\overline{T}^*$ are conjugate in $\overline{G}$, thus $\overline{T} = g^{-1}\overline{T}^* g$ for some g in $\overline{G}$. For any t in $\overline{T}$ there is t^* in $\overline{T}^*$ with $t = g^{-1}t^*g$. For t in T, we have $\sigma g^{-1}\sigma t^* \sigma g = \sigma t = t = g^{-1}t^*g$, hence $\sigma t^* = g_\sigma^{-1} t^* g_\sigma \in \overline{T}^*$, and $g_\sigma \in \overline{N}$. Taking regular t (and t^*), g_σ is uniquely determined modulo $\overline{T}^*$, namely in W. For any t^* in $\overline{T}^*$ we then have $\sigma(g^{-1}t^*g) = g^{-1}(g\sigma(g^{-1}))\sigma(t^*)(\sigma(g)g^{-1})g$, hence the induced action on $\overline{T}^*$ is given by $\sigma^*(t^*) = g_\sigma \sigma(t^*) g_\sigma^{-1}$. The cocycle $\rho = \rho(T): , \to W$, given by $\rho(\sigma) = g_\sigma \bmod \overline{T}^*$, determines $\mathbf{T}$ up to stable conjugacy. Conversely, a $\{g_\sigma\}$ in $\ker[H^1(F,\mathbf{N}) \to H^1(F,\mathbf{G})]$ determines an action

$\sigma^*(t^*) = g_\sigma \sigma(t^*) g_\sigma^{-1}$ on $\overline{T}^*$. A well known theorem of Steinberg asserts that when $\mathbf{G}$ is quasi split over F, a conjugacy class over F in $\overline{G}$ of a regular t^* contains a rational element $g^{-1}t^*g$ (in G); its centralizer is an F-torus which defines g_σ. □

Let us now specialize to our situation. Put $J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$, and introduce an action of the Galois group $\mathrm{Gal}(\overline{F}/F)$ on $GL(3)$ by $\tau((g_{ij})) = (\tau g_{ij})$ if the restriction of τ to E is trivial, and $\tau((g_{ij})) = J^t(\tau g_{ij})^{-1}J$ if $\tau|E$ is the generator σ of $\mathrm{Gal}(E/F)$. Then $\mathbf{G}$ is $GL(3)$ with this $\mathrm{Gal}(\overline{F}/F)$-action, and its group G of F-rational points is $G = \{g \in GL(3,E); gJ^t\overline{g} = J\}$. Here $\overline{(g_{ij})} = (\overline{g}_{ij})$, and $\overline{a} = \sigma a$ for $a \in E$. Fix $\mathbf{T}^*$ to be its diagonal subgroup. The Weyl group W is the symmetric group S_3 on 3 variables, and $\mathrm{Gal}(\overline{F}/F)$ acts on W via $\mathrm{Gal}(E/F)$, σ mapping the reflection (12) to (23), and (23) to (12), thus fixing only 1 and (13). It is easy to classify the stable conjugacy classes of F-tori in $\mathbf{G}$, but we consider only those which split over E, resp. the biquadratic extension EL of F; in the other cases the stable conjugacy class consists of a single conjugacy class. The stable classes are determined by $H^1(\mathrm{Gal}(E/F), W)$, resp. $H^1(\mathrm{Gal}(EL/F), W)$. Put $NE^\times$ for $\{x\sigma(x); x \in E^\times\}$.

2. Proposition. *There are two stable conjugacy classes of F-tori in $\mathbf{G}$ which split over E. One consists of a single conjugacy class, represented by the torus $\mathbf{T}^*$ ($T^* = \{\mathrm{diag}(a, b, \sigma a^{-1})$; $a \in E^\times, b \in E^1 = \{x \in E^\times; x\sigma x = 1\}$). The other consists of tori $\mathbf{T}$ with $T = (E^1)^3$, and $D(T/F) = (F^\times/NE^\times)^2$.*

The stable conjugacy classes of F-tori in $\mathbf{G}$ whose splitting fields are quadratic extensions of E, are parametrized by the (ramified) quadratic extensions L of F which are not isomorphic to E. Each stable class consists of tori $\mathbf{T}$ with $T = (EL)^1 \times E^1$, and $D(T/F) = \mathbb{Z}/2$.

Proof. A cocycle in $H^1(\mathrm{Gal}(E/F), W)$ is determined by w_σ in W, with $1 = w_{\sigma^2} = w_\sigma \sigma(w_\sigma)$, thus w_σ is 1 or (13), or (12)(23) or (23)(12). As $\sigma((23))(12)(23)(23) = 1 = \sigma((12))(23)(12)(12)$, the last two are cohomologous to 1. The cocycle $w_\sigma = 1$ defines the action $\sigma^*(t^*) = \sigma(t^*)$ on $\overline{T}^*$. To determine $D(T^*/F)$, note that $H^1(F, \mathbf{T}^*) = H^1(\mathrm{Gal}(E/F), \mathbf{T}^*(E))$ is the quotient of the cocycles $t_\sigma = \mathrm{diag}(a, b, c) \in \mathbf{T}^*(E) = E^{\times 3}$, $t_\sigma \sigma(t_\sigma) = t_{\sigma^2} = 1$, thus $t_\sigma = \mathrm{diag}(a, b, \sigma a)$, $a \in E^\times$, $b \in F^\times$, by the coboundaries $t_\sigma \sigma(t_\sigma^{-1}) = \mathrm{diag}(a\sigma c, b\sigma b, c\sigma a)$. Since $\mathbf{G}^{sc}$ is the subgroup of $\mathbf{G}$ of elements of determinant 1, the cocycles which come from $H^1(F, \mathbf{T}^{*sc})$ have the form $t_\sigma = \mathrm{diag}(a, 1/a\sigma a, \sigma a)$. These are coboundaries $(u_\sigma \sigma(u_\sigma^{-1})$, with $u_\sigma = (a, 1/a, 1))$, hence $D(T^*/F)$ is trivial.

The cocycle $w_\sigma = (13)$ defines the action $\sigma^*(\mathrm{diag}(a, b, c)) = (\sigma a^{-1}, \sigma b^{-1}, \sigma c^{-1})$ on $\overline{T}^*$. Then $\mathbf{T} = g^{-1}\mathbf{T}^*g$ for some g in $\overline{G}$ with $g\sigma(g^{-1}) = J \pmod{\overline{T}^*}$, and $T = \mathbf{T}(F) = (E^1)^3$. A cocycle $t_\sigma = \mathrm{diag}(a, b, c) \in (E^\times)^3$ of $\mathrm{Gal}(E/F)$ in $\mathbf{T}(E)$ satisfies $1 = t_{\sigma^2} = t_\sigma \sigma(t_\sigma) = \mathrm{diag}(a/\sigma a, b/\sigma b, c/\sigma c)$, thus $a, b, c \in F^\times$ and it comes from $\mathbf{T}^{sc}(E)$ if $abc = 1$. The coboundaries take the form $t_\sigma \sigma(t_\sigma)^{-1} = \mathrm{diag}(a\sigma a, b\sigma b, c\sigma c)$, hence $D(T/F) = (F^\times/NE^\times)^2$.

Consider next an F-torus $\mathbf{T}$ in $\mathbf{G}$ which splits over a quadratic extension L_1 of E, but not over E. The involution $\iota(x) = J^t\overline{x}J$ stabilizes $T = \mathbf{T}(F)$, and its centralizer $L_1^\times \times E^\times$ in $GL(3, E)$; it induces on L_1 an automorphism whose restriction to E generates $\mathrm{Gal}(E/F)$. Hence L_1/F is Galois. But it is not $\mathbb{Z}/4$. Indeed, if $\mathrm{Gal}(L_1/F) = \mathbb{Z}/4$ were generated by τ, then τ^2 be trivial on E, $(w_{\tau^2})^2 = 1$ implies $w_{\tau^2} = 1$ or (13) up to coboundaries, but $(13) = w_{\tau^2} = w_\tau \tau(w_\tau) = w_\tau (13) w_\tau (13)$ implies $w_\tau^2 = (13)$, which has no solutions, and $w_{\tau^2} = 1$ implies that T splits over E. Then $\mathrm{Gal}(L_1/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$, and L_1 is the compositum of E and a quadratic extension L of F, not isomorphic to E. Since $p > 2$, there are two such L (up to isomorphism), both ramified (since E/F is unramified). The Galois group $\mathrm{Gal}(LE/F)$ is generated by σ whose restriction to L is trivial, and τ whose restriction to E is trivial. Up to coboundaries, w_τ is 1 or (13). If $w_\sigma = (13)$, then $w_\tau \neq 1$ is of order 2. Up to coboundary which does not change w_σ, we have $w_\tau = (13)$, and replacing σ by $\sigma\tau$ (thus changing L) we may assume $w_\sigma = 1$. If $w_\sigma = 1$, $w_\tau w_\sigma = w_{\tau\sigma} = w_{\sigma\tau} = w_\sigma \sigma(w_\tau) = w_\sigma (13) w_\tau (13)$ implies that w_τ $(\neq 1)$ commutes with (13), hence $w_\tau = (13)$. Up to isomorphism, T consists of $(a, b, c) \in (LE)^{\times 3}$ which are fixed by $\sigma(a, b, c) = (\sigma c^{-1}, \sigma b^{-1}, \sigma a^{-1})$ and $\tau(a, b, c) = (\tau c, \tau b, \tau a)$. Thus $b = \tau b = \sigma b^{-1}$ lies in E^1, and $c = \sigma a^{-1} = \tau a$, namely $T \simeq \{(a, b, \sigma a^{-1}); b \in E^1, a\sigma\tau a = 1, a \in (EL)^\times\}$.

It is simplest to compute $D(T/F)$ using Tate-Nakayama duality. The image of

$$\hat{H}^{-1}(F, X_*(T^{sc})) = \{X = (x, y, z) \in \mathbb{Z}^3; x + y + z = 0\}/\langle X - \sigma X, X - \tau X\rangle$$

in

$$\hat{H}^{-1}(F, X_*(T)) = \mathbb{Z}^3/\langle X - \sigma X = (2x, 2y, 2z), \quad X - \tau X = (x - z, 0, z - x)\rangle$$

is $\mathbb{Z}/2$. □

To compute our integrals we need explicit realizations of the tori $T = (E^1)^3$ and $T = (EL)^1 \times E^1$.

3. Proposition. *Put* $T_0 = \{t_0 = \mathrm{diag}(a, b, c); a, b, c \in E^1\}$, $h = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ -1 & & 1 \end{pmatrix}$, $r = \mathrm{diag}(\mathfrak{r}, 1, 1)$, *with* $\mathfrak{r} \in F - NE$, $T_1 = h^{-1}T_0 h$ *and* $T_2 = (hr)^{-1}T_0 hr$. *Then* T_1 *and* T_2 *are tori in* G, *and a complete set of representatives for the conjugacy classes within the stable conjugacy class of a regular* $t_1 = h^{-1}\mathrm{diag}(a, b, c)h$ *in* T_1 *(thus* $a \neq b \neq c \neq a$*), is given by* t_1, $t_2 = r^{-1}h^{-1}\mathrm{diag}(a, b, c)hr$, $t_3 = r^{-1}h^{-1}\mathrm{diag}(a, c, b)hr$, *and* $t_4 = r^{-1}h^{-1}\mathrm{diag}(b, a, c)hr$.

A set of representatives for the conjugacy classes of tori $\simeq (LE)^1 \times E^1$ *is given by*

$$T_H = \{\delta^{-1} \begin{pmatrix} \alpha & \boldsymbol{\pi}\beta/\sqrt{D} \\ \beta\sqrt{D} & \alpha \end{pmatrix}; \delta \in E^1, \alpha^2 - \boldsymbol{\pi}\beta^2 = 1\} \times E^1$$

$$\subset H = Z_G(\mathrm{diag}(1, -1, 1)) = U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times E^1 \subset G = U(J),$$

where $D \in R^\times - R^{\times 2}$, *and*

$$T_{H'} = \{\delta^{-1} \begin{pmatrix} \alpha & \boldsymbol{\pi}\beta \\ \beta & \alpha \end{pmatrix}; \delta \in E^1, \alpha^2 - \boldsymbol{\pi}\beta^2 = 1\} \times E^1$$

$$\subset H' = Z_{G'}(\mathrm{diag}(1, 1, -1)) = U\begin{pmatrix} \boldsymbol{\pi} & 0 \\ 0 & -1 \end{pmatrix} \times E^1 \subset G' = U(J'),$$

where $J' = \mathrm{diag}(\boldsymbol{\pi}, -1, -\boldsymbol{\pi}^{-1})$, *and* $J = gJ'{}^t\overline{g}$, *with* $g = \begin{pmatrix} 1/2\boldsymbol{\pi} & 0 & -1/2 \\ 0 & 1 & 0 \\ 1 & 0 & \boldsymbol{\pi} \end{pmatrix}$, *so that* $G' = g^{-1}Gg$.

Proof. An F-torus $\mathbf{T}$ within the stable conjugacy class defined by the cocycle $\{\sigma \mapsto (13)\}$ in $H^1(\mathrm{Gal}(E/F), W)$ takes the form $h^{-1}\mathbf{T}^*h$, with h in $\mathbf{G}(E) = GL(3, E)$ such that $h_\sigma = h\sigma(h^{-1})$ is (13) in W. The h of the proposition satisfies $\sigma(h^{-1}) = \mathrm{diag}(1/2, 1, 1/2)h$, and $h^2 = \begin{pmatrix} 2 & & 0 \\ & -1 & \\ 0 & & -2 \end{pmatrix}\begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}$. Then $t_1 = \begin{pmatrix} \frac{1}{2}(a+c) & & \frac{1}{2}(a-c) \\ & b & \\ \frac{1}{2}(a-c) & & \frac{1}{2}(a+c) \end{pmatrix}$.

