MOOCULUS

massive open online calculus

sequences and series

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This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. David Guichard's text is available at http://www.whitman.edu/mathematics/calculus/ under a Creative Commons license.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at http://www.math.wisc.edu/~keisler/calc.html under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material. Thanks to Walter Nugent, Nicholas J. Roux, Dan Dimmitt, Donald Wayne Fincher, Nathalie Dalpé, chas, Clark Archer, Sarah Smith, MithrandirAgain, AlmeCap, and hrzhu for proofreading and contributing on GitHub.

This book is typeset in the Kerkis font, Kerkis © Department of Mathematics, University of the Aegean.

We will be glad to receive corrections and suggestions for improvement at fowler@math.osu.edu or snapp@math.osu.edu.

Contents

How to ~~read~~ do mathematics

Reading mathematics is **not** the same as reading a novel—it's more fun, and more interactive! To read mathematics you need

(a) a pen,

(b) plenty of blank paper, and

(c) the courage to write down everything—even "obvious" things.

As you read a math book, you work along with me, the author, trying to anticipate my next thoughts, repeating many of the same calculations I did to write this book. You must **write** down each expression, **sketch** each graph, and constantly **think** about what you are doing. You should work examples. You should fill-in the details I left out. This is not an easy task; it is **hard** work, but, work that is, I very much hope, rewarding in the end.

Mathematics is not a passive endeavor. I may call you a "reader" but you are not reading; you are writing this book for yourself.

—the so-called "author"

Acknowledgments

This text is a modification of David Guichard's open-source calculus text which was itself a modification of notes written by Neal Koblitz at the University of Washington and includes exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals* by H. Jerome Keisler. I am grateful to David Guichard for choosing a Creative Commons license. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material. The stylesheet, based on `tufte-latex`, was designed by Bart Snapp.

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—Jim Fowler

Introduction, or. . . what is this all about?

Consider the following sum:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}+\cdots$$

The dots at the end indicate that the sum goes on forever. Does this make sense? Can we assign a numerical value to an infinite sum? While at first it may seem difficult or impossible, we have certainly done something similar when we talked about one quantity getting "closer and closer" to a fixed quantity. Here we could ask whether, as we add more and more terms, the sum gets closer and closer to some fixed value. That is, look at

$$\begin{aligned}\frac{1}{2}&=\frac{1}{2}\\ \frac{3}{4}&=\frac{1}{2}+\frac{1}{4}\\ \frac{7}{8}&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\\ \frac{15}{16}&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\end{aligned}$$

and so on, and ask whether these values have a limit. They do; the limit is 1. In fact, as we will see,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}=\frac{2^i-1}{2^i}=1-\frac{1}{2^i}$$

and then

$$\lim_{i\to\infty}1-\frac{1}{2^i}=1-0=1.$$

This is less ridiculous than it appears at first. In fact, you might already believe that

$$0.3333\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \frac{1}{3},$$

which is similar to the sum above, except with powers of ten instead of powers of two. And this sort of thinking is needed to make sense of numbers like π, considering

$$3.14159\ldots = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \cdots = \pi.$$

Before we investigate infinite sums—usually called **series**—we will investigate limits of **sequences** of numbers. That is, we officially call

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^i} + \cdots$$

a series, while

$$\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{7}{8}, \quad \frac{15}{16}, \quad \ldots, \quad \frac{2^i - 1}{2^i}, \quad \ldots$$

is a sequence. The value of a series is the limit of a particular sequence, that is,

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{i\to\infty} \frac{2^i - 1}{2^i}.$$

If this all seems too obvious, let me assure you that there are twists and turns aplenty. And if this all seems too complicated, let me assure you that we'll be going over this in much greater detail in the coming weeks. In either case, I hope that you'll join us on our journey.

1 Sequences

1.1 Notation

A "sequence" of numbers is just a list of numbers. For example, here is a list of numbers:

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad \ldots$$

Note that numbers in the list can repeat. And consider those little dots at the end! The dots "..." signify that the list keeps going, and going, and going—forever. Presumably the sequence continues by following the pattern that the first few "terms" suggest. But what's that pattern?

To make this talk of "patterns" less ambiguous, it is useful to think of a sequence as a function. We have up until now dealt with functions whose domains are the real numbers, or a subset of the real numbers, like $f(x) = \sin(1/x)$.

A real-valued function with domain the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a **sequence**.

Other functions will also be regarded as sequences: the domain might include 0 alongside the positive integers, meaning that the domain is the non-negative integers, $\mathbb{Z}^{\geq 0} = \{0, 1, 2, 3, \ldots\}$. The range of the function is still allowed to be the real numbers; in symbols, the function $f \colon \mathbb{N} \to \mathbb{R}$ is a sequence.

Sequences are written down in a few different, but equivalent, ways; you might

Maybe you are feeling that this formality is unnecessary, or even ridiculous; why can't we just list off a few terms and pick up on the pattern intuitively? As we'll see later, that might be very hard—nay, impossible—to do! There might be very different—but equally reasonable—patterns that start the same way.

To resolve this ambiguity, it is perhaps not so ridiculous to introduce the formalism of "functions." Functions provide a nice language for associating numbers (terms) to other numbers (indices).

see a sequence written as

$$
\begin{aligned}
&a_1, \quad a_2, \quad a_3, \quad \ldots, \\
&a_n \\
&(a_n)_{n\in\mathbb{N}}\,, \\
&\{a_n\}_{n=1}^{\infty}\,, \\
&\{f(n)\}_{n=1}^{\infty}\,, \quad \text{or} \\
&(f(n))_{n\in\mathbb{N}}\,,
\end{aligned}
$$

depending on which author you read. Worse, depending on the situation, the same author (and this author) might use various notations for a sequence! In this textbook, I will usually write (a_n) if I want to speak of the sequence as a whole (think *gestalt*) and I will write a_n if I am speaking of a specific term in the sequence.

Let's summarize the preceding discussion in the following definition.

Definition A **sequence** (a_n) is, formally speaking, a real-valued function with domain

$$\{n \in \mathbb{Z} : n \geq N\}, \quad \text{for some integer } N.$$

Stated more humbly, a sequence assigns a real number to the integers starting with an index N.

The "outputs" of a sequence are the **terms** of the sequence; the "n^{th} term" is the real number that the sequence associates to the natural number n, and is usually written a_n. The n in the phrase "n^{th} term" is called an **index**; the plural of index is either indices or indexes, depending on who you ask. The first index N is called the **initial index**.

Recall that the natural numbers $\mathbb{N}$ are the counting numbers $1, 2, 3, 4, \ldots$. If we want our sequence to start at zero, we use $\mathbb{Z}^{\geq 0}$ as the domain instead. The fancy symbols $\mathbb{Z}^{\geq 0}$ refer to the non-negative integers, which include zero (since zero is neither positive nor negative) and also positive integers (since they certainly aren't negative).

To confuse matters further, some people—especially computer scientists—might include zero in the natural numbers $\mathbb{N}$. Mathematics is cultural.

Warning Usually the "domain" of a sequence is $\mathbb{N}$ and $\mathbb{Z}^{\geq 0}$. But depending on the context, it may be convenient for a sequence to start somewhere else—perhaps with some negative number. We shouldn't let the usual situation of $\mathbb{N}$ or $\mathbb{Z}^{\geq 0}$ get in the way of making the best choice for the problem at hand.

As you can tell, there is a deep tension between precise definition and a vague flexibility; as instructors, how we navigate that tension will be a big part of whether we are successful in teaching the course. We need to invoke precision when we're tempted to be too vague, and we need to reach for an extra helping of vagueness when the formalism is getting in the way of our understanding. It can be a tough balance.

1.2 Defining sequences

1.2.1 Defining sequences by giving a rule

Just as real-valued functions from Calculus One were usually expressed by a formula, we will most often encounter sequences that can be expressed by a formula. In the Introduction to this textbook, we saw the sequence given by the rule $a_i = f(i) = 1 - 1/2^i$. Other examples are easy to cook up, like

$$a_i = \frac{i}{i+1},$$
$$b_n = \frac{1}{2^n},$$
$$c_n = \sin(n\pi/6), \text{ or}$$
$$d_i = \frac{(i-1)(i+2)}{2^i}.$$

Frequently these formulas will make sense if thought of either as functions with domain $\mathbb{R}$ or $\mathbb{N}$, though occasionally the given formula will make sense only for integers. We'll address the idea of a real-valued function "filling in" the gaps between the terms of a sequence when we look at graphs in Section 1.5.

Warning A common misconception is to confuse the sequence with the rule for generating the sequence. The sequences (a_n) and (b_n) given by the rules $a_n = (-1)^n$ and $b_n = \cos(\pi n)$ are, despite appearances, different rules which give rise to the *same* sequence. These are just different names for the same object.

Let's give a precise definition for "the same" when speaking of sequences.

Compare this to equality for functions: two functions are the same if they have same domain and codomain, and they assign the same value to each point in the domain.

Definition Suppose (a_n) and (b_n) are sequences starting at 1. These sequences are **equal** if for all natural numbers n, we have $a_n = b_n$.

More generally, two sequences (a_n) and (b_n) are **equal** if they have the same initial index N, and for every integer $n \geq N$, the n^{th} terms have the same value, that is,

$$a_n = b_n \quad \text{for all } n \geq N.$$

In other words, sequences are the same if they have the same set of valid indexes, and produce the same real numbers for each of those indexes—regardless of whether the given "rules" or procedures for computing those sequences resemble each other in any way.

1.2.2 Defining sequences using previous terms

You might be familiar with *recursion* from a computer science course.

Another way to define a sequence is *recursively*, that is, by defining the later outputs in terms of previous outputs. We start by defining the first few terms of the sequence, and then describe how later terms are computed in terms of previous terms.

Example 1.2.1 Define a sequence recursively by

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 10,$$

and the rule that $a_n = a_{n-1} - a_{n-3}$. Compute a_5.

Solution First we compute a_4. Substituting 4 for n in the rule $a_n = a_{n-1} - a_{n-3}$, we find

$$a_4 = a_{4-1} - a_{4-3} = a_3 - a_1.$$

But we have values for a_3 and a_1, namely 10 and 1, respectively. Therefore $a_4 = 10 - 1 = 9$.

Now we are in a position to compute a_5. Substituting 5 for n in the rule

$a_n = a_{n-1} - a_{n-3}$, we find

$$a_5 = a_{5-1} - a_{5-3} = a_4 - a_2.$$

We just computed $a_4 = 9$; we were given $a_2 = 3$. Therefore $a_5 = 9 - 3 = 6$.

You can imagine some very complicated sequences defined recursively. Make up your own sequence and share it with your friends! Use the hashtag #sequence.

1.3 Examples

Tons of entertaining sequences are listed in the The On-Line Encyclopedia of Integer Sequences.

Mathematics proceeds, in part, by finding precise statements for everyday concepts. We have already done this for sequences when we found a precise definition ("function from $\mathbb{N}$ to $\mathbb{R}$") for the everyday concept of "a list of real numbers." But all the formalisms in the world aren't worth the paper they are printed on if there aren't some interesting *examples* of those precise concepts. Indeed, mathematics proceeds not only by generalizing and formalizing, but also by focusing on specific, concrete instances. So let me share some specific examples of sequences.

But before I can share these examples, let me address a question: how can I hand you an example of a sequence? It is not enough just to list off the first few terms. Let's see why.

Example 1.3.1 Consider the sequence (a_n)

$$a_1 = 41, \quad a_2 = 43, \quad a_3 = 47, \quad a_4 = 53, \quad \ldots$$

What is the next term a_5? Can you identify the sequence?

Solution In spite of many so-called "intelligence tests" that ask questions just like this, this question simply doesn't have an answer. Or worse, it has too many answers!

This particular polynomial $n^2 - n + 41$ is rather interesting, since it outputs many prime numbers. You can read more about it at the OEIS.

This sequence might be "the prime numbers in order, starting at 41." If that's the case, then the next term is $a_5 = 59$. But maybe this sequence is the sequence given by the polynomial $a_n = n^2 - n + 41$. If that's the case, then the next term is $a_5 = 61$. Who is to say which is the "better" answer?

Recall that a **prime number** is an integer greater than one that has no positive divisors besides itself and one.

Now let's consider two popular "families" of sequences.

1.3.1 Arithmetic sequences

The first family[1] we consider are the "arithmetic" sequences. Here is a definition.

[1] Mathematically, the word **family** does not have an entirely precise definition; a family of things is a **collection** or a **set** of things, but family also has a connotation of some sort of relatedness.

Definition An **arithmetic progression** (sometimes called an arithmetic sequence) is a sequence where each term differs from the next by the same, fixed quantity.

Example 1.3.2 An example of an arithmetic progression is the sequence (a_n) which begins

$$a_1 = 10, \quad a_2 = 14, \quad a_3 = 18, \quad a_4 = 22, \quad \ldots$$

and which is given by the rule $a_n = 6 + 4n$. Each term differs from the previous by four.

In general, an arithmetic progression in which subsequent terms differ by m can be written as

$$a_n = m(n-1) + a_1.$$

Alternatively, we could describe an arithmetic progression recursively, by giving a starting value a_1, and using the rule that $a_n = a_{n-1} + m$.

Why are arithmetic progressions called *arithmetic?* Note that every term is the **arithmetic mean**, that is, the **average**, of its two neighbors.

An arithmetic progression can decrease; for instance,

$$17, \quad 15, \quad 13, \quad 11, \quad 9, \quad \ldots$$

is an arithmetic progression.

1.3.2 Geometric sequences

The second family we consider are geometric progressions.

Definition A **geometric progression** (sometimes called a geometric sequence) is a sequence where the ratio between subsequent terms is the same, fixed quantity.

Example 1.3.3 An example of a geometric progression is the sequence (a_n) starting

$$a_1 = 10, \quad a_2 = 30, \quad a_3 = 90, \quad a_4 = 270, \quad \ldots$$

and given by the rule $a_n = 10 \cdot 3^{n-1}$. Each term is three times the preceding term.

In general, a geometric progression in which the ratio between subsequent terms is r can be written as

$$a_n = a_1 \cdot r^{n-1}.$$

Alternatively, we could describe a geometric progression recursively, by giving a starting value a_1, and using the rule that $a_n = r \cdot a_{n-1}$.

A geometric progression needn't be increasing. For instance, in the following geometric progression

$$\frac{7}{5}, \quad \frac{7}{10}, \quad \frac{7}{20}, \quad \frac{7}{40}, \quad \frac{7}{80}, \quad \frac{7}{160}, \quad \ldots$$

the ratio between subsequent terms is one half, and each term is smaller than the previous.

Why are geometric progressions called *geometric?* Note that every term is the **geometric mean** of its two neighbors. The geometric mean of two numbers a and b is defined to be $\sqrt{ab}$.

Of course, that raises another question: why is the geometric mean called *geometric?* One geometric interpretation of the geometric mean of a and b is this: the geometric mean is the side length of a square whose area is equal to that of the rectangle having side lengths a and b.

1.3.3 Triangular numbers

The sequence of **triangular numbers** (T_n) is a sequence of integers counting the number of dots in increasingly large "equilateral triangles" built from dots. The term T_n is the number of dots in a triangle with n dots to a side.

There are a couple of ways of making this discussion more precise. Given an equilateral triangle with n dots to a side, how many more dots do you need to build the equilateral triangle with $n + 1$ dots to a side? All you need to do to transform the smaller triangle to the larger triangle is an additional row of $n + 1$ dots placed

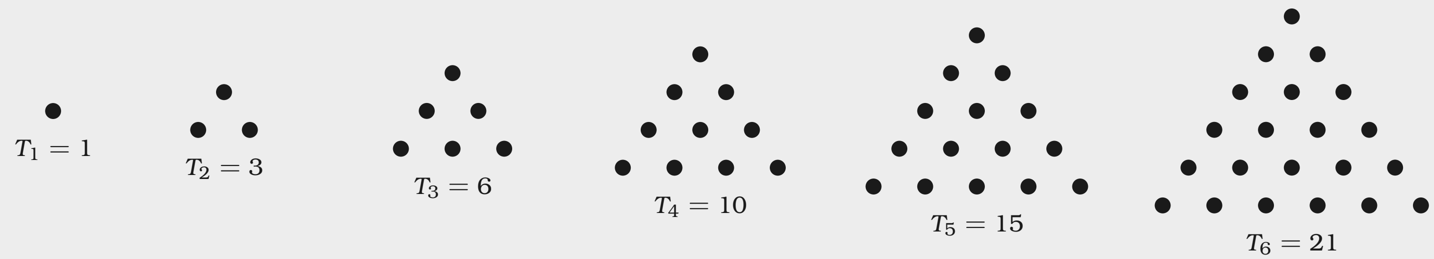

Figure 1.1: The first six triangular numbers

along any side. Therefore,

$$T_{n+1} = T_n + (n+1).$$

Since $T_1 = 1$, this recursive definition suffices to determine the whole sequence.

But there are other ways of computing T_n. Indeed, you may recall the explicit formula

$$T_n = \frac{n \cdot (n+1)}{2}$$

from Calculus One.

1.3.4 Fibonacci numbers

The Fibonacci numbers are interesting enough that a journal, The Fibonacci Quarterly is published four times yearly entirely on topics related to the Fibonacci numbers.

The **Fibonacci numbers** are defined recursively, starting with

$$F_0 = 0 \text{ and } F_1 = 1$$

and the rule that $F_n = F_{n-1} + F_{n-2}$. We can restate this formula in words, instead of symbols; stated in words, each term is the sum of the previous two terms. So the sequence of Fibonacci numbers begins

$$0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad \ldots$$

and continues.

This is certainly not the last time we will see the Fibonacci numbers.

1.3.5 Collatz sequence

Here is a fun sequence with which to amuse your friends—or distract your enemies. Let's start our sequence with $a_1 = 6$. Subsequent terms are defined using the rule

$$a_n = \begin{cases} a_{n-1}/2 & \text{if } a_{n-1} \text{ is even, and} \\ 3\,a_{n-1} + 1 & \text{if } a_{n-1} \text{ is odd.} \end{cases}$$

Let's compute a_2. Since a_1 is even, we follow the instructions in the first line, to find that $a_2 = a_1/2 = 3$. To compute a_3, note that a_2 is odd so we follow the instruction in the second line, and $a_3 = 3\,a_2 + 1 = 3 \cdot 3 + 1 = 10$. Since a_3 is even, the first line applies, and $a_4 = a_3/2 = 10/2 = 5$. But a_4 is odd, so the second line applies, and we find $a_5 = 3 \cdot 5 + 1 = 16$. And a_5 is even, so $a_6 = 16/2 = 8$. And a_6 is even, so $a_7 = 8/4 = 4$. And a_7 is even, so $a_8 = 4/2 = 2$, and then $a_9 = 2/2 = 1$. Oh, but a_9 is odd, so $a_{10} = 3 \cdot 1 + 1 = 4$. And it repeats. Let's write down the start of this sequence:

$$6,\quad 3,\quad 10,\quad 5,\quad 16,\quad 8,\quad 4,\quad 2,\quad 1,\quad 4,\quad 2,\quad 1,\quad \overbrace{4,\quad 2,\quad 1,}^{\text{repeats}}\quad 4,\quad \ldots$$

What if we had started with a number other than six? What if we set $a_1 = 25$ but then we used the same rule? In that case, since a_1 is odd, we compute a_2 by finding $3\,a_1 + 1 = 3 \cdot 25 + 1 = 76$. Since 76 is even, the next term is half that, meaning $a_3 = 38$. If we keep this up, we find that our sequence begins

$$\begin{array}{l} 25,\quad 76,\quad 38,\quad 19,\quad 58,\quad 29,\quad 88,\quad 44,\quad 22,\quad 11,\quad 34,\quad 17,\quad 52,\quad 26, \\ 13,\quad 40,\quad 20,\quad 10,\quad 5,\quad 16,\quad 8,\quad 4,\quad 2,\quad 1,\quad \ldots \end{array}$$

and then it repeats "4, 2, 1, 4, 2, 1, ..." just like before.

Does this always happen? Is it true that no matter which positive integer you start with, if you apply the half-if-even, $3x + 1$-if-odd rule, you end up getting stuck in the "4, 2, 1, ..." loop? That this is true is the **Collatz conjecture**; it has been verified for all starting values below 5×2^{60}. Nobody has found a value which doesn't return to one, but for all anybody knows there *might* well be a very large initial value which doesn't return to one; nobody knows either way. It is an unsolved problem[2] in mathematics.

