Solution.

1. \( \pm \mathbf{v}_1 \) (\( v=1,2,3 \)) the vertices of \( P \).

Since \( f \) swaps each long diagonal of \( P \) to itself, \( f(\pm \mathbf{v}_1) = \pm \mathbf{v}_1 \)

\( f(\pm \mathbf{v}_2) = \pm \mathbf{v}_2 \) and \( f(\pm \mathbf{v}_3) = \pm \mathbf{v}_3 \)

\( f \in \text{Sym}(P) \Rightarrow f \) preserves distances and angles.

Let \( f(\mathbf{v}_1) = \mathbf{v}_1 \Rightarrow f(\mathbf{v}_2) = \mathbf{v}_2 \) or \( -\mathbf{v}_3 \Rightarrow f(\mathbf{v}_3) = \mathbf{v}_3 \)

\( \Rightarrow f(\mathbf{v}_3) = \mathbf{v}_3 \) or \( \mathbf{v}_4 \Rightarrow f(\mathbf{v}_4) = \mathbf{v}_4 \)

\( f(-\mathbf{v}_1) = -\mathbf{v}_1 \) or \( \mathbf{v}_2 \Rightarrow f(\mathbf{v}_1) = -\mathbf{v}_1 \)
\[ f(-v_2) = -v_2 \text{ or } v_3 \Rightarrow f(-v_2) = -v_2 \]

\[ f(-v_3) = -v_3 \text{ or } -v_1 \Rightarrow f(v_3) = -v_3 \]

\[ f = \mathbb{I}. \]

If \( f(v_1) = -v_1 \Rightarrow f(v_2) = -v_2 \text{ or } v_3 \Rightarrow f(v_2) = -v_2 \)

\[ f(v_3) = -v_3 \text{ or } -v_1 \Rightarrow f(v_3) = -v_3 \]

\[ f(-v_4) = v_4 \text{ or } -v_2 \Rightarrow f(v_4) = v_4 \]

\[ f(-v_2) = v_2 \text{ or } -v_3 \Rightarrow f(-v_2) = v_2 \]

\[ f(-v_3) = v_4 \text{ or } v_2 \Rightarrow f(-v_3) = v_3 \]

\[ f = -\mathbb{I}. \]
2) $|X| = 4! = 24$

The group acting on $X$ is the group of rotations of $T$. (Call it $G$.)

$G = \{ \text{id}, 8 \text{ rotations of } 120^\circ, 3 \text{ rotations of } 180^\circ \}$

- Rot's of $120^\circ$ correspond to permutations with cycle structure $3,1,1$ of the vector.
- Rot's of $180^\circ$ correspond to permutations with cycle structure $2,2,2$ of the vector.

Call $\text{id} = \text{id}eometry$

$\phi_i$: rotation of $120^\circ$ ($i=1, \ldots, 8$)

$\sigma_i$: rotation of $180^\circ$ ($i=1, \ldots, 3$).
\( f(10) = 24 \)

\( f(6) = 0 \quad (c = 2, \ldots, 8) \)

\( f(62) = 0 \quad (c = 1, 2, 3) \)

Using orbit counting formula:

\[
\text{# orbits} = \frac{1}{12} \cdot (24) = 2.
\]
3) \( \mathbb{Q}[x] \) \( \cong \mathbb{E} \) \( \frac{(x^2-2)}{} \\

Every equivalence class in \( \mathbb{E} \) is represented by a linear polynomial \( [x] \) with \( a, b \in \mathbb{Q} \).

We prove \( [x] \cong [x^2] = [2] \).

\( \Rightarrow \forall [p(x)], [q(x)] \in \mathbb{Q}[x] \)

\( \text{with } p(x), q(x) \in \mathbb{Q}[x] \).

\( [p(x)] \cdot [q(x)] = [x + \beta x] \cdot \mathbb{E} \) \( x + \beta x = \)

\( = [2\beta + (2\beta + \beta x) + \beta x] = \)

\( = [2\beta + (2 + \beta x)] \) \( x + \beta x = \)

\( \Rightarrow x^2 - 2 = 0 \quad \Leftrightarrow \quad x = \pm \sqrt{2} \)

\( \Rightarrow x^2 - 2 \text{ is irreducible in } \mathbb{Q}[x], \)

\( \Rightarrow (x^2-2) \text{ maximal ideal } \Rightarrow \mathbb{Q}[x] \big/ (x^2-2) \text{ field.} \)
\( i \times j \) is such that \( i \times j^2 = 2 \)

\( \Rightarrow i \times j \) is the square root of Two.
4) \(|G| = 12\) and \(G\) abelian group.

Choose \(g \in G\) with \(g \neq \text{id}\) (identity).

Consider \(<g>\) (subgroup generated by \(g\))

by Lagrange Theorem \(\Rightarrow\) \(|<g>| \mid |G| = 12\)

\(\Rightarrow\) if \(|<g>| < 12\) then \(<g>\) is a proper normal subgroup and \(G\) is not simple.

If \(|<g>| = 12\), take \(g^3\) \(\Rightarrow\) \(g^4\) has order 4 \(\Rightarrow\) \(|<g^4>| = 4\) and \(2 <g^4>\) is a proper normal subgroup of \(G\)

and \(G\) is not simple.
5) \( f : \mathbb{Z}_5 \to \mathbb{Z}_5 \), \( f(0) = 0^5 \)

Since field homomorphisms are injective, it is enough to check that \( f \) is a field homomorphism:

- \( f([0]) = [0]^5 = [0] \) on \( \mathbb{Z}_5 \)

- \( f([x]) = [x^5] = [x] \) on \( \mathbb{Z}_5 \)

- \( f([x^2 + xy]) = ([x] + [xy])^5 = [x^5 + 5xy + 10x^2y + 10xy^2 + 5x^4y + y^5] = [x^5] + [5xy + 10x^2y + 10xy^2 + 5x^4y + y^5] = [x^5] + [y^5] = [x^5] + [y^5] = [x^5 + y^5] \)

\( = f([x] + f([y]) \)

\( = f([x] + [y]) \)

- \( f([x + y]) = ([x] + [y])^5 = [x^5 + y^5] = f([x]) \cdot f([y]) \)