A stably conjugate $t_2 = g_2^{-1}t_1g_2 = (hg_2)^{-1}t_0hg_2$ is defined by $g_2 \in \mathbf{G}(E)$ such that $g_{2\sigma} = g_2\sigma(g_2)^{-1} = h^{-1}a_{2\sigma}h$, where $a_{2\sigma} = \mathrm{diag}(\mathfrak{r}, 1, \mathfrak{r}^{-1})$ (we take the elements of $D(T_1/F)$ to be represented by $g_\sigma = 1$, $a_{2\sigma}$, $a_{3\sigma} = \mathrm{diag}(\mathfrak{r}, \mathfrak{r}^{-1}, 1)$, $a_{4\sigma} = \mathrm{diag}(1, \mathfrak{r}, \mathfrak{r}^{-1})$, $\mathfrak{r} \in F - NE$). Thus we need to solve $hg_2J^t(h\overline{g}_2) = hg_2\sigma(hg_2)^{-1}J = a_{2\sigma}h\sigma(h^{-1})J = a_{2\sigma}\,\mathrm{diag}(2, -1, -2) = \mathrm{diag}(2\mathfrak{r}, -1, -2/\mathfrak{r})$ (bar indicates componentwise action of σ). Clearly $g_2 = r$ is a solution.

The next stably conjugate element is $t_3 = g_3^{-1}t_1g_3 = (hg_3)^{-1}t_0hg_3$, where g_3 satisfies $g_{3\sigma} = g_3\sigma(g_3^{-1}) = h^{-1}a_{3\sigma}h \in T_1$. Thus we need to solve $gh_3J^t(h\overline{g}_3) = hg_3\sigma(hg_3)^{-1}J = a_{3\sigma}h\sigma(h)^{-1}J = \mathrm{diag}(2\mathfrak{r}, -2/\mathfrak{r}, -2)$. Since E/F is unramified, there is $x \in E$ with $x\overline{x} = 2$. Define g_3 by $hg_3 = \begin{pmatrix} 1 & & 0 \\ & x^{-1} & \\ 0 & & x \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \end{pmatrix}gh_2$, for which

$$hg_3J^t(h\overline{g}_3) = \begin{pmatrix} 1 & & 0 \\ & x^{-1} & \\ 0 & & x \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 2\mathfrak{r} & & 0 \\ & -1 & \\ 0 & & -2/\mathfrak{r} \end{pmatrix}\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ & \overline{x}^{-1} & \\ 0 & & \overline{x} \end{pmatrix} = \begin{pmatrix} 2\mathfrak{r} & & 0 \\ & -1/\mathfrak{r} & \\ 0 & & -2 \end{pmatrix}.$$

For the last case, replace the index 3 by 4, and note that a solution to $hg_4J^t(h\overline{g}_4) = \mathrm{diag}(2, -\mathfrak{r}, -2/\mathfrak{r})$ is given by g_4 defined by $hg_4 = \begin{pmatrix} y & & 0 \\ & y^{-1} & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}hg_2$, with $y \in E$ such that $y\overline{y} = -2$.

To exhibit non conjugate (in G) tori $\simeq (LE)^1 \times E^1$ in G, we construct one (T_H) in the quasi-split subgroup $H = U(1,1) \times U(1)$ of G, and another $(T_{H'})$ in the anisotropic subgroup $H' = U(2) \times U(1)$ of G. To simplify the notations, we omit the factor E^1 from the notations. To describe T_H, consider the ramified torus $\tilde{T}_1 = \{\begin{pmatrix} \alpha & \beta\boldsymbol{\pi} \\ \beta & \alpha \end{pmatrix} \in GL(2, F)\}$. Put $GL(2, E/F) = \{x \in GL(2, F); \det x \in NE^\times = R^\times\boldsymbol{\pi}^{2\mathbb{Z}}\}$. Then $T \cap GL(2, E/F) = Z\tilde{T}_0$, where $\tilde{T}_0 = \tilde{T}_1 \cap SL(2, F)$, and $Z = F^\times$ is the center of $GL(2, F)$. We have $E^\times GL(2, E/F) = E^\times U_2$, where $U_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, hence the corresponding torus in U_2 is $E^1\tilde{T}_0$. But $H = U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D_1^{-1}U_2D_1$, where $D_1 = \mathrm{diag}(\sqrt{D}, 1)$. Then T_H is as asserted.

To describe $T_{H'}$ and H', note that there is only one form of the unitary group in 3 variables associated with a quadratic extension E/F of p-adic fields. We then work with $G' = U(J')$, which is $g^{-1}Gg$ as stated in the proposition, as the anisotropic H' is easily specified as the centralizer $Z_{G'}(\mathrm{diag}(1, 1, -1))$. Note that we could work with $H'' = gH'g^{-1} = Z_G\begin{pmatrix} 0 & & 1/2\boldsymbol{\pi} \\ & 1 & \\ 2\boldsymbol{\pi} & & 0 \end{pmatrix}$. Now H' consists of $\mathrm{diag}(A, e)$, $e \in E^1$, and $A \in GL(2, E)$

with $A\,\mathrm{diag}(\boldsymbol{\pi},-1)^t\overline{A}=\mathrm{diag}(\boldsymbol{\pi},-1)$. Clearly $\det A=\overline{u}/u$ for some $u\in E^\times$, and solving the equation we see that $A=u^{-1}\left(\begin{smallmatrix}a & c\boldsymbol{\pi}\\ \overline{c} & \overline{a}\end{smallmatrix}\right)$ with $a\overline{a}-\boldsymbol{\pi}c\overline{c}=u\overline{u}$, or alternatively $A=\left(\begin{smallmatrix}a & uc\boldsymbol{\pi}\\ \overline{c} & u\overline{a}\end{smallmatrix}\right)$ with $a\overline{a}-\boldsymbol{\pi}c\overline{c}=1$, $u\in E^1$. A maximal torus splitting over EL, in H', is given by $\{\delta^{-1}\left(\begin{smallmatrix}\alpha & \beta\boldsymbol{\pi}\\ \beta & \alpha\end{smallmatrix}\right);\delta\overline{\delta}=\alpha^2-\boldsymbol{\pi}\beta^2,\alpha,\beta\in F;\delta\in E^\times\}$. Since $\alpha^2-\boldsymbol{\pi}\beta^2=\delta\overline{\delta}\in NE^\times=R^\times\boldsymbol{\pi}^{2\mathbb{Z}}$, we have that both sides are squares, say r^2, $r\in F^\times$, and dividing α,β,δ by r we conclude that $\alpha^2-\boldsymbol{\pi}\beta^2=1=\delta\overline{\delta}$. Then $T_{H'}$ is as asserted. $\square$

Remark. The Weyl group $W(T)$ of $T=T_1$ in G is S_3; for example, $h^{-1}\begin{pmatrix}y & & 0\\ & y^{-1} & \\ 0 & & 1\end{pmatrix}\begin{pmatrix} & & 1\\ & 1 & 0\\ 1 & & \end{pmatrix}h$ lies in G ($y\overline{y}=-2$) and represents the reflection (12). The Weyl group $W(T^*)$ of T^* in G consists of 1 and (13) only.

C. Decompositions.

Let K be the maximal compact subgroup $\mathbf{G}(R)$ of $\mathbf{G}$ (its entries are in the ring R_E of integers of E). Denote by 1_K the characteristic function of K in G, and fix the Haar measure on G which assigns K the volume 1. Our aim is to compute the orbital integrals

$$\int_{T_\rho\backslash G}1_K(x^{-1}t_\rho x)dx\,,\qquad t_\rho=\begin{pmatrix}\frac{a+c}{2} & & \frac{a-c}{2}\rho\\ & b & \\ \frac{a-c}{2\rho} & & \frac{a+c}{2}\end{pmatrix},$$

where ρ is 1 or $\boldsymbol{\pi}$, thus $T_\rho=T_1$ if $\rho=1$ and $T_\rho=T_2$ if $\rho=\boldsymbol{\pi}$. We shall also compute the integrals $\int_{T_H\backslash G}1_K(x^{-1}tx)dx$ and $\int_{T_{H'}\backslash G}1_K(x^{-1}t'x)dx$. The measure on each compact torus is chosen to assign it the volume 1, and we define $\overline{\rho}$ by $\rho=\boldsymbol{\pi}^{\overline{\rho}}$ ($\overline{\rho}=0$ or 1). Put H for the centralizer of $\mathrm{diag}(1,-1,1)$ in G; it contains T_ρ and T_H. Let N denote the unipotent upper triangular subgroup of G; it contains

$$u_0'=\begin{pmatrix}1&1&\frac12\\0&1&1\\0&0&1\end{pmatrix}\text{ and }u_0=\begin{pmatrix}1&x&1\\0&1&\overline{x}\\0&0&1\end{pmatrix}=\begin{pmatrix}x&&0\\&1&\\0&&\overline{x}^{-1}\end{pmatrix}u_0'\begin{pmatrix}x&&0\\&1&\\0&&\overline{x}^{-1}\end{pmatrix}^{-1}\quad(x\overline{x}=2)\,.$$

Our computation of the orbital integral is based on the following decomposition.

4. Proposition. *We have* $G=\bigcup_{m\geq0}Hu_mK$, *where* $u_m=u_0d_m$, $d_m=\mathrm{diag}(t,1,t^{-1})$, $t=\boldsymbol{\pi}^m$. *Further,* $H_m^K=H\cap u_mKu_m^{-1}$ *consists of* $\begin{pmatrix}a_1-b+ta_2 & 0 & b-ta_2+tb_3+2a_3t^2\\ 0 & a_1 & 0\\ b & 0 & a_1-b-tb_3\end{pmatrix}\in H$ *with* a_1, a_2, a_3, b, b_3 *in* R_E.

Also $G=\cup_{m\geq0}H''d_mK$, *and* $H'_m=H'\cap g^{-1}d_mKd_m^{-1}g$ *consists of* $\mathrm{diag}(u^{-1}\left(\begin{smallmatrix}a & c\boldsymbol{\pi}\\ \overline{c} & \overline{a}\end{smallmatrix}\right),e)$, $e\in E^1$, $u\in E^\times$, $a,c\in E$ *with* $a\overline{a}-\boldsymbol{\pi}c\overline{c}=u\overline{u}$ *and* $|a/u-e|\leq|\boldsymbol{\pi}|^{1+2m}$, $|c/u|\leq|\boldsymbol{\pi}|^m$, *or equivalently of scalar multiples by* E^1 *of* $\mathrm{diag}(e\left(\begin{smallmatrix}a & uc\boldsymbol{\pi}\\ \overline{c} & u\overline{a}\end{smallmatrix}\right),1)$, $e,u\in E^1$, $a,c\in R_E$ *with* $1=a\overline{a}-\boldsymbol{\pi}c\overline{c}$, $|a-1|\leq|\boldsymbol{\pi}|^{1+2m}$, $|c|\leq|\boldsymbol{\pi}|^m$. *Both decompositions are disjoint.*

Proof. For the decomposition: $G=T^*NK=HNK=\bigcup_{m\geq0}\bigcup_{\varepsilon\in R_E^\times}H\begin{pmatrix}1&\varepsilon t^{-1}&\frac12\varepsilon\overline{\varepsilon}t^{-2}\\0&1&\overline{\varepsilon}t^{-1}\\0&0&1\end{pmatrix}K=$

$\bigcup_{m,\varepsilon} H \begin{pmatrix} \varepsilon t^{-1} & & 0 \\ & 1 & \\ 0 & & \overline{\varepsilon}^{-1} t \end{pmatrix} u'_0 \begin{pmatrix} \varepsilon^{-1} t & & 0 \\ & 1 & \\ 0 & & \overline{\varepsilon} t^{-1} \end{pmatrix} K = \bigcup_{m \geq 0} H u'_m K$, $u'_m = u'_0 d_m$. It is disjoint since (by matrix multiplication) ${u'_m}^{-1} h u'_m$ lies in K for some h in H only if $n = m$.

The intersection ${H'}^K_m = H \cap u'_m K {u'_m}^{-1}$ consists of (a_i, b_i, c_i in R_E):

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t & & 0 \\ & 1 & \\ 0 & & t^{-1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} t^{-1} & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & t a_2 & t^2 a_3 \\ t^{-1} b_1 & b_2 & t b_3 \\ t^{-2} c_1 & t^{-1} c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

in H, thus $c_1 = -t b_1$ and $c_1 = t c_2$, and we define $b \in E$ by $b_1 = -2bt$. Thus $c_1 = 2bt^2$, $c_2 = 2bt$, and we continue with

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & t a_2 & t^2 a_3 \\ -2b & b_2 & t b_3 \\ 2b & 2b & c_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & t a_2 - a_1 & \frac{1}{2} a_1 - t a_2 + t^2 a_3 \\ -2b & b_2 + 2b & -b - b_2 + t b_3 \\ 2b & 0 & c_3 - b \end{pmatrix}$$

$$= \begin{pmatrix} a_1 - b & X & \frac{1}{2} b - \frac{1}{2} t a_2 + \frac{1}{2} t b_3 + t^2 a_3 \\ 0 & a_1 - t a_2 & Y \\ 2b & 0 & a_1 - b - t a_2 - t b_3 \end{pmatrix} = \begin{pmatrix} x & & 0 \\ & 1 & \\ 0 & & \overline{x}^{-1} \end{pmatrix}^{-1} \begin{pmatrix} a_1 - b & 0 & b - t a_2 + t b_3 + 2 a_3 t^2 \\ 0 & a_1 - t a_2 & 0 \\ b & 0 & a_1 - b - t a_2 - t b_3 \end{pmatrix} \begin{pmatrix} x & & 0 \\ & 1 & \\ 0 & & \overline{x}^{-1} \end{pmatrix}.$$

Since this has to be in H, we obtained the relation $X = 0$, thus $a_1 - t a_2 = b_2 + 2b$, which implies that $b \in R_E$, and $Y = 0$, thus $c_3 - b = b + b_2 - t b_3 = a_1 - b - t a_2 - t b_3$. Replacing $a_1$by $a_1 + t a_2$, and noting that $H^K_m = \operatorname{diag}(x, 1, \overline{x}^{-1}) {H'}^K_m \operatorname{diag}(x, 1, \overline{x}^{-1})^{-1}$, the first part of the proposition follows.