If you think you have an argument that answers the Collatz conjecture, I challenge you to try your hand at the $5x + 1$ conjecture, that is, use the rule

$$a_n = \begin{cases} a_{n-1}/2 & \text{if } a_{n-1} \text{ is even, and} \\ 5\,a_{n-1} + 1 & \text{if } a_{n-1} \text{ is odd.} \end{cases}$$

[2] This is not the last unsolved problem we will encounter in this course. There are many things which humans do not understand.

1.4 Where is a sequence headed? Take a limit!

We've seen a lot of sequences, and already there are a few things we might notice. For instance, the arithmetic progression

$$1, \quad 8, \quad 15, \quad 22, \quad 29, \quad 36, \quad 43, \quad 50, \quad 57, \quad 64, \quad 71, \quad 78, \quad 85, \quad 92, \quad \ldots$$

just keeps getting bigger and bigger. No matter how large a number you think of, if I add enough 7's to 1, eventually I will surpass the giant number you thought of. On the other hand, the terms in a geometric progression where each term is half the previous term, namely

$$\frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \frac{1}{32}, \quad \frac{1}{64}, \quad \frac{1}{128}, \quad \frac{1}{256}, \quad \frac{1}{512}, \quad \frac{1}{1024}, \quad \ldots,$$

are getting closer and closer to zero. No matter how close you stand near but not at zero, eventually this geometric sequence gets even closer than you are to zero.

These two sequences have very different stories. One shoots off to infinity; the other zooms in towards zero. Mathematics is not just about numbers; mathematics provides tools for talking about the qualitative features of the numbers we deal with. What about the two sequences we just considered? They are qualitatively very different. The first "goes to" infinity; the second "goes to" zero.

If you were with us in Calculus One, you are perhaps already guessing that by "goes to," I actually mean "has limit."

In short, given a sequence, it is helpful to be able to say something qualitative about it; we may want to address the question such as "what happens after a while?" Formally, when faced with a sequence, we are interested in the limit

$$\lim_{i \to \infty} f(i) = \lim_{i \to \infty} a_i.$$

In Calculus One, we studied a similar question about

$$\lim_{x \to \infty} f(x)$$

when x is a variable taking on real values; now, in Calculus Two, we simply want to restrict the "input" values to be integers. No significant difference is required in the definition of limit, except that we specify, perhaps implicitly, that the variable is an integer.

Definition Suppose that (a_n) is a sequence. To say that $\lim_{n\to\infty} a_n = L$ is to say that

for every $\varepsilon > 0$,
there is an $N > 0$,
so that whenever $n > N$,
we have $|a_n - L| < \varepsilon$.

If $\lim_{n\to\infty} a_n = L$ we say that the sequence **converges**. If there is no finite value L so that $\lim_{n\to\infty} a_n = L$, then we say that the limit **does not exist**, or equivalently that the sequence **diverges**.

The definition of limit is being written as if it were poetry, what with line breaks and all. Like the best of poems, it deserves to be memorized, performed, internalized. Humanity struggled for millenia to find the wisdom contained therein.

Warning In the case that $\lim_{n\to\infty} a_n = \infty$, we say that (a_n) diverges, or perhaps more precisely, we say (a_n) diverges to infinity. The only time we say that a sequence converges is when the limit exists and is equal to a *finite* value.

One way to compute the limit of a sequence is to compute the limit of a function.

Theorem 1.4.1 *Let $f(x)$ be a real-valued function. If $a_n = f(n)$ defines a sequence (a_n) and if $\lim_{x\to\infty} f(x) = L$ in the sense of Calculus One, then $\lim_{n\to\infty} a_n = L$ as well.*

Example 1.4.2 Since $\lim_{x\to\infty}(1/x) = 0$, it is clear that also $\lim_{n\to\infty}(1/n) = 0$; in other words, the sequence of numbers

$$\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \ldots$$

get closer and closer to 0, or more precisely, as close as you want to get to zero, after a while, all the terms in the sequence are that close.

More precisely, no matter what $\varepsilon > 0$ we pick, we can find an N big enough so that, whenever $n > N$, we have that $1/n$ is within ε of the claimed limit, zero. This can be made concrete: let's suppose we set $\varepsilon = 0.17$. What is a suitable choice for N in response? If we choose $N = 5$, then whenever $n > 5$ we have $0 < 1/n < 0.17$.

But it is important to note that the converse[3] of this theorem is not true. To show the converse is not true, it is enough to provide a single example where it fails. Here's the counterexample[4].

[3] The **converse** of a statement is what you get when you swap the assumption and the conclusion; the converse of "if it is raining, then it is cloudy" is the statement "if it is cloudy, then it is raining." Which of those statements is true?

[4] An instance of (a potential) general rule being broken is called a **counterexample**. This is a popular term among mathematicians and philosophers.

Example 1.4.3 Consider the sequence (a_n) given by the rule $a_n = f(n) = \sin(n\pi)$. This is the sequence

$$\sin(0\pi), \quad \sin(1\pi), \quad \sin(2\pi), \quad \sin(3\pi), \quad \ldots,$$

which is just the sequence $0, 0, 0, 0, \ldots$ since $\sin(n\pi) = 0$ whenever n is an integer. Since the sequence is just the constant sequence, we have

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} 0 = 0.$$

But $\lim_{x\to\infty} f(x)$, when x is real, does not exist: as x gets bigger and bigger, the values $\sin(x\pi)$ do not get closer and closer to a single value, but instead oscillate between -1 and 1.

Here's some general advice. If you want to know $\lim_{n\to\infty} a_n$, you might first think of a function $f(x)$ where $a_n = f(n)$, and then attempt to compute $\lim_{x\to\infty} f(x)$. If the limit of the function exists, then it is equal to the limit of the sequence. But, if for some reason $\lim_{x\to\infty} f(x)$ does not exist, it may nevertheless still be the case that $\lim_{n\to\infty} f(n)$ exists—you'll just have to figure out another way to compute it.

1.5 Graphs

It is occasionally useful to think of the graph of a sequence. Since the function is defined only for integer values, the graph is just a sequence of dots. In Figure 1.2 we see the graph of a sequence and the graph of a corresponding real-valued function.

There are lots of real-valued functions which "fill in" the missing values of a sequence.

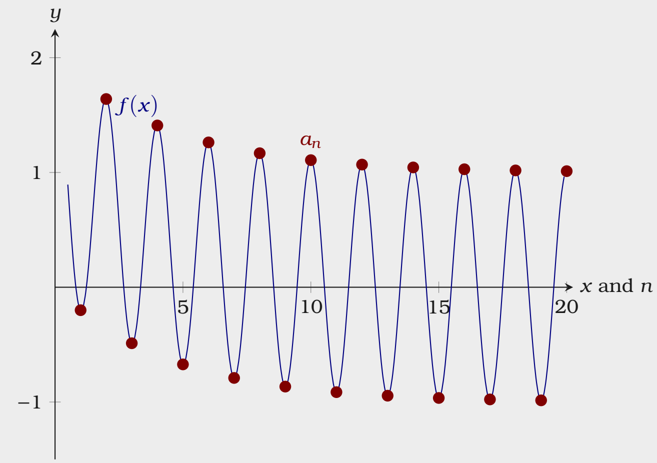

Figure 1.2: Plots of $f(x) = \cos(\pi x) + (4/5)^x$ and the sequence $a_n = (-1)^n + (4/5)^n$.

Example 1.5.1 Here's a particularly tricky example of "filling in" the missing values of a sequence. Consider the sequence

$$1, \quad 2, \quad 6, \quad 24, \quad 120, \quad 720, \quad 5040, \quad 40320, \quad 362880, \quad \ldots,$$

where the n^{th} term is the product of the first n integers. In other words $a_n = n!$, where the exclamation mark denotes the **factorial** function. Explicitly describe a function f of a real variable x, so that $a_n = f(n)$ for natural numbers n.

Solution There are lots of solutions. Here is a solution:

$$f(x) = \lfloor x \rfloor !.$$

In that definition, $\lfloor x \rfloor$ denotes the "greatest integer less than or equal to x" and is called the **floor function**. This is shown in Figure 1.3.

On the other hand, there are much trickier things that you could try to do. If you define the **Gamma function**

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$$

then it is perhaps very surprising to find out that $g(x) = \Gamma(x+1)$ is a function so that $g(n) = n!$ for natural numbers n. A graph is shown in Figure 1.4. Unlike f, which fails to be continuous, the function g is continuous.

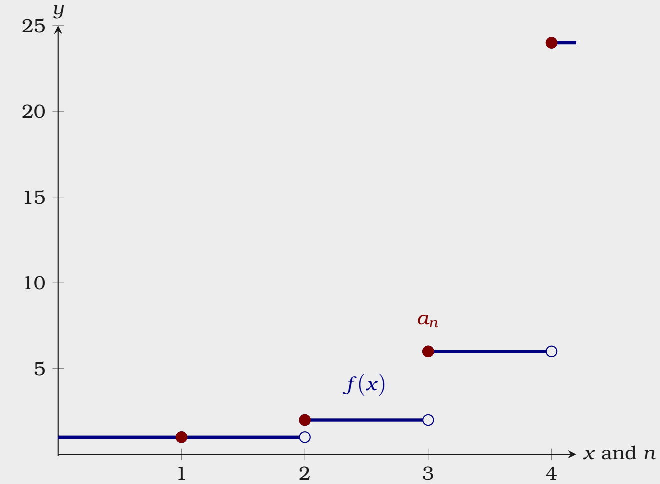

Figure 1.3: A plot of $f(x) = \lfloor x \rfloor !$ and $a_n = n!$. Recall that, by convention, $0! = 1$.

It is hard to define the "greatest integer" function, because they are all pretty great.

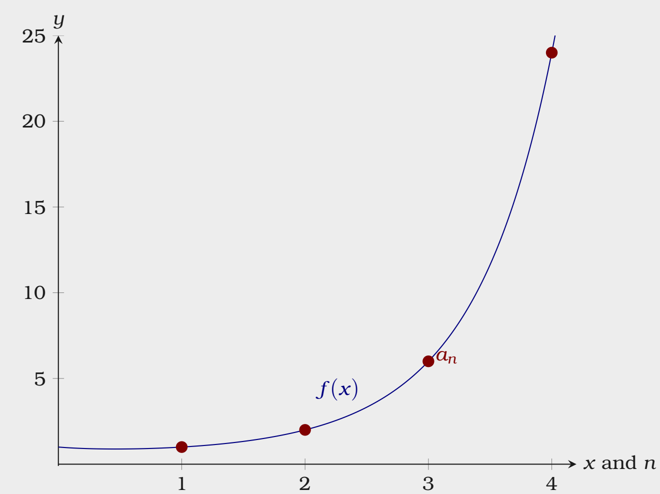

Figure 1.4: Plots of $f(x) = \int_0^\infty t^z e^{-t} \, dt.$ and $a_n = n!$.

1.6 New sequences from old

Given a sequence, one way to build a new sequence is to start with the old sequence, but then throw away a whole bunch of terms. For instance, if we started with the sequence of perfect squares

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad \ldots$$

we could throw away all the odd-indexed terms, and be left with

$$4, \quad 16, \quad 36, \quad 64, \quad 100, \quad 144, \quad 196, \quad 256, \quad 324, \quad 400, \quad 484, \quad \ldots$$

We say that this latter sequence is a **subsequence** of the original sequence. Here is a precise definition.

Definition Suppose (a_n) is a sequence with initial index N, and suppose we have a sequence of integers (n_i) so that

$$N \leq n_1 < n_2 < n_3 < n_4 < n_5 < \cdots$$

Then the sequence (b_i) given by $b_i = a_{n_i}$ is said to be a **subsequence** of the sequence a_n.

Limits are telling the story of "what happens" to a sequence. If the terms of a sequence can be made as close as desired to a limiting value L, then the subsequence must share that same fate.

Theorem 1.6.1 *If (b_i) is a subsequence of the convergent sequence (a_n), then* $\lim_{i\to\infty} b_i = \lim_{n\to\infty} a_n$.

Of course, just because a subsequence converges does not mean that the larger sequence converges, too. We'll see this again in more detail when we get to Example 1.7.6, but we'll discuss it briefly now.

Example 1.6.2 Find a convergent subsequence of the sequence (a_n) given by the rule $a_n = (-1)^n$.

Solution Note that the sequence (a_n) does not converge. But by considering the sequence of indexes $n_i = 2 \cdot i$, we can build a subsequence

$$b_i = a_{n_i} = a_{2i} = (-1)^{2i} = 1,$$

which is a constant sequence, so it converges to 1.

There are other subsequences of $a_n = (-1)^n$ which converge but do *not* converge to one. For instance, the subsequence of odd indexed terms is the constant sequence $c_n = -1$, which converges to -1. For that matter, the fact that there are convergent subsequences with distinct limits perhaps explains why the original sequence (a_n) does not converge. Let's formalize this.

Corollary 1.6.3

Suppose (b_i) and (c_i) are convergent subsequences of the sequence (a_n), but

$$\lim_{i\to\infty} b_i \neq \lim_{i\to\infty} c_i.$$

Then the sequence (a_n) does not converge.

Proof *Suppose, on the contrary, the sequence (a_n) did converge. Then by Theorem 1.6.1, the subsequence (b_i) would converge, too, and*

$$\lim_{i\to\infty} b_i = \lim_{n\to\infty} a_n.$$

Again by Theorem 1.6.1, the subsequence (c_i) would converge, too, and

$$\lim_{i\to\infty} c_i = \lim_{n\to\infty} a_n.$$

But then $\lim_{i\to\infty} b_i = \lim_{i\to\infty} c_i$, which is exactly what we are supposing doesn't happen! To avoid this contradiction, it must be that our original assumption that (a_n) converged was incorrect; in short, the sequence (a_n) does not converge.

1.7 Helpful theorems about limits

Not surprisingly, the properties of limits of real functions translate into properties of sequences quite easily.

Theorem 1.7.1 *Suppose that* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$ *and* k *is some constant. Then*

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL,$$
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M,$$
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M,$$
$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM, \text{ and}$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}, \text{ provided } M \neq 0.$$

1.7.1 Squeeze Theorem

Likewise, there is an analogue of the squeeze theorem for functions.

Theorem 1.7.2 *Suppose there is some* N *so that for all* $n > N$*, it is the case that* $a_n \leq b_n \leq c_n$*. If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

, then $\lim_{n\to\infty} b_n = L$.

And a final useful fact:

Theorem 1.7.3 $\lim_{n\to\infty} |a_n| = 0$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

This says simply that the size of a_n gets close to zero if and only if a_n gets close to zero.

Sometimes people write "iff" as shorthand for "if and only if."

1.7.2 Examples

Armed with these helpful theorems, we are now in a position to work a number of examples.

Example 1.7.4 Determine whether the sequence (a_n) given by the rule $a_n = \dfrac{n}{n+1}$ converges or diverges. If it converges, compute the limit.

Solution Consider the real-valued function

$$f(x) = \frac{x}{x+1}.$$

Since $a_n = f(n)$, it will be enough to find $\lim_{x\to\infty} f(x)$ in order to find $\lim_{n\to\infty} a_n$. We compute, as in Calculus One, that

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{x+1} &= \lim_{x\to\infty} \frac{(x+1)-1}{x+1} \\
&= \lim_{x\to\infty} \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right) \\
&= \lim_{x\to\infty} \left(1 - \frac{1}{x+1}\right) \\
&= \lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x+1} \\
&= 1 - \lim_{x\to\infty} \frac{1}{x+1} = 1 - 0 = 1.
\end{aligned}$$

We therefore conclude that $\lim_{n\to\infty} a_n = 1$.

And this is reasonable: by choosing n to be a large enough integer, I can make $\frac{n}{n+1}$ as close to 1 as I would like. Just imagine how close $\frac{10000000000}{10000000001}$ is to one.

Example 1.7.5 Determine whether the sequence (a_n) given by $a_n = \dfrac{\log n}{n}$ converges or diverges. If it converges, compute the limit.

Solution By l'Hôpital's rule, we compute

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\log n}{n} = 0.$$

I'm not too fond of l'Hôpital's rule, so I would have been happier if I had given a solution that didn't involve it; you could avoid mentioning l'Hôpital's rule in Example 1.7.5 if you used, say, the squeeze theorem and the fact that $\log n \le \sqrt{n}$.

Example 1.7.6 Determine whether the sequence (a_n) given by the rule $a_n = (-1)^n$ converges or diverges. If it converges, compute the limit.

Solution Your first inclination might be to consider the "function" $f(x) = (-1)^x$, but you'll run into trouble when trying to tell me the value of $f(1/2)$.

How does the sequence $a_n = (-1)^n$ begin? It starts

$$-1, \quad 1, \quad -1, \quad 1, \quad -1, \quad 1, \quad -1, \quad 1, \quad -1, \quad 1, \quad \ldots,$$

so the sequence isn't getting close to any number in particular.

Intuitively, the above argument is probably pretty convincing. But if you want an airtight argument, you can reason like this: suppose—though we'll soon see that this is a ridiculous assumption—that the sequence $a_n = (-1)^n$ did converge to L. Then any subsequence would also converge to L, by Theorem 1.6.1 which stated that the limit of a subsequence is the same as the limit of the original sequence. If I throw away every other term of the sequence (a_n), I am left with the constant sequence

$$-1, \quad -1, \quad -1, \quad -1, \quad -1, \quad -1, \quad -1, \quad \ldots,$$

which converges to -1, and so L must be -1.

On the other hand, if I throw away all the terms with odd indices and keep only those terms with even indices, I am left with the constant subsequence

$$1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad \ldots,$$

so L must be 1. Since L can't be both -1 and 1, it couldn't have been the case that $\lim_{n\to\infty} a_n = L$ for a real number L. In other words, the limit does not exist.

I imagine that the "airtight argument" in the solution to Example 1.7.6 is difficult to understand. Please don't worry if you find the argument confusing now—we'll have more opportunities for doing these sorts of proofs by contradiction in the future.

Example 1.7.7 Determine whether the sequence $a_n = (-1/2)^n$ converges or diverges. If it converges, compute the limit.

In this problem, you must be very careful to recognize the difference between $(-1/2)^n$ and $-(1/2)^n$. The former flip-flops between being positive and being negative, while the latter is always negative.

Solution Let's use the Squeeze Theorem. Consider the sequences $b_n = -(1/2)^n$ and $c_n = (1/2)^n$. Then $b_n \leq a_n \leq c_n$. And $\lim_{n\to\infty} c_n = 0$ because $\lim_{x\to\infty}(1/2)^x = 0$. Since $b_n = -c_n$, we have $\lim_{n\to\infty} b_n = -\lim_{n\to\infty} c_n = -0 = 0$. Since b_n and c_n converge to zero, the squeeze theorem tells us that $\lim_{n\to\infty} a_n = 0$ as well.

If you don't want to mention the Squeeze Theorem, you could instead apply Theorem 1.7.3. In that case, we would again consider the sequence $c_n = |a_n|$ and observe that $\lim_{n\to\infty} c_n = 0$. But then Theorem 1.7.3 steps in, and tells us that $\lim_{n\to\infty} a_n = 0$ as well. Of course, a convincing argument for why Theorem 1.7.3 works at all goes via the squeeze theorem, so this second method is not so different from the first.

Example 1.7.8 Determine whether $a_n = \dfrac{\sin n}{\sqrt{n}}$ converges or diverges. If it converges, compute the limit.

Solution Since $-1 \leq \sin n \leq 1$, we have

$$\frac{-1}{\sqrt{n}} \leq \frac{\sin n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}},$$

and can therefore apply the Squeeze Theorem. Since $\lim_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$, we get

$$\lim_{n\to\infty} \frac{-1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

and so by squeezing, we conclude $\lim_{n\to\infty} a_n = 0$.

You might be wondering why I love the Squeeze Theorem so much; one reason is that the Squeeze Theorem gets you into the idea of "comparing" one sequence to another, and this "comparison" idea will be big when we get to convergence tests in Chapter 3.

Example 1.7.9 A particularly common and useful sequence is the geometric progression $a_n = r^n$ for a fixed real number r. For which values of r does this sequence converge?

Solution It very much does depend on r.

If $r = 1$, then $a_n = (1)^n$ is the constant sequence

$$1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1, \quad \ldots,$$

so the sequence converges to one. A similarly boring fate befalls the case $r = 0$, in which case $a_n = (0)^n$ converges to zero.

If $r = -1$, we are reprising the sequence which starred in Example 1.7.6; as we saw, that sequence diverges.

If either $r > 1$ or $r < -1$, then the terms $a_n = r^n$ can be made as large as one likes by choosing n large enough (and even), so the sequence diverges.

If $0 < r < 1$, then the sequence converges to 0.

If $-1 < r < 0$ then $|r^n| = |r|^n$ and $0 < |r| < 1$, so the sequence $\{|r|^n\}_{n=0}^{\infty}$ converges to 0, so also $\{r^n\}_{n=0}^{\infty}$ converges to 0.

