Exercises for midterm 1

Math 4581 Autumn 2013

1) Let $H$ be the plane in $\mathbb{R}^3$ whose equation is $2x + y - 3z = 0$.
   a) Verify that $H$ is a vector subspace of $\mathbb{R}^3$.
   b) Give a basis $B$ for $H$.
   c) Extend $B$ to a basis $B^*$ for $\mathbb{R}^3$.

2) Let $H$ be the plane in $\mathbb{R}^3$ whose equation is $x = 0$.
   a) Verify that $H$ is a vector subspace of $\mathbb{R}^3$.
   b) Give a basis $B$ for $H$.
   c) Extend $B$ to a basis $B^*$ for $\mathbb{R}^3$.

3) Let $H$ and $K$ be the vector subspaces of $\mathbb{R}^3$ defined as follows:
   - $H = \{(x, y, z) \in \mathbb{R}^3 | x - y - z = 0\}$
   - $K = \{(x, y, z) \in \mathbb{R}^3 | x - y = z = 0\}$

   Is $H \cup K$ a vector subspace of $\mathbb{R}^3$? (Explain)

4) Let $V$ be a vector space and $T : V \rightarrow V$ be a linear operator. Prove that $\text{Ker}(T)$ is a vector subspace of $V$.

5) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear operator defined by the following matrix (with respect to the standard basis in $\mathbb{R}^3$):

$$
\begin{pmatrix}
0 & 5 & \frac{1}{2} \\
2 & 10 & 1 \\
1 & 15 & \frac{3}{2}
\end{pmatrix}
$$
Is $T$ invertible? (Explain)

6) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Assume that $T$ is surjective. Is $T$ injective? (Explain)

7) Show that $SO(n)$ is a normal subgroup of $O(n)$.

8) Consider the following matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & -\sqrt{2} & -\sqrt{2} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & -\sqrt{3} & \sqrt{3} \end{pmatrix}$$

Answer the following questions providing an explanation.

- Is $A$ invertible?
- Is $B$ invertible?
- What is the determinant of $C$?
- What is the determinant of $D$?
- Does $C$ belong to $SO(n)$?
- Does $D$ belong to $SO(n)$?
- Does $C$ belong to $O(n)$?
- Does $B$ belong to $O(n)$?
- Does $A$ belong to $P_n$?
- Does $C$ belong to $P_n$?
- Does $C$ belong to $GL(n, \mathbb{R})$?
- Does $A$ belong to $GL(n, \mathbb{R})$?
9) Prove: Let $G$ be a finite group of permutations of the set $X$. Suppose that $G$ acts transitively on $X$ (i.e. $X$ is a single $G$-orbit). Then $X$ is a finite set, and $|X|$ divides $|G|$.

10) Consider the following permutation in $S_9$:

$$
\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 2 & 7 & 8 & 9 & 4 & 5 & 6 \end{pmatrix}
$$

$$
\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 5 & 4 & 3 & 7 & 8 & 6 & 9 \end{pmatrix}
$$

$$
\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 2 & 1 & 7 & 8 & 9 & 6 \end{pmatrix}
$$

$$
\sigma_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 6 & 5 & 4 & 8 & 9 & 3 & 1 \end{pmatrix}
$$

- Write the cycle decomposition of $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\sigma_4$.
- Compute $\sigma_1 \circ \sigma_2$.
- Compute $\sigma_1 \circ \sigma_3$.
- Compute $\sigma_3 \circ \sigma_2$.
- Compute $\sigma_3 \circ \sigma_1$.
- Compute $\sigma_1 \circ \sigma_2 \circ \sigma_3$.
- Compute $\sigma_1 \circ \sigma_2 \circ \sigma_4$.
- Compute $\sigma_2 \circ \sigma_1 \circ \sigma_3$.
- Compute $\sigma_1 \circ \sigma_2 \circ \sigma_2$.
- Compute the order of $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\sigma_4$.
- Are $\sigma_1$ and $\sigma_2$ conjugate? (Explain)
- Are $\sigma_1$ and $\sigma_3$ conjugate? (Explain)
- Are $\sigma_2$ and $\sigma_3$ conjugate? (Explain)
• Are \( \sigma_1 \) and \( \sigma_4 \) conjugate? (Explain)
• Are \( \sigma_2 \) and \( \sigma_4 \) conjugate? (Explain)
• Are \( \sigma_3 \) and \( \sigma_4 \) conjugate? (Explain)

11) Consider the following permutation \( \sigma \) in \( S_6 \):

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 5 & 6 & 3 & 4
\end{pmatrix}
\]

Show that

\[ H = \{id, \sigma\} \]

is a subgroup of \( S_6 \).