Recall that $G' = g^{-1} G g$, and note that $H' = Z_{G'}(\operatorname{diag}(1,1,-1))$ is $\operatorname{Stab}_{G'}(v'_0) = \{x' \in G'; v'_0 x' = \lambda v'_0,\ \lambda \in E^1\}$, where $v'_0 = (0,0,1)$. Put $v_0 = v'_0 g^{-1} = (-1, 0, 1/2\boldsymbol{\pi})$. Then $H'' = g H' g^{-1} = Z_G \begin{pmatrix} 0 & & 1/2\boldsymbol{\pi} \\ & 1 & \\ 2\boldsymbol{\pi} & & 0 \end{pmatrix}$ is $\operatorname{Stab}_G(v_0) = \{x \in G; v_0 x = \lambda v_0,\ \lambda \in E^1\}$. Embed $H'' \backslash G \hookrightarrow S = \{v \in E^3; v J {}^t\overline{v} = v_0 J {}^t\overline{v}_0 = -\boldsymbol{\pi}^{-1}\}$ by $x \mapsto v = v_0 x$. We have a disjoint decomposition $S = \cup_{m \geq 0} v_0 d_m K$, as $v_0 d_m = (-\boldsymbol{\pi}^m, 0, 1/2\boldsymbol{\pi}^{m+1})$, and $v_0 d_m K = \{v \in S; \|v\| = |\boldsymbol{\pi}|^{-m-1}\}$. Here $\|(x,y,z)\| = \max\{|x|, |y|, |z|\}$, and the union ranges only over $m \geq 0$ since $\{m, -m-1\} = \{n, -n-1\}$ if $n + m = -1$. The decomposition $G = \cup_{m \geq 0} H'' d_m K$ follows.

To describe H'_m, consider the elements of $d_m^{-1} g H' g^{-1} d_m$ in K. Thus

$$\begin{pmatrix} 1/t & & 0 \\ & 1 & \\ 0 & & t \end{pmatrix} \begin{pmatrix} 1/2\boldsymbol{\pi} & & -1/2 \\ & 1 & \\ 1 & & \boldsymbol{\pi} \end{pmatrix} \begin{pmatrix} a/u & c\boldsymbol{\pi}/u & 0 \\ \overline{c}/u & \overline{a}/u & 0 \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi} & & 1/2 \\ & 1 & \\ -1 & & 1/2\boldsymbol{\pi} \end{pmatrix} \begin{pmatrix} t & & 0 \\ & 1 & \\ 0 & & 1/t \end{pmatrix}$$

$$= \begin{pmatrix} (a/u+e)/2 & c/2ut & (a/u-e)/4\boldsymbol{\pi} t^2 \\ \boldsymbol{\pi} t \overline{c}/u & \overline{a}/u & \overline{c}/2ut \\ (a/u-e)\boldsymbol{\pi} t^2 & \boldsymbol{\pi} t c/u & (a/u+e)/2 \end{pmatrix}$$

lies in K precisely when $|c/u| \leq |\boldsymbol{\pi}|^m$, $|a/u - e| \leq |\boldsymbol{\pi}|^{1+2m}$. □

Note that the integrals $\int_{G/K} dx$ and $\int_{H/K^H} dg$ are independent of the choice of the Haar measures dx on G and dh on H. Also, $\int_{H/K_1^H} dh = [K^H : K_1^H] \int_{H/K^H} dh$ for a compact open subgroup K_1^H of K^H. It is convenient to normalize the measures dx and dh to assign K and K^H the volume one. Then $[K^H : K_1^H] = |K_1^H|^{-1}$.

5. Proposition. *The orbital integral of* 1_K *at a regular* $t \in T \subset H$ $(T = T_\rho$ *or* $T_H)$ *can be expressed as*

$$\int_{G/K} 1_K(x^{-1}tx)dx = \sum_{m\geq 0} \int_{H/H_m^K} 1_K(u_m^{-1}h^{-1}thu_m)dh = \sum_{m\geq 0} \int_{H/H_m^K} 1_{H_m^K}(h^{-1}th)dh\,.$$

At a regular $t = gt'g^{-1} \in G$, *where* $t' \in T_{H'} \subset H' \subset G' = g^{-1}Gg$, *we have*

$$\int_{G/K} 1_K(x^{-1}tx)dx = \sum_{m\geq 0} \int_{H'/H'_m} 1_{H'_m}(h^{-1}t'h)dh.$$

Proof. For the last equality of the first assertion, note that $u_m^{-1}h^{-1}thu_m \in K$ implies that $h^{-1}th \in H \cap u_m K u_m^{-1} = H_m^K$.

For the last claim, the left side equals

$$\sum_{m\geq 0} \int_{H''/H''\cap d_m K d_m^{-1}} 1_K(d_m^{-1}h^{-1}thd_m)dh$$

$$= \sum_{m\geq 0} \int_{H'/H'\cap g^{-1}d_m K d_m^{-1} g} 1_K(d_m^{-1}gh'^{-1}t'h'g^{-1}d_m)dh;$$

the displayed equality follows on writing $h = gh'g^{-1}$ and $t' = g^{-1}tg$. The right side is equal to the right side of the equality of the proposition. □

We then need a decomposition for $T_\rho\backslash H/K \cap H$ and $T_H\backslash H/K \cap H$. Note that $H = U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times E^1$, where the first factor is the unitary group in two variables which consists of the g in $GL(2, E)$ with $g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Correspondingly we write $T_\rho = T_{H\rho} \times E^1$ and $K \cap H = K_H \times E^1$. Put $r_j^\rho = \mathrm{diag}(\boldsymbol{\pi}^{-(j-\overline{\rho})/2}, \boldsymbol{\pi}^{(j-\overline{\rho})/2})$ for $j \geq 0$, $j \equiv \overline{\rho} \pmod 2$. In the following statement the factors E^1 and $R^\times$ – whose volume is 1 – can be ignored for our purposes. Write $[x]$ for the largest integer $\leq x$.

6. Proposition. *We have* $H = \bigcup_{j\geq 0} T_{H\rho} \cdot r_j^\rho \cdot K_H \times E^1$ $(j \equiv \overline{\rho}(2),\ j \geq 0)$, *and* $(r_j^\rho)^{-1} T_{H\rho} r_j^\rho \cap K_H = (R + \boldsymbol{\pi}^j R_E)^\times / R^\times \times E^1$. *Further we have* $H = \bigcup_{j\geq 0} T_H \cdot r_j \cdot K_H$, *and* $r_j^{-1} T_H r_j \cap K_H$ *is* $R_L(j)^1 = E^1 \cap R_L(j)$, $R_L(j) = R + \sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^j R$, *where* $r_j = \begin{pmatrix} 0 & \boldsymbol{\pi} \\ 1 & 0 \end{pmatrix}^{j-2[\frac{j}{2}]} \boldsymbol{\pi}^{-[\frac{j+1}{2}]} \begin{pmatrix} 1 & 0 \\ 0 & -\boldsymbol{\pi} \end{pmatrix}^j$.

Proof. Note that $E = F(\sqrt{D})$, $D \in R - R^2$. Put $D_1 = \mathrm{diag}(\sqrt{D}, 1)$. Then $U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D_1^{-1} U_2 D_1$, where U_2 is the unitary group $U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $\mathrm{diag}(a, \overline{a}^{-1}) = a\,\mathrm{diag}(1, 1/a\overline{a})$, we have $E^\times U_2 = E^\times GL(2, E/F)$, where $GL(2, E/F) = \{g \in GL(2, F); \det g \in NE^\times\}$;

note that $NE^\times = \boldsymbol{\pi}^{2\mathbb{Z}}R^\times$. Note that $T_{1\rho} = \left\{\begin{pmatrix} u & vD\rho \\ v/\rho & u \end{pmatrix} \in GL(2,F)\right\}$ lies in $GL(2,E/F)$, as $u^2 - v^2D = \alpha\overline{\alpha} \in NE^\times$ (for $\alpha = u + v\sqrt{D}$ in $E^\times$). The corresponding torus in U_2 is $T_{2\rho} = \left\{\frac{\beta}{\alpha}\begin{pmatrix} u & v\rho D \\ v/\rho & u \end{pmatrix};\ \beta \in E^1\right\}$, and $T_{H\rho} = D_1^{-1}T_{2\rho}D_1$ is the torus $\left\{\frac{\beta}{\alpha}\begin{pmatrix} u & v\rho\sqrt{D} \\ v\sqrt{D}/\rho & u \end{pmatrix}\right\}$ in $D_1^{-1}U_2D_1 = U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus the map $T_{1\rho} \to T_{H\rho}$ takes an element with eigenvalues $\{\alpha, \overline{\alpha}\}$ to one with eigenvalues $\{\beta, \beta\overline{\alpha}/\alpha\}$. From the well known (see the Remark following the present proof) decomposition $GL(2,F) = \bigcup_{j\geq 0} T_{1\rho}\operatorname{diag}(1,\boldsymbol{\pi}^j)GL(2,R)$ we obtain $GL(2,E/F) = \bigcup_j T_{1\rho}r_j^\rho GL(2,R)$ ($j \geq 0$, $j \equiv \overline{\rho}(2)$). Hence $U_2 = \cup T_{2\rho}r_j^\rho K_2$, where $K_2 = U_2 \cap GL(2,R_E)$. Conjugating by D_1 we get the decomposition of the proposition. Finally,

$$(r_j^\rho)^{-1} \cdot T_{H\rho} \cdot r_j^\rho \cap K_H = \left\{\frac{\beta}{\alpha}\begin{pmatrix} u & v\boldsymbol{\pi}^j\sqrt{D} \\ v\boldsymbol{\pi}^{-j}\sqrt{D} & u \end{pmatrix} \in K_H;\ \alpha = u + v\sqrt{D}\right\}.$$

The last matrix has eigenvalues $\beta \in E^1$ and $\beta\overline{\alpha}/\alpha$. Since E/F is unramified, $E^\times/F^\times = R_E^\times/R^\times$, we may assume that $\alpha \in R_E^\times$ and conclude that $u \in R$, $v \in \boldsymbol{\pi}^jR$. Thus our intersection is isomorphic to $(R + \boldsymbol{\pi}^jR_E)^\times/R^\times \times E^1$, as asserted.

For the last claim, in the notations of Proposition 3 in the ramified case ($T = (LE)^1 \times E^1$), we have that $GL(2,F) = \cup_{j\geq 0}T_1\operatorname{diag}(1,(-\boldsymbol{\pi})^j)K = \cup_{j\geq 0}T_1r_jK$, $r_j = t_j\operatorname{diag}(1,(-\boldsymbol{\pi})^j)$, where t_j is $\boldsymbol{\pi}^{-j/2}$ if j is even, and $\boldsymbol{\pi}^{-(j+1)/2}\begin{pmatrix} 0 & \boldsymbol{\pi} \\ 1 & 0 \end{pmatrix}$ if j is odd. Then $GL(2,E/F) = \cup_{j\geq 0}ZT_0r_jK$, and $U = U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \cup_{j\geq 0}E^1T_0r_jK_U$, and $H = U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D_1^{-1}UD_1$ with $D_1 = \operatorname{diag}(\sqrt{D},1)$ has $H = \cup_{j\geq 0}T_Hr_jK_H$, where T_H is as described in Proposition 3.

Now $r_j^{-1}T_Hr_j \cap K_H$ consists of $\delta^{-1}\begin{pmatrix} \alpha & \beta\boldsymbol{\pi}(-\boldsymbol{\pi})^j/\sqrt{D} \\ \beta\sqrt{D}/(-\boldsymbol{\pi})^j & \alpha \end{pmatrix} \in K_H$ in the case where j is even (replace D by $1/D$ when j is odd), namely $|\beta| \leq |\boldsymbol{\pi}|^j$. Thus $r_j^{-1}T_Hr_j \cup K_H$ is $R_L(j)^1 = E^1 \cap R_L(j)$, $R_L(j) = R + \sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^jR$, up to factors of the form E^1, whose volume is 1 and is ignored here. □

Remark. A proof of the well-known decomposition $GL(2,F) = \bigcup_{j\geq 0} T\operatorname{diag}(1,\boldsymbol{\pi}^j)GL(2,R)$ – extracted from a letter of J.G.M. Mars – is as follows. For another proof see [F5], Lemma I.I.1. Let E/F be a separable quadratic extension of non archimedean local fields. Let V be E considered as a two dimensional vector space over F. Multiplication in E gives an embedding $E \subset \operatorname{End}_F(V)$ and $E^\times \subset GL(V)$. The ring of integers R_E is a lattice in V and $K = \operatorname{Stab}(R_E)$ is a maximal compact subgroup of $GL(V)$.

Let Λ be a lattice in V. Then $R(\Lambda) = \{x \in E; x\Lambda \subset \Lambda\}$ is an order. The orders in E are $R_E(j) = R + \boldsymbol{\pi}^jR_E$, $j \geq 0$ ($\boldsymbol{\pi} = \boldsymbol{\pi}_F$). Note that $R_E(j)/R_E(j+1)$ is a one dimensional vector space over $R/\boldsymbol{\pi}$. If $R(\Lambda) = R_E(j)$, then $\Lambda = zR_E(j)$ for some $z \in E^\times$. Choose a basis 1, w of E such that $R_E = R + Rw$. Define d_j in $GL(V)$ by $d_j(1) = 1$, $d_j(w) = \boldsymbol{\pi}^jw$. Then $R_E(j) = d_jR_E$. It follows immediately that $GL(V) = \cup_{j\geq 0}E^\times d_jK$, or, in coordinates with respect to 1, w: $GL(2,F) = \bigcup_{j\geq 0} T\operatorname{diag}(1,\boldsymbol{\pi}^j)GL(2,R)$, with $T = \{\begin{pmatrix} a & \alpha b \\ b & a+\beta b \end{pmatrix};\ a,\ b \in F$, not both $0\}$, where $w^2 = \alpha + \beta w$, α, $\beta \in R$.

7. Proposition. *If* $R_E(j) = R + \boldsymbol{\pi}^j R_E$, $j \geq 0$, *then* $[R_E^\times : R_E(j)^\times]$ *is* 1 *if* $j = 0$, *and* $(1+q^{-1})q^j$ *if* $j \geq 1$. *Further,* $[(R+\sqrt{\boldsymbol{\pi}}R)^1 : (R+\sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^j R)^1] = q^j$.