That last example of a geometric progression is involved enough that it deserves to be summarized as a theorem.

Theorem 1.7.10 *The sequence $a_n = r^n$ converges when $-1 < r \leq 1$, and diverges otherwise. In symbols,*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1, \\ 1 & \text{if } r=1, \text{ and} \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

1.8 Qualitative features of sequences

Sometimes we will not be able to determine the limit of a sequence, but we still would like to know whether or not it converges to some unspoken number. In many cases, we can determine whether a limit exists, without needing to—or without even being able to—compute that limit.

Your first exposure to mathematics might have been about **constructions**; you might have been asked to compute a numeric answer or to propose a solution to a problem. But much of mathematics is concerned with showing **existence**, even if the thing that is being shown to exist cannot be exhibited itself.

1.8.1 Monotonicity

And sometimes we don't even care about limits, but we'd simply like some terminology with which to describe features we might notice about sequences. Here is some of that terminology.

For instance, how much money I have on day n is a sequence; I probably hope that sequence is an increasing sequence.

Definition A sequence is called **increasing** (or sometimes **strictly increasing**) if $a_n < a_{n+1}$ for all n. It is called **non-decreasing** if $a_n \leq a_{n+1}$ for all n.

Similarly a sequence is **decreasing** (or, by some people, **strictly decreasing**) if $a_n > a_{n+1}$ for all n and **non-increasing** if $a_n \geq a_{n+1}$ for all n.

To make matters worse, the people who insist on saying "strictly increasing" may—much to everybody's confusion—insist on calling a non-decreasing sequence "increasing." I'm not going to play their game; I'll be careful to say "non-decreasing" when I mean a sequence which is getting larger or staying the same.

To make matters better, lots of facts are true for sequences which are either increasing or decreasing; to talk about this situation without constantly saying "either increasing or decreasing," we can make up a single word to cover both cases.

Definition If a sequence is increasing, non-decreasing, decreasing, or non-increasing, it is said to be **monotonic**.

Let's see some examples of sequences which are monotonic.

Example 1.8.1 The sequence $a_n = \dfrac{2^n - 1}{2^n}$ which starts

$$\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{7}{8}, \quad \frac{15}{16}, \quad \ldots,$$

is increasing. On the other hand, the sequence $b_n = \frac{n+1}{n}$, which starts

$$\frac{2}{1}, \quad \frac{3}{2}, \quad \frac{4}{3}, \quad \frac{5}{4}, \quad \ldots,$$

is decreasing.

1.8.2 Boundedness

Sometimes we can't say exactly which number a sequence approaches, but we can at least say that the sequence doesn't get too big or too small.

Definition A sequence (a_n) is **bounded above** if there is some number M so that for all n, we have $a_n \leq M$. Likewise, a sequence (a_n) is **bounded below** if there is some number M so that for every n, we have $a_n \geq M$.

If a sequence is both bounded above and bounded below, the sequence is said to be **bounded**.

If a sequence $\{a_n\}_{n=0}^{\infty}$ is increasing or non-decreasing it is bounded below (by a_0), and if it is decreasing or non-increasing it is bounded above (by a_0).

Finally, with all this new terminology we can state the most important theorem of Chapter 1.

Theorem 1.8.2 *If the sequence a_n is bounded and monotonic, then $\lim_{n\to\infty} a_n$ exists.*

In short, bounded monotonic sequences converge—though we can't necessarily describe the number to which they converge.

We will not prove this theorem in the textbook.[5] Nevertheless, it is not hard to believe: suppose that a sequence is increasing and bounded, so each term is larger than the one before, yet never larger than some fixed value M. The terms must then get closer and closer to some value between a_0 and M. It certainly need not be M, since M may be a "too generous" upper bound; the limit will be the smallest number

[5] Proving this theorem is, honestly, the purview of a course in *analysis*, the theoretical underpinnings of calculus. That's not to say it couldn't be done in this course, but I intend this to be a "first glance" at sequences—so much will be left unsaid.

that is above[6] all of the terms a_n. Let's try an example!

Example 1.8.3 All of the terms $(2^i - 1)/2^i$ are less than 2, and the sequence is increasing. As we have seen, the limit of the sequence is 1—1 is the smallest number that is bigger than all the terms in the sequence. Similarly, all of the terms $(n+1)/n$ are bigger than $1/2$, and the limit is 1—1 is the largest number that is smaller than the terms of the sequence.

We don't actually need to know that a sequence is monotonic to apply this theorem—it is enough to know that the sequence is "eventually" monotonic,[7] that is, that at some point it becomes increasing or decreasing. For example, the sequence 10, 9, 8, 15, 3, 21, 4, 3/4, 7/8, 15/16, 31/32, . . . is not increasing, because among the first few terms it is not. But starting with the term 3/4 it is increasing, so if the pattern continues and the sequence is bounded, the theorem tells us that the "tail" $3/4, 7/8, 15/16, 31/32, \ldots$ converges. Since convergence depends only on what happens as n gets large, adding a few terms at the beginning can't turn a convergent sequence into a divergent one.

Example 1.8.4 Show that the sequence (a_n) given by $a_n = n^{1/n}$ converges.

Solution We might first show that this sequence is decreasing, that is, we show that for all n,

$$n^{1/n} > (n+1)^{1/(n+1)}.$$

[6] This concept of the "smallest number above all the terms" is an incredibly important one; it is the idea of a least upper bound that underlies the real numbers.

[7] After all, the limit only depends on what is happening after some large index, so throwing away the beginning of a sequence won't affect its convergence or its limit.

You may be worried about my saying that $\log 3 > 1$. If log were the common (base 10) logarithm, this would be wrong, but as far as I'm concerned, there is only one log, the natural log. Since $3 > e$, we may conclude that $\log 3 > 1$.

But this isn't true! Take a look

$$\begin{aligned} a_1 &= 1, \\ a_2 &= \sqrt{2} \approx 1.4142, \\ a_3 &= \sqrt[3]{3} \approx 1.4422, \\ a_4 &= \sqrt[4]{4} \approx 1.4142, \\ a_5 &= \sqrt[5]{5} \approx 1.3797, \\ a_6 &= \sqrt[6]{6} \approx 1.3480, \\ a_7 &= \sqrt[7]{7} \approx 1.3205, \\ a_8 &= \sqrt[8]{8} \approx 1.2968, \text{ and} \\ a_9 &= \sqrt[9]{9} \approx 1.2765. \end{aligned}$$

But it does seem that this sequence perhaps is decreasing after the first few terms. Can we justify this?

Yes! Consider the real function $f(x) = x^{1/x}$ when $x \geq 1$. We compute the derivative—perhaps via "logarithmic differentiation"—to find

$$f'(x) = \frac{x^{1/x}(1 - \log x)}{x^2}.$$

Note that when $x \geq 3$, the derivative $f'(x)$ is negative. Since the function f is decreasing, we can conclude that the sequence is decreasing—well, at least for $n \geq 3$.

Since all terms of the sequence are positive, the sequence is decreasing and bounded when $n \geq 3$, and so the sequence converges.

As it happens, you could compute the limit in Example 1.8.4, but our given solution shows that it converges even without knowing the limit!

Example 1.8.5 Show that the sequence $a_n = \dfrac{n!}{n^n}$ converges.

Solution Let's get an idea of what is going on by computing the first few terms.

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{2}{9} \approx 0.22222, \quad a_4 = \frac{3}{32} \approx 0.093750,$$
$$a_5 = \frac{24}{625} \approx 0.038400, \quad a_6 = \frac{5}{324} \approx 0.015432,$$
$$a_7 = \frac{720}{117649} \approx 0.0061199, \quad a_8 = \frac{315}{131072} \approx 0.0024033.$$

The sequence appears to be decreasing. To formally show this, we would need to show $a_{n+1} < a_n$, but we will instead show that

$$\frac{a_{n+1}}{a_n} < 1,$$

which amounts to the same thing. It is helpful trick here to think of the ratio between subsequent terms, since the factorials end up canceling nicely. In particular,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} \\ &= \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} \\ &= \frac{n+1}{n+1} \left(\frac{n}{n+1}\right)^n \qquad = \left(\frac{n}{n+1}\right)^n < 1. \end{aligned}$$

Note that the sequence is bounded below, since every term is positive.

Because the sequence is decreasing and bounded below, it converges. Indeed, Exercise 2 asks you to compute the limit.

These sorts of arguments involving the ratio of subsequent terms will come up again in a big way in Section 3.1. Stay tuned!

Exercises for Section 1.8

(1) Compute $\lim_{x\to\infty} x^{1/x}$.

(2) Use the squeeze theorem to show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

(3) Determine whether $\{\sqrt{n+47} - \sqrt{n}\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

(4) Determine whether $\left\{\frac{n^2+1}{(n+1)^2}\right\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

(5) Determine whether $\left\{\frac{n+47}{\sqrt{n^2+3n}}\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit.

(6) Determine whether $\left\{\frac{2^n}{n!}\right\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

2 Series

We've only just scratched the surface of sequences, and already we've arrived in Chapter 2. Series will be the main focus of our attention for the rest of the course. If that's the case, then why did we bother with sequences? Because **a series is what you get when you add up the terms of a sequence, *in order***. So we needed to talk about sequences to provide the language with which to discuss series.

Suppose (a_n) is a sequence; then the associated series[1]

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

I might be thinking that I feel just fine, but I have woken a terrible beast! What does that innocuous looking "$\cdots$" mean? What does it mean to add up infinitely many numbers? It's not as if I'll ever be done with all the adding, so how can I ever attach a "value" to a series? How can I do infinitely many things, yet live to share the answer with you?

I can't.

But I can do a large, but finite, number of things, and then see if I'm getting close to anything in particular. In other words, I can take a limit.

[1] The Σ symbol may look like an E, but it is the Greek letter *sigma*, and it makes an S sound—just like the first letter of the word "series." If you see "GRΣΣK," say "grssk."

2.1 Definition of convergence

But a limit. . . of what? From a sequence, we can consider the associated series, and associated to that series is a yet another sequence—the **sequence of partial sums**. Here's a formal definition which unwinds this tangled web.

Definition Suppose (a_n) is a sequence with associated series $\sum_{k=1}^{\infty} a_k$. The **sequence of partial sums** associated to these objects is the sequence

$$s_n = \sum_{k=1}^{n} a_k.$$

Working this out, we have

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \\ s_5 &= a_1 + a_2 + a_3 + a_4 + a_5, \text{ and so on.} \end{aligned}$$

Instead of adding up the infinite sequence a_n, which we can't do, we will instead look at the sequence of partial sums, and ask whether that sequence of partial sums converges. And if it converges to L, then we'll call L the **value** of the series.

This might seem overly complicated, but it solves a serious problem: we no longer are confronted with the supertask of adding up infintely many numbers but living to tell the tale. To take the limit of the sequence of partial sums *is* to add up lots—but not all!—of the terms in the original sequence to see if we're staying close to a particular number—the limit. That particular limiting value is then, by definition, *declared* to be the result of adding up all the terms in the original sequence *in order*.

That the order matters will be a major theme in Chapter 4.

Definition Consider the series $\sum_{k=1}^{\infty} a_k$. This series **converges** if the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges. More precisely, if $\lim_{n\to\infty} s_n = L$, we then write

$$\sum_{k=1}^{\infty} a_k = L$$

and say, "the series $\sum_{k=1}^{\infty} a_k$ converges to L."

If the sequence of partial sums diverges, we say that the series **diverges**.

Remember, infinity is not a number. So if it happens that $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \infty$, then we might write $\sum_{k=1}^{\infty} a_k = \infty$ but nevertheless we still say that the series diverges.

Sometimes, to emphasize that the series involves adding up infinitely many terms, we will say **"infinite series"** instead of just "series."

2.2 Geometric series

Armed with the official definition of convergence in general, we focus in on the specific example: a sequence of the form $a_n = a_0\, r^n$ is called a **geometric progression** as we learned back in Subsection 1.3.2. What happens when we add up the terms of a geometric progression?

A geometric series was used by Archimedes—who lived more than two thousand years ago!—to compute the area between a parabola and a straight line. Humans had the first inklings of calculus a very long time ago.

Definition (Geometric Series) A series of the form

$$\sum_{k=0}^{\infty} a_0\, r^k$$

is called a **geometric series**.

We can't simply "add" up the infinitely many terms in the geometric series. What we can do, instead, is add up the a first handful of terms. Pick a big value for n, and instead compute

$$s_n = \sum_{k=0}^{n} a_0\, r^k = a_0 + a_0\, r + a_0\, r^2 + a_0\, r^3 + \cdots + a_0\, r^n.$$

This is the n^{th} partial sum. Concretely,

$$\begin{aligned} s_0 &= a_0, \\ s_1 &= a_0 + a_0\, r, \\ s_2 &= a_0 + a_0\, r + a_0\, r^2, \\ s_3 &= a_0 + a_0\, r + a_0\, r^2 + a_0\, r^3 \\ &\vdots \\ s_n &= a_0 + a_0\, r + a_0\, r^2 + a_0\, r^3 + \cdots + a_0\, r^n. \end{aligned}$$

In our quest to assign a value to the infinite series $\sum_{k=0}^{\infty} a_0\, r^k$, we instead[2] consider

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_0\, r^k.$$

[2] Replacing the actual "∞" by a limit (that is, a "potential" infinity) shouldn't seem all that surprising; we encountered the same trick in Calculus One.

We can perform some algebraic manipulations on the partial sum. The manipulation begins with our multiplying s_n by $(1-r)$ to cause some convenient cancellation, specifically,

$$\begin{aligned} s_n(1-r) &= a_0\left(1 + r + r^2 + r^3 + \cdots + r^n\right)(1-r) \\ &= a_0\left(1 + r + r^2 + r^3 + \cdots + r^n\right) 1 - a_0\left(1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n\right) r \\ &= a_0\left(1 + r + r^2 + r^3 + \cdots + r^n\right) - a_0\left(r + r^2 + r^3 + \cdots + r^n + r^{n+1}\right) \\ &= a_0\left(1 + r + r^2 + r^3 + \cdots + r^n - r - r^2 - r^3 - \cdots - r^n - r^{n+1}\right) \\ &= a_0(1 - r^{n+1}). \end{aligned}$$

Dividing both sides[3] by $(1-r)$ shows

$$s_n = a_0 \cdot \frac{1 - r^{n+1}}{1-r}.$$

[3] Here, we tacitly assume $r \neq 1$. But can you see what happens to the geometric series when $r = 1$ without going through this argument?

Therefore,

$$\sum_{k=0}^{\infty} a_0\, r^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(a_0 \cdot \frac{1 - r^{n+1}}{1-r} \right).$$

The limit depends very much on what r is.

Suppose $r \geq 1$ or $r \leq -1$. In those cases, $\lim_{n\to\infty} r^{n+1}$ does not exist, and likewise $\lim_{n\to\infty} s_n$ does not exist. So the series diverges if $r \geq 1$ or if $r \leq -1$. One quicker way of saying this is that the series diverges when $|r| \geq 1$.

On the other hand, suppose $|r| < 1$. Then $\lim_{n\to\infty} r^{n+1} = 0$, and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_0 \frac{1-r^{n+1}}{1-r} = \frac{a_0}{1-r}.$$

Thus, when $|r| < 1$ the geometric series converges to $a_0/(1-r)$. This is important enough that we'll summarize it as a theorem.

Theorem 2.2.1 *Suppose $a_0 \neq 0$. Then for a real number r such that $|r| < 1$, the geometric series*

$$\sum_{k=0}^{\infty} a_0 r^k$$

converges to $\frac{a_0}{1-r}$.

For a real number r where $|r| \geq 1$, the aforementioned geometric series diverges.

Example 2.2.2 When, for example, $a_0 = 1$ and $r = 1/2$, this means

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2,$$

which makes sense. Consider the partial sum

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n}.$$

This partial sum gets as close to two as you'd like—as long as you are willing to choose n large enough. And it doesn't take long to get close to two! For example, even just $n = 6$, we get

$$s_6 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{127}{64}$$

which is close to two.

Example 2.2.3 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k}.$$

This does not *quite* fit into the preceding framework, because this series starts with $k = 1$ instead of $k = 0$.

Nevertheless, we can work out what happens. The series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is just like series $\sum_{k=0}^{\infty} \frac{1}{2^k}$ except the former is missing an initial $k = 0$ term, which is $1/2^0 = 1$. So each partial sum of the former series is one less than the corresponding partial sum for the latter series, so the limit is also one less than the value of the geometric series. In symbols,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \right) - 1 = \frac{1}{1 - \frac{1}{2}} - 1 = 1.$$

If you don't find this argument convincing, look at Figure 2.1, which displays a visual argument that the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges to one.

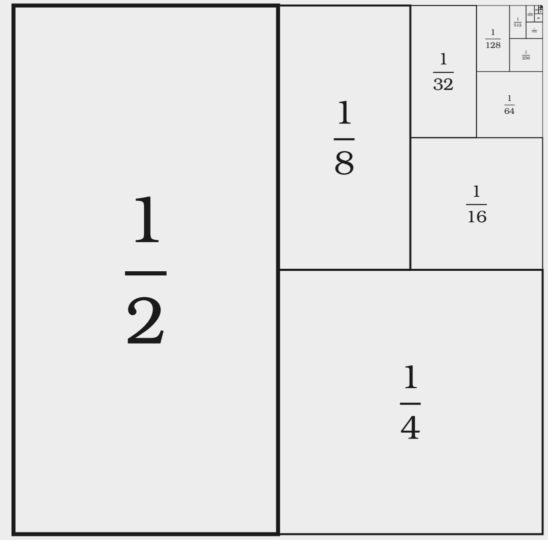

Figure 2.1: Visual evidence that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Begin with a $\frac{1}{2} \times 1$ rectangle, and build each subsequent rectangle by cloning and halving the previous rectangle; all these rectangles fit together to fill up a unit square.

2.3 Properties of series

Theorem 1.7.1 presented many properties of sequences. Since the value of a series is the limit of the sequence of partial sums, properties of limits can be reformulated into properties of series.

2.3.1 Constant multiple

Theorem 2.3.1 *Suppose* $\sum_{k=0}^{\infty} a_k$ *is a convergent series, and* c *is a constant. Then* $\sum_{k=0}^{\infty} c\, a_k$ *converges, and*

$$\sum_{k=0}^{\infty} c\, a_k = c \sum_{k=0}^{\infty} a_k.$$

Proof *If you know just enough algebra to be dangerous, you may remember that for any real numbers* a*,* b*, and* c*,*

$$c(a+b) = c\,a + c\,b.$$

This is the ***distributive law*** *for real numbers. By using the distributive law more than once,*

$$c(a_0 + a_1 + a_2) = c\,a_0 + c\,a_1 + c\,a_2,$$

or more generally, for some finite n*,*

$$c(a_0 + a_1 + a_2 + \cdots + a_n) = c\,a_0 + c\,a_1 + c\,a_2 + \cdots + c\,a_n.$$

So what's the big deal with proving Theorem ? Can't we just scream "distributive law!" and be done with it? After all, Theorem amounts to

$$c\,a_0 + c\,a_1 + c\,a_2 + \cdots + c\,a_n + \cdots = c(a_0 + a_1 + a_2 + \cdots + a_n + \cdots).$$

But hold your horses: you can only apply the distributive law finitely many times! The distributive law, without some input from calculus, will not succeed in justifying Theorem .

Let's see how to handle this formally. By hypothesis, $\sum_{k=0}^{\infty} a_k$ *is a convergent series, meaning its associated sequence of partial sums,*

$$s_n = \sum_{k=0}^{n} a_k,$$

converges. In other words, $\lim_{n\to\infty} s_n$ *exists. Then, by Theorem ,*

$$\lim_{n\to\infty}(c\,s_n) = c\lim_{n\to\infty} s_n.$$

But $c\,s_n$ *is the sequence of partial sums for the series* $\sum_{k=0}^{\infty} c\,a_k$*, because*

$$c\,s_n = c\,a_0 + c\,a_1 + c\,a_2 + \cdots + c\,a_n.$$

Consequently,

$$\sum_{k=0}^{\infty} c\,a_k = \lim_{n\to\infty}(c\,s_n) = c\lim_{n\to\infty} s_n,$$

which is what we wanted to prove.

That theorem addresses the case of multiplying a convergent series by a constant c; what about divergence?

Example 2.3.2 Suppose that $\sum_{k=0}^{\infty} a_k$ diverges; does $\sum_{k=0}^{\infty} c\,a_k$ also diverge?

Solution If $c = 0$, then $\sum_{k=0}^{\infty} c\,a_k = \sum_{k=0}^{\infty} 0$ which does converge, to zero.

