MID TERM 1. SAMPLE

Problem 1. Let G be a finite group, $N \triangleleft G$ a normal subgroup, and p be a prime. Prove or disprove: number of Sylow p-subgroups of N is equal to the number of Sylow p-subgroups of G.

Problem 2. Prove that there are no simple groups with 36 elements. (Recall: G is simple if it has only 2 normal subgroups: trivial and G.)

Problem 3. Consider the group $G = \mathrm{GL}_2(\mathbb{F}_5)$, acting on itself by conjugation: $g \bullet x = gxg^{-1}$. What is the order of the stabilizer of $x = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$?

Problem 4. Prove that the following subset of S_4 is a normal subgroup.

$$\{e, (12)(34), (13)(24), (14)(23)\}$$

Problem 5. Assume that a finite group G acts on a finite set X transitively. Assume further that $|X| \geq 2$. Prove that there exists $g \in G$ such that $|X^g| = \emptyset$.

Problem 6. How many group homomorphisms are there from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/15\mathbb{Z}$?

Problem 7. Let G be a finite abelian group, which we write additively. Let $H := \{g \in G : 2g = 0\}$. Prove that

$$\sum_{g \in G} g = \sum_{h \in H} h$$

Further prove that the sum above is zero if $|H| \neq 2$.