Proof. The first index is the quotient of $[R_E^\times : 1+\boldsymbol{\pi}^j R_E] = (q^2-1)q^{2(j-1)}$ by $[R^\times : 1+\pi^j R] = (q-1)q^{j-1}$ when $j \geq 1$. When $j = 0$, $R_E(j) = R_E$. The last claim follows from the fact that $u^2 - \boldsymbol{\pi} v^2 = 1$ implies $u = 1 + \boldsymbol{\pi} v^2/2 + \dots$, up to a sign. □

8. Proposition. *We have* $K_H \times E^1 = P_H H_m^K$, *where* $P_H = \left\{ \begin{pmatrix} u & & 0 \\ & w & \\ 0 & & \overline{u}^{-1} \end{pmatrix} \begin{pmatrix} 1 & & v\sqrt{D} \\ & 1 & \\ 0 & & 1 \end{pmatrix}; u \in R_E^\times, w \in E^1, v \in R \right\}$, *and* $[P_H : P_H \cap H_m^K]$ *is* 1 *if* $m = 0$ *and* $(1-q^{-2})q^{4m}$ *if* $m \geq 1$.

Proof. Define $u \in R^\times$, $v \in R$, by the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & cD \\ c & d \end{pmatrix}$ in $GL(2,R)$. Hence K_H consists of $\begin{pmatrix} u & 0 \\ 0 & \overline{u}^{-1} \end{pmatrix} \begin{pmatrix} 1 & v\sqrt{D} \\ 0 & 1 \end{pmatrix} \frac{1}{\alpha} \begin{pmatrix} d & c\sqrt{D} \\ c\sqrt{D} & d \end{pmatrix}$ ($u \in R_E^\times$, $v \in R$; $\alpha = d + c\sqrt{D} \in R_E^\times$), and $K_H \times E^1 = P_H H_m^K$. The intersection $P_H \cap H_m^K$ is P_H when $m = 0$, but when $m \geq 1$ and $t = \boldsymbol{\pi}^m$, it consists of

$$\begin{pmatrix} a_1 + ta_2 & & -ta_2 + tb_3 + 2a_3t^2 \\ & a_1 & \\ 0 & & a_1 - tb_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 + ta_2' & & -ta_2' + tb_3' + 2a_3't^2 \\ & 1 & \\ 0 & & 1 - tb_3' \end{pmatrix},$$

where $a_2' = a_2/a_1$, $b_3' = b_3/a_1$, $a_3' = a_3/a_1$, $a_1\overline{a}_1 = 1$. These satisfy $1 = (1+t\overline{a}_2')(1-tb_3')$, namely $b_3' = \overline{a}_2'/(1+t\overline{a}_2')$. Thus $t(b_3' - a_3') = t(\overline{a}_2'/(1+t\overline{a}_2') - a_2') = t(\overline{a}_2' - a_2' - ta_2'\overline{a}_2')/(1+t\overline{a}_2')$. Erasing the prime from a_2, and the middle entry 1, $P_H \cap H_m^K$ consists of the product of $E^1 = \{a_1\}$ and the matrices

$$\begin{pmatrix} 1+ta_2 & t(\overline{a}_2 - a_2 - ta_2\overline{a}_2)(1+t\overline{a}_2)^{-1} + t^2 2a_3' \\ 0 & 1 - t\overline{a}_2(1+t\overline{a}_2)^{-1} \end{pmatrix} = \begin{pmatrix} 1+ta_2 & t(\overline{a}_2 - a_2)/(1+t\overline{a}_2) \\ 0 & 1 - t\overline{a}_2/(1+t\overline{a}_2) \end{pmatrix} \begin{pmatrix} 1 & t^2 a_3'' \sqrt{D} \\ 0 & 1 \end{pmatrix}.$$

then $[P_H : P_H \cap H_m^K]$ is the product of $[R_E^\times : 1 + \boldsymbol{\pi}^m R_E] = (q^2-1)q^{2(m-1)}$ (for a_2) and $[R : \boldsymbol{\pi}^{2m} R] = q^{2m}$ (for a_3). □

Definition. Put $\delta(X) = 1$ if "X" holds, and $\delta(X) = 0$ if "X" does not hold.

Note that $\int_{P_H/P_H \cap K_m^K} f(p)dp = [P_H : P_H \cap H_m^K] \int_{P_H} f(p)dp$, if the measure dp assigns the compact P_H the volume one.

9. Corollary. *The orbital integral* $\int_{T_\rho \backslash G} 1_K(x^{-1}t_\rho x)dx$ *is equal to*

$$\sum_{j \geq 0,\ j \equiv \overline{\rho}(2)} [\delta(j=0) + (1+q^{-1})q^j \delta(j \geq 1)] \sum_{m \geq 0} \int_{P_H/P_H \cap H_m^K} 1_{H_m^K}(p^{-1}(r_j^\rho)^{-1} t_\rho r_j^\rho p) dp.$$

For a regular $t \in T_H$, *the orbital integral* $\int_{T_H \backslash G} 1_K(x^{-1}tx)dx$ *is equal to*

$$\sum_{m \geq 0} |H_m^K|^{-1} \sum_{j \geq 0} \int_{K_H \cap r_j^{-1} T_H r_j \backslash K_H} 1_{H_m^K}(k^{-1} r_j^{-1} t r_j k) dk$$

$$= \sum_{j\geq 0} q^j \sum_{m\geq 0} \int_{P_H/H_m^K\cap P_H} 1_{H_m^K}(p^{-1}r_j^{-1}tr_jp)dp.$$

□

D. Computations: $j \geq 1$.

In computing the integrals $\int_{P_H} 1_{H_m^K}(p^{-1}(r_j^\rho)^{-1}t_\rho r_j^\rho p)dp$ at $t_\rho = r_\rho^{-1}h^{-1}\operatorname{diag}(a,b,c)hr_\rho$, we put $a' = \frac{a}{b} - 1$, $c' = \frac{c}{b} - 1$, define N_1 by $a' \in \boldsymbol{\pi}^{N_1}R_E^\times$, N_2 by $c' \in \boldsymbol{\pi}^{N_2}R_E^\times$, N by $a' - c' \in \boldsymbol{\pi}^N R_E^\times$ and N^+ by $a' + c' \in \boldsymbol{\pi}^{N^+}R_E^\times$. Since γ_ρ is regular, N, N_1 and N_2 are finite non-negative integers. Put $M = \max(N_1, N_2)$. We shall distinguish between two cases. If $|a' - c'| < |a'|$, then $|a'| = |c'| = |a' + c'|$, thus $N^+ = N_1 = N_2 < N$. If $|a'| \leq |a' - c'|$, then either $|a'| < |a' - c'|$ $(= |c'| = |a' + c'|$, thus $N^+ = N_2 = N < N_1)$, or $|a'| = |a' - c'|$ $(\geq |a' + c'|, |c'|$, thus $N^+, N_2 \geq N_1 = N)$, namely $N \leq N^+$. Put $\mu = N - j$, and denote – as usual – by $[x]$ the maximal integer $\leq x$.

10. Proposition. *If $j \geq 1$, then $\int_{P_H/P_H\cap H_m^K} 1_{H_m^K}(p^{-1}(r_j^\rho)^{-1}t_\rho r_j^\rho p)dp$ is 1 if $m = 0$, $(1 - q^{-2})q^{4m}$ if $1 \leq m \leq \min\left(\left[\frac{\mu}{2}\right], \left[\frac{N^+}{2}\right]\right)$, and $(1 - q^{-2})q^{4m} \cdot (q-1)^{-1}q^{\mu+1-2m} = (1 + q^{-1})q^{\mu+2m}$ if $\mu = N^+ < 2m \leq 2\mu$. For all other $m \geq 0$ the integral is zero.*

For a regular $t = \operatorname{diag}(\delta^{-1}\begin{pmatrix} \alpha & \beta\boldsymbol{\pi}/\sqrt{D} \\ \beta\sqrt{D} & \alpha \end{pmatrix}, \upsilon)$ in $T_H \subset H$, the integral $\int_{P_H/P_H\cap H_m^K} 1_{H_m^K}(p^{-1}r_j^{-1}tr_jp)dp$ is 1 if $m = 0$, $(1 - q^{-2})q^{4m}$ if $1 \leq m \leq \min([\mu/2], [(1 + N_2)/2])$, and $(1 + q^{-1})q^{\mu+2m}$ if $\mu = 1 + N_2 < 2m \leq 2 + 2N_2$, and $N_2 < N$. For all other $m \geq 0$ the integral is zero. Here $\beta = B\boldsymbol{\pi}^N$ $(B \in R^\times)$, and $\delta = \delta_1 + i\delta_2 \in E^1$ with $\delta_2 = D_2\boldsymbol{\pi}^{N_2}$, $\delta_1, D_2 \in R^\times$.

Proof. As $P_H \subset H_m^K$ when $m = 0$, we assume $m \geq 1$. We need to compute the volume of solutions in $u \in R_E^\times/(1 + tR_E)$ and $\upsilon \in R/t^2R$ $(t = \boldsymbol{\pi}^m)$, of the equation

$$\begin{pmatrix} 1 & & -\upsilon\sqrt{D} \\ & 1 & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & & (u-\overline{u})/u \\ & 1 & \\ 0 & & \overline{u} \end{pmatrix}\begin{pmatrix} \frac12(a+c) & & \frac12(a-c)\boldsymbol{\pi}^j \\ & b & \\ \frac12(a-c)\boldsymbol{\pi}^{-j} & & \frac12(a+c) \end{pmatrix}\begin{pmatrix} u & & (\overline{u}-u)/\overline{u} \\ & 1 & \\ 0 & & \overline{u}^{-1} \end{pmatrix}\begin{pmatrix} 1 & & \upsilon\sqrt{D} \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1-b_1+ta_2 & & b_1-ta_2+tb_3+2a_3t^2 \\ & a_1 & \\ b_1 & & a_1-b_1-tb_3 \end{pmatrix},$$

for $a_1 \in E^1$; $b_1, a_2, a_3, b_3 \in R_E$. To have a solution, a_1 must be equal to b. We then replace a by a/b, c by c/b on the left, and b_1, a_2, b_3, a_3 by their quotients by a_1 on the right, to assume that $a_1 = b = 1$. Put $w = \upsilon\sqrt{D} + (\overline{u} - u)/u\overline{u}$, erase 2nd row and column of our matrices, write b for b_1, define $B \in R_E^\times$ by $\frac12(a-c)\boldsymbol{\pi}^{-j} = B\boldsymbol{\pi}^\mu$ $(\mu = N - j \leq N)$, to express our identity as the equality of

$$\begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac12(a+c) & \frac12(a-c)\boldsymbol{\pi}^j/u\overline{u} \\ \frac12(a-c)u\overline{u}\boldsymbol{\pi}^{-j} & \frac12(a+c) \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac12(a+c)-wu\overline{u}B\boldsymbol{\pi}^\mu & B\boldsymbol{\pi}^\mu u\overline{u}(\boldsymbol{\pi}^{2j}/(u\overline{u})^2-w^2) \\ B\boldsymbol{\pi}^\mu u\overline{u} & \frac12(a+c)+wB\boldsymbol{\pi}^\mu u\overline{u} \end{pmatrix}$$

and

$$\begin{pmatrix} 1-b+ta_2 & b-ta_2+tb_3+2a_3t^2 \\ b & 1-b-tb_3 \end{pmatrix}.$$

Since $b \in R_E$, to have solutions we must have that $\mu \geq 0$ (consider the entry (row, column) $= (2,1)$ in our identity). This is congruent to $\begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix}$ modulo $\boldsymbol{\pi}^m$. Considering the entries $(1,1)$ and $(2,2)$, we deduce that $w\boldsymbol{\pi}^\mu \equiv 0 \pmod{\boldsymbol{\pi}^m}$. If $m > \mu$, considering the entries $(1,2)$ and $(2,1)$ we conclude that $j = 0$.

Since $j \geq 1$, we may now assume that $1 \leq m \leq \mu$. Then $b \equiv \boldsymbol{\pi}^\mu \equiv 0\ (\boldsymbol{\pi}^m)$, and from the equality of the entries $(1,1)$ or $(2,2)$, we obtain $\frac{1}{2}(a+c) \equiv 1(\boldsymbol{\pi}^m)$. Put $a' = a-1$, $c' = c-1$. Then $a' + c' \equiv 0(\boldsymbol{\pi}^m)$. Since also $a' - c' \equiv 0(\boldsymbol{\pi}^m)$, we have $a', c' \equiv 0(\boldsymbol{\pi}^m)$, and we have $a'' = a'\boldsymbol{\pi}^{-m}$, $c'' = c'\boldsymbol{\pi}^{-m}$, $b' = b\boldsymbol{\pi}^{-m}$ in R_E. Put $\mu' = \mu - m \geq 0$. The matrix identity translates to 4 equations, the first 3 define b, a_2, b_3 and hence are always solvable:

$$B\boldsymbol{\pi}^{\mu'} u\overline{u} = b'\,,\ \ \frac{1}{2}(a''+c'') + (1-w)u\overline{u}B\boldsymbol{\pi}^{\mu'} = a_2\,,\ \ \frac{1}{2}(a''+c'') + (1+w)u\overline{u}B\boldsymbol{\pi}^{\mu'} = -b_3\,,$$

$$B''\boldsymbol{\pi}^{\mu''} + B\boldsymbol{\pi}^{\mu'}u\overline{u}(1 - Dv_1^2 + \boldsymbol{\pi}^{2j}/(u\overline{u})^2) = 2a_3\boldsymbol{\pi}^m \ \ (\text{where } B''\boldsymbol{\pi}^{\mu''} = a''+c''\,,\ v_1 = w/\sqrt{D} \in R)\,.$$

If $m \leq \mu', \mu''$, namely $2m \leq \mu$, N^+, any $u \in R_E^\times$, $v_1 \in R$, make a solution (a_3 is defined by the 4th equation). This proves the proposition for m $(1 \leq m \leq \min\left(\left[\frac{\mu}{2}\right], \left[\frac{N^+}{2}\right]\right))$.