On the other hand, provided $c \neq 0$, then, yes, $\sum_{k=0}^{\infty} c\,a_k$ also diverges. How do we know?

We are working under the hypothesis that $\sum_{k=0}^{\infty} a_k$ diverges. Suppose now, to the contrary, that $\sum_{k=0}^{\infty} c\,a_k$ did converge; applying Theorem 2.3.1 (albeit with a_k replaced by $c\,a_k$ and c replaced by $(1/c)$), the series

$$\sum_{k=0}^{\infty}\left(\frac{1}{c}\right)c\,a_k$$

converges, but that is ridiculous, since

$$\sum_{k=0}^{\infty} \left(\frac{1}{c}\right) c\, a_k = \sum_{k=0}^{\infty} a_k$$

and the latter, under our hypothesis, diverged. A series cannot both converge and diverge, so our assumption (that $\sum_{k=0}^{\infty} c\, a_k$ did converge) must have been mistaken—it must be that $\sum_{k=0}^{\infty} c\, a_k$ diverges.

But what about the case where $c = 0$? In that case, the series converged! Did we make a mistake? In the argument for divergence, we multiplied by $1/c$, which is something we are not permitted to do when $c = 0$.

In light of this example, we have actually proved something stronger.

We can connect this discussion back to the discussion of geometric series in Section 2.2. If you believe that $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, then by Theorem 2.3.3, you believe $\sum_{k=0}^{\infty} c\, r^k = \frac{c}{1-r}$, which is part of Theorem 2.2.1.

Theorem 2.3.3 *Consider the series* $\sum_{k=0}^{\infty} a_k$*, and suppose* c *is a nonzero constant. Then* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} c\, a_k$ *share a common fate: either both series converge, or both series diverge.*

Moreover, when $\sum_{k=0}^{\infty} a_k$ *converges,*

$$\sum_{k=0}^{\infty} c\, a_k = c \cdot \sum_{k=0}^{\infty} a_k.$$

2.3.2 Sum of series

Suppose $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are convergent series. What can be said of $\sum_{k=0}^{\infty} (a_k + b_k)$?

Addition is **associative**[4] and **commutative**[5], so for any real numbers a, b, c,

[4] To say that addition is "associative" is, intuitively, to say that how the expression is parenthesized doesn't matter; formally, "associativity" means that for any a, b, and c, we have $a + (b + c) = (a + b) + c$.

[5] To say that addition is "commutative" is, intuitively, to say that the order in which the adding is done doesn't matter; formally, "commutativity" means that for a and b, we have $a + b = b + a$.

and d,

$$(a+b)+(c+d)=(a+c)+(b+d).$$

More generally, for real numbers a_0, a_1, a_2, ..., a_n and real numbers b_0, b_1, b_2, ..., b_n,

$$(a_0+a_1+a_2+\cdots+a_n)+(b_0+b_1+b_2+\cdots+b_n)=(a_0+b_0)+(a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n).$$

But this finite statement can be beefed up into a statement about series. What we want to prove is

$$\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} (a_k + b_k).$$

From above, we already know

$$\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_k + b_k).$$

Take the limit of both sides.

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k\right) = \lim_{n\to\infty}\sum_{k=0}^{n} (a_k + b_k).$$

But the limit of a sum is the sum of the limits[6], so

$$\left(\lim_{n\to\infty}\sum_{k=0}^{n} a_k\right) + \left(\lim_{n\to\infty}\sum_{k=0}^{n} b_k\right) = \lim_{n\to\infty}\sum_{k=0}^{n} (a_k + b_k).$$

Those three limits of partial sums can each be replaced by the series, which shows

$$\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} (a_k + b_k).$$

This can be summarized in a theorem.

[6] I like to call this a **chiastic rule**, since it has the rhetorical pattern of a chiasmus.

Theorem 2.3.4 *Suppose* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *are convergent series. Then* $\sum_{k=0}^{\infty} (a_k + b_k)$ *is convergent, and*

$$\sum_{k=0}^{\infty} (a_k + b_k) = \left(\sum_{k=0}^{\infty} a_k\right) + \left(\sum_{k=0}^{\infty} b_k\right).$$

That covers sums of convergent series.

Example 2.3.5 Now suppose that $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ diverge; does $\sum_{k=0}^{\infty} (a_k + b_k)$ diverge?

Solution Not necessarily. Let $a_k = 1$ and $b_k = -1$, so $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ diverge. But

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} ((1) + (-1)) = \sum_{k=0}^{\infty} 0 = 0.$$

This is *not* to say that the term-by-term sum of divergent series *necessarily* converges, either. It is entirely possible that $\sum_{k=0}^{\infty} (a_k + b_k)$ will also diverge. For istance, if $a_k = b_k = 1$, then

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} (1 + 1) = \sum_{k=0}^{\infty} 2$$

also diverges. So the term-by-term sum of divergent series might converge or might diverge, depending on the situation.

2.4 Telescoping series

For most of this course, we will be happy if we can show that a series converges or that a series diverges; we will not, usually, be too concerned with finding the value of a series. Why not? Usually it is just too hard to determine the value; we would if we could, but since it is often too hard, we don't bother.

Nevertheless, there is one family of series for which we can calculate the value with relative ease: the **telescoping series.** Here is a first example which suggests what we mean by "telescoping."

Example 2.4.1 Compute $\sum_{k=1}^{n} \frac{1}{k\cdot(k+1)}$.

Solution Note that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k\cdot(k+1)} - \frac{k}{k\cdot(k+1)} = \frac{k+1-k}{k\cdot(k+1)} = \frac{1}{k\cdot(k+1)}$$

Consequently,

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k\cdot(k+1)} &= \left(\frac{1}{1} - \frac{1}{1+1}\right) + \left(\frac{1}{2} - \frac{1}{2+1}\right) + \left(\frac{1}{3} - \frac{1}{3+1}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{1} + \left(-\frac{1}{1+1} + \frac{1}{2}\right) + \left(-\frac{1}{2+1} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{(n-1)+1} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= \frac{1}{1} - \frac{1}{n+1}\end{aligned}$$

since most of these terms end up canceling.

You might know about a method called **partial frac-ns** to rewrite $\frac{1}{k\cdot(k+1)}$ as a combination of $\frac{1}{k}$ l $\frac{1}{k+1}$.

In general, we say that a series **telescopes** if, after some simplification, there is a formula for the sequence of partial sums with a fixed number of terms. The name suggests the way the cancellation happens: just as the nesting rings in an expandable spyglass fit together, so too do the neighboring terms in a telescoping series fit together and collapse.

Armed with a formula for the sequence of partial sums, we can attack the corresponding infinite series.

Example 2.4.2 Compute $\sum_{k=1}^{\infty} \frac{1}{k \cdot (k+1)}$.

Solution We just computed that

$$\sum_{k=1}^{n} \frac{1}{k \cdot (k+1)} = 1 - \frac{1}{n+1}.$$

Rewriting the infinite series as the limit of the sequence of partial sums yields

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k \cdot (k+1)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k \cdot (k+1)} \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\ &= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} \\ &= 1 - 0 = 1.\end{aligned}$$

So the value of this series is 1.

2.5 A test for divergence

Usually, the sequence of partial sums $s_n = a_0 + a_1 + \cdots + a_n$ is harder to understand and analyze than the sequence of terms a_k. It would be helpful if we could say something about the complicated sequence s_n by studying the easier-to-understand sequence a_k.

Specifically, if the sequence s_n converges, what can be said about the sequence a_k? If adding up more and more terms from the sequence a_k gets closer and closer to some number, then the size of the terms of a_k had better be getting very small. Let's make this precise.

Theorem 2.5.1 *If* $\sum_{k=0}^{\infty} a_k$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Proof *Intuitively, this should seem reasonable: after all, if the terms in the sequence* a_n *were getting very large (that is, not converging to zero), then adding up those very large numbers would prevent the series* $\sum_{k=0}^{\infty} a_k$ *from converging.*

We can put this intuitive thinking on a firm foundation. Say $\sum_{k=0}^{\infty} a_k$ *converges to L, meaning* $\lim_{n\to\infty} s_n = L$. *But then also* $\lim_{n\to\infty} s_{n-1} = L$, *because that sequence amounts to saying the same thing, but with the terms renumbered. By Theorem* **??**,

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

Replacing s_n *with* $a_0 + \cdots + a_n$, *we get*

$$s_n - s_{n-1} = (a_0 + a_1 + a_2 + \cdots + a_n) - (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_n,$$

and therefore, $\lim_{n\to\infty} a_n = 0$.

The contrapositive of Theorem 2.5.1 can be used as a divergence test.

Theorem 2.5.2 *Consider the series* $\sum_{k=0}^{\infty} a_k$. *If the limit* $\lim_{n\to\infty} a_n$ *does not exist or has a value other than zero, then the series diverges.*

We'll usually call this theorem the "n^{th} term test."

Warning The converse of Theorem 2.5.1 is *not* true: even if $\lim_{n\to\infty} a_n = 0$, the series could diverge.

This is a *very* common mistake: you might be tempted to show that a series converges by showing $\lim_{n\to\infty} a_n = 0$, but that doesn't work. The n^{th} term test **either says "diverges!" or says nothing at all.** It is not possible to show that anything converges by using the n^{th} term test.

For an example, see Section 2.6.

One analogy that can be helpful is think about weather: whenever it is raining, it is cloudy. Yet it is possible for there to be clouds, even on a rainless day. [7]

Likewise, whenever the series $\sum_{k=0}^{\infty} a_k$ converges ("it is raining"), the sequence a_k converges to zero ("it is cloudy"). If it isn't cloudy, then we can be sure it isn't raining—and this is the statement of Theorem 2.5.2. Let's use this "divergence test" to show that a particular series diverges.

[7] We first introduced this idea on Page 24.

Example 2.5.3 Show that $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Solution We apply the n^{th} term test: all we need to do is to compute the limit

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Since the limit exists but is not zero (i.e., "it is not cloudy"), the series must diverge ("it can't be raining.").

Looking at the first few terms perhaps makes it clear that the series has no chance of converging:

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

will just get larger and larger; indeed, after a bit longer the series starts to look very much like $\cdots + 1 + 1 + 1 + 1 + \cdots$, and if we add up many numbers which are very close to one, then we can make the sum as large as we desire.

2.6 Harmonic series

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ has a special name.

Definition The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is called the **harmonic series**.

The main question for this section—indeed, the question which should always be our first question anytime we see an unknown series—is the following question:

Does the harmonic series converge. . . or diverge?

How can we begin to explore this question?

2.6.1 The limit of the terms

When confronted with the question of whether a series diverges or converges, the first thing to check is the limit of the terms—if that limit doesn't exist, or does exist but equals a number other than zero, then the series diverges.

In general, the easiest way to prove that a series diverges is to apply the n^{th} term test from Theorem 2.5.2. What does the n^{th} term test tell us for the harmonic series? We calculate

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

so the n^{th} term test is *silent*; since the limit of the terms of the series exists and is equal to zero, the n^{th} term test does not tell us any information.

The harmonic series passed the first gauntlet—but that does not mean the harmonic series will survive the whole game. All we know is that the harmonic does doesn't diverge for the most obvious reason, but whether it diverges or converges is yet to be determined.

2.6.2 Numerical evidence

If you have the fortitude[8] to add up the first hundred terms, you will find that

[8] lacking fortitude, software will suffice

$$\sum_{n=1}^{100} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{100}$$
$$= \frac{14466636279520351160221518043104131447711}{2788815009188499086581352357412492142272} \approx 5.19.$$

If we add up the first thousand terms, we will find that

$$\sum_{n=1}^{1000} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{1000} \approx 7.49.$$

If we add up the first ten thousand terms, we will find that

$$\sum_{n=1}^{10000} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{10000} \approx 9.79$$

The partial sums are getting bigger, but not very quickly. Maybe the series converges. Maybe it converges to... about ten?

2.6.3 An analytic argument

That numeric evidence might have made us think otherwise, but **the harmonic series diverges.**

But in fact the partial sums do get arbitrarily large; they just get big very, very slowly. Consider the following:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{16} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

and so on. By swallowing up more and more terms we can always manage to add at least another $1/2$ to the sum, and by adding enough of these we can make the partial sums as big as we like. In fact, it's not hard to see from this pattern that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} > 1 + \frac{n}{2},$$

so to make sure the sum is over 100, for example, we'd add up terms until we get to around $1/2^{198}$, that is, about $4 \cdot 10^{59}$ terms.

2.7 Comparison test

A bounded, monotonic sequence necessarily converges (Theorem 1.8.2). How does this fact about sequences relate to series? When is the sequence of partial sums monotonic? If the terms of a series are non-negative, then the associated sequence of partial sums is non-decreasing.

Corollary 2.7.1 *Consider the series* $\sum_{k=0}^{\infty} a_k$*. Assume the terms* a_k *are non-negative. If the sequence of partial sums* $s_n = a_0 + \cdots + a_n$ *is bounded, then the series converges.*

So we can show that a series of positive terms converges, provided we can bound the sequence of partial sums.

2.7.1 Statement of the Comparison Test

But how can we manage to do that? One way to ensure that the sequence of partial sums is bounded is by **comparing** the series to another series. Consider two series

$$\sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k.$$

Suppose, for all k, that $b_k \geq a_k \geq 0$. Then

$$a_0 + a_1 + \cdots + a_n \leq b_0 + b_1 + \cdots + b_n.$$

Suppose that $\sum_{k=0}^{\infty} b_k$ converges to L. Then

$$a_0 + a_1 + \cdots + a_n \leq b_0 + b_1 + \cdots + b_n \leq L,$$

so the sequence of partial sums $s_n = a_0 + a_1 + \cdots + a_n$ is bounded. But we just won the game: each term a_k is nonnegative, so the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ is increasing. Theorem 1.8.2 guarantees that the sequence (s_n) converges.

Let's summarize what just happened: if a series with positive terms is, termwise, less than a convergent series, it converges. We have just proved half of the following theorem.

Theorem 2.7.2 *Suppose that a_n and b_n are non-negative for all n and that, for some N, whenever $n \geq N$, we have $a_n \leq b_n$.*

If $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.

If $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.

This is usually called the **Comparison Test**; we might summarize it like this:

- A non-negative series, overestimated by a convergent series, converges.
- A non-negative series, underestimated by a divergent series, diverges.

Warning Being less than a divergent series does not help: the comparison test is silent in that case.

Similarly, being larger than a convergent series does not help. The Comparison Test only says something when a series (with non-negative terms!) is less than a convergent series, or greater than a divergent series.

2.7.2 Applications of the Comparison Test

Like the n^{th} term test (Theorem 2.5.2), we can use the Comparison Test (Theorem 2.7.2) to show that a series diverges.

Example 2.7.3 Does the series $\sum_{n=2}^{\infty} \frac{\log n}{n}$ converge?

Solution Our first inclination might be to apply the n^{th} term test, but in this case,

$$\lim_{n\to\infty} \frac{\log n}{n} = 0,$$

so the n^{th} term test is silent in this case. As far as we know at this point, the series may diverge or converge.

Instead, we'll try the Comparison Test. Set $a_n = \frac{1}{n}$ and $b_n = \frac{\log n}{n}$. Note that whenever $n \geq 3$, we have

$$0 \leq a_n \leq b_n,$$

but the series $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, and so by the Comparison Test, the given series (which is even bigger!) must likewise diverge.

Recall that the n^{th} term test *cannot* be used to prove that a series converges; if the n^{th} term test does not answer "diverges!" then the test is silent. In wonderful contrast, the Comparison Test *can* be used to show that a series converges.

Example 2.7.4 Does the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ converge?

Solution Yes. Set

$$a_n = \frac{\sin^2 n}{2^n} \text{ and } b_n = \frac{1}{2^n}$$

Note that $0 \leq a_n \leq b_n$. But the series $\sum_{n=1}^{\infty} b_n$ converges, since it is a geometric series with common ratio $1/2$, as in Example 2.2.2. Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges by the comparison test.

2.7.3 Cauchy Condensation Test

Remember in Section 2.6 when we considered the harmonic series? We showed that it diverged by comparing it with the divergent series $\sum_{n=1}^{\infty} \frac{1}{2}$, but we couldn't make that comparison right away—first we had to group together the terms in a somewhat complicated seeming way.

We can generalize that "grouping together" trick; this is called the Cauchy Condensation Test.

If you have already seen some convergence tests before—perhaps you have already been through Calculus Two!—you might be wondering why "condensation" is making an appearance. It is perhaps less popular than other tests, but I like it. Pedagogically, it is nice to see that the "trick" in the harmonic series can be generalized and applied to lots of other series. In particular, condensation permits the study of p-series without going through the usual route of the Integral Test.

Theorem 2.7.5 *Suppose (a_n) is a non-increasing sequence of positive numbers. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} (2^n a_{2^n})$ converges.*

The series $\sum_{n=0}^{\infty} (2^n a_{2^n})$ is often called the **condensed** series associated to the series $\sum_{n=1}^{\infty} a_n$.

Proof *Let's suppose that $\sum_{n=0}^{\infty} (2^n a_{2^n})$ converges; the goal then is to show that $\sum_{n=1}^{\infty} a_n$ also converges.*

Since the sequence (a_n) is decreasing, we have that

$$
\begin{aligned}
a_2 + a_3 &\leq a_2 + a_2 \\
a_4 + a_5 + a_6 + a_7 &\leq a_4 + a_4 + a_4 + a_4 \\
a_8 + \cdots + a_{15} &\leq 8\, a_8 \\
&\vdots \\
a_{2^n} + \cdots + a_{2^{n+1}-1} &\leq 2^n\, a_{2^n}.
\end{aligned}
$$

Therefore,

$$\sum_{n=1}^{2^k-1} a_n \leq \sum_{n=0}^{k-1} 2^n\, a_{2^n}.$$

As a result, the sequence of partial sums $s_k = \sum_{n=1}^{k} a_n$ *is bounded above by* $\sum_{n=0}^{\infty} 2^n\, a_{2^n}$. *Moreover, the sequence of partial sums* (s_k) *is increasing. Therefore, by the Monotone Convergence Theorem, the series* (s_k) *converges.*

On the other hand, suppose that $\sum_{n=1}^{\infty} a_n$ *converges; let's show that* $\sum_{n=0}^{\infty} (2^n a_{2^n})$ *also converges. Once we have done so, we will have shown that* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=0}^{\infty} (2^n a_{2^n})$ *converges.*

Since the sequence (a_n) *is decreasing, we have that*

$$\begin{aligned}
a_1 + a_2 &< a_1 + a_1 \\
a_2 + 3\,a_4 &< a_2 + a_2 + a_3 + a_3 \\
a_4 + 7\,a_8 &< a_4 + a_4 + a_5 + a_5 + a_6 + a_6 + a_7 + a_7 \\
a_8 + 15\,a_{16} &< 2(a_8 + \cdots + a_{15}) \\
&\vdots \\
a_{2^n} + (2^{n+1} - 1)\,a_{2^{n+1}} &< 2(a_{2^n} + \cdots + a_{2^{n+1}-1}).
\end{aligned}$$

So the sequence of partial sums for the series $\sum_{n=0}^{k} (2^n a_{2^n})$ *are bounded above by* $2 \cdot \sum_{n=1}^{\infty} a_n$. *Moreover, that sequence of partial sums is increasing, and therefore, by the Monotone Convergence Theorem, the series* $\sum_{n=0}^{\infty} (2^n a_{2^n})$ *converges—which is what we wanted to show.*

What we've shown is a bit stronger than simply that the original series and the condensed series share the same fate—converging or diverging together. In fact, we have an estimate on the value of the original series, in terms of the value of the condensed series. We have shown that

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=0}^{k} (2^n a_{2^n}) \leq 2 \cdot \sum_{n=1}^{\infty} a_n.$$

2.7.4 Examples of condensation

Example 2.7.6 Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

This is *not* a geometric series: we already know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, but this is asking about something very different, namely $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution To get some intuition for what is going on, let's do some numerical calculations.

$$\begin{aligned} \sum_{n=1}^{10} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \\ &= \frac{1968329}{1270080} \approx 1.5498, \end{aligned}$$

or going out a bit farther,

$$\sum_{n=1}^{100} \frac{1}{n^2} = \frac{1}{1^2} + \cdots + \frac{1}{100^2} \approx 1.6350 \text{ and}$$

$$\sum_{n=1}^{1000} \frac{1}{n^2} = \frac{1}{1^2} + \cdots + \frac{1}{1000^2} \approx 1.6439.$$

From this numerical evidence, it certainly *looks* like this series converges. And indeed, it does—quite surprisingly,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is the so-called Basel problem.

