PROBLEMS FOR PRACTICE

Problem 1. Prove or disprove: S_5 is a semidirect product of A_5 and $\mathbb{Z}/2\mathbb{Z}$. *Recall: S_n is the group of permutations of $\{1, 2, \ldots, n\}$ and A_n is the kernel of the sign homomorphism $\varepsilon : S_n \to \{\pm 1\}$.*

Problem 2. Determine whether the following statements are true or false.

(1) If G is a cyclic group, then $\mathrm{Aut}_{gp}(G)$ is cyclic.

(2) If G is a cyclic group, then $\mathrm{Aut}_{gp}(G)$ is abelian.

(3) Let N_1, N_2 be two subgroups of a group G. Assume that N_1 is normal in G and $N_2 \subset N_1$ is normal in N_1. Then N_2 is normal in G.

(4) Every group G admits at least one Jordan-Hölder series. *Recall: a Jordan-Hölder series is a finite, strictly descending chain of normal subgroups which cannot be further refined. Equivalently, each graded piece must be simple.*

Problem 3. Let H and N be two groups and $\alpha, \beta : H \to \mathrm{Aut}_{gp}(N)$ two group homomorphisms. Consider the semidirect products $A = N \rtimes_\alpha H$, and $B = N \rtimes_\beta H$.

(1) Assume that there exists $T \in \mathrm{Aut}_{gp}(N)$ such that $\beta(h)(n) = T(\alpha(h)(T^{-1}(n)))$ for every $h \in H$ and $n \in N$. Write an isomorphism $F_T : A \to B$.

(2) Assume that there exists $\psi \in \mathrm{Aut}_{gp}(H)$ such that $\alpha(h) = \beta(\psi(h))$, for every $h \in H$. Write an isomorphism $F_\psi : A \to B$.

(3) Assume that there exists a group homomorphism $j : H \to N$ such that

$$\alpha(h)(n) = j(h) \cdot (\beta(h)(n)) \cdot \left(\beta(h)\left(j(h)^{-1}\right)\right),$$

for every $h \in H$ and $n \in N$. Write an isomorphism $F_j : A \to B$.

Bonus. Given any isomorphism $F : A \to B$, is it true that we should be able to write F (up to certain trivial identifications) in terms of F_T, F_ψ, F_j as above?

Problem 4. Let G be a group and consider the group homomorphism $C : G \to \mathrm{Aut}_{gp}(G)$ given by $C(g)(x) = gxg^{-1}$, for every $g, x \in G$. Prove that $G \rtimes_C G$ is isomorphic to $G \times G$.

Problem 5. Let H be any group with 27 elements, and let $f : A_5 \to H$ be a group homomorphism. Prove that $f(x) = e$ (unit of H) for every $x \in A_5$. *Recall: A_5 is simple.*

Problem 6. What does it mean for two composition series Σ_1 and Σ_2 of a group G to be equivalent? (Just write the definition.)

Problem 7. Prove or disprove: two groups with equivalent Jordan-Hölder series must be isomorphic.

Problem 8. What is the center of the group $G = \mathrm{GL}_2(\mathbb{F}_p)$? Here $\mathbb{F}_p$ is the field with p elements, where p is a prime.

Problem 9. Compute the commutator series, and the central series for the following group:

$$B = \left\{ \begin{pmatrix} d_1 & x & z \\ 0 & d_2 & y \\ 0 & 0 & d_3 \end{pmatrix} \text{ where } d_1, d_2, d_3 \in \mathbb{C} \setminus \{0\}, \text{ and } x, y, z \in \mathbb{C} \right\}$$

Recall: commutator series is defined inductively as: $G^{(0)} = G$ *and* $G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}]$. *Central series is defined inductively as* $C^1(G) = G$ *and* $C^{\ell+1}(G) = [G, C^\ell(G)]$.

Problem 10. Write the Jordan-Hölder series for the dihedral group D_{12} with 12 elements.