If $\mu'' < \mu'$, m, there are no solutions in u, v_1.

If $\mu' < \mu''$, m, since $j \geq 1$ and $1 - Dv_1^2 \in R^\times$, there are no solutions either.

It remains to consider the case where $\mu' = \mu'' < m$ $(\leq \mu)$. Write $\varepsilon^{-1} = -u\overline{u}(1 - Dv_1^2)B/B''$. Then our equation can be written in the form

$$1 - 2a_3\boldsymbol{\pi}^{m-\mu'}/B'' = -u\overline{u}B/B''(1 - Dv_1^2 + \boldsymbol{\pi}^{2j}(u\overline{u})^{-2}) = \varepsilon^{-1}(1 + \zeta\boldsymbol{\pi}^{2j}\varepsilon^2)\,,$$

where $\zeta = (B/B'')^2(1 - Dv_1^2)$, namely

$$\varepsilon \equiv 1 + \zeta\boldsymbol{\pi}^{2j}\varepsilon^2 \equiv 1 + \zeta\boldsymbol{\pi}^{2j}(1 + 2\zeta\boldsymbol{\pi}^{2j}\varepsilon^2 + \rho^2\boldsymbol{\pi}^{4j}\varepsilon^4) = 1 + \zeta\boldsymbol{\pi}^{2j} + 2\zeta^2\boldsymbol{\pi}^{4j}\varepsilon^2 + \zeta^3\boldsymbol{\pi}^{6j}\varepsilon^4 \pmod{\boldsymbol{\pi}^{m-\mu'}}\,,$$

so that ε is uniquely determined modulo $\boldsymbol{\pi}^{m-\mu'}$. Thus a choice of v_1 in R determines ζ, and ε in $R^\times/1+\boldsymbol{\pi}^{m-\mu'}R$, hence $u\overline{u} \in R^\times/1+\boldsymbol{\pi}^{m-\mu'}R$. The volume of one coset mod $\boldsymbol{\pi}^{m-\mu'}$ in $R^\times$ is $[R^\times : 1+\boldsymbol{\pi}^{m-\mu'}R]^{-1} = 1/[(q-1)q^{2m-\mu-1}]$. Multiplying by $[P_H : P_H \cap H_m^K] = (1-q^{-2})q^{4m}$ we get $(1+q^{-1})q^{2m+\mu}$.

In the ramified case, the case $m = 0$ is again trivial, so we assume $m \geq 1$. Putting $B_1 = B\overline{\delta}\sqrt{D}(-1)^j \in R_E^\times$, in analogy with the previous case we are led to solve in u and $v_1 = w/\sqrt{D}$ the equation

$$\begin{pmatrix} \alpha\overline{\delta} - wu\overline{u}B_1\boldsymbol{\pi}^\mu & u\overline{u}B_1\boldsymbol{\pi}^\mu(\boldsymbol{\pi}^{2j+1}/D(u\overline{u})^2 - Dv_1^2) \\ u\overline{u}B_1\boldsymbol{\pi}^\mu & \alpha\overline{\delta} + u\overline{u}B_1\boldsymbol{\pi}^\mu \end{pmatrix} = \begin{pmatrix} 1-b+ta_2 & b-ta_2+tb_3+2a_3t^2 \\ b & 1-b-tb_3 \end{pmatrix} \equiv \begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix} (\operatorname{mod} \boldsymbol{\pi}^m).$$

As $b \in R_E$, using $(2,1)$ we have $0 \leq \mu \leq N$. From $(1,1)$ and $(2,2)$, $w\boldsymbol{\pi}^\mu \equiv 0(\boldsymbol{\pi}^m)$. If $\mu < m$ then $|w| < 1$, but this contradicts $(1,2)$ and $(2,1)$. Hence $1 \leq m \leq \mu \leq N$. Put $b' = b\boldsymbol{\pi}^{-m}$, $\mu' = \mu - m$. Then $B_1u\overline{u}\boldsymbol{\pi}^{\mu'} = b'$, $\alpha'' + (1-w)u\overline{u}B_1\boldsymbol{\pi}^{\mu'} = a_2$, $\alpha'' + (1+w)u\overline{u}B_1\boldsymbol{\pi}^{\mu'} = -b_3$, define $b,\ a_2,\ b_3$. Here $\alpha' = \alpha\overline{\delta} - 1 \equiv 0(\boldsymbol{\pi}^m)$ is used to define $\alpha'' = \alpha'\boldsymbol{\pi}^{-m}$. The remaining equation (add all four entries in the matrix equality) is

$$B''\boldsymbol{\pi}^{\mu''} + u\overline{u}B_1\boldsymbol{\pi}^{\mu'}(1 - Dv_1^2 + \boldsymbol{\pi}^{1+2j}/D(u\overline{u})^2) = 2a_3\boldsymbol{\pi}^m,$$

where $2\alpha'' = B''\boldsymbol{\pi}^{\mu''}$, $B'' \in R_E^\times$. If $2\alpha'' = B''\boldsymbol{\pi}^{N^+}$, $N^+ = \mu'' + m$, then $N^+ = \min(1 + N_2, 1 + 2N)$, since

$$\alpha' = \alpha\overline{\delta} - 1 = (1 + B^2\boldsymbol{\pi}^{1+2N}/2 + \dots)(1 + DD_2^2\boldsymbol{\pi}^{2+2N_2}/2 + \dots - \sqrt{D}D_2\boldsymbol{\pi}^{1+N_2}) - 1$$

$$= -\sqrt{D}D_2\boldsymbol{\pi}^{1+N_2} + B^2\boldsymbol{\pi}^{1+2N}/2 + \dots \equiv 0(\boldsymbol{\pi}^m).$$

Of course $\alpha \equiv \delta(\boldsymbol{\pi}^m)$ implies $\delta_2 \equiv 0(\boldsymbol{\pi}^m)$, and $m \leq 1 + N_2$.

Returning to the remaining equation, if $1 \leq m \leq \mu', \mu''$, thus $2m \leq \mu, N^+$, and $\mu \leq N$ implies $1 \leq m \leq \min([\mu/2], [(1 + N_2)/2])$, any $u \in R_E^\times$ and $v_1 \in R$ make a solution, a_3 is defined by the equation, and the number of solutions is as stated in the proposition.

If $\mu'' < \mu', m$, or $\mu' < \mu'', m$, there are no solutions, as $1 - Dv_1^2 \in R^\times$.

If $\mu' = \mu'' < m \leq \mu$, namely $\mu = \min(1 + N_2, 1 + 2N) < 2m \leq 2\mu$, but $\mu \leq N$ implies $\mu = 1 + N_2$, so $N_2 < N$, and the number of solutions is computed as in the unramified case to be as asserted in the proposition. □

11. Proposition. *When $\overline{\rho} = 1$ the orbital integral $\int_{T_\rho\backslash G} 1_K(x^{-1}t_\rho x)dx$ is equal to $\frac{q+1}{q^4-1}\left(q^{4\left[\frac{N+1}{2}\right]} - 1\right)$ if $N \leq N_1$, and to*

$$-\frac{q+1}{q^4-1}(1 + q^{2+4[N_1/2]}) + \frac{(-q)^{N+N_1}}{q-1} + \delta \cdot \frac{q+1}{q-1}q^{N+2N_1}$$

if $N > N_1$. Here $\delta = \delta(2 \mid N - 1 - N_1)$ (is 1 if $N - N_1 - 1$ is even, 0 if $N - N_1$ is even).

The orbital integral $\int_{T_H\backslash G} 1_K(x^{-1}tx)dx$ is equal to: (1) if $N \leq N_2$, it is $(q^{2N+2} - 1)/((q^2 + 1)(q - 1))$ if N is odd, and $(q^{2N+4} - 1)/((q^2 + 1)(q - 1)) - q^{1+2N}$ if N is even, and (2) if $N_2 < N$, it is $q^{N+2N_2+3}/(q - 1) - (q^{2N_2+2} + 1)/((q^2 + 1)(q - 1))$ if N_2 is even, and $-(q^{2N_2+4} + 1)/((q^2 + 1)(q - 1)) + q^{N+2N_2+3}/(q - 1)$ if N_2 is odd.

Proof. It suffices to prove the first statement with N_1 replaced by N^+, since $N > N_1$ if and only if $N > N^+$, in which case $N_1 = N^+$. The contribution from the terms $j \geq 1$ is

$$\sum_{\substack{1\leq j\leq N\\ j\equiv\overline{\rho}(2)}} (1 + q^{-1})q^j \left(1 + \sum_{1\leq m\leq \min\left(\left[\frac{\mu}{2}\right],\left[\frac{N^+}{2}\right]\right)} (1 - q^{-2})q^{4m} + \sum_{\frac{\mu}{2}=\frac{N^+}{2}<m\leq\mu} (1 + q^{-1})q^{\mu+2m}\right).$$

If $\overline{\rho}=1$, this is the entire orbital integral. In this case we replace j by $2j+1$, and let j range over $0\le j\le (N-1)/2$. If $N\le N^+$, $\mu=N-1-2j$ is smaller than N^+, and we get

$$\begin{aligned}
&(q+1)\sum_{0\le j\le[(N-1)/2]} q^{2j}\left(1+\sum_{1\le m\le[(N-1)/2]-j}(1-q^{-2})q^{4m}\right)\\
=&(q+1)\sum_j q^{2j}(1+(1-q^{-2})q^4(q^{4[(N-1)/2]-4j}-1)/(q^4-1))\\
=&\frac{q+1}{q^2+1}\sum_j q^{2j}(1+q^{2+4[(N-1)/2]-4j})\\
=&\frac{q+1}{q^2+1}\left(\frac{q^{2[(N+1)/2]}-1}{q^2-1}+q^{2+4[(N-1)/2]}\cdot\frac{1-q^{-2[(N+1)/2]}}{1-q^{-2}}\right),
\end{aligned}$$

which is equal to the asserted expression.

If ($\overline{\rho}\equiv 1$ and) $N>N^+$, then $\mu=N-1-2j$, and $\frac{\mu}{2}=\frac{N-1}{2}-j>\frac{N^+}{2}$ precisely when $\frac{1}{2}(N-1-N^+)>j$ (same with $<$ or $=$). Note that $\delta(N^+=\mu)$ is δ. Put min $=\min\left(\left[\frac{\mu}{2}\right],\left[\frac{N^+}{2}\right]\right)$. Our integral is then

$$\begin{aligned}
&(q+1)\sum_{0\le j\le[(N-1)/2]} q^{2j}\left(\frac{1}{q^2+1}+\frac{q^{2+4\min}}{q^2+1}\right)+\delta\frac{q^{N^++1}}{q-1}(q^{2N^+}-q^{2[N^+/2]})\\
=&\frac{q+1}{q^2+1}\frac{q^{2[(N+1)/2]}-1}{q^2-1}+\frac{q^2(q+1)}{q^2+1}\\
&\cdot\left(\sum_{0\le j\le[(N-1-N^+)/2]} q^{4[N^+/2]}q^{2j}+\sum_{[(N-1-N^+)/2]<j\le[(N-1)/2]} q^{4[(N-1)/2]}q^{-2j}\right)+\delta *\\
=&\frac{q+1}{q^4-1}(q^{2[(N+1)/2]}-1)+\frac{q^2(q+1)}{q^2+1}\\
&\cdot\left(q^{4[N^+/2]}\frac{q^{2[(N+1-N^+)/2]}-1}{q^2-1}+q^{4[(N-1)/2]}\frac{q^{-2([(N-1-N^+)/2]+1)}-q^{-2([(N-1)/2]+1)}}{1-q^{-2}}\right)+\delta *\\
=&\frac{q+1}{q^4-1}(-1-q^{2+4[N^+/2]}+q^{2+4[N^+/2]+2[(N+1-N^+)/2]}+q^{4[(N+1)/2]-2[(N+1-N^+)/2]})\\
&+\delta\frac{q+1}{q-1}(q^{N+2N^+}-q^{N+2[N^+/2]}).
\end{aligned}$$

If $\delta=0$, then N is even iff N^+ is even, and $\left[\frac{1}{2}(N+1-N^+)\right]=\frac{1}{2}(N-N^+)=[N/2]-[N^+/2]$. Hence we obtain

$$\begin{aligned}
&-\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{q+1}{q^4-1}q^{2[N^+/2]+2[N/2]}(q^2+q^{4[(N+1)/2]-4[N/2]})\\
&=-\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{q^{N^++N}}{q-1}.
\end{aligned}$$

If $\delta = 1$, then N is even iff N^+ is odd, and $\left[\frac{1}{2}(N-1-N^+)\right] = \frac{1}{2}(N-1) - \frac{1}{2}N^+ = \left[\frac{1}{2}(N-1)\right] - \left[\frac{1}{2}N^+\right]$. We get

$$-\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]}) + \frac{q+1}{q^4-1}(q^{2+2[N^+/2]+2[(N+1)/2]} + q^{2[(N+1)/2]+2[N^+/2]})$$
$$-\frac{q+1}{q-1}q^{N+2[N^+/2]} + \frac{q+1}{q-1}q^{N+2N^+}$$
$$= -\frac{q+1}{q^4-1}(1+q^{2+4[N^+/2]}) + \frac{q^{2[N^+/2]}}{q-1}(q^{2[(N+1)/2]} - (q+1)q^N) + \frac{q+1}{q-1}q^{N+2N^+}.$$

The middle term is $-q^{N+N^+}/(q-1)$ since $N+1$ is even iff N^+ is even.

In the ramified case we compute as follows. Suppose that $N \leq N_2$. Then the integral is

$$\sum_{0\leq\mu\leq N} q^{N-\mu}(1+\sum_{1\leq m\leq[\mu/2]}(q^4-q^2)q^{4(m-1)}) = \sum_{0\leq\mu\leq N} q^\mu/(q^2+1) +$$
$$q^{2+N}\sum_{0\leq\mu\leq N} q^{4[\mu/2]-\mu}/(q^2+1) = \frac{q^{N+1}-1}{(q^2+1)(q-1)} + \frac{q^{N+2}}{q^2+1}(\sum_{0\leq\mu_1\leq[N/2],\mu=2\mu_1} q^{2\mu_1}$$
$$+\sum_{0\leq\mu_1\leq[(N-1)/2],\mu=2\mu_1+1} q^{2\mu_1-1}) = \frac{q^{N+2[N/2]+4} + q^{N+2[(N-1)/2]+3} - q - 1}{q^4-1},$$

as asserted.