We do not yet have the tools necessarily to show that the value of the series is $\pi^2/6$, but do we have the tools needed to show that the series converges. By condensation, it suffices to show that $\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^2}$ converges. But

$$\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{1}{(2^n)} = 1,$$

and since the "condensed" series converges, so too must the original series converge.

Example 2.7.7 Does $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ converge?

Solution We can't apply Cauchy condensation here, because the terms of this series are not decreasing. But we can apply the Comparison Test. Moments ago, we saw that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2},$$

because $|\sin n| \leq 1$. The partial sums are non-decreasing and bounded above by $\sum_{n=1}^{\infty} 1/n^2 = L$, so the series converges.

2.7.5 Convergence of p-series

Let us consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Such a series is called a **p-series**. Does a p-series converge? Diverge? It depends on p.

If we think of this as a function of p, then we have the **Riemann zeta function**, that is,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The Riemann zeta function is quite important: it plays a key role in number theory via the **Riemann hypothesis** and also has applications in physics. Something that connects the physical world to number theory must be pretty incredible.

Example 2.7.8 Let $p \leq 1$. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Solution When $p = 1$, this series is the harmonic series we already proved to diverge in Section 2.6.

But more generally, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges whenever $p \leq 1$. We will show this by comparing to a harmonic series. Since $p \leq 1$, then $n^p \leq n$, and so

$$\frac{1}{n^p} \geq \frac{1}{n}.$$

But the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and so by comparison, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Example 2.7.9 Let $p > 1$. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Solution It converges. For this, we use Cauchy condensation: consider the "condensed" series

$$\sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p}.$$

But this series simplifies to

$$\sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^{p-1})^n},$$

▮ which converges.

Exercises for Section 2.7

(1) Explain why $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ diverges.

(2) Explain why $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^{1/n}+14}$ diverges.

(3) Explain why $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n}$ diverges.

(4) Compute $\displaystyle\sum_{n=0}^{\infty} \frac{4}{(-3)^n} - \frac{3}{3^n}$.

(5) Compute $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} + \frac{4}{5^n}$.

(6) Compute $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$.

(7) Compute $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^{n+1}}$.

(8) Compute $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$.

(9) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$.

3 Convergence tests

It is generally quite difficult—indeed, often impossible—to determine the value of a series exactly. Even if we can't compute the value of a series, in many cases it is possible to determine whether or not the series converges. We will spend most of our time on this problem.

3.1 Ratio tests

Does the series $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converge? It is possible, but a bit unpleasant, to approach this with the comparison test.

Mathematics is more than just about getting answers; a goal of mathematics is not only to find truth, but to package the resulting "truth" in a format that permits another human being to understand the reasons for its being true. Arguments like this—which are perfectly convincing but seem entirely unmotivated—are, arguably, missing the point. What has been gained if we find something is true but the reason for its being true remains inscrutable?

Example 3.1.1 The series $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converges.

Solution As long as $n \geq 23$, we have $2^n \geq n^5$. Therefore, as long as $n \geq 23$, we have

$$\frac{n^5}{5^n} \leq \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$$

This idea—that the first handful of terms do not affect convergence at all—will be discussed formally in Section 5.1.

But the geometric series $\sum_{n=23}^{\infty} \left(\frac{2}{5}\right)^n$ converges, since the ratio between subsequent terms is less than one. So by comparison, the smaller series

$$\sum_{n=23}^{\infty} \frac{n^5}{5^n}$$

must also converge. The first handful of terms when $n < 23$ doesn't affect convergence at all, so we are justified to conclude that the original series $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converges.

That worked—but it invoked an unmotivated fact: how do I know that $2^n \geq n^5$ whenever $n \geq 23$? Invoking that fact seems a bit random—yes, yes, a proof, but perhaps not a proof that conveys exactly what is going on. It is a valid argument, but missing some motivation.

3.1.1 Theory

Instead, consider what happens as we move from one term to the next term in this series, that is, consider two neighboring terms

$$\cdots + \frac{n^5}{5^n} + \frac{(n+1)^5}{5^{n+1}} + \cdots.$$

The denominator goes up by a factor of 5, $5^{n+1} = 5 \cdot 5^n$, but the numerator goes up by much less: $(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$, which is much less than $5n^5$ when n is large, because $5n^4$ is much less than n^5. (This sort of thinking is *why* it was worth comparing n^5 to an exponential 2^n for n large.)

So we might guess that in the long run—when n is very large—it begins to look as if each term is about $1/5$ of the previous term. We have seen series that behave like this, namely the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{5}{4}.$$

We are beginning to see why it made sense to compare the given series to a geometric series as in the initially very unmotivated argument above.

But we can do better! Instead of an *ad hoc* argument which compared n^5 to 2^n, we can try to make rigorous the idea that each term is "eventually" about a fifth as big as the previous term. The key is to notice that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{5^{n+1}} \frac{5^n}{n^5} = \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \frac{1}{5} = 1 \cdot \frac{1}{5} = \frac{1}{5}.$$

This is a more formal version of what we noticed about the ratio of subsequent terms: in the long run, each term is one fifth of the previous term. Pick some number just slightly bigger than $\frac{1}{5}$; let's call that number $\frac{1}{5} + \varepsilon$.

The symbol ε is the Greek letter *epsilon*, and, conventionally, denotes a small but positive number.

Because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5},$$

then by choosing n big enough, say $n \geq N$ for some N, we can guarantee that $\frac{a_{n+1}}{a_n}$ is as close as we'd like to $\frac{1}{5}$, say within ε of $\frac{1}{5}$. More succinctly, there must be some N so that whenever $n \geq N$, we have

$$\frac{a_{n+1}}{a_n} < \frac{1}{5} + \varepsilon$$

Multiplying both sides by a_n, we find whenever $n \geq N$ that

$$a_{n+1} < \left(\frac{1}{5} + \varepsilon\right) a_n.$$

We can say the same thing when n is replaced by $N + 1$, meaning

$$a_{N+2} < \left(\frac{1}{5} + \varepsilon\right) a_{N+1},$$

which together with the previous statement when n is N implies

$$a_{N+2} < \left(\frac{1}{5} + \varepsilon\right)^2 a_N.$$

And we can repeat this again! Since

$$a_{N+3} < \left(\frac{1}{5} + \varepsilon\right) a_{N+2},$$

we also learn that

$$a_{N+3} < \left(\frac{1}{5} + \varepsilon\right)^3 a_N,$$

or in general,

$$a_{N+k} < \left(\frac{1}{5} + \varepsilon\right)^k a_N.$$

This is setting up a comparison. The geometric series

$$\sum_{k=0}^{\infty} \left(\frac{1}{5} + \varepsilon\right)^k a_N = \left(\sum_{k=0}^{\infty} \left(\frac{1}{5} + \varepsilon\right)^k\right) a_N$$

converges as long as ε had been chosen so small that $\frac{1}{5} + \varepsilon < 1$. But if that series converges, then the series

$$\sum_{k=0}^{\infty} a_{N+k} = a_N + a_{N+1} + a_{N+2} + \cdots$$

also converges, because each term of that series is smaller than the corresponding term of the convergent geometric series. But if *that* series converges, then

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots$$

converges since we've just added the number $a_0 + a_1 + \cdots + a_{N-1}$ to the convergent series $\sum_{k=0}^{\infty} a_{N+k}$.

Under what circumstances could we do this? What was crucial was that the limit of a_{n+1}/a_n, say L, was less than 1 so that we could pick a value ε so that $L + \varepsilon < 1$ and then compare to the convergent series

$$\sum_{k=0}^{\infty} (L + \varepsilon)^k \cdot a_N.$$

That's really all that is required to make the argument work. We also made use of the fact that the terms of the series were positive. Let's summarize the situation when this works.

In general, we can consider instead the absolute values of the terms, and end up testing for **absolute convergence.** We'll discuss this topic in Section 4.1.

Theorem 3.1.2 *(The Ratio Test) Consider the series* $\sum_{n=0}^{\infty} a_n$ *where each term* a_n *is positive. Suppose that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

If $L < 1$ *the series* $\sum_{n=0}^{\infty} a_n$ *converges. If* $L > 1$ *the series diverges. If* $L = 1$ *this test is inconclusive.*

Proof *Example 3.1.1 essentially proves the first part of this, if we simply replace* $1/5$ *by* L *and* $1/2$ *by* r*. So when* $L < 1$*, the series converges.*

Let's consider the other situation. Suppose that $L > 1$*, and pick* r *so that* $1 < r < L$*. Then for* $n \geq N$*, for some* N*,*

$$\frac{|a_{n+1}|}{|a_n|} > r \quad \text{and} \quad |a_{n+1}| > r|a_n|.$$

This implies that $|a_{N+k}| > r^k|a_N|$*, but since* $r > 1$ *this means that* $\lim_{k\to\infty} |a_{N+k}| \neq 0$*, which means also that* $\lim_{n\to\infty} a_n \neq 0$*. By the divergence test, the series diverges.*

Finally, when $L = 1$*, the test truly is inconclusive—not because we aren't clever enough, but because there are both convergent and divergent series with* $L = 1$*. For example, the p-series* $\sum_{n=1}^{\infty} 1/n^2$ *converges, and the harmonic series* $\sum_{n=1}^{\infty} 1/n$ *diverges—but in both cases,* $L = 1$*.*

3.1.2 Practice

The ratio test is particularly useful for series involving the factorial function.

"It works in practice—but does it work in theory?" The ratio test is awfully useful in practice, but I think the theory behind it—that we're taking a limit of the ratio between neighboring terms in order to compare with a geometric series—is quite lovely.

Example 3.1.3 Does the series

$$\sum_{n=0}^{\infty} 5^n/n!$$

converge or diverge?

Remember our convention that $0! = 1$. Despite appearances, we are not dividing by zero.

Solution Let's name the terms $a_n = 5^n/n!$ so we are considering $\sum_{n=0}^{\infty} a_n$. To apply the ratio test, we consider

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} \\ &= \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} 5 \frac{1}{(n+1)} = 0.\end{aligned}$$

Since $0 < 1$, the series converges by the ratio test.

Exercises for Section 3.1

(1) Compute $\lim_{n\to\infty} |a_{n+1}/a_n|$ for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(2) Compute $\lim_{n\to\infty} |a_{n+1}/a_n|$ for the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Determine whether each of the following series converges or diverges.

(3) $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^n}$

(4) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(5) $\sum_{n=1}^{\infty} \frac{n^5}{n^n}$

(6) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$

3.2 Integral test

If all of the terms a_n in a series are non-negative, then clearly the sequence of partial sums s_n is non-decreasing. This means that if we can show that the sequence of partial sums is bounded, the series must converge. We know that if the series converges, the terms a_n approach zero, but this does not mean that $a_n \geq a_{n+1}$ for every n. Many useful and interesting series do have this property, however, and they are among the easiest to understand. Let's look at an example.

3.2.1 An example

Example 3.2.1 Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Solution We have actually already seen this very series—it appeared in Example 2.7.6, where we attacked this series via condensation. Here, we consider a different approach, namely integration.

The terms $1/n^2$ are positive and decreasing, and since $\lim_{x\to\infty} 1/x^2 = 0$, the terms $1/n^2$ approach zero. We seek an upper bound for all the partial sums, that is, we want to find a number N so that $s_n \leq N$ for every n. The upper bound is provided courtesy of integration.

Figure 3.1 shows the graph of $y = 1/x^2$ together with some rectangles that lie completely below the curve and that all have base length one. Because the heights of the rectangles are determined by the height of the curve, the areas of the rectangles are $1/1^2$, $1/2^2$, $1/3^2$, and so on—in other words, exactly the terms of the series. The partial sum s_n is simply the sum of the areas of the first n rectangles. Because the rectangles all lie between the curve and the x-axis, any sum of rectangle areas is less than the corresponding area under the curve, and so of course any sum of rectangle areas is less than the area under the entire curve, that is, all the way to infinity. There is a bit of trouble at the left end, where there is an asymptote, but we can work around that

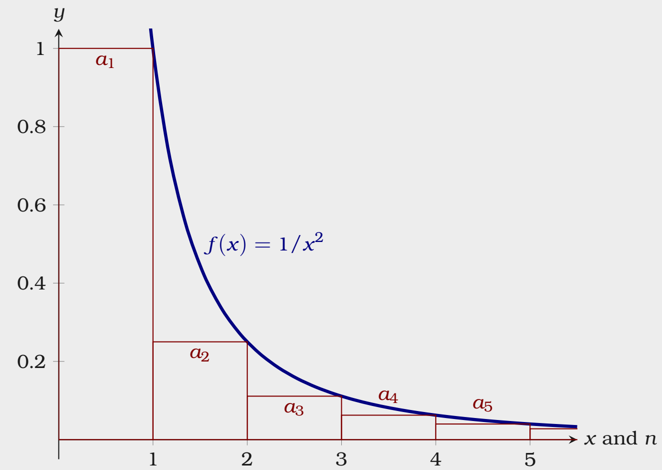

Figure 3.1: Plot of $f(x) = 1/x^2$ alongside boxes representing $a_n = 1/n^2$.

easily. Here it is:

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \int_1^n \frac{1}{x^2}\,dx < 1 + \int_1^\infty \frac{1}{x^2}\,dx = 1 + 1 = 2,$$

recalling how to compute this improper integral. Since the sequence of partial sums s_n is increasing and bounded above by 2, we know that $\lim_{n\to\infty} s_n = L < 2$, and so the series converges to some number less than 2.

In fact, it is possible, though difficult, to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

3.2.2 Harmonic series

We already know that $\sum 1/n$ diverges. What goes wrong if we try to apply this technique to it? We find

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx < 1 + \int_1^\infty \frac{1}{x}\,dx$$

but this amounts to saying nothing, because the improper integral $\int_1^\infty \frac{1}{x}\,dx$ does not converge, and claiming that s_n is bounded by something divergent is to make no claim at all: every real number is less than infinity! In other words, this does *not* prove that $\sum 1/n$ diverges; it is just that this particular calculation fails to prove that it converges. A slight modification, however, allows us to prove in a second way that $\sum 1/n$ diverges.

Example 3.2.2 Consider a slightly altered version of Figure 3.1, shown in Figure 3.2. Explain how to use the figure to see that the harmonic series diverges.

Solution The rectangles this time are above the curve, that is, each rectangle completely contains the corresponding area under the curve. This means that

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \ln x\Big|_1^{n+1} = \ln(n+1).$$

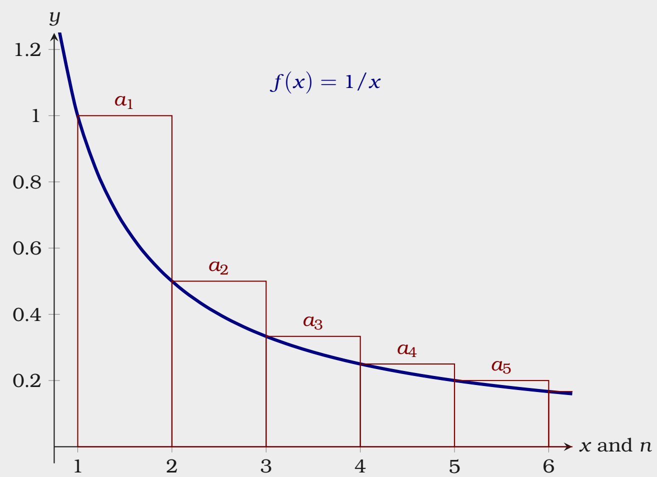

Figure 3.2: Plot of $f(x) = 1/x$ alongside boxes representing $a_n = 1/n$.

As n gets bigger, $\ln(n+1)$ goes to infinity, so the sequence of partial sums s_n must also go to infinity, so the harmonic series diverges.

3.2.3 Statement of integral test

The important fact that clinches this example is that

$$\lim_{n\to\infty}\int_1^{n+1}\frac{1}{x}\,dx=\infty,$$

which we can rewrite as

$$\int_1^{\infty}\frac{1}{x}\,dx=\infty.$$

So these two examples taken together indicate that we can prove that a series converges or prove that it diverges with a single calculation of an improper integral. This is known as the **integral test**, which we state as a theorem.

Theorem 3.2.3 (Integral test) *Suppose that $f(x)>0$ and is decreasing on the infinite interval $[k,\infty)$ (for some $k\geq 1$) and that $a_n=f(n)$. Then the series $\sum_{n=k}^{\infty}a_n$ converges if and only if the improper integral $\int_k^{\infty}f(x)\,dx$ converges.*

3.2.4 p-series

The two examples we have seen are examples of p-series, which we first encountered in Subsection 2.7.5. Recall that a p-series is any series of the form $\sum_{n=1}^{\infty}1/n^p$.

Theorem 3.2.4 *A p-series converges if and only if $p>1$.*

We already proved this theorem using condensation in Example 2.7.8 and Example 2.7.9. Nevertheless, we provide a second proof, using the integral test.

Proof *We use the integral test; the case $p = 1$ is that of the harmonic series, which we know diverges, so without loss of generality, we may assume that $p \neq 1$. If $p \leq 0$, $\lim_{n\to\infty} 1/n^p \neq 0$, so the series diverges by Theorem 2.5.2. So we may also assume that $p > 0$.*

Then we compute

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{N\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^N = \lim_{N\to\infty} \frac{N^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $p > 1$ then $1 - p < 0$ and $\lim_{N\to\infty} N^{1-p} = 0$, so the integral converges. If $0 < p < 1$ then $1 - p > 0$ and $\lim_{N\to\infty} N^{1-p} = \infty$, so the integral diverges.

Example 3.2.5 Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Solution We could of course use the integral test, but now that we have the theorem we may simply note that this is a p-series with $p = 3 > 1$.

Example 3.2.6 Show that $\sum_{n=1}^{\infty} \frac{5}{n^4}$ converges.

Solution We know that if $\sum_{n=1}^{\infty} 1/n^4$ converges then $\sum_{n=1}^{\infty} 5/n^4$ also converges, by Theorem 2.3.1. Since $\sum_{n=1}^{\infty} 1/n^4$ is a convergent p-series, $\sum_{n=1}^{\infty} 5/n^4$ converges also.

Example 3.2.7 Show that $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$ diverges.

Solution This also follows from Theorem 2.3.4. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, it diverges, and so does $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$.

3.2.5 Integrating for approximations

Since it is typically difficult to compute the value of a series exactly, a good approximation is frequently required. In a real sense, a good approximation is only as good as we know it is, that is, while an approximation may in fact be good, it is only valuable in practice if we can guarantee its accuracy to some degree[1]. This guarantee is usually easy to come by for series with decreasing positive terms.

[1] After all, $\pi \approx 17$ just with very bad error bounds. It is better to make a statement like $|\pi - 17| < 14$, which is not only saying that π is "close" to 17, but is quantifying exactly how close (within 14—so perhaps not all that close).

Example 3.2.8 Approximate $\sum_{n=1}^{\infty} 1/n^2$ to two decimal places.

It turns out that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, so your approximation of this series will also—in a roundabout way—yield an approximate value for π—and one which will be better than 17.

Solution Referring to Figure 3.1, if we approximate the sum by $\sum_{n=1}^{N} 1/n^2$, the error we make is the total area of the remaining rectangles, all of which lie under the curve $1/x^2$ from $x = N$ out to infinity. So we know the true value of the series is larger than the approximation, and no bigger than the approximation plus the area under the curve from N to infinity. Roughly, then, we need to find N so that

$$\int_N^{\infty} \frac{1}{x^2}\,dx < 1/100.$$

We can compute the integral:

$$\int_N^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N},$$

so $N = 100$ is a good starting point. Adding up the first 100 terms gives approximately 1.634983900, and that plus $1/100$ is 1.644983900, so approximating the series by the value halfway between these will be at most $1/200 = 0.005$

in error. The midpoint is 1.639983900, but while this is correct to ±0.005, we can't tell if the correct two-decimal approximation is 1.63 or 1.64. We need to make N big enough to reduce the guaranteed error, perhaps to around 0.004 to be safe, so we would need $1/N \approx 0.008$, or $N = 125$. Now the sum of the first 125 terms is approximately 1.636965982, and that plus 0.008 is 1.644965982 and the point halfway between them is 1.640965982. The true value is then 1.640965982 ± 0.004, and all numbers in this range round to 1.64, so 1.64 is correct to two decimal places.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, our estimate yields

$$1.63 < \frac{\pi^2}{6} < 1.65,$$

and so

$$9.78 < \pi^2 < 9.90.$$

which means that

$$3.127 < \pi < 3.147,$$

which is better than $\pi \approx 3$. And we have explicit bounds on the error.

Exercises for Section 3.2

Determine whether each series converges or diverges.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi/4}}$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

(7) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

(8) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(9) Find an N so that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is between $\displaystyle\sum_{n=1}^{N} \frac{1}{n^4}$ and $\displaystyle\sum_{n=1}^{N} \frac{1}{n^4} + 0.005$.

