PROBLEMS FOR PRACTICE - MID TERM 3

All rings considered below are commutative and unital. All rings homomorphisms are assumed to preserve identities.

Problem 1. Let R be a ring. Prove that the following are equivalent: (1) R is a field. (2) The zero ideal is a maximal ideal. (3) The set of ideals in R is $\{(0), (1) = R\}$.

Problem 2. Let $f : R \to S$ be a ring homomorphis. Prove or disprove:

(1) If R is a field, then f is injective.
(2) If S is a field, then f is injective.

Problem 3. Prove that $\mathbb{Q}[x]$ is a principal ideal domain.

Problem 4. Let R be a principal ideal domain. Prove that every non-zero prime ideal of R is maximal.

Problem 5. Let R be a principal ideal domain. Prove that the set of ideals in R is same as $R/R^\times$. *Here $R^\times$ is the group of units in R.*

Problem 6. What are the prime ideals in $\mathbb{Q}[x]/(x^3)$?

Problem 7. Let R be a ring and $I, J \subset R$ be two ideals. Prove that $IJ \subset I \cap J$. Give an example when this inclusion is proper.

Problem 8. Keep the set up of Problem 7 above. Further assume that I and J are coprime (that is, $I + J = R$). Prove that

(1) $IJ = I \cap J$.
(2) The ring homomorphism $R \to R/I \times R/J$ is surjective.

Problem 9. Let R be a ring, $I, J \subset R$ two ideals and let P be a prime ideal such that $I \cap J \subset P$. Prove that either $I \subset P$ or $J \subset P$.

Problem 10. Prove that every finite integral domain is a field.

Problem 11. Prove that every maximal ideal is prime. Give an example of a prime ideal that is not maximal.

Problem 12. Prove that $\mathbb{Q}[x, y]$ is not a principal ideal domain.

Problem 13. Let R be a ring. Prove that R is a local ring if, and only if $R \setminus R^\times$ is an ideal.

Problem 14. Compute the greatest common divisor $d(x)$ of the following two elements of $\mathbb{Q}[x]$:

$$a(x) = x^3 + 4x^2 + x - 6 \qquad\qquad b(x) = x^5 - 6x + 5$$

Write $d(x) = \alpha a(x) + \beta b(x)$ for $\alpha, \beta \in \mathbb{Q}[x]$.

Problem 15. Let $I \subset \mathbb{Z}[x]$ be a maximal ideal. Prove or disprove: $\mathbb{Z}[x]/I$ is finite.

Problem 16. Prove that $R = \mathbb{Z}/36\mathbb{Z}$ is a principal ideal ring. Compute each of the following in R: (1) Prime ideals in R. (2) Maximal ideals in R. (3) Units in R.

Problem 17. Let R be a ring. Define the *characteristic* of R. Let $n = \text{Characteristic}(R)$. Prove or disprove: the map $R \to R$ that sends $a \in R$ to $a^n \in R$ is a ring homomorphism.

Problem 18. Let $f : R \to S$ be a ring homomorphism. Recall that for an ideal $J \subset S$, we defined

$$f^*(J) := \{a \in R \text{ such that } f(a) \in J\}$$

Prove, or provide a counterexample to each of the following statements.

(1) $f^*(J) \subset R$ is an ideal.
(2) If $J \subset S$ is a prime ideal, then $f^*(J) \subset R$ is also prime.
(3) If $J \subset S$ is a maximal ideal, then $f^*(J) \subset R$ is also maximal.

Problem 19. Let $R \subset \mathbb{Q}$ be the set consisting of all rational functions whose denominator is an odd number. Prove that R is a local ring. Describe its unique maximal ideal.

Problem 20. Let R be a ring. For $a \in R$, define the following subset of R.

$$I(a) := \{r \in R \text{ such that } ra = 0\}$$

(1) Prove that $I(a) \subset R$ is an ideal.
(2) Prove that $I(a) \subset I(ab)$ for every $a, b \in R$.
(3) Prove that $I(a) = R$ if, and only if $a = 0$.

Problem 21. Let R be a ring and $S \subset R$ be a multiplicatively closed set. *Recall that this means:* $1 \in S$, $0 \notin S$, *and for every* $a, b \in S$, *we have* $ab \in S$.

(1) What is the kernel of the ring homomorphism $j : R \to S^{-1}R$? Recall that $j(r) = \frac{r}{1}$ for every $r \in R$.

(2) Prove that every ideal in $S^{-1}R$ is of the following form:

$$\left\{\frac{a}{s} \text{ such that } a \in I, s \in S\right\}$$

where $I \subset R$ is an ideal.

Problem 22. Let R be a ring. Consider the following subset (of nilpotent elements of R):

$$\mathcal{N}(R) = \{r \in R \text{ such that there exists } n \geq 1 \text{ so that } r^n = 0\}$$

Prove that $\mathcal{N}(R)$ is a proper ideal of R. Let $P \subset R$ be a prime ideal. Prove that $\mathcal{N}(R) \subset P$.

Problem 23. Let R be a ring. Recall that for an ideal $I \subset R$, we defined $\text{Rad}\,(I) \subset R$ as

$$\text{Rad}\,(I) = \{r \in R \text{ such that } r^n \in I \text{ for some } n \geq 1\}$$

(1) Prove that $\mathcal{N}(R) = \text{Rad}\,((0))$.
(2) Prove that $\text{Rad}\,(I) = \pi^*(\mathcal{N}(R/I))$ where $\pi : R \to R/I$ is the natural surjective ring homomorphism to the quotient ring. (See problem 18 above for the notation π^*.)
(3) Prove that I is a proper ideal if, and only if $\text{Rad}\,(I)$ is a proper ideal.
(4) For any two ideals $I, J \subset R$, prove that $\text{Rad}\,(I \cap J) = \text{Rad}\,(I) \cap \text{Rad}\,(J)$.
(5) Let $P \subset R$ be a prime ideal. Prove that $\text{Rad}\,(P) = P$.
(6) Let $I \subset P \subset R$, where I is an ideal and P is a prime ideal. Prove that $\text{Rad}\,(I) \subset P$.
(7) Give an example of a proper ideal $I \subset R$ such that $\text{Rad}\,(I) = I$ but I is not a prime ideal.