Suppose that $N_2 < N$. Then the integral is

$$\sum_{0\leq\mu\leq 1+N_2} q^{N-\mu}(1+\sum_{1\leq m\leq[\mu/2]}(1-q^{-2})q^{4m}) + q^{N-N_2-1}\sum_{[(1+N_2)/2]<m\leq 1+N_2}(1+q^{-1})q^{2m+1+N_2}$$
$$+\sum_{1+N_2<\mu\leq N} q^{N-\mu}(1+\sum_{1\leq m\leq[(1+N_2)/2]}(1-q^{-2})q^{4m}).$$

This is the sum of

$$\frac{q^{N+2}}{q^2+1}\sum_{0\leq\mu_1\leq[(N_2+1)/2],\mu=2\mu_1} q^{2\mu_1} + \frac{q^{N+1}}{q^2+1}\sum_{0\leq\mu_1\leq[N_2/2],\mu=2\mu_1+1} q^{2\mu_1} + \frac{q^N}{q^2+1}\cdot\frac{q^{-N_2-2}-1}{q^{-1}-1}$$

and

$$(1+q^{-1})q^N\frac{q^{2(N_2+2)} - q^{2[(1+N_2)/2]+2}}{q^2-1} + \frac{q^{4[(1+N_2)/2]+2}+1}{q^2+1}\cdot\frac{q^{N-N_2-1}-1}{q-1}.$$

Adding, we get the expression of the proposition. □

12. Proposition. *When $\overline{\rho}=0$, the contribution to the orbital integral of 1_K at t_ρ from the terms indexed by $j>0$ is*

$$\frac{(q+1)q}{q^4-1}(q^{4[N/2]}-1)$$

if $N\le N^+$; when $N>N^+$, if $N-N^+$ is odd ($\delta=\delta(n\mid N-N^+>0)$ is 0) we obtain

$$-\frac{(q+1)q}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{q^{N+N^+}}{q-1},$$

while if $\delta=1$ ($N-N^+>0$ is even) we obtain

$$-\frac{(q+1)q}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{q^{1+2[N^+/2]+2[N/2]}}{q-1}+\frac{q+1}{q-1}q^{N+2N^+}-\frac{q+1}{q-1}q^{N+2[N^+/2]}.$$

Proof. Put $\mu=N-2j$, $1\le j\le[N/2]$. The sum over j is

$$(1+q^{-1})\sum_{1\le j\le[N/2]}q^{2j}\left(\frac{1}{q^2+1}+\frac{q^{2+4\min}}{q^2+1}+\delta\sum_{\frac{\mu}{2}=\frac{N^+}{2}<m\le\mu}(1+q^{-1})q^{\mu+2m}\right).$$

If $N\le N^+$, then $\min=[\mu/2]=[N/2]-j$ and $\delta=0$, so we get

$$\frac{q+1}{q(q^2+1)}\sum_{1\le j\le[N/2]}(q^{2j}+q^{2+4[N/2]-2j})=\frac{(q+1)q}{q^2+1}\left(\frac{q^{2[N/2]}-1}{q^2-1}+q^{4[N/2]}\frac{q^{-2}-q^{-2([N/2]+1)}}{1-q^{-2}}\right),$$

which is the asserted expression.

If $N>N^+$, then $\mu/2=N/2-j>N^+/2$ iff $\frac{1}{2}(N-N^+)>j$, in which case $\min([\mu/2],[N^+/2])$ is $[N^+/2]$ (it is $[N/2]-j$ when $>$ is replaced by $<$). Thus we obtain the sum of

$$\begin{aligned}&\frac{(q+1)q}{q^2+1}\ \frac{q^{2[N/2]}-1}{q^2-1}+\frac{(q+1)q^2}{q(q^2+1)}\left(q^{4[N^+/2]}\sum_{1\le j\le[(N-N^+)/2]}q^{2j}+q^{4[N/2]}\sum_{(N-N^+)/2<j\le[N/2]}q^{-2j}\right)\\
=&\frac{(q+1)q}{q^2+1}\ \frac{q^{2[N/2]}-1}{q^2-1}\\
&+\frac{(q+1)q^2}{q(q^2+1)}\left(q^{4[N^+/2]}\frac{q^{2[(N-N^+)/2]+2}-q^2}{q^2-1}+q^{4[N/2]}\frac{q^{-2[(N-N^+)/2]-2}+q^{-2[N/2]-2}}{1-q^{-2}}\right)\\
=&\frac{(q+1)q}{q^4-1}(-1+q^{2+4[N^+/2]+2[(N-N^+)/2]}-q^{2+4[N^+/2]}+q^{4[N/2]-2[(N-N^+)/2]})\end{aligned}$$

and

$$\delta(q+1)^2q^{N-2}\sum_{N^+/2<m\le N^+}q^{2m}=\delta\frac{q+1}{q-1}q^N(q^{2N^+}-q^{2[N^+/2]}).$$

When $\delta=0$, $2[(N-N^+)/2]=N-N^+-1$, and noting that N is even iff N^+ is odd, the asserted expression is obtained. When $\delta=1$, N is even iff so is N^+, hence $2[(N-N^+)/2]=N-N^+=2[N/2]-2[N^+/2]$, and again we obtain the asserted expression. □

E. Computations: $j = 0$**.**

To complete the computation of the orbital integral of 1_K at t_ρ, it remains to compute the contribution from the term indexed by $j = 0$, which exists only when $\overline{\rho} = 0$.

13. Proposition. *When $\overline{\rho} = 0 = j$, the non zero values of the integral $\int_{P_H/P_H\cap H_m^K} 1_{K_m^K}(p^{-1}t_\rho p)dp$ are: 1 if $m = 0$,*

(a) $(1-q^{-2})q^{4m}$ if $1 \le m \le \min([N/2],[N^+/2])$,

(b) $(1+q^{-1})q^{2m+2[N/2]}$ if $[N/2]+1 \le m \le \min(N,[M/2])$ (thus $N \le N^+$; recall: $M = \max(N_1,N_2)$),

(c) $(1+q^{-1})^2q^{2m+N}$ if $[M/2]+1 \le m \le N$ (thus $N \le N^+$) and $M-N$ is even,

(d) $(1+q^{-1})q^{2m+2[N/2]}$ if $N+1 \le m \le [M/2]$, and

(e) $(1+q^{-1})^2q^{2m+N}$ if $\max(N+1,[M/2]+1) \le m \le [(M+N)/2]$ and $M-N$ is even.

Proof. As in Proposition 10, we may assume that $m \ge 1$, and compute the volume of solutions in $u \in R_E^\times/1+\boldsymbol{\pi}^m R_E$ and $v \in R/\boldsymbol{\pi}^{2m}R$, $w = v\sqrt{D}$, of the equation (for some $a_2, a_3, b \in R_E$):

$$\begin{pmatrix} \frac{1}{2}(a+c) - wu\overline{u}B\boldsymbol{\pi}^N & u\overline{u}B\boldsymbol{\pi}^N((u\overline{u})^{-2} - Dv^2) \\ u\overline{u}B\boldsymbol{\pi}^N & \frac{1}{2}(a+c) + wu\overline{u}B\boldsymbol{\pi}^N \end{pmatrix} = \begin{pmatrix} 1-b+ta_2 & b-ta_2+tb_3+2a_3t^2 \\ b & 1-b-tb_3 \end{pmatrix}.$$

Consider first the case where $m > N$. Since the matrix on the right is congruent mod $\boldsymbol{\pi}^m$ to $\begin{pmatrix} 1-b & b \\ b & 1-b \end{pmatrix}$, considering the entries $(1,1)$ and $(2,2)$ of the equality, we get that $w = v\sqrt{D}$, $v = v_1\boldsymbol{\pi}^{m-N}$, $v_1 \in R$. The identities of the entries $(1,2)$ and $(2,1)$ imply that $u\overline{u} \equiv \pm 1(\boldsymbol{\pi}^{m-N})$. If $u\overline{u} \equiv 1(\boldsymbol{\pi}^{m-N})$, put $u\overline{u} = 1+\varepsilon'\boldsymbol{\pi}^{m-N}$. The matrix identity becomes four equations: $b = (a'-c')/2+\varepsilon' B\boldsymbol{\pi}^m$ (always solvable, defines b), $a_2 = a''+\varepsilon' B - B\sqrt{D}v_1u\overline{u}$ (is solvable precisely when $a'' = a'\boldsymbol{\pi}^{-m} \in R_E$, namely $m \le N_1$), $-b_3 = a''+\varepsilon' B + B\sqrt{D}v_1u\overline{u}$ (solvable when $m \le N_1$), and $2a' + B\boldsymbol{\pi}^N u\overline{u}(1+(u\overline{u})^{-2} - 2(u\overline{u})^{-1} - Dv_1^2\boldsymbol{\pi}^{2m-2N}) = 2a_3\boldsymbol{\pi}^{2m}$. Thus the 2nd and 3rd equations are solvable when $N < m \le N_1$ if $u\overline{u} \equiv 1$, and when $N < m \le N_2$ if $u\overline{u} \equiv -1$. Hence we are led to consider m in the range $N = N^+ = \min(N_1,N_2) < m \le M = \max(N_1,N_2)$. Defining $\varepsilon_1 \in R$ by $(u\overline{u})^{-1} = 1+\varepsilon_1\boldsymbol{\pi}^{m-N}$, the remaining, 4th equation, takes the form $2a''/B + (2a''/B)\varepsilon_1\boldsymbol{\pi}^{m-N} + \boldsymbol{\pi}^{m-N}(\varepsilon_1^2 - Dv_1^2) \in \boldsymbol{\pi}^m R_E$, or $2a''/B + \boldsymbol{\pi}^{m-N}((\varepsilon_1+a''/B)^2 - (a''/B)^2 - Dv_1^2) \in \boldsymbol{\pi}^m R_E$, and finally $(2a''/B)(1-(a''/2B)\boldsymbol{\pi}^{m-N}) + \boldsymbol{\pi}^{m-N}(\varepsilon^2 - Dv_1^2) \in \boldsymbol{\pi}^m R_E$, where $\varepsilon = \varepsilon_1 + a''/B$. Note that when $u\overline{u} \equiv -1$, a has to be replaced by c in these equations.

We claim that to have a solution, we must have $2m \le N+M$. Indeed, $\varepsilon^2 - Dv_1^2 \in R$. Put $\operatorname{Im} x = x - \overline{x}$ for $x \in R_E$. Recall that $a\overline{a} = 1 = c\overline{c}$. Then $\operatorname{Im}(a-1)/(a-c) = -a'c'/(a'-c') \in \boldsymbol{\pi}^M R_E^\times$, hence $\operatorname{Im}(a''/B) = \boldsymbol{\pi}^{N-m}\operatorname{Im}(a'/(a'-c')) \in \boldsymbol{\pi}^{M+N-m}R_E^\times$, and our equation will have no solution unless $M+N-m \ge m$. For such m we may regard a''/B as lying in R, rather then R_E. There are two subcases.

If $N < m \le M/2$, thus $m \le M-m$, our equation reduces to $\varepsilon^2 - Dv_1^2 \in \boldsymbol{\pi}^N R$. Then ε, $v_1 \in \boldsymbol{\pi}^{[(N+1)/2]}R$, thus $(u\overline{u})^{-1} = 1 + (\varepsilon - a''/B)\boldsymbol{\pi}^{m-N} \in 1 + \alpha\boldsymbol{\pi}^{M-N} + \boldsymbol{\pi}^{m-N+[(N+1)/2]}R$.

Let us compute the number of solutions u, v. First, note that for $0 < k \leq m$ we have

$$\#\{u \in R_E^\times/1+\boldsymbol{\pi}^m R_E\,;\ u\overline{u} \in 1+\boldsymbol{\pi}^k R\} = \frac{[R_E^\times : 1+\boldsymbol{\pi}^m R_E]}{[R^\times : 1+\boldsymbol{\pi}^m R]}[\boldsymbol{\pi}^k R : \boldsymbol{\pi}^m R] = (1+q^{-1})q^m \cdot q^{m-k}\,.$$

Hence

$$\#\{u \in R_E^\times/1+\boldsymbol{\pi}^m R_E\,;\ (u\overline{u})^{-1} \in 1+\alpha\boldsymbol{\pi}^{M-N}+\boldsymbol{\pi}^{m-N+[(N+1)/2]}R\} = (1+q^{-1})q^{m+N-[(N+1)/2]}\,.$$

Further,

$$\#\{v \in R/\pi^{2m}R;\ v = v_1\boldsymbol{\pi}^{m-N}\,,\ v_1 \in \pi^{[(N+1)/2]}R\,,\ \text{thus } v \in \boldsymbol{\pi}^{m-N+[(N+1)/2]}R\}$$

is $q^{m+N-[(N+1)/2]}$. Hence the number of solutions is $(1+q^{-1})q^{2m+2N-2[(N+1)/2]}$, as asserted in *case* (d) of the proposition.