(10) Find an N so that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ is between $\displaystyle\sum_{n=0}^{N} \frac{1}{e^n}$ and $\displaystyle\sum_{n=0}^{N} \frac{1}{e^n} + 10^{-4}$.

(11) Find an N so that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is between $\displaystyle\sum_{n=1}^{N} \frac{\ln n}{n^2}$ and $\displaystyle\sum_{n=1}^{N} \frac{\ln n}{n^2} + 0.005$.

(12) Find an N so that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is between $\displaystyle\sum_{n=2}^{N} \frac{1}{n(\ln n)^2}$ and $\displaystyle\sum_{n=2}^{N} \frac{1}{n(\ln n)^2} + 0.005$.

3.3 More comparisons

Armed with the ratio test and the integral test, it can be tempting to apply them frequently. But when faced with a new series $\sum_{n=1}^{\infty} a_n$, what should we do? First, consider $\lim_{n\to\infty} a_n$ and if that limit is nonzero, then by Theorem 2.5.2 the series diverges. If the series passes that first test, then it is worth considering other tests like the ratio test (perhaps if a_n has factorials and powers) or the integral test (if $a_n = f(n)$ and $\int f(x)\,dx$ is not too hard to compute).

But the ratio test and integral test are not the only tools in our toolbox. The comparison test, which we discussed in Section 2.7, is extremely useful.

Example 3.3.1 Does $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ converge?

When I write log, I mean the natural logarithm. I prefer writing $\log n$ to writing $\ln n$.

Solution The obvious first approach, based on what we know, is the integral test. Unfortunately, we can't compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2 \log n} < \frac{1}{n^2},$$

when $n \ge 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ converges if and only if $\sum_{n=3}^{\infty} \frac{1}{n^2 \log n}$ converges—all we've done is dropped the initial term. We know that $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges. Looking at two typical partial sums:

$$s_n = \frac{1}{3^2 \log 3} + \frac{1}{4^2 \log 4} + \frac{1}{5^2 \log 5} + \cdots + \frac{1}{n^2 \log n} < \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} =$$

Since the p-series converges, say to L, and since the terms are positive, $t_n < L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n < t_n < L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges.

Sometimes, even when the integral test applies, comparison to a known series is easier, so it is a good idea to think about doing a comparison before doing the integral test.

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

Example 3.3.2 Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$ converge?

Solution We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2} = 1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n},$$

so that

$$s_n = \frac{1}{\sqrt{2^2-3}} + \frac{1}{\sqrt{3^2-3}} + \cdots + \frac{1}{\sqrt{n^2-3}} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n = 2$ instead of $n = 1$). Since $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} s_n = \infty$ as well.

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series. This is more of an art than a science, which is part of what makes these sorts of problems so much fun to do.

Example 3.3.3 Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ converge?

Solution Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately,

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{n},$$

so we can't compare the series directly to the harmonic series. A little thought leads us to

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n},$$

so if $\sum 1/(2n)$ diverges then the given series diverges. But since $\sum 1/(2n) = (1/2)\sum 1/n$, theorem **??** implies that it does indeed diverge.

Exercises for Section 3.3

Determine whether the series converge or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+3n+5}$

(2) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-3n-5}$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+4}{2n^2+3n+5}$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^2+3n+5}$

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^3}$

(8) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log n}$

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n}$

(10) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$

3.4 The mostly useless root test

There is another convergence test called the **root test**, which can be justified with an argument not so different[2] from that which justified the ratio test in Subsection 3.1.1. The root test is *very occasionally* easier to apply, but usually not as good as choosing to use the ratio test.

[2] Indeed, justifying the root test makes a good exercise for you, the reader, so it is included among the exercises for this section.

With those disparaging remarks out of the way, let us now state the root test.

Theorem 3.4.1 (The Root Test) *Consider the series* $\sum_{n=0}^{\infty} a_n$ *where each term* a_n *is positive. Suppose that* $\lim_{n\to\infty} (a_n)^{1/n} = L$. *Then,*

- *if* $L < 1$ *the series* $\sum a_n$ *converges,*
- *if* $L > 1$ *the series diverges, and*
- *if* $L = 1$, *then the root test is inconclusive.*

Let's apply the root test to analyze the convergence of a series.

Example 3.4.2 Analyze $\sum_{n=0}^{\infty} \frac{5^n}{n^n}$.

Solution Usually, the ratio test is a good choice when the series involves n^{th} powers; in this case, the ratio test turns out to be a bit difficult on this series, since we have to calculate

$$\lim_{n\to\infty} \frac{5^{n+1}/(n+1)^{n+1}}{5^n/n^n}$$

and that may not be entirely obvious. So we, begrudgingly, apply the root test, which asks us to calculate

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left(\frac{5^n}{n^n}\right)^{1/n} \\
&= \lim_{n\to\infty} \frac{(5^n)^{1/n}}{(n^n)^{1/n}} \\
&= \lim_{n\to\infty} \frac{5}{n} = 0.
\end{aligned}
$$

Since $L = 0 < 1$, we may conclude that the given series converges.

The root test is frequently useful when n appears as an exponent in the general term of the series—though the ratio test is also useful in that case. Technically, whenever the ratio test is conclusive (i.e., whenever $\lim\limits_{n\to\infty} a_{n+1}/a_n = L \neq 1$), so is the root test—but not vice versa. In other words, the root test *does* work on some series that the ratio test fails on.

Example 3.4.3 Find a series for which the ratio test is inconclusive, but the root test determines that the series converges.

Solution Here is such a situation. Try using the ratio test on $\sum\limits_{n=1}^{\infty} a_n$ where (a_n) is a sequence which "stutters" like

$$
a_n = \begin{cases} 1/2^{n/2} & \text{if } n \text{ is even, and} \\ 1/2^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}
$$

Note that $a_1 = a_2$ and $a_3 = a_4$ and $a_5 = a_6$, so a_{n+1}/a_n is often 1, which messes up the ratio test—indeed, $\lim\limits_{n\to\infty} a_{n+1}/a_n$ does not exist in this case. Nevertheless,

$$
\sqrt[n]{a_n} = \begin{cases} 1/2^{1/2} & \text{if } n \text{ is even, and} \\ 1/2^{(n+1)/(2n)} & \text{if } n \text{ is odd,} \end{cases}
$$

and so $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1/\sqrt{2} < 1$, which means the sequence converges by the root test. Admittedly, we didn't need the root test: this series is just a geometric series where the terms repeat, so it definitely converges. Still, it proves the point that the ratio test can fail while the root test succeeds.

Exercises for Section 3.4

(1) Prove theorem 3.4.1, the root test.

(2) Compute $\lim_{n\to\infty} |a_n|^{1/n}$ for the series $\sum 1/n^2$.

(3) Compute $\lim_{n\to\infty} |a_n|^{1/n}$ for the series $\sum 1/n$.

4 Alternating series

4.1 Absolute convergence

Roughly speaking there are two ways for a series to converge: As in the case of $\sum_{n=1}^{\infty} 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum_{n=1}^{\infty} (-1)^{n+1}/n$, the terms don't get small fast enough ($\sum_{n=1}^{\infty} 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough to do the job, then whether or not some terms are negative and some positive the series converges.

Theorem 4.1.1 *If* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof *Note that* $0 \le a_n + |a_n| \le 2|a_n|$ *so by the comparison test* $\sum_{n=1}^{\infty}(a_n + |a_n|)$ *converges. Now*

$$\sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} a_n$$

converges by Theorem 2.3.4.

So given a series $\sum_{n=1}^{\infty} a_n$ with both positive and negative terms, you should first ask whether $\sum_{n=1}^{\infty} |a_n|$ converges. This may be an easier question to answer, because we have tests that apply specifically to series with non-negative terms. If $\sum_{n=1}^{\infty} |a_n|$ converges then you know that $\sum_{n=1}^{\infty} a_n$ converges as well. If $\sum |a_n|$ diverges then it still may be true that $\sum_{n=1}^{\infty} a_n$ converges—you will have to do more work to decide the question. Another way to think of this result is: it is (potentially) easier for $\sum_{n=1}^{\infty} a_n$ to converge than for $\sum_{n=1}^{\infty} |a_n|$ to converge, because the latter series cannot take advantage of cancellation.

If $\sum_{n=1}^{\infty} |a_n|$ converges we say that $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**; to say that $\sum_{n=1}^{\infty} a_n$ converges absolutely is to say that any cancellation that happens to come along is not really needed, as the terms already get small so fast that convergence is guaranteed by that alone. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not, we say that

With this terminology, Theorem 4.1.1 is saying that an absolutely convergent series is also a plain old convergent series.

$\sum_{n=1}^{\infty} a_n$ converges **conditionally**. For example, $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2}$ converges absolutely, while $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ converges conditionally.

Example 4.1.2 Does $\sum_{n=2}^{\infty} \frac{\sin n}{n^2}$ converge?

Solution In Example 2.7.7, we saw that $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ converges, so the given series converges absolutely.

Example 4.1.3 Does $\sum_{n=1}^{\infty} (-1)^n \frac{3n+4}{2n^2+3n+5}$ converge?

Solution Taking the absolute value, $\sum_{n=1}^{\infty} \frac{3n+4}{2n^2+3n+5}$ diverges by comparison to $\sum_{n=1}^{\infty} \frac{3}{10n}$, so if the series converges it does so conditionally. It is true that $\lim_{n\to\infty} (3n+4)/(2n^2+3n+5) = 0$, so to apply the alternating series test we need to know whether the terms are decreasing. If we let $f(x) = (3x+4)/(2x^2+3x+5)$ then $f'(x) = -(6x^2+16x-3)/(2x^2+3x+5)^2$, and it is not hard to see that this is negative for $x \geq 1$, so the series is decreasing and by the alternating series test it converges.

Exercises for Section 4.1

Determine whether each series converges absolutely, converges conditionally, or diverges.

(1) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n^2+3n+5}$

(2) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n^2+4}{2n^2+3n+5}$

(3) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$

(4) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n^3}$

(5) $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln n}$

(6) $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+5^n}$

(7) $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+3^n}$

(8) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan n}{n}$

4.2 Alternating series test

Next we consider series with both positive and negative terms, but in a regular pattern: the signs alternate, as in the **alternating harmonic series** for example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \cdots$$
$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

In this series the sizes of the terms decrease, that is, $|a_n|$ forms a decreasing sequence, but this is not required in an alternating series. As with positive term series, however, when the terms do have decreasing sizes it is easier to analyze the series, much easier, in fact, than positive term series. Consider pictorially what is going on in the alternating harmonic series, shown in Figure 4.1. Because the sizes of the terms a_n are decreasing, the partial sums s_1, s_3, s_5, and so on, form a decreasing sequence that is bounded below by s_2, so this sequence must converge. Likewise, the partial sums s_2, s_4, s_6, and so on, form an increasing sequence that is bounded above by s_1, so this sequence also converges. Since all the even numbered partial sums are less than all the odd numbered ones, and since the "jumps" (that is, the a_i terms) are getting smaller and smaller, the two sequences must converge to the same value, meaning the entire sequence of partial sums $s_1, s_2, s_3, \ldots$ converges as well.

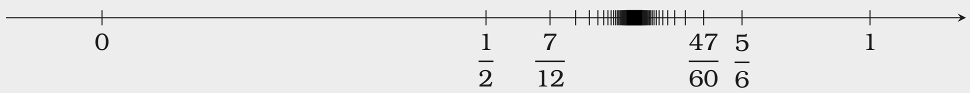

Figure 4.1: Partial sums of the alternating harmonic series

There's nothing special about the alternating harmonic series—the same argument works for any alternating sequence with decreasing size terms. The alternating series test is worth calling a theorem.

Theorem 4.2.1 *Suppose that (a_n) is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

Proof *The odd numbered partial sums, s_1, s_3, s_5, and so on, form a decreasing sequence, because $s_{2k+3} = s_{2k+1} - a_{2k+2} + a_{2k+3} \le s_{2k+1}$, since $a_{2k+2} \ge a_{2k+3}$. This sequence is bounded below by s_2, so it must converge, say $\lim_{k\to\infty} s_{2k+1} = L$. Likewise, the partial sums s_2, s_4, s_6, and so on, form an increasing sequence that is bounded above by s_1, so this sequence also converges, say $\lim_{k\to\infty} s_{2k} = M$. Since $\lim_{n\to\infty} a_n = 0$ and $s_{2k+1} = s_{2k} + a_{2k+1}$,*

$$L = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty}(s_{2k} + a_{2k+1}) = \lim_{k\to\infty} s_{2k} + \lim_{k\to\infty} a_{2k+1} = M + 0 = M,$$

so $L = M$, the two sequences of partial sums converge to the same limit, and this means the entire sequence of partial sums also converges to L.

We have considered alternating series with first index 1, and in which the first term is positive, but a little thought shows this is not crucial. The same test applies to any similar series, such as $\sum_{n=0}^{\infty}(-1)^n a_n$, $\sum_{n=1}^{\infty}(-1)^n a_n$, $\sum_{n=17}^{\infty}(-1)^n a_n$, etc.

We have shown more than convergence: if we are careful about thinking about the previous argument, we can find error bounds. Let's see how. Suppose that

$$L = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

and that we approximate L by a finite part of this sum, say

$$L \approx \sum_{n=1}^{N}(-1)^{n+1}a_n.$$

Because the terms are decreasing in size, we know that the true value of L must be between this approximation and the next one, that is, between

$$\sum_{n=1}^{N}(-1)^{n+1}a_n \quad \text{and} \quad \sum_{n=1}^{N+1}(-1)^{n+1}a_n.$$

Depending on whether N is odd or even, the second will be larger or smaller than the first. This is important enough that it deserves to be highlighted as a theorem.

Theorem 4.2.2 *Suppose that* $\{a_n\}_{n=1}^{\infty}$ *is a decreasing sequence of positive numbers and* $\lim_{n\to\infty} a_n = 0$. *By Theorem 4.2.1, we then know that* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges to some value, say L. Moreover, L is between*

$$\sum_{n=1}^{N} (-1)^{n+1} a_n \quad \text{and} \quad \sum_{n=1}^{N+1} (-1)^{n+1} a_n.$$

Example 4.2.3 Approximate the alternating harmonic series to one decimal place.

Solution We need to go roughly to the point at which the next term to be added or subtracted is $1/10$. Adding up the first nine and the first ten terms we get approximately 0.746 and 0.646. These are $1/10$ apart, but it is not clear how the correct value would be rounded. It turns out that we are able to settle the question by computing the sums of the first eleven and twelve terms, which give 0.737 and 0.653, so correct to one place the value is 0.7.

Exercises for Section 4.2

Determine whether the following series converge or diverge.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+5}$

(2) $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n-3}}$

(3) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n-2}$

(4) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

(5) Approximate $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$ to two decimal places.

(6) Approximate $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ to two decimal places.

5 Another comparison test

We've covered a ton of material thus far in this course; there is one more comparison test that comes in quite handy—the Limit Comparison Test—which we will meet in Section 5.2. The purpose of this chapter, however, runs deeper than "just" another comparison test.

The emphasis on series has been almost entirely on the question of their convergence; we have not paid much heed to the value of the series, but we've developed a lot of techniques to analyze their convergence. The question is always "Does it converge?" and the answer is "yes, it converges!" or "no, it does not converge." Considering how qualitative our answer is, we might hope that there are equally qualitative methods for analyzing series. If convergence is just a yes-or-no matter, one might hope that the methods for analyzing series are equally loose and qualitative.

But that hasn't been our experience. Convergence is a tricky business, requiring precision and careful analysis. There have been hints, though, that things are easier than they seem: the comparison test is perhaps the best example of that. In your past mathematical life, you've probably been given "expressions" or "equations" to which you apply various rules in order to derive an answer. With the comparison test, the situation is less about rules, and more about creatively ignoring parts of the expression in order to find a useful bound. More than being rule-based, testing convergence via the comparison test requires some guesswork, and a willingness to ignore the parts of the expression that don't matter, in order to get at the part that does.

Let me be more precise. Suppose we wanted to analyze the convergence of a series such as

$$\sum_{n=52}^{\infty} \frac{n^4 - 3n + 5}{2n^5 + 5n^3 - n^2}$$

This is a complicated series, but the given expression is the ratio between a fifth degree polynomial and a fourth degree polynomial, so this series "is more or less" the same as the series

$$\sum_{n=52}^{\infty} \frac{n^4}{n^5} = \sum_{n=52}^{\infty} \frac{1}{n}$$

which is the harmonic series, and diverges! This is how we'd like to think, but we need to justify the concept of "is more or less." This is what Section 5.2 will teach us to do.

5.1 Convergence depends on the tail

The harmonic series diverges, but so does the series

$$\sum_{n=100}^{\infty} \frac{1}{n} = \frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \frac{1}{103} + \cdots$$

Convergence doesn't depend on the beginning of the series; whether or not I include the first 99 terms

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{97} + \frac{1}{98} + \frac{1}{99}$$

does affect whether or the sum of *all* the terms diverges. In short, convergence depends not on how a series begins, but on how a series ends. The end of a series is sometimes called the **tail** of the series.

Definition Let $N > 1$ be an integer, and consider a series $\sum_{n=1}^{\infty} a_n$. The series we get by removing the first $N - 1$ terms, namely

$$\sum_{n=N}^{\infty} a_n$$

is called a **tail** of the given series.

Here is the theorem that describes how tails relate to convergence.

Theorem 5.1.1 *Let $N > 1$ be an integer. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges.*

This could be shortened to "The series converges iff a tail of the series converges," or even just to the slogan that convergence depends on the tail.

Proof *Suppose $\sum_{n=1}^{\infty} a_n$ converges to L, meaning*

$$\lim_{M\to\infty} \sum_{n=1}^{M} a_n = L.$$

In that case, applying limit laws reveals

$$\begin{aligned}
\lim_{M\to\infty} \sum_{n=N}^{M} a_n &= \lim_{M\to\infty} \left(\left(\sum_{n=1}^{M} a_n \right) - \left(\sum_{n=1}^{N-1} a_n \right) \right) \\
&= \lim_{M\to\infty} \left(\sum_{n=1}^{M} a_n \right) - \lim_{M\to\infty} \left(\sum_{n=1}^{N-1} a_n \right) \\
&= \lim_{M\to\infty} \left(\sum_{n=1}^{M} a_n \right) - \left(\sum_{n=1}^{N-1} a_n \right) \\
&= L - \left(\sum_{n=1}^{N-1} a_n \right),
\end{aligned}$$

which means the series $\sum_{n=N}^{M} a_n$ *converges.*

The other direction is left to you, the reader.

Example 5.1.2 Does the series $\sum_{n=153}^{\infty} \frac{1}{n^2}$ converge?

Solution Yes! This series is a tail of the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$; in this case, $p = 2$.

You might recall Example 3.1.1, which we'll redo here.

Example 5.1.3 Show that $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converges by using the comparison test and Theorem 5.1.1.

Solution The given series converges iff $\sum_{n=23}^{\infty} \frac{n^5}{5^n}$ by Theorem 5.1.1. But whenever $n \geq 23$, we have

$$\frac{n^5}{5^n} \leq \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$$

The series $\sum_{n=23}^{\infty} \left(\frac{2}{5}\right)^n$ converges, since it is a geometric series with common ratio $2/5$, so by comparison, the smaller series $\sum_{n=23}^{\infty} \frac{n^5}{5^n}$ also converges.

In light of Theorem 5.1.1, many textbooks will choose to write

$$\sum_{n} a_n$$

instead of $\sum_{n=1}^{\infty} a_n$ when discussing convergence. Whether or not the series converges doesn't depend on the initial index, so if we want to state theorems about convergence, we can avoid potentially distracting details by simply not speaking about where the series begins.

5.2 Limit comparison test

This test is usually called the **limit comparison test**,

Theorem 5.2.1 *Suppose $a_n \geq 0$ and $b_n \geq 0$. Then if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0,$$

the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

The idea is that this is basically a comparison test, but instead of asking for something on the nose, like $0 \leq a_n \leq b_n$, I am instead asking only that a_n and b_n be similar in size.

5.2.1 Proof of the Limit Comparison Test

Assuming $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then we may also conclude that $\lim_{n\to\infty} \frac{b_n}{a_n} = 1/L > 0$.

Theorem 5.2.2 *Suppose $a_n \geq 0$ and $b_n \geq 0$. Then if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0,$$

the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

5.2.2 How to apply the Limit Comparison Test

Example 5.2.3 Does the series

$$\sum_{n=52}^{\infty} \frac{n^4 - 3n + 5}{2n^5 + 5n^3 - n^2}$$

converges or diverge?

