PROBLEMS ABOUT RINGS AND IDEALS

Definitions. Let R be a commutative ring. For an ideal $I \subset R$, we defined

$$\mathrm{Rad}\,(I) := \{a \in R \text{ such that } a^n \in I \text{ for some } n \geq 1\}$$

A proper ideal $Q \subsetneq R$ is called *primary* if $a, b \in Q$ and $a \notin Q$ implies that $b^n \in Q$ for some $n \geq 1$. Equivalently:

$$Q \text{ is primary} \iff \mathcal{N}\,(R/Q) = \{ \text{ zero divisors in } R/Q\}$$

where $\mathcal{N}\,(R)$ is the ideal consisting of nilpotent elements of R, called the *nilradical* of R. In this perspective, the radical of a proper ideal $I \subsetneq R$ can be written as:

$$\mathrm{Rad}\,(I) = \pi^*(\mathcal{N}\,(R/I)) := \{a \in R : \pi(a) \text{ is nilpotent in } R/I\}$$

where $\pi : R \to R/I$ is the natural projection homomorphism.

An ideal $I \subset R$ is called *irreducible* if it cannot be written as an intersection of two ideals properly containing it. That is, if $I_1, I_2 \subset R$ are two ideals such that $I = I_1 \cap I_2$, then $I = I_1$ or $I = I_2$.

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(1) Let R be a commutative ring and $I_1, I_2 \subset R$ be two ideals in R. Prove that $\mathrm{Rad}\,(I_1 \cap I_2) = \mathrm{Rad}\,(I_1) \cap \mathrm{Rad}\,(I_2)$.

(2) Let $Q \subsetneq R$ be a primary ideal. Prove that $\mathrm{Rad}\,(Q) = P$ is a prime ideal.

(3) Assume that R is Noetherian. For any ideal I, prove that there exists $N \geq 1$ such that $\mathrm{Rad}\,(I)^N \subset I$.

(4) Let R be a commutative ring. Prove that $\mathcal{N}\,(R) = \mathrm{Rad}\,(\{0\})$.

(5) Let R be a commutative ring, $I \subsetneq R$ a proper ideal, and $P \subsetneq R$ a prime ideal such that $I \subset P$. Prove that $\mathrm{Rad}\,(I) \subset P$.

(6) Let R be a commutative ring and $P \subsetneq R$ be a prime ideal. Prove that $\mathrm{Rad}\,(P^n) = P$ for any $n \geq 1$.

(7) Let K be a field, $R = K[x, y, z]/(z^2 - xy)$ and $P = (x, z) \subset R$. Prove that P is a prime ideal, but P^2 is not a primary ideal (and hence also not irreducible, because for Noetherian rings, irreducible implies primary).

(8) Prove the following equality is true in the ring $K[x, y]$, where, K is again a field.

$$(x^2, y) \cap (x, y^2) = (x, y)^2.$$

Prove that $(x, y)^2 \subset K[x, y]$ is a primary ideal. (*Hence, primary does not imply irreducible.*)

(9) Prove that $(4, t) \subset \mathbb{Z}[t]$ is a primary ideal. Verify that $\mathrm{Rad}\,((4, t)) = (2, t)$ which is a maximal ideal in $\mathbb{Z}[t]$. Prove that $(2, t)^2 \subsetneq (4, t) \subsetneq (2, t)$. *Hence, a primary ideal need not be power of a prime..*

(10) Let $Q_1, \ldots, Q_\ell$ be primary ideals in R such that $\mathrm{Rad}\,(Q_j) = P$ for every $1 \leq j \leq \ell$. Prove that $Q = Q_1 \cap \cdots \cap Q_\ell$ is again primary and $\mathrm{Rad}\,(Q) = P$.

(11) Let $R = K[x, y]$, $I = (x^2, xy) \subset R$. Take $P = (x)$ and $Q_n = (x^2, xy, y^n)$ for each $n \geq 2$. Prove that
  (a) P is a prime ideal. Each Q_n is primary and $\mathrm{Rad}\,(Q_n) = (x, y)$.
  (b) $P \cap Q_n = I$.

Hence, we have infinitely many distinct primary decompositions. Notice that they all have the same set of primes $\{(x), (x, y)\}$.

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(12) Let R be a commutative ring. For an ideal $I \subset R$ and an element $x \in R$, define:

$$(I : x) := \{r \in R \text{ such that } rx \in I\}$$

(a) Prove that $(I : x)$ is an ideal.

(b) Let $I_1, I_2 \subset R$ be two ideals. Prove that $(I_1 \cap I_2 : x) = (I_1 : x) \cap (I_2 : x)$.

(13) Again, let R be a commutative ring. Let $Q \subsetneq R$ be a primary ideal, and let $P = \mathrm{Rad}\,(Q)$ be its *associated* prime. For a given $x \in R$, prove the following:

(a) If $x \in Q$, then $(Q : x) = R$.

(b) If $x \notin Q$, then $(Q : x)$ is also primary and $\mathrm{Rad}\,((Q : x)) = P$.

(c) If $x \notin P$, then $(Q : x) = Q$.