If $M-m < m$, thus $2N$, $M < 2m \leq M+N$, we need to solve the equation $\varepsilon^2 - Dv_1^2 \in \alpha\boldsymbol{\pi}^{M+N-2m} + \boldsymbol{\pi}^N R = \alpha\boldsymbol{\pi}^{M+N-2m}(1+\boldsymbol{\pi}^{2m-M}R)$. Since $F(\sqrt{D})/F$ is unramified, there is a solution precisely when $M+N$ is even. Put $\varepsilon = \boldsymbol{\pi}^{\frac{1}{2}(M+N)-m}\varepsilon_2$, $v_1 = \boldsymbol{\pi}^{\frac{1}{2}(M+N)-m}v_2$. So we need to solve $\varepsilon_2^2 - Dv_2^2 \in 1+\boldsymbol{\pi}^{2m-M}R$. Namely we count the pairs

$$\{(u \in R_E^\times/1+\boldsymbol{\pi}^m R_E\,;\ v = v_1\boldsymbol{\pi}^{m-N} = \boldsymbol{\pi}^{(M-N)/2}v_2 \in R/\boldsymbol{\pi}^{2m}R)\}$$

such that $(u\overline{u})^{-1} = 1+\varepsilon_1\boldsymbol{\pi}^{m-N} = 1+(\varepsilon - a''/B)\boldsymbol{\pi}^{m-N} + \pi^{(M-N)/2}\varepsilon_2$ and $\varepsilon_2^2 - Dv_2^2 \in 1+\boldsymbol{\pi}^{2m-M}R$. The relation $\varepsilon_2^2 - Dv_2^2 \in 1+\boldsymbol{\pi}^{2m-M}R$ can be replaced by $\varepsilon_2^2 - Dv_2^2 \in R^\times$ if we multiply the cardinality by $[R^\times : 1+\boldsymbol{\pi}^{2m-M}R]^{-1}$, and it can be replaced by $\varepsilon_2 \in R$ and $v_2 \in R$ if we further multiply by the quotient $[R_E : R_E^\times]$ of the volume of R_E by that of $R_E^\times$. Then the number of u is $([R_E^\times : 1+\boldsymbol{\pi}^m R_E]/[R^\times : 1+\boldsymbol{\pi}^m R])[\boldsymbol{\pi}^{(M-N)/2}R : \boldsymbol{\pi}^m R]$, and the number of v is $[\boldsymbol{\pi}^{(M-N)/2}R : \boldsymbol{\pi}^{2m}R]$. The product is

$$\begin{aligned}
=&([R_E^\times : 1+\boldsymbol{\pi}^m R_E]/[R^\times : 1+\boldsymbol{\pi}^m R])[\boldsymbol{\pi}^{(M-N)/2}R : \boldsymbol{\pi}^m R]\\
&\cdot[\boldsymbol{\pi}^{(M-N)/2}R : \boldsymbol{\pi}^{2m}R][R_E : R_E^\times][R^\times : 1+\boldsymbol{\pi}^{2m-M}R]^{-1}\\
=&(1+q^{-1})q^m \cdot q^{m-(M-N)/2} \cdot q^{2m-(M-N)/2} \cdot (1-q^{-2}) \cdot ((1-q^{-1})q^{2m-M})^{-1}\\
=&(1+q^{-1})^2 q^{2m+N}\,.
\end{aligned}$$

This completes *case* (e) of the proposition.

It remains to consider $1 \leq m \leq N$. Then $\boldsymbol{\pi}^N \equiv 0\ (\boldsymbol{\pi}^m)$, thus $a'-c' \equiv 0\ (\boldsymbol{\pi}^m)$. Considering the entries (1,1) and (2,2) of our matrix identity, we get $(a+c)/2 \equiv 1(\boldsymbol{\pi}^m)$ (since $b \equiv 0\ (\boldsymbol{\pi}^m)$). Then $a'+c' \equiv 0\ (\boldsymbol{\pi}^m)$, and $a'' = a'\boldsymbol{\pi}^{-m}$, $c'' = c'\boldsymbol{\pi}^{-m} \in R_E$. Denoting $b' = b\boldsymbol{\pi}^{-m}$, $N' = N-m$, we see that the first three equations are always solvable: $b' = u\overline{u}B\boldsymbol{\pi}^{N'}$, $a_2 = (a''+c'')/2+u\overline{u}B\boldsymbol{\pi}^{N'}(1-w)$, $-b_3 = (a''+c'')/2+u\overline{u}B\boldsymbol{\pi}^{N'}(1+w)$ (these equations simply define b, a_2, b_3). The remaining equation is $a'+c'+\frac{1}{2}(a'-c')u\overline{u}(1+(u\overline{u})^{-2}-Dv^2) = 2a_3\boldsymbol{\pi}^{2m}$.

When $2m \le N, N^+$ every u, v makes a solution. This completes *case* (a) of the proposition. If $N^+ < N$, $2m$, then there are no solutions.

It remains to deal with the case where $N \le N^+$ and $N < 2m$. Put $\varepsilon = (u\overline{u})^{-1} \in R^\times$, $x = (a'+c')/(a'-c')$. We have to solve the equation $\varepsilon^2 + 1 - Dv^2 + 2\varepsilon x \in \boldsymbol{\pi}^{2m-N} R_E$. Note that $\mathrm{Im}(x) \in \boldsymbol{\pi}^{N_1+N_2-N} R_E^\times$. Since $N \le N^+$, we have $N = \min(N_1, N_2)$, and $2m \le 2N \le N_1 + N_2 = N + M$. Hence $\mathrm{Im}(x) \in \boldsymbol{\pi}^{2m-N} R_E$, and we may assume that $x \in R$. Thus we need to solve $(\varepsilon + x)^2 - Dv^2 \in x^2 - 1 + \boldsymbol{\pi}^{2m-N} R$, for a fixed $x \in \boldsymbol{\pi}^{N^+-N} R^\times \subset R$. Once we find a solution, in $\varepsilon \in R$, then $\varepsilon \in R^\times$; otherwise $\varepsilon \in \boldsymbol{\pi} R$, hence $Dv^2 \in 1 + \boldsymbol{\pi} R$, but $D \notin R^{\times 2}$. Note that $x \pm 1$ is $2a'/(a'-c')$ or $2c'/(a'-c')$, so $x^2 - 1 = 4a'c'/(a'-c')^2 \in \boldsymbol{\pi}^{N_1+N_2-2N} R_E^\times = \boldsymbol{\pi}^{M-N} R_E^\times$. We distinguish between two cases.

If $N/2 < m \le \min(N, [M/2])$ and $N \le N^+$, then $M - N \ge 2m - N > 0$, and we must have $N = N^+$ (thus $|x| = 1$). Thus we need to count the $\varepsilon = (u\overline{u})^{-1} \in -x + \boldsymbol{\pi}^{m-[N/2]} R$ and $v \in \boldsymbol{\pi}^{m-[N/2]} R / \boldsymbol{\pi}^{2m} R$. Then $\#\{u \in R_E^\times / 1 + \boldsymbol{\pi}^m R_E\,;\ u\overline{u} \in 1 + \boldsymbol{\pi}^{m-[N/2]} R\}$ is $(1+q^{-1}) q^{m+[N/2]}$, while the number of the v is $q^{m+[N/2]}$. This completes *case* (b) of the proposition.

If $M/2 < m \le N (\le N^+)$, thus $M - N < 2m - N$, we need to solve $(\varepsilon + x)^2 - Dv^2 \in \alpha \boldsymbol{\pi}^{M-N} + \boldsymbol{\pi}^{2m-N} R = \alpha \boldsymbol{\pi}^{M-N}(1 + \boldsymbol{\pi}^{2m-M} R)$ (for some $\alpha \in R^\times$). There is a solution precisely when $M - N$ is even (as $N R_E^\times = R^\times$). As noted above, given a solution, ε must be in $R^\times$. To compute the volume of solutions, fix measures with $\int_{R_E^\times} d^\times u = \int_{R^\times} d^\times \varepsilon$ and $d^\times \varepsilon = (1 - q^{-1})^{-1} d\varepsilon$ (thus $\int_{R^\times} d^\times \varepsilon = \int_R d\varepsilon$). Then the volume is

$$
\begin{aligned}
&(1-q^{-2}) q^{4m} \int_{u \in R_E^\times} \int_{v \in R} \delta(\{(u\overline{u} + x)^2 - Dv^2 \in \alpha \boldsymbol{\pi}^{M-N}(1 + \boldsymbol{\pi}^{2m-M} R)\}) d^\times u\, dv \\
&= (1-q^{-2}) q^{4m} (1-q^{-1})^{-1} \int_{\varepsilon \in R} \int_{v \in R} \delta(\{\varepsilon^2 - Dv^2 \in \boldsymbol{\pi}^{M-N} \alpha (1 + \boldsymbol{\pi}^{2m-M} R)\}) d\varepsilon\, dv \\
&= (1-q^{-2})(1-q^{-1})^{-1} q^{4m} q^{-(M-N)} \int_{z \in R_E} \delta(\{Nz \in 1 + \boldsymbol{\pi}^{2m-M} R\}) dz\,.
\end{aligned}
$$

The last integral ranges only over $R_E^\times$, and there $dz/|z| = (1-q^{-2}) d^\times z$. Now $\int_{R^\times} \delta(\{z \in 1 + \boldsymbol{\pi}^{2m-M} R\}) d^\times z$ is the inverse of

$$[R^\times : 1 + \boldsymbol{\pi}^{2m-M} R] = (1 - q^{-1}) q^{2m-M}\,.$$

Altogether we get $(1-q^{-2})^2 (1-q^{-1})^{-2} q^{4m+N-M-2m+M} = (1+q^{-1})^2 q^{2m+N}$, completing *case* (c), and the proposition.

An alternative volume computation is as follows. The cardinality of $\{(u \in R_E^\times / 1 + \boldsymbol{\pi}^m R_E\,,\ v \in R/\boldsymbol{\pi}^{2m} R)\,;\ (u\overline{u} + x)^2 - Dv^2 \in \alpha \boldsymbol{\pi}^{M-N}(1 + \boldsymbol{\pi}^{2m-M} R)\}$ is $(1+q^{-1}) q^m$ times $\#\{(\varepsilon \in R^\times / 1 + \boldsymbol{\pi}^m R,\ v \in \dots)\,;\ (\varepsilon + x)^2 - Dv^2 \in \dots\}$, and since ε must be in $R^\times$ to have a solution, this $\#$ is equal to $\#\{(\varepsilon \in R/\boldsymbol{\pi}^m R,\ v \in R/\boldsymbol{\pi}^{2m} R)\,;\ \varepsilon^2 - Dv^2 \in \alpha \boldsymbol{\pi}^{M-N}(1 + \cdots)\}$. As $\varepsilon = \varepsilon_1 \boldsymbol{\pi}^{(M-N)/2}$, $v = v_1 \boldsymbol{\pi}^{(M-N)/2}$, this product is

$$(1+q^{-1}) q^m \cdot q^{m-(M-N)/2} \cdot q^{2m-(M-N)/2} \cdot \mathrm{vol}\{z \in R_E\,;\ Nz \in 1 + \boldsymbol{\pi}^{2m-M} R\} = (1+q^{-1})^2 q^{2m+N}\,,$$

as required. □

14. Proposition. *When $\overline{\rho}=0$ the orbital integral $\int_{T_\rho\backslash G} 1_K(g^{-1}t_\rho g)dg$ is equal to*

$$-\frac{q+1}{q^4-1}(1+q^{2+4[N_1/2]})-\frac{(-q)^{N+N_1}}{q-1}+\delta(2\mid N+N^+)\frac{q+1}{q-1}q^{2N_1+N}\,, \qquad \text{if } N_1<N\,,$$

in which case $N^+=N_1=N_2$, and to

$$-\frac{q+1}{q^4-1}(1+q^{2+4[N/2]})-\frac{(-q)^{M+N}}{q-1}+\delta(2\mid M-N)\frac{q+1}{q-1}q^{2N+M}\,, \qquad \text{if } N\le N_1\,.$$

Proof. It suffices to prove this with N_1 replaced by N^+, as $N_1<N$ precisely when $N^+<N$, in which case $N^+=N_1$. If $N^+<N$, $j=0$ contributes

$$1+\sum_{1\le m\le \min([N/2],[N^+/2])}(1-q^{-2})q^{4m}=\frac{q^2-1}{q^4-1}(1+q^{2+4[N^+/2]})\,.$$

The $j>0$ contributes, when $\delta=0$, thus $N+N^+$ is odd, the expression:

$$-\frac{q^2+q}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{q^{N+N^+}}{q-1}\,,$$

while when $\delta=1$, thus $N+N^+$ is even, the $j>0$ contribute to the orbital integral:

$$-\frac{q^2+q}{q^4-1}(1+q^{2+4[N^+/2]})+\frac{1}{q-1}(q^{1+2[N^+/2]+2[N/2]}+(q+1)q^{N+2N^+}-(q+1)q^{N+2[N^+/2]})\,.$$

The sum is as stated in the proposition.

If $N\le N^+$, the sum is (when $M/2<N$ and also when $M/2\ge N$)

$$\begin{aligned}
&\frac{q^2+q}{q^4-1}(q^{4[N/2]}-1)+1+q^2(q^2-1)\sum_{0\le m<[N/2]}q^{4m}+(1+q^{-1})q^{2[N/2]}\sum_{[N/2]+1\le m\le[M/2]}q^{2m}\\
&\qquad+\delta(2\mid M-N)(1+q^{-1})^2q^N\sum_{[M/2]+1\le m\le[(M+N)/2]}q^{2m}\\
&=-\frac{q+1}{q^4-1}+\frac{q^4+q}{q^4-1}q^{4[N/2]}+q^{2[N/2]+1}\cdot\frac{q^{2[M/2]}-q^{2[N/2]}}{q-1}+\delta\frac{q+1}{q-1}q^N(q^{M+N}-q^{2[M/2]})\,,
\end{aligned}$$

which is easily seen to be the expression of the proposition (consider separately the cases of even ($\delta=1$) and odd ($\delta=0$) values of $M-N$). □

F. Conclusion.

Put $\Phi(t)=\int_{Z(t)\backslash G}1_K(g^{-1}tg)dg$. In the notations of Proposition 3 for anisotropic tori which split over E, the κ-orbital integral is $\Phi^\kappa_{1_K}(t_0)=\Phi(t_1)+\Phi(t_2)-\Phi(t_3)-\Phi(t_4)$. The tori $T_1=Z(t_1)$ and $T_2=Z(t_2)$ ($Z(t)$ is the centralizer of t in G) embed as tori in H. Denote by K_H the maximal compact subgroup $H\cap K$ of H, by 1_{K_H} its characteristic function in H, choose on H the Haar measure which assigns K_H the volume 1, introduce the stable orbital integral $\Phi^{st}_{1_{K_H}}(t_0)=\Phi^H(t_1)+\Phi^H(t_2)$, where $\Phi^H(t)=\int_{Z_H(t)\backslash H}1_{K_H}(h^{-1}th)dh$ and $Z_H(t)$ is the centralizer in H of a regular t in H. It is well known (see, e.g., [F1], Proposition 5) that $\Phi^{st}_{1_{K_H}}(t_0)=(q^N(q+1)-2)/(q-1)$ (where E/F is unramified).