Solution We will invoke the Limit Comparison Test, that is, Theorem 5.2.2. Set

$$a_n = \frac{n^4 - 3n + 5}{2n^5 + 5n^3 - n^2} \text{ and } b_n = \frac{1}{n},$$

so in this case,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^4 - 3n + 5}{2n^5 + 5n^3 - n^2} \cdot \frac{n}{1} \\
&= \lim_{n\to\infty} \frac{n^5 - 3n^2 + 5n}{2n^5 + 5n^3 - n^2} \\
&= \lim_{n\to\infty} \frac{n^5 - 3n^2 + 5n}{2n^5 + 5n^3 - n^2} \cdot \frac{1/n^5}{1/n^5} \\
&= \lim_{n\to\infty} \frac{1 - \frac{3}{n^3} + \frac{5}{n^4}}{2 + \frac{5}{n^2} - \frac{1}{n^3}} \\
&= \frac{\lim_{n\to\infty}\left(1 - \frac{3}{n^3} + \frac{5}{n^4}\right)}{\lim_{n\to\infty}\left(2 + \frac{5}{n^2} - \frac{1}{n^3}\right)} \\
&= \frac{1}{2} > 0,
\end{aligned}$$

and so $\sum_{n=52}^{\infty} a_n$ and $\sum_{n=52}^{\infty} b_n$ share the same fate. But $\sum_{n=52}^{\infty} b_n$ diverges, and so too does $\sum_{n=52}^{\infty} a_n$.

Of course, we could have just used a direct comparison test; the big benefit here is that it is relatively painless to compute the limit, but it requires significant creativity to set up a direct comparison test. Think back, for instance, to Example 3.1.1.

6 Power series

6.1 Definitions

At first, we studied the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n,$$

but then we replaced $\frac{1}{2}$ with a variable, so we were able to analyze all geometric series simultaneously. In the end, we discovered that

$$\sum_{n=0}^{\infty} kx^n = \frac{k}{1-x} \text{ if } |x| < 1,$$

and that the series diverges when $|x| \geq 1$. At the time, we thought of x as an unspecified constant, but we could just as well think of it as a variable. A number which depends on another number is just a function, so we may write

$$f(x) = \sum_{n=0}^{\infty} kx^n$$

and then observe that it is also the case that $f(x) = k/(1-x)$, as long as $|x| < 1$. While $k/(1-x)$ is a reasonably easy function to deal with, the more complicated $\sum kx^n$ does have its attractions: it appears to be an infinite version of one of the simplest sorts of functions—a polynomial. Do other functions have representations as series? Is there an advantage to viewing them in this way?

You may recall that we used r instead of x; I think r was a good choice back then, since r evokes "ratio" and stood for the common ratio between the terms in a geometric series. Now we use x, which I think is a good choice now. Why the change? To me, x suggests "independent variable of a function" much more generally than just a parameter in a geometric series, and we will be thinking very broadly about how quite general functions can be represented as a power series. One may regard power series as a natural generalization of polynomials, and you may already be comfortable thinking of x as the variable in a polynomial.

Usually the coefficients aren't all the same in a polynomial; the geometric series is somewhat unusual in that all the coefficients of the powers of x are the same, namely k. We will need to allow more general coefficients if we are to get anything other than the geometric series.

Definition Let (a_n) be a sequence of real numbers starting with a_0. Then the **power series** associated to (a_n) is

$$\sum_{n=0}^{\infty} a_n x^n.$$

Sometimes people are confused when considering, say,

$$\sum_{n=0}^{\infty} (\sin x) \cdot x^n.$$

Despite appearances, we will not be regarding such series as "power series." Since (a_n) is a sequence of real numbers, the coefficients a_n cannot depend on x.

6.2 Convergence of power series

Example 6.2.1 Consider the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

For which x does this converge?

In Example 6.2.1, note that we are no longer considering convergence of a single series; instead, by regarding x as a parameter, we are considering convergence for a whole family of series—namely all those of the form $\sum_{n=1}^{\infty} x^n/n$. It often happens in mathematics that it is wiser to study a family of objects simultaneously than to study a single object in isolation.

Another way to think of this question is this: we are being asked to determine the domain of f.

Solution We can investigate the convergence of this series using the ratio test. The difficulty now is that, instead of a single limit, we must now compute a limit than involves a variable x which is *not* the variable with respect to which we are taking a limit.

Another issue is that x may be negative, so we will actually check absolute convergence; in other words, we are actually considering

$$f(x) = \sum_{n=1}^{\infty} \left| \frac{x^n}{n} \right|.$$

Bravely,

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{n+1}}{\frac{|x|^n}{n}} \\ &= \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{1}{\frac{|x|^n}{n}} \\ &= \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} \\ &= \lim_{n\to\infty} \frac{|x|^{n+1} \cdot n}{(n+1) \cdot |x|^n} \\ &= \lim_{n\to\infty} |x| \frac{n}{n+1} = |x| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |x|. \end{aligned}$$

So when $L = |x| < 1$, the ratio test says that the series converges absolutely. When $|x| > 1$, the series does not converge absolutely—in fact, when $|x| > 1$, we have that

$$\lim_{n\to\infty} \frac{x^n}{n} \neq 0$$

and so the series diverges.

So when $|x| < 1$ the series converges and when $|x| > 1$ it diverges, which leaves only two values in doubt. When $x = 1$, the series is the harmonic series and diverges; when $x = -1$, it is the alternating harmonic series (rather, the negative of the "usual" alternating harmonic series) and converges.

In other words, we may regard f as a function with domain $[-1, 1)$.

We analyzed this power series by invoking the ratio test, but that was no accident. We will see that the ratio test applied to a power series will always have the same nice form. Feeling confident from our display of bravery before the preceding example, let's attack the general case of a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

When does this series converge absolutely? Applying the ratio test again, we find

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|a_{n+1}| \cdot |x|^{n+1}}{|a_n| \cdot |x|^n} \\ &= \lim_{n\to\infty} |x| \cdot \frac{|a_{n+1}|}{|a_n|} \\ &= |x| \cdot \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}. \end{aligned}$$

So the series converges absolutely whenever $L < 1$, but in this case, L depends in a rather uncomplicated way on $|x|$. The whole story is controlled by a limit that *does not* depend on x, namely

$$1/R = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

which you see I have presciently related to a hitherto unmentioned variable, R. So the series converges absolutely whenever $|x|/R < 1$, meaning the series converges absolutely whenever $|x| < R$. When $|x| > R$, the series diverges[1].

[1] This is a consequent of the ratio test, but perhaps we haven't emphasized it enough; if $L > 1$ in the ratio test, then the limit of the terms is not zero, so not only does the series not converge absolutely—the original series diverges, too, by the limit test.

Once again, only the two values $x = \pm R$ require further investigation. In any case, if we begin with a sequence (a_n) for which

$$1/R = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

the associated power series $\sum_{n=0}^{\infty} a_n x^n$ will converge for values of x in the interval $(-R, R)$, and it may even converge (or not, depending on the specific situation) at $x = R$ or at $x = -R$, so sometimes the interval on which the series makes sense will be $[-R, R)$ or $(-R, R]$ or even $[-R, R]$.

Definition The set of values of x for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges is the **interval of convergence**.

As a consequence of the above discussion, the interval of convergence of a power series always have a nice form.

Theorem 6.2.2 *For a power series, the interval of convergence is, in fact, an interval. It has the form $(-R, R)$ or $[-R, R)$ or $(-R, R]$ or $[-R, R]$. In short, it is centered around* 0.

Because the interval of convergence is, indeed, centered around 0, and because we often don't care about what happens at the endpoint, it is often convenient to just describe R.

Definition In the interval of convergence of a power series, the value R is called the **radius of convergence** of the series.

Two special cases deserve mention. If

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0,$$

then we might say "$R = \infty$" since no matter what x is, the product

$$|x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0,$$

and therefore for all values of x, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges. So the radius of convergence, R, is infinite.

The opposite may happen as well. It may happen that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty,$$

in which case the only way that

$$|x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

is when $|x| = 0$. The interval of convergence is just the single point $\{0\}$. And in this case, we say $R = 0$.

Warning People often confuse "radius of convergence" with "interval of convergence." For starters, the radius of convergence is a single number, while the interval of convergence is, well, an interval (though it was defined as just a set—the fact that the set where the series converges *is* an interval was a theorem and was certainly not obvious—perhaps the first of many lovely surprises with power series).

There is another difference between the radius of convergence and the interval of convergence; it is not simply that they are the same information in different packages. The radius of convergence cannot distinguish between, say, $(-R, R)$ versus $[-R, R]$. The interval of convergence contains the extra information about what is happening at the endpoints. So a homework question which asks you to find the radius R is much easier than a homework question asking you to find the interval.

6.3 Power series centered elsewhere

If we are speaking of the "radius" of convergence, you might wonder about the "center" of convergence. Thus far, our intervals of convergence have been centered around the origin, but by changing the series, we can move the interval of convergence. Let's see how.

Consider once again the geometric series,

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } x \in (-1, 1).$$

Whatever benefits there might be in using the series form of this function are only available to us when x is between -1 and 1. We can address this shortcoming by modifying the power series slightly.

Example 6.3.1 Find a series representation for $\frac{1}{1-x}$ valid on the interval $(-5, 1)$.

Solution Consider that the series

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x+2}{3}\right)^n = \frac{1}{1-\frac{x+2}{3}} = \frac{3}{1-x},$$

because this is just a geometric series with x replaced by $(x+2)/3$. Multiplying both sides by $1/3$ gives

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \frac{1}{1-x},$$

the same function as before. For what values of x does this series converge? Since it is a geometric series, we know that it converges when

$$\begin{aligned} |x+2|/3 &< 1 \\ |x+2| &< 3 \\ -3 < x+2 &< 3 \\ -5 < x &< 1. \end{aligned}$$

So we have a series representation for $1/(1-x)$ that works on a larger interval than before, at the expense of a somewhat more complicated series. The endpoints of the interval of convergence now are -5 and 1, but note that those two endpoints can be described as -2 ± 3. We say that 3 is the radius of convergence, and we now say that the series is centered at -2.

Let's capture this in a definition.

It is worth contrasting Definition 18 with Definition 15, our original description of power series. In particular, $\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at zero.

Definition Let (a_n) be a sequence of real numbers starting with a_0. Then the **power series** centered at c and associated to (a_n) is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

In Example 6.3.1, we formed a series that involved $(x+2)^n$, meaning that in that case $c = -2$.

You are now in a position to try your hand at finding the radius of convergence and sometimes even the interval of convergence for some power series. I encourage you to cook up your own power series, and then see what you can say about the radius and interval of convergence.

Exercises for Section 6.3

Find the radius and interval of convergence for each series. In exercises 3 and 4, do not attempt to determine whether the endpoints are in the interval of convergence.

(1) $\sum_{n=0}^{\infty} nx^n$

(2) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(3) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

(4) $\sum_{n=1}^{\infty} \frac{n!}{n^n} (x-2)^n$

(5) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n} (x-2)^n$

(6) $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$

(7) Find a power series with radius of convergence 0.

6.4 Calculus with power series

Now we know that some functions can be expressed as power series, which look like infinite polynomials. Since calculus, that is, computation of derivatives and antiderivatives, is easy for polynomials, the obvious question is whether the same is true for infinite series. The answer is yes!

Theorem 6.4.1 *Suppose the power series* $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ *has radius of convergence R. Then*

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1},$$
$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1},$$

for values of x in the interval $(c-R, c+R)$. *The two new series have radius of convergence R, just like the original series.*

Example 6.4.2 Find an alternating series for $\log\left(\frac{3}{2}\right)$. Approximate this value to two decimal places.

Solution Begin with the geometric series, namely

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Then integrate term-by-term to find

$$\int \frac{1}{1-x}\,dx = -\log|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$
$$\log|1-x| = \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1}$$

for real numbers x such that $|x| < 1$. To compute $\log(3/2)$, we should choose $x = -1/2$. In that case,

$$\begin{aligned}\log(3/2) &= \log(1 - -1/2)\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \frac{1}{2^{n+1}}\end{aligned}$$

and so

$$\begin{aligned}\log(3/2) &\approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} = \frac{909}{2240}\\ &\approx 0.406.\end{aligned}$$

Because this is an alternating series with decreasing terms, we know that the true value is between $909/2240$ and $909/2240 - 1/2048 = 29053/71680 \approx .4053$, so $\log(3/2)$ is, to two decimal places, 0.41.

Example 6.4.3 What about $\log(9/4)$? Find an approximate value for it.

Solution Since $9/4$ is larger than 2 we cannot use the series directly, but

$$\log(9/4) = \log((3/2)^2) = 2\log(3/2) \approx 0.82,$$

so in fact we get a lot more from this one calculation than first meets the eye. To estimate the true value accurately we actually need to be a bit more careful.

When we multiply by two we know that the true value is between 0.8106 and 0.812, and so, rounded to two decimal places, the true value is 0.81.

Exercises for Section 6.4

(1) Find a series representation for $\log 2$.

(2) Find a power series representation for $1/(1-x)^2$.

(3) Find a power series representation for $2/(1-x)^3$.

(4) Find a power series representation for $1/(1-x)^3$. What is the radius of convergence?

(5) Find a power series representation for $\int \log(1-x)\,dx$.

7 Taylor series

We have seen that some functions can be represented as series. But thus far, our only examples have been those that result from manipulation of our one fundamental example, the geometric series. We might start with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ when } |x| < 1,$$

and then, say, integrate term-by-term to get a formula for a logarithm, as we did in Example 6.4.2. Instead of starting with a series representing a function, and then messing around with the series to find more functions represented by series, we should *start* with a function, and then try to find a series that represents it—if that is possible!

7.1 Finding Taylor series

The easiest case is when the function f is given to us as a power series already! Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval of convergence, say, when $|x| < R$. Then we know, by Theorem 6.4.1, that we can compute derivatives of f by

It might seem that considering this case is pointless: who cares about representing a function by a power series if the function is *given* to us as a power series? The point is not to come up with a new power representation: the point is to relate the derivatives of the function to the coefficients of the power series, which is the sort of thing that generalizes. By studying a case we already understand completely, we are seeking insights which will apply even in cases we don't fully understand.

differentiating the series term-by-term. Let's look at the first few derivatives.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots$$

By examining these derivatives, we can discern the general pattern. The k^{th} derivative must be

Recall that, for a function f, we write its k^{th} derivative by writing $f^{(k)}$. So the k^{th} derivative evaluated at x is written $f^{(k)}(x)$.

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k} \\ &= k(k-1)(k-2)\cdots(2)(1)a_k + (k+1)(k)\cdots(2)a_{k+1}x + \\ &\quad + (k+2)(k+1)\cdots(3)a_{k+2}x^2 + \cdots, \end{aligned}$$

but we can write this more easily by using factorials, namely

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = k!a_k + (k+1)!a_{k+1}x + \frac{(k+2)!}{2!} a_{k+2}x^2 + \cdots.$$

Now, substituting $x = 0$ yields

$$f^{(k)}(0) = k!a_k + \sum_{n=k+1}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = k!a_k,$$

and solving for a_k gives

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

So if a function f can be represented by a series, we know just what series it is! We know the series because we know the derivatives of f.

Note that $a_k = f^{(k)}(0)/k!$ makes sense even when $k = 0$, since in that special case, we have $a_0 = f^{(0)}(0)/0! = f(0)$ because the zeroth derivative of f is f itself, and $0! = 1$.

Definition Given a function f, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called the **Maclaurin series** for f, or often just the **Taylor series** for f centered around zero, since it is a power series centered around zero in the sense of Section 6.3.

Let me warn you that, in order to write down this power series, you had better be able to take first, second, third—indeed, n^{th}—derivatives at zero, because the given formula involves $f^{(n)}(0)$. And there is a worse warning.

Warning Even if f is infinitely differentiable at zero (meaning $f^{(n)}(0)$ makes sense), even if I then write down

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

does not mean that I have a power series for f valid on any open interval! All I have done is written down the power series that must represent f *assuming it has a power series representation at all!* Nobody is promising you that some function you find—even if it is infinitely differentiable—actually has a power series representation. In other words, I am **not** claiming that there is an $R \neq 0$ so that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \text{ for } x \in (-R, R).$$

All I am claiming is that if f has such a power series, then I can find it by taking derivatives.

Let's see this worked out in an example we already understand, namely, for the function $f(x) = 1/(1-x)$.

Example 7.1.1 Find a Taylor series for $f(x) = 1/(1-x)$ centered around zero.

Solution We need to compute the derivatives of f, and then hope to spot a pattern. Here we go:

$$\begin{aligned} f(x) &= (1-x)^{-1}, \\ f'(x) &= (1-x)^{-2}, \\ f''(x) &= 2(1-x)^{-3}, \\ f'''(x) &= 6(1-x)^{-4}, \\ f^{(4)} &= 4!(1-x)^{-5}, \\ &\vdots \\ f^{(n)} &= n!(1-x)^{-n-1}. \end{aligned}$$

I see the pattern!

$$\frac{f^{(n)}(0)}{n!} = \frac{n!(1-0)^{-n-1}}{n!} = 1,$$

and the Taylor series centered around zero is

$$\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n,$$

which we already knew—it is just the geometric series.

So given a function f, we may be able to differentiate it around zero, spot a pattern, and thereby compute the Taylor series, but that does not mean we have found a series representation for f. Worse, we don't even know if the series we wrote down converges anywhere, let alone converges to the function f! The miracle is that for many popular functions[1] the Taylor series does converge to f on some interval. But this is certainly not true of all functions—or even true of most[2] functions!

As a practical matter, if we are interested in using a series to approximate a value of a function at some point, we will need some finite number of terms of the series.

[1] Arguably, this is not so surprising: maybe these are popular functions precisely because they have nice properties. That the sky is sky-colored is not so surprising as the fact that it is blue.

[2] Most functions are probably not even continuous, let alone infinitely differentiable, let alone having the property of having a power series representation! The latter mouthful is usually termed **real analytic**.

Even for functions with terrible looking derivatives, we can compute the initial terms using computer software like Sage. If we want to know the whole series—that is, the n^{th} term in the series for an arbitrary n—then we need a function whose derivatives fall into a pattern that we can discern. A few of the most popular functions have nice patterns. Let's see some now!

Example 7.1.2 Find a Taylor series for $\sin x$.

Solution The derivatives are quite easy, namely

$$\begin{aligned} f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f'''(x) &= -\cos x, \\ f^{(4)}(x) &= \sin x, \end{aligned}$$

and then the pattern repeats. We want to know the derivatives at zero, so those are

$$\begin{aligned} f'(0) &= \cos 0 = 1 \\ f''(0) &= -\sin 0 = 0 \\ f'''(0) &= -\cos 0 = -1, \\ f^{(4)}(0) &= \sin 0 = 0, \end{aligned}$$

and then the pattern repeats, namely it goes—starting from the zeroth derivative—like this:

$$0, \quad 1, \quad 0, \quad -1, \quad 0, \quad 1, \quad 0, \quad -1, \quad 0, \quad \dots.$$

And so, the Taylor series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Sometimes people are confused by the fact that the exponent in this power series is $2n+1$ instead of just n. It turns out that setting it up this way, with x^{2n+1}, nicely manages to kill all the terms $x^{\text{even number}}$ which have a zero coefficient, since differentiating f an even number of times results in $\pm \sin x$, which is zero when $x = 0$.

But there is more to be anxious about. Before worrying whether this series converges to $\sin x$, we should ask the prior question: does this series converge anywhere? Let's determine the radius of convergence by using the ratio test, which asks us to consider

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} \\ &= \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0, \end{aligned}$$

and so the series converges regardless of what x is. It will turn out later, by applying Theorem 7.2.1, that this series does converge to $\sin x$.

Sometimes the formula for the n^{th} derivative of a function f is difficult to discover, but a combination of a known Taylor series and some algebraic manipulation leads easily to the Taylor series for f.

Example 7.1.3 Find the Taylor series for $f(x) = x\sin(-x)$ centered at zero.

Solution To get from $\sin x$ to $x\sin(-x)$ we substitute $-x$ for x and then multiply by x. Let's do the same thing to the series for $\sin x$, namely

$$\begin{aligned} x\sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n+1}}{(2n+1)!} &= x\sum_{n=0}^{\infty}(-1)^n(-1)^{2n+1}\frac{x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+2}}{(2n+1)!}. \end{aligned}$$

Of course, we *could* have differentiated $f(x)$, which would have yielded

$$\begin{aligned} f'(x) &= -x\cos(-x) + \sin(-x) = 0, \\ f''(x) &= -x\sin(-x) - 2\cos(-x) = -2, \\ f'''(x) &= x\cos(-x) - 3\sin(-x) = 0, \\ f^{(4)}(x) &= x\sin(-x) + 4\cos(-x) = 4, \end{aligned}$$

but maybe it would have been harder to pick up on the pattern that way.