Problem 11. Assume that G is a (non-trivial) nilpotent group. Prove that $Z(G) \neq \{e\}$. Here $Z(G)$ is the center of G. *Recall:* G *is nilpotent iff* $C^n(G) = \{e\}$ *for some* $n \geq 0$*; iff* G *admits a composition series* $G = H_0 \rhd \ldots \rhd H_m = \{e\}$ *such that* $[G, H_\ell] \subset H_{\ell+1}$ *for every* ℓ.

Problem 12. Prove that every p-group is nilpotent. *Recall: a group* G *is nilpotent iff* $G/Z(G)$ *is nilpotent.*

Problem 13. Give an example of a group G and a normal subgroup $N \lhd G$ such that both N and G/N are nilpotent, but G is not.

Problem 14. Compute $\mathrm{Aut}_{gp}(\mathbb{Z}/2Z \times \mathbb{Z}/4\mathbb{Z})$.

Problem 15. How many abelian groups of size 24 are there?

Problem 16. Compute $\mathrm{Aut}_{gp}(\mathbb{Z}/48\mathbb{Z})$.

Problem 17. Let G be a group and $N \lhd G$ be a normal subgroup. Prove that G/N is abelian if, and only if $[G, G] \subset N$.

Problem 18. True or false: center of a solvable group is always non-trivial. *Recall:* G *is solvable iff* $G^{(n)} = \{e\}$ *for some* $n \geq 0$*; iff* G *admits a composition series* $G = H_0 \rhd \ldots \rhd H_m = \{e\}$ *such that* $H_\ell/H_{\ell+1}$ *is abelian for every* ℓ.

Problem 19. Let B and $N \leq B$ be the following groups:

$$B = \left\{ \begin{pmatrix} d_1 & x \\ 0 & d_2 \end{pmatrix} \text{ where } d_1, d_2 \neq 0 \text{ and } x \text{ is arbitrary} \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ where } x \text{ is arbitrary} \right\}$$

Prove that N is normal in B. Prove that B is a semidirect product of N and H, where H is the subgroup of B consisting of diagonal matrices.

Problem 20. Prove or disprove: $\mathrm{Aut}_{gp}(S_5) \cong S_5$.

Recall: every finite group admits a Jordan-Hölder series. Any two Jordan-Hölder series are equivalent.

Problem 21. For a finite group H, let $\ell(H)$ be the length of a Jordan-Hölder series of H. How do we know that this number does not depend on the choice of the Jordan-Hölder series?

Problem 22. Let G be a finite group and $N \triangleleft G$ be a normal subgroup. Prove that $\ell(G) = \ell(N) + \ell(G/N)$.

Problem 23. Fix a finite simple group S. For a finite group G, choose a Jordan-Hölder series $\Sigma : G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{e\}$. Let $\mathrm{Mult}(S; G)$ be defined as:

$$\mathrm{Mult}(S; G) := \#\{j : G_j/G_{j+1} \cong S\}$$

Prove that the number $\mathrm{Mult}(S; G)$ does not depend on the choice of the Jordan-Hölder series Σ.

Problem 24. Fix a finite simple group S. Let G be a finite group and $N \triangleleft G$ be a normal subgroup. Prove that $\mathrm{Mult}(S; G) = \mathrm{Mult}(S; N) + \mathrm{Mult}(S; G/N)$.

Problem 25. (Bonus). Prove or disprove: $\mathrm{Mult}(S; G)$ as defined above, can be computed as cardinality of the set of group homomorphisms $S \to G$ counted up to conjugation by elements of $\mathrm{Aut}_{gp}(S)$ and $\mathrm{Aut}_{gp}(G)$. More precisely, is the following equation true?

$$\mathrm{Mult}(S; G) = |\mathrm{Aut}_{gp}(S) \setminus \mathrm{Hom}_{gp}(S, G) / \mathrm{Aut}_{gp}(G)|$$