Lecture 2

(2.0) Recall: (1) A group G is a set together with an associative binary operation, an identity element, such that every element has an (the) inverse.

(2) A subset (non-empty) $H \subset G$ is a subgroup, denoted by $H \leq G$, if $a, b \in H \Rightarrow a * b^{-1} \in H$.

(3) A subset $A \subset G$ generates G if the smallest subgroup of G containing A is G itself.

"subgroup generated by A"

(4) Cyclic groups = groups that admit one generator

$$= \langle a \mid a^k = e \rangle \begin{cases} \xrightarrow{k=0} \mathbb{Z} \\ \xrightarrow{k \geq 1} \mathbb{Z}/k\mathbb{Z} \end{cases} \Bigg\} \text{ list of all cyclic groups}$$

(2.1) Presentation of a group. -

Written as $\langle \text{generators} \mid \text{relations} \rangle$

$$G = \langle \underbrace{a_1, a_2, \ldots}_{\text{symbols/alphabets}} \mid r_1, r_2, r_3, \ldots \rangle$$

- G consists of words in the alphabet $\{a_1, a_2, \dots\}$ [subject to rules below]

  e.g. $a_3^{-2} a_1^{4} a_2^{2}$

  [A typical word $w = x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_\ell^{n_\ell}$ where $x_1, x_2, \dots, x_\ell \in \{a_1, a_2, \dots\}$ $n_1, n_2, \dots, n_\ell \in \mathbb{Z}$.]

- $r_1, r_2, r_3, \dots$ are words in the ~~alphabets~~ alphabet $\{a_1, a_2, \dots\}$

- Rules : $r_1 = e$, $r_2 = e$, ...

  [If one of r_j appears in w, say $w = w_1 r_j w_2$, then $w = w_1 w_2$.]

(2.2) Example. Let $G = \langle a, b \mid a^2, b^2 \rangle$

Then a typical element of G has the following form $\left\{ \begin{array}{l} a\,b\,a\,b\,a\,b \cdots \\ \text{or } b\,a\,b\,a\,b\,a \cdots \end{array} \right.$

(2.3) Definition. Let G be a group and $a \in G$. Order of a, sometimes abbreviated as $\mathrm{ord}(a) \in \mathbb{Z}_{\geq 1}$, (or ∞)

- $\mathrm{ord}(a) :=$ smallest positive integer $\underset{\ell}{\downarrow}$ such that

$$a^{\ell} = e$$

$\mathrm{ord}(a) = 1 \iff a = e$; $\mathrm{ord}(a) = \infty$ means

$$\langle \{a\} \rangle \cong \mathbb{Z}$$

(subgp. of G generated by a)

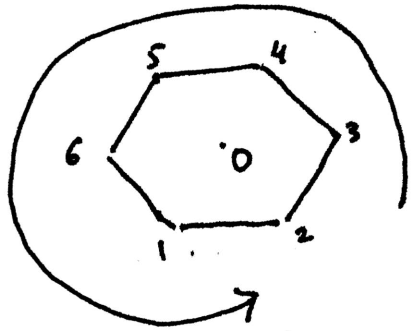
See example (2.2) : product of elements of finite order, need not have finite order.

Note : if $a, b \in G$ are such that $\underline{a * b = b * a}$.
Then $\mathrm{ord}(a*b) \leq \mathrm{l.c.m.}(\mathrm{ord}(a), \mathrm{ord}(b))$ [Ex.].

(2.4) Presentations of the dihedral group.

D_{2n} = symmetries of a regular n-gon.

e.g. $n = 6$

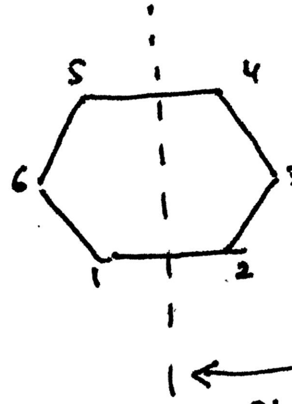


$$\rho : \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 3 \\ \vdots \\ 6 \mapsto 1 \end{array}$$

ρ = rotation around 0 by $\frac{\pi}{3}$.

$$\boxed{\rho^6 = e}$$

$$\sigma : \begin{array}{l} 1 \longleftrightarrow 2 \\ 3 \longleftrightarrow 6 \\ 4 \longleftrightarrow 5 \end{array}$$



$$\sigma^2 = e$$

σ = reflection about y-axis

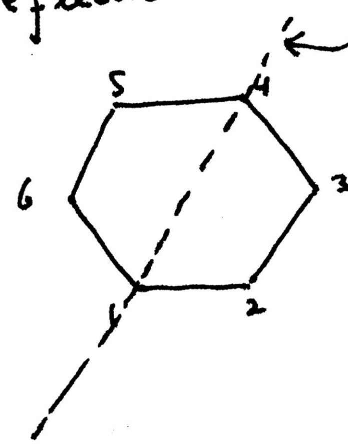
What is the relation between σ and ρ?

$$\boxed{\sigma \rho \sigma = \rho^{-1}}$$

Proof:

$$\sigma \rho : \begin{array}{l} 1 \longmapsto 1 \\ 2 \longmapsto 6 \\ 6 \longmapsto 2 \\ 3 \longmapsto 5 \\ 5 \longmapsto 3 \\ 4 \longmapsto 4 \end{array}$$

= reflection about



So $(\sigma \rho)^2 = e$

$$\Rightarrow \sigma \rho \sigma \rho = e$$

$$\Rightarrow \sigma \rho \sigma = \rho^{-1}$$

□

Proposition (2.5)

$$D_{2n} = \langle \sigma, \rho \mid \sigma^2, \rho^n, \sigma\rho\sigma\rho \rangle$$

Proof. Take $\rho = (1\,2\,3 \ldots n)$

$\sigma = (12)(34) \quad \ldots \quad (n-1, n)$ if n is even

$\quad = (12)(34) \quad \ldots \quad (n-2, n-1)$ if n is odd

Check: $\sigma^2 = e$, $\rho^n = e$, $\sigma\rho\sigma\rho = e$. [Ex.]