As we have seen in Section 6.3, a general power series can be centered at a point other than zero, and the method that produces the Taylor series can also produce such series.

Definition Given a function f, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the **Taylor series** for f centered around c.

Let's see an example, reminiscent of Example 6.3.1 but attacked with a different method.

Example 7.1.4 Find a Taylor series centered at -2 for $1/(1-x)$.

Solution If the series is $\sum_{n=0}^{\infty} a_n(x+2)^n$ then looking at the kth derivative:

$$k!(1-x)^{-k-1} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x+2)^{n-k}$$

and substituting $x = -2$ we get $k!3^{-k-1} = k!a_k$ and $a_k = 3^{-k-1} = 1/3^{k+1}$, so the series is

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}.$$

Exercises for Section 7.1

For each function, find the Taylor series centered at c, and the radius of convergence. Do not worry about whether or not the series converges to the given function—that will be our concern in Section 7.2.

(1) $\cos x$ around $c = 0$,

(2) e^x around $c = 0$,

(3) $1/x$ around $c = 5$

(4) $\log x$ around $c = 1$

(5) $\log x$ around $c = 2$

(6) $1/x^2$ around $c = 1$

(7) $1/\sqrt{1-x}$ around $c = 0$

(8) Find the first four terms of the Taylor series for $\tan x$ centered at zero. By "first four terms" I mean up to and including the x^3 term.

(9) Use a combination of Taylor series and algebraic manipulation to find a series centered at zero for $x\cos(x^2)$.

(10) Use a combination of Taylor series and algebraic manipulation to find a series centered at zero for xe^{-x}.

7.2 Taylor's Theorem

In Section 7.1, we were finding Taylor series without worrying at all that what we were finding actually related to the original function f. We now remedy this oversight.

Our whole mission in this, Chapter 7, was to start with a function, then write down a power series converging to f, with the plan that we might finally use that power series to, say, approximate values of the function, among other things. What we have done thus far is *assumed* a function had such a power series representation, and then deduced that the coefficients are somehow related to the derivatives of f. So does the power series converge to the function?

And if the power series we got in Section 7.1 does converge to f, how good is the approximation? I mean, if I just consider the first N terms of the power series, am I close to f at all? How close? Remember our experience with alternating series? For alternating series, Theorem 4.2.2 not only gave a test for convergence: it also bounded how far the N^{th} partial sum could be from the true value of the series. Let's see a similar sort of "error estimate" for power series.

Theorem 7.2.1 (Taylor's theorem) *Suppose that f is defined on some open interval $I = (a - R, a + R)$ around a and suppose the function f is $(N+1)$-times differentiable on I, meaning that $f^{(N+1)}(x)$ exists for $x \in I$. Then for each $x \neq a$ in I there is a value z between x and a so that*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

The upshot here is that, by bounding the function $f^{(N+1)}$ on the interval between x and a, we manage to bound the difference between $f(x)$ and the partial sum. See Example 7.2.2 for an example. But before we get to the example, let's prove Theorem 7.2.1; the proof perhaps seems unmotivated, since we'll be "clever" in setting things up, but I hope you will be able to follow the argument, even if you don't trust that you could have created the proof *ex nihilo*.

Proof *Define the function $F(t)$ by*

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}(x-t)^n + B(x-t)^{N+1} \text{ for } t \text{ between } a \text{ and } x.$$

Here we have replaced a by t in the first $N+1$ terms of the Taylor series, and added a carefully chosen term on the end, with B to be determined. Note that we are temporarily keeping x fixed, so the only variable in this equation is t, and we will be interested only in t between a and x. If we set $t = a$, then we get

$$F(a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x-a)^{N+1}.$$

Set this equal to $f(x)$:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x-a)^{N+1}.$$

Since $x \neq a$, we can solve this for B, which is a "constant"—it depends on x and a but those are temporarily fixed. Now we have defined a function $F(t)$ with the property that $F(a) = f(x)$. Consider also $F(x)$: all terms with a positive power of $(x-t)$ become zero when we substitute x for t, so we are left with $F(x) = f^{(0)}(x)/0! = f(x)$. So $F(t)$ is a function with the same value on the endpoints of the interval $[a,x]$. By Rolle's theorem, we know that there is a value $z \in (a,x)$ such that $F'(z) = 0$. Let's look at $F'(t)$. Each term in $F(t)$, except the first term and the extra term involving B, is a product, so to take the derivative we use the product rule on each of these terms. It will help to write out the first few terms of the definition:

$$\begin{aligned} F(t) = f(t) &+ \frac{f^{(1)}(t)}{1!}(x-t)^1 + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \frac{f^{(3)}(t)}{3!}(x-t)^3 + \cdots \\ &+ \frac{f^{(N)}(t)}{N!}(x-t)^N + B(x-t)^{N+1}. \end{aligned}$$

Now take the derivative:

$$\begin{aligned} F'(t) = f'(t) &+ \left(\frac{f^{(1)}(t)}{1!}(x-t)^0(-1) + \frac{f^{(2)}(t)}{1!}(x-t)^1 \right) \\ &+ \left(\frac{f^{(2)}(t)}{1!}(x-t)^1(-1) + \frac{f^{(3)}(t)}{2!}(x-t)^2 \right) \\ &+ \left(\frac{f^{(3)}(t)}{2!}(x-t)^2(-1) + \frac{f^{(4)}(t)}{3!}(x-t)^3 \right) + \cdots + \\ &+ \left(\frac{f^{(N)}(t)}{(N-1)!}(x-t)^{N-1}(-1) + \frac{f^{(N+1)}(t)}{N!}(x-t)^N \right) \\ &+ B(N+1)(x-t)^N(-1). \end{aligned}$$

Now most of the terms in this expression cancel out, leaving just

$$F'(t) = \frac{f^{(N+1)}(t)}{N!}(x-t)^N + B(N+1)(x-t)^N(-1).$$

At some z*,* $F'(z) = 0$ *so*

$$\begin{aligned} 0 &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N + B(N+1)(x-z)^N(-1) \\ B(N+1)(x-z)^N &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N \\ B &= \frac{f^{(N+1)}(z)}{(N+1)!}. \end{aligned}$$

Now we can write

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}(x-t)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-t)^{N+1}.$$

Recalling that $F(a) = f(x)$ we get

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1},$$

which is what we wanted to show.

It may not be immediately obvious that this is particularly useful, so let's look at some examples of Theorem 7.2.1 in action.

Example 7.2.2 Suppose $x \in [-\pi/2, \pi/2]$. Find a polynomial approximation for $\sin x$ accurate to ± 0.005.

Solution Note that if we can compute $\sin x$ for $x \in [-\pi/2, \pi/2]$, then we can compute $\sin x$ for all x. So what we are asking for here is actually quite general! Once we figure out how to calculate $\sin x$ for $x \in [-\pi/2, \pi/2]$, we can shift any other value of x into that interval, and thereby compute $\sin x$.

Let's get started. From Theorem 7.2.1, we have that

$$\sin x = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1} \text{ for some } z \text{ between } a \text{ and } x.$$

What can we say about the size of that "error" term? In other words, how big could

$$\frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}$$

possibly be when z is between x and a? Every derivative of $\sin x$ is either $\pm\sin x$ or $\pm\cos x$, so $|f^{(N+1)}(z)| \leq 1$. The factor $(x-a)^{N+1}$ is a bit more difficult, since $x-a$ could be quite large. To simplify matters, let's pick $a = 0$. We need to pick N so that

$$\left|\frac{x^{N+1}}{(N+1)!}\right| < 0.005.$$

Since we are only considering the case when $x \in [-\pi/2, \pi/2]$, we have

$$\left|\frac{x^{N+1}}{(N+1)!}\right| < \frac{2^{N+1}}{(N+1)!}.$$

The quantity on the right decreases with increasing N, so all we need to do is find an N so that

$$\frac{2^{N+1}}{(N+1)!} < 0.005.$$

A little trial and error shows that $N = 8$ works, and in fact $2^9/9! < 0.0015$. Consequently,

$$\begin{aligned}\sin x &= \sum_{n=0}^{8} \frac{f^{(n)}(0)}{n!} x^n \pm 0.0015 \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \pm 0.0015.\end{aligned}$$

Figure 7.1 shows the graphs of $\sin x$ and and the approximation on $[0, 3\pi/2]$. As x gets larger, the approximation heads to negative infinity very quickly, since it is essentially acting like $-x^7$.

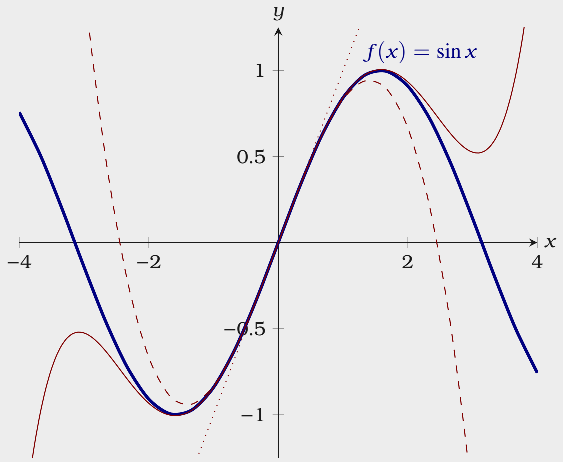

Figure 7.1: A plot of $f(x) = \sin x$ with a thick line, placed alongside the dotted line $y = x$, a dashed plot of $y = x - \frac{x^3}{6}$, and a thin solid plot of $y = x - \frac{x^3}{6} + \frac{x^5}{120}$. Note how successively higher partial sums of the Taylor series are hugging the graph of $\sin x$ increasingly well.

We can extract a bit more information from this example. If we do not restrict the value of x to lie in the interval $x \in [-\pi/2, \pi/2]$, we still know that

$$\left|\frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}\right| \le \left|\frac{x^{N+1}}{(N+1)!}\right|$$

because the derivative $f^{(N+1)}(z)$ is either $\pm \sin z$ or $\pm \cos z$. We can use this to prove the following result.

Theorem 7.2.3 *For all real numbers x,*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Proof *If we can show that*

$$\lim_{N\to\infty} \left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

for each x, then we know that the error term

$$\left|\frac{f^{(N+1)}(z)}{(N+1)!}x^{N+1}\right| \leq \left|\frac{x^{N+1}}{(N+1)!}\right|,$$

is, in the limit, zero, and so we may conclude that

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\,x^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.\end{aligned}$$

In other words, we will conclude that the sine function is actually equal to its Taylor series around zero for all x. But how can we prove that the limit is zero?

Suppose that N is larger than $|x|$, and let M be the largest integer less than $|x|$. If $M = 0$, then x^{N+1} is small, and even smaller after dividing by $(N+1)!$. On the other hand, if $M > 0$, then just a bit more work is called for. We compute that

$$\begin{aligned}\frac{|x^{N+1}|}{(N+1)!} &= \frac{|x|}{N+1}\frac{|x|}{N}\frac{|x|}{N-1}\cdots\frac{|x|}{M+1}\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1} \\ &\leq \frac{|x|}{N+1}\cdot 1\cdot 1\cdots 1\cdot\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1} \\ &= \frac{|x|}{N+1}\frac{|x|^M}{M!}.\end{aligned}$$

The quantity $|x|^M/M!$ is a constant, so

$$\lim_{N\to\infty}\frac{|x|}{N+1}\frac{|x|^M}{M!} = 0$$

and by squeezing via Theorem 1.7.2,

$$\lim_{N\to\infty}\left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

as desired.

Essentially the same argument works for $\cos x$, which yields the following.

Theorem 7.2.4 *For all real numbers x,*

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

And similarly, the function e^x is **real analytic**, meaning that the Taylor series for e^x converges to e^x.

Theorem 7.2.5 *For all real numbers x,*

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Example 7.2.6 Find a polynomial approximation for e^x near $x = 2$ accurate to ± 0.005.

Solution From Taylor's theorem:

$$e^x = \sum_{n=0}^{N} \frac{e^2}{n!}(x-2)^n + \frac{e^z}{(N+1)!}(x-2)^{N+1},$$

since $f^{(n)}(x) = e^x$ for all n. We are interested in x near 2, and we need to control the size of $|(x-2)^{N+1}|$, so we may as well specify that $|x-2| \leq 1$, meaning $x \in [1, 3]$. Also

$$\left|\frac{e^z}{(N+1)!}\right| \leq \frac{e^3}{(N+1)!},$$

so we need to find an N that makes $e^3/(N+1)! \le 0.005$. This time $N = 6$ makes $e^3/(N+1)! < 0.004$, so the approximating polynomial is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \frac{e^2}{24}(x-2)^4 + \frac{e^2}{120}(x-2)^5 + \frac{e^2}{720}$$

This presents an additional problem for approximation, since we also need to approximate e^2, and any approximation we use will increase the error, but we will not pursue this complication.

Note well that in these examples we found polynomials of a certain accuracy only on a small interval, even though the series for $\sin x$ and e^x converge for all x. This is usually how things work out. To get the same accuracy on a larger interval, what can you do? Use more terms!

Exercises for Section 7.2

(1) Find a polynomial approximation for $\cos x$ on $[0, \pi]$, accurate to $\pm 10^{-3}$

(2) How many terms of the series for $\log x$ centered at 1 are required so that the guaranteed error on $[1/2, 3/2]$ is at most 10^{-3}? What if the interval is instead $[1, 3/2]$?

(3) Find the first three nonzero terms in the Taylor series for $\tan x$ on $[-\pi/4, \pi/4]$, and compute the guaranteed error term as given by Taylor's theorem. (You may want to use Sage or a similar aid.)

(4) Prove Theorem 7.2.4, that is, show that $\cos x$ is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.

(5) Prove Theorem 7.2.5, that is, show that e^x is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.

Review

These problems require various techniques, and are in no particular order. Many problems may be done in more than one way.

I also encourage you to try your hand at crafting your own problems: mathematics is more than answer-getting—it is also about problem-posing! Building your own problems will give you a strong sense of the limitations of the machinery we have built in this book.

Exercises for Section 7.2

Determine whether the series converges.

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+4}$

(2) $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\cdots$

(3) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n^2+4)^2}$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{8^n}$

(5) $\displaystyle 1-\frac{3}{4}+\frac{5}{8}-\frac{7}{12}+\frac{9}{16}+\cdots$

(6) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+4}}$

(7) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^3(n)}{n^2}$

(8) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{e^n}$

(9) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

(10) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

(11) $\displaystyle\frac{1}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\frac{3}{4\cdot 5\cdot 6}+\frac{4}{5\cdot 6\cdot 7}+\cdots$

(12) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n)!}$

(13) $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{n!}$

(14) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

(15) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^{n-1}}{n!}$

(16) $1 + \dfrac{5^2}{2^2} + \dfrac{5^4}{(2\cdot 4)^2} + \dfrac{5^6}{(2\cdot 4\cdot 6)^2} + \dfrac{5^8}{(2\cdot 4\cdot 6\cdot 8)^2} + \cdots$

(17) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$

Find the interval and radius of convergence; you need not check the endpoints of the intervals.

(18) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

(19) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{1+3^n}$

(20) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

(21) $x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{x^7}{7} + \cdots$

(22) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2} x^n$

(23) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n} x^{2n}$

(24) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$

Find a series for each function, using the formula for Maclaurin series and algebraic manipulation as appropriate.

(25) 2^x

(26) $\ln(1+x)$

(27) $\ln\left(\dfrac{1+x}{1-x}\right)$

(28) $\sqrt{1+x}$

(29) $\dfrac{1}{1+x^2}$

(30) $\arctan(x)$

(31) Use the answer to the previous problem to discover a series for a well-known mathematical constant.

Epilogue (or. . . what happens to Harry?)

The worst part about reading a great[3] novel is that last page. The book gets thinner and thinner, and then, poof! Not just the characters, but the whole world that the author has crafted for them is gone! And how frequently I want to stay in that world just a bit longer.

[3] Or even a bad novel.

Humanity has found an antidote to novel-endings; this antidote is the *sequel*. With a name like "Calculus Two" you might be getting the idea that there is a Calculus Three, and who knows. . . maybe!

A-hundred-and-some-odd[4]-pages ago, I pointed out that reading mathematics is **not** the same as reading a novel. This book is done, but you are not. There is more mathematics yet to learn about, and more mathematics yet to create. And I don't mean to say that you should write go and write fan fiction. I'm no author, and you are no mere reader. You have worked through the exercises, you have thought about this material in your own way—so you are the author of your own understanding, and you must keep writing.

[4] Or even.

—the so-called "author"

Answers to Exercises

Answers for 1.8

1. 1 **3.** 0 **4.** 1 **5.** 1 **6.** 0

Answers for 2.7

1. $\lim_{n\to\infty} n^2/(2n^2+1) = 1/2$ **2.** $\lim_{n\to\infty} 5/(2^{1/n}+14) = 1/3$ **3.** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty} 3\frac{1}{n}$ diverges **4.** $-3/2$ **5.** 11 **6.** 20 **7.** 3/4 **8.** 3/2 **9.** 3/10

Answers for 3.1

1. 1 **2.** 1 **3.** converges **4.** converges **5.** converges **6.** diverges

Answers for 3.2

1. diverges **2.** diverges **3.** converges **4.** converges **5.** converges **6.** converges **7.** diverges **8.** converges **9.** $N = 5$ **10.** $N = 10$ **11.** $N = 1687$ **12.** any integer greater than e^{200}

Answers for 3.3

1. converges **2.** converges **3.** converges **4.** diverges **5.** diverges **6.** diverges **7.** converges **8.** diverges **9.** converges **10.** diverges

Answers for 3.4

Answers for 4.1

1. converges absolutely **2.** diverges **3.** converges conditionally **4.** converges absolutely **5.** converges conditionally **6.** converges absolutely **7.** diverges **8.** converges conditionally

Answers for 4.2

1. converges **2.** converges **3.** diverges **4.** converges **5.** 0.90 **6.** 0.95

Answers for 6.3

1. $R = 1$, $I = (-1, 1)$ **2.** $R = \infty$, $I = (-\infty, \infty)$ **3.** $R = e$, $I = (-e, e)$ **4.** $R = e$, $I = (2-e, 2+e)$ **5.** $R = 0$, converges only when $x = 2$ **6.** $R = 1$, $I = [-6, -4]$ **7.** There are many choices—for instance, see Exercise 5—but $\sum_{n=0}^{\infty} n! \cdot x^n$ works.

Answers for 6.4

1. the alternating harmonic series **2.** $\sum_{n=0}^{\infty}(n+1)x^n$ **3.** $\sum_{n=0}^{\infty}(n+1)(n+2)x^n$ **4.** $\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$, $R = 1$ **5.** $C + \sum_{n=0}^{\infty}\frac{-1}{(n+1)(n+2)}x^{n+2}$

Answers for 7.1

1. $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$, $R = \infty$ **2.** $\sum_{n=0}^{\infty}x^n/n!$, $R = \infty$ **3.** $\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^{n+1}}$, $R = 5$ **4.** $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$, $R = 1$ **5.** $\log(2) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^n}{n2^n}$, $R = 2$ **6.** $\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$, $R = 1$ **7.** $1 + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^n}x^n = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!}x^n$, $R = 1$ **8.** $x + x^3/3$ **9.** $\sum_{n=0}^{\infty}(-1)^n x^{4n+1}/(2n)!$ **10.** $\sum_{n=0}^{\infty}(-1)^n x^{n+1}/n!$

Answers for 7.2

1. $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots+\frac{x^{12}}{12!}$ **2.** 1000; 8 **3.** $x+\frac{x^3}{3}+\frac{2x^5}{15}$, error ± 1.27.

Answers for 7.2

1. diverges **2.** converges **3.** converges **4.** diverges **5.** diverges **6.** diverges **7.** converges **8.** converges **9.** converges **10.** converges **11.** converges **12.** converges **13.** converges **14.** converges **15.** converges **16.** converges **17.** diverges **18.** $(-\infty,\infty)$ **19.** $(-3,3)$ **20.** $(-3,3)$ **21.** $(-1,1)$ **22.** radius is 0—it converges only when $x=0$ **23.** $(-\sqrt{3},\sqrt{3})$ **24.** $(-\infty,\infty)$ **25.** $\sum_{n=0}^{\infty}\frac{(\ln(2))^n}{n!}x^n$ **26.** $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$ **27.** $\sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}$ **28.** $1+x/2+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^n$ **29.** $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ **30.** $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$ **31.** $\pi=\sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}$

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