Remark. A proof of the last equality – extracted from Mars' letter mentioned in the Remark following the proof of Proposition 6 – is as follows. Thus $G=GL(V)$ and $K=\mathrm{Stab}(R_E)$, dg on G assigns K the volume 1, dt on $E^\times$ assigns $R_E^\times$ the volumes 1, and $\gamma\in E^\times-F^\times$. Then $\int_{E^\times\backslash G}1_K(g^{-1}\gamma g)dg/dt$ is $\sum_{E^\times\backslash G/K}|K|/|E^\times\cap gKg^{-1}|1_K(g^{-1}\gamma g)$. But $E^\times\backslash G/K$ is the set of $E^\times$-orbits on the set of all lattices in E. Representatives are the lattices $R_E(j)$, $j\geq 0$. So our sum is the sum of $|R_E^\times|/|R_E(j)^\times|=[R_E^\times:R_E(j)^\times]$ over the $j\geq 0$ such that $\gamma\in R_E(j)^\times$. As $[R_E^\times:R_E(j)^\times]$ is 1 if $j=0$ and $q^{j+1-f}(q^f-1)/(q-1)$ if $j>0$, putting N for the maximum of the j with $\gamma\in R_E(j)^\times$, the integral equals $(q^N(q+1)-2)/(q-1)$ if $e=1$, and $(q^{N+1}-1)/(q-1)$ if $e=2$ ($ef=2$). Of course, the integral vanishes for γ not in $R_E^\times$. If $\gamma=a+bw\in R_E^\times$, then N is the order of b. Note that the stable orbital integral on the unitary group H in two variables is just the orbital integral on $GL(2)$.

Put $\Delta_{G/H}(t_0)=(-q)^{-N_1-N_2}$. The fundamental lemma is the following.

15. Theorem. *For a regular t_0 we have $\Delta_{G/H}(t_0)\Phi^\kappa_{1_K}(t_0)=\Phi^{st}_{1_{K_H}}(t_0)$.*

Proof. Note that $\Phi(t_2)$ depends only on N_1, N_2, N, so we write $\Phi(t_2)=\varphi(N_1,N_2,N)$, and so $\Phi(t_3)=\varphi(N,N_2,N_1)$ and $\Phi(t_4)=\varphi(N_1,N,N_2)$. If $N=N_2<N_1$, $\Phi(t_2)=\Phi(t_4)$, hence $\Phi^K(t_0)=\Phi(t_1)-\Phi(t_3)$, and this difference is

$$-\frac{2}{q-1}(-q)^{N_2+N_1}+(\delta(2\mid N_1-N_2)-\delta(2\mid N_1-1-N_2))\frac{q+1}{q-1}q^{N_1+2N_2}\,,$$

as required.

If $N=N_1\leq N_2$, $\Phi(t_2)=\Phi(t_3)$, hence $\Phi^\kappa(t_0)=\Phi(t_1)-\Phi(t_4)$, and this difference is

$$-\frac{2}{q-1}(-q)^{N_1+N_2}+(\delta(2\mid N_2-N_1)-\delta(2\mid N_2-1-N_1))\frac{q+1}{q-1}q^{N_2+2N_1}\,,$$

as required.

If $N_1=N_2<N$, $\Phi^\kappa(t_0)$ is the sum of

$$\Phi(t_1)=-\frac{q+1}{q^4-1}(1+q^{2+4[N_1/2]})-\frac{(-q)^{N+N_1}}{q-1}+\delta(2\mid N+N_1)\frac{q+1}{q-1}q^{N+2N_1}\,,$$

$$\Phi(t_2)=-\frac{q+1}{q^4-1}(1+q^{2+4[N_1/2]})+\frac{(-q)^{N+N_1}}{q-1}+\delta(2\mid N-1-N_1)\frac{q+1}{q-1}q^{N+2N_1}\,,$$

and

$$-\Phi(t_3)-\Phi(t_4)=-2\frac{q+1}{q^4-1}(q^{4[N_1+2)/2]}-1)\,.$$

This sum is $-\frac{2q^{2N_1}}{q-1}+\frac{q+1}{q-1}q^{N+2N_1}$, as required.

Since the two minimal numbers among N_1, N_2, N are equal, we are done. □

We now turn to the ramified case. It remains to deal with regular t' in the torus $T_{H'}\subset H'\subset G'$ of Proposition 3.

16. Proposition. *The integral $\int_{H'/H'_m}1_{H'_m}(h^{-1}t'h)dh$ of Proposition 5 is equal to $(q+1)q^{4m}$ if $0\le m\le\min([N/2],[N_2/2])$, and to $(q+1)q^{N+2m}$ if $N\le N_2$ and $[N/2]<m\le N$. Here $t'=\operatorname{diag}(\delta^{-1}\left(\begin{smallmatrix}\alpha & \beta\boldsymbol{\pi}\\ \beta & \alpha\end{smallmatrix}\right),1)$, $\delta\overline{\delta}=\alpha^2-\boldsymbol{\pi}\beta^2=1$, $\beta=B\boldsymbol{\pi}^N$ and $\delta=\delta_1+\delta_2\sqrt{D}$, $\delta_2=D_2\boldsymbol{\pi}^{1+N_2}$, and B, D_2, δ_1, $\alpha\in R^\times$.*

Proof. We need to compute the number of $c\in R_E/\boldsymbol{\pi}^mR_E$, and $a\in R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E$, for which

$$\left(\begin{smallmatrix}\overline{a} & -c\boldsymbol{\pi}\\ -\overline{cu} & a\overline{u}\end{smallmatrix}\right)\overline{\delta}\left(\begin{smallmatrix}\alpha & \beta\boldsymbol{\pi}\\ \beta & \alpha\end{smallmatrix}\right)\left(\begin{smallmatrix}a & uc\boldsymbol{\pi}\\ \overline{c} & u\overline{a}\end{smallmatrix}\right)=\overline{\delta}\left(\begin{smallmatrix}\alpha+\boldsymbol{\pi}\beta(\overline{ac}-ac) & \beta\boldsymbol{\pi}u(\overline{a}^2-\boldsymbol{\pi}c^2)\\ a^2\beta\overline{u}-\boldsymbol{\pi}\beta\overline{c}^2\overline{u} & \alpha+\boldsymbol{\pi}\beta(ac-\overline{ac})\end{smallmatrix}\right)$$

lies in H'_m. Using the description of H'_m in Proposition 4, this is equivalent to solving two equations: $|\beta(a^2-\boldsymbol{\pi}\overline{c}^2)|\le|\boldsymbol{\pi}|^m$, which means $0\le m\le N$ since $a\in R_E^\times$, $c\in R_E$, $\beta\in\boldsymbol{\pi}^NR^\times$ (note that there is no constraint on $u\in E^1$, and the volume of E^1 is 1), and $|\alpha+\boldsymbol{\pi}\beta(\overline{ac}-ac)-\delta|\le|\boldsymbol{\pi}|^{1+2m}$. Replacing c by c/a, the equations simplify to $a\overline{a}-\boldsymbol{\pi}c\overline{c}/a\overline{a}=1$, and $|\alpha+\boldsymbol{\pi}\beta(\overline{c}-c)-\delta|\le|\boldsymbol{\pi}|^{1+2m}$. The last equation implies $\alpha-\delta_1\in\boldsymbol{\pi}^{1+2m}R$. Since $\alpha^2=1+B^2\boldsymbol{\pi}^{1+2N}$, and $1=\delta\overline{\delta}=\delta_1^2-D\delta_2^2$, we conclude that $\delta_2^2\in\boldsymbol{\pi}^{1+2m}R$, hence $\delta_2=D_2\boldsymbol{\pi}^{1+N_2}\in\boldsymbol{\pi}^{1+m}R$, and $m\le N_2$. Put $c=c_1+c_2i$, $i=\sqrt{D}$, $\overline{c}-c=-2ic_2$, $c_2=C_2\boldsymbol{\pi}^{n_2}$ ($C_2\in R^\times$). Then our equation becomes $-2BC_2\boldsymbol{\pi}^{N+n_2}-D_2\boldsymbol{\pi}^{N_2}\in\boldsymbol{\pi}^{2m}R$.

We shall now determine the number of c. If $0\le m\le[N/2]$, then $2m\le N$, hence $2m\le N_2$ (if there are solutions to our equation), namely $m\le[N_2/2]$, and any (C_2 and) c is a solution. The number of c is $\#R_E/\boldsymbol{\pi}^mR_E=q^{2m}$. If $[N/2]<m\le N$, thus $m\le N<2m$, we consider two subcases. If $m\le[N_2/2]$, or $2m\le N_2$, then $N<N_2$, and there are solutions C_2 precisely when $n_2\ge 2m-N$, and any C_2 is a solution. Then $c_2=C_2\boldsymbol{\pi}^{n_2}\in\boldsymbol{\pi}^{2m-N}R/\boldsymbol{\pi}^mR\simeq R/\boldsymbol{\pi}^{N-m}R$ has q^{N-m} possibilities, $c_1\in R/\boldsymbol{\pi}^mR$ has q^m, and $\#c=q^N$. If $m>[N_2/2]$, or $N_2<2m$, there are solutions only when $n_2=N_2-N$ ($n_2\ge 0$ implies $N\le N_2$), and the solutions are given by $C_2\in -D_2/2B+\boldsymbol{\pi}^{2m-N_2}R$, and again c_2 is determined modulo $\boldsymbol{\pi}^{n_2}\boldsymbol{\pi}^{2m-N_2}R/\boldsymbol{\pi}^mR=R/\boldsymbol{\pi}^{N-m}R$.

Given $c\in R_E/\boldsymbol{\pi}^mR_E$, we need to solve in $a\in R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E$ the equation $(a\overline{a})^2-a\overline{a}+1/4=1/4-\boldsymbol{\pi}c\overline{c}$, namely $(a\overline{a}-1/2)^2=(1-2\boldsymbol{\pi}c\overline{c}+\dots)^2/4$, or $a\overline{a}=1/2\pm(1-2\boldsymbol{\pi}c\overline{c}+\dots)/2$. There are no solutions for the negative sign, and there exists a solution for the positive sign. The number of $a\in R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E$ with $a\overline{a}\in v+\boldsymbol{\pi}^{1+2m}R$ ($v\in R^\times$) is $\#(R_E^\times/1+\boldsymbol{\pi}^{1+2m}R_E)/\#(R^\times/1+\boldsymbol{\pi}^{1+2m}R)=((q^2-1)q^{2\cdot 2m}/(q-1)q^{2m})=(q+1)q^{2m}$, as asserted. □

17. Proposition. *The last orbital integral of Proposition 5, of* 1_K *at a regular* $t = gt'g^{-1} \in G$*, where* $t' \in T_{H'} \subset H' \subset G'$*, is*

$$(q^{4+4\min} - 1)/((q^2+1)(q-1)) + \delta(N \le N_2)q^N(q^{2N+2} - q^{2[N/2]+2})/(q-1).$$

Here $\min = \min([N/2], [N_2/2])$*, and* N*,* N_2 *are defined in Proposition 16.*

Proof. The integral is equal to

$$\sum_{0 \le m \le \min} (q+1)q^{4m} + \delta(N \le N_2) \sum_{[N/2] < m \le N} (q+1)q^{N+2m},$$

which is equal to the asserted expressions. □

The κ-orbital integral $\Phi^{\kappa}_{1_K}(t)$ of 1_K on the stable conjugacy class of a regular $t \in T_H \subset H \subset G$ is the difference of $\Phi(t) = \int_{T_H \backslash G} 1_K(x^{-1}tx)dx$ and $\Phi'(t) = \int_{Z_G(t'') \backslash G} 1_K(x^{-1}t''x)dx$, where $t'' = gt'g^{-1} \in G$ is stably conjugate to t (and $t' \in T_{H'} \subset H' \subset G' = g^{-1}Gg$). The stable conjugacy class of t in H consists of a single conjugacy class, and it is well known (see Remark before Theorem 15) that $\Phi^{st}_{1_{K_H}}(t) = \Phi^H(t) = (q^N - 1)/(q-1)$, where N is defined in Proposition 16. The transfer factor $\Delta_{G/H}(t)$ is $(-q)^{-n}$, where if $t = (t_1, 1) \in (EL)^1 \times E^1$, the n is defined by $t_1 - 1 \in \boldsymbol{\pi}^n_{EL} R^\times_{EL}$.

18. Theorem. *For a regular* t *we have* $\Delta_{G/H}(t)\Phi^{\kappa}_{1_K}(t) = \Phi^{st}_{1_{K_H}}(t)$.

Proof. Since $t = (\alpha + \beta\sqrt{\boldsymbol{\pi}})(\delta_1 - i\delta_2)$ is $(1 + B^2\boldsymbol{\pi}^{1+2N}/2 + \cdots + B\sqrt{\boldsymbol{\pi}}\boldsymbol{\pi}^N)$ times $(1 + DD_2^2\boldsymbol{\pi}^{2+2N_2} + \cdots - \sqrt{D}D_2\boldsymbol{\pi}^{1+N_2})$, namely $1 + B\boldsymbol{\pi}^{N+1/2} - \sqrt{D}D_2\boldsymbol{\pi}^{1+N_2} + \dots$, we have that $n = \min(1+2N, 2+2N_2)$. If $N \le N_2$, we then need to show that $\Phi^{\kappa}_{1_K}(t) = -q^{1+2N}(q^{N+1} - 1)/(q-1)$. When $N_2 < N$, we have to show that $\Phi^{\kappa}_{1_K}(t) = q^{2+2N_2}(q^{N+1} - 1)/(q-1)$. Proposition 11 gives an explicit expression for $\Phi(t)$. Proposition 17 gives an explicit expression for $\Phi'(t)$. The difference, $\Phi^{\kappa}_{1_K}(t)$, is easily seen to be equal to $\Phi^H(t)$. □

Remark. Reference [FH] is missing in [F1]; it is supplied below.

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