Now $\langle \sigma, \rho \mid \sigma^2 = e, \rho^n = e, \sigma\rho\sigma\rho = e \rangle$

$$= \left\{ e, \begin{array}{l} \sigma, \sigma\rho, \sigma\rho^2, \ldots, \sigma\rho^{n-1} \\ \rho, \rho^2, \ldots \rho^{n-1} \end{array} \right\} \quad 2n\text{-elements}$$

i.e. subgroup of D_{2n} generated by σ, ρ has same size (subject to only 3 relations)

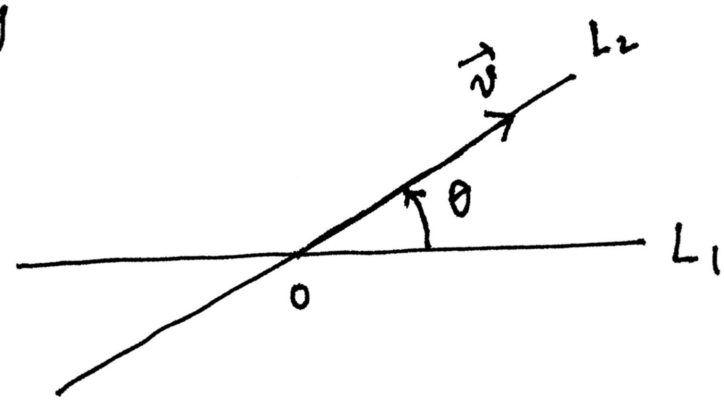
as D_{2n}. So it has to be D_{2n} $\square$

(2.6) Lesson. - product of two reflections is a rotation.

Let us take two lines in two dimensional plane $\mathbb{R}^2$.

$L_1 = x$-axis for simplicity

$$L_2 = \mathbb{R} . \underbrace{\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}}_{\vec{v}}$$



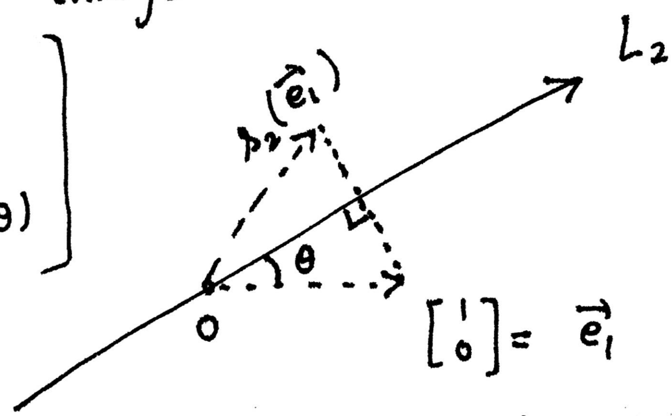
s_1, s_2 = reflections about L_1 and L_2.

In matrix notation $s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (sorry for the mess $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$)

X-axis doesn't change

y-axis $\mapsto$ -ve y-axis.

Ex. $s_2 = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$



[reflection of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ about L_2]

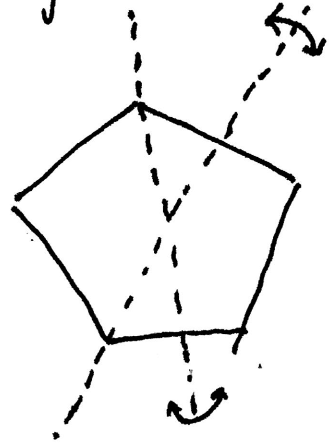
(use: reflection about a line $L = \mathbb{R}\cdot\vec{v}$ (unit-vector) is given by $\vec{w} \longmapsto -\vec{w} + 2(\vec{v}\cdot\vec{w})\,\vec{v}$)

So $s_2 s_1 = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$

= counter clockwise rotation by 2θ.

(2.7) D_{2n} can be generated by 2 reflections.

$$D_{2n} = \left\langle s_1, s_2 \;\middle|\; \begin{array}{l} s_1^2 = e = s_2^2 \\ (s_1 s_2)^n = e \end{array} \right\rangle$$



Reflections generating D_{10}

[From Prop (2.5) above,

$s_1 \longleftrightarrow \sigma$

$s_2 \longleftrightarrow \sigma\rho$ $\qquad$ $\left(s_1 s_2 \longleftrightarrow \sigma\sigma\rho = \sigma^2\rho = \rho \right)$]

(2.8) What is "wrong" with group presentations?

example. – $\langle x, y \mid xy^2 = y^3x \,;\, yx^2 = x^3y \rangle$

This group is actually trivial.

<u>Hint</u>: $xy^2 = y^3x \implies x^2y^8x^{-2} = y^{18}$

$\qquad\qquad\qquad\qquad\quad\;\; x^3y^8x^{-3} = y^{27}$

$yx^2y^{-1} = x^3$ (2nd rel^n) $\implies y^{18} = y^{27} \implies y^9 = e$

so, $e = x^{-1}y^9x^{+1} = (x^{-1}\underset{\uparrow\, y^2}{y^3}x^{+1})^3 = y^6$

$\implies y^3 = e \overset{\text{rel}^n\,1}{\implies} y^2 = e \implies y = e$ $\qquad \square$