Lecture 3

(3.0) Recall : presentation of a group :

$$G = \langle \underbrace{a_1, a_2, \ldots}_{\text{generators}} \mid \underbrace{r_1, r_2, r_3, \ldots}_{\text{relations}} \rangle$$

e.g. $\mathbb{Z}/_{k\mathbb{Z}} = \langle a \mid a^k = e \rangle$ (cyclic gps.)

$$D_{2n} = \left\langle s, r \;\middle|\; \begin{array}{l} s^2 = e = r^n \\ srs = r^{-1} \end{array} \right\rangle = \left\langle s_1, s_2 \;\middle|\; \begin{array}{l} s_1^2 = e = s_2^2 \\ (s_1 s_2)^n = e \end{array} \right\rangle$$

(3.1) A <u>free group</u> on a set $\mathcal{A}$ is the group

$$\underline{\mathrm{Free}(\mathcal{A})} = \langle \text{Generators} = \text{elements of } \mathcal{A} \mid \text{Relations} = \text{none} \rangle$$

not a standard notation

If $\mathcal{A} = \{a_1, \ldots, a_n\}$ is finite (with n elements)

we just write $\mathrm{Free}(n)$ = free group on n letters.

Note : - every element $w \in \mathrm{Free}(\mathcal{A})$ is written <u>uniquely</u> as

$$w = x_1^{m_1} x_2^{m_2} \cdots x_\ell^{m_\ell} \quad \begin{cases} x_1, \ldots, x_\ell \in \mathcal{A} \\ m_1, \ldots, m_\ell \in \mathbb{Z}_{\neq 0} \end{cases}$$

$x_1 \neq x_2$, $x_2 \neq x_3$, ... , $x_{\ell-1} \neq x_\ell$ (avoid redundancies)

For a group with relations, the word "uniquely" needs to be dropped.

e.g. $D_8 = \left\langle s_1, s_2 \;\middle|\; \begin{array}{l} s_1^2 = e = s_2^2 \\ (s_1 s_2)^4 = e \end{array} \right\rangle$

$$= \left\{ e, \begin{array}{ccccc} s_1 & s_1 s_2 & s_1 s_2 s_1 & s_1 s_2 s_1 s_2 \\ & & & \shortparallel \\ s_2 & s_2 s_1 & s_2 s_1 s_2 & s_2 s_1 s_2 s_1 \end{array} \right\}$$

not unique expression

(3.2) A standard counterexample to
"Subgroup of a finitely generated group is finitely generated".

$\text{Free}(2) = \langle a, b \mid \text{no rel}^{\underline{n}} \rangle$

$\cup$

H = subgroup gen. by $\{ x y x^{-1} y^{-1} \mid x, y \in \text{Free}(2) \}$

Prop. – H is not finitely generated.

Proof. Define a function

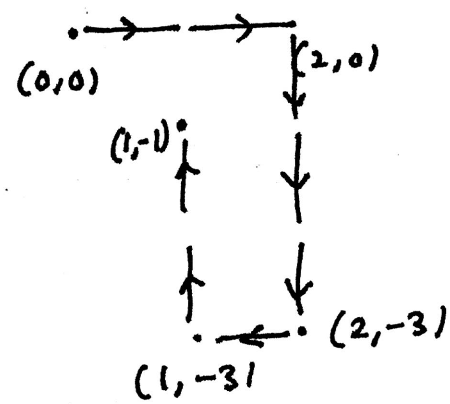
$$d : \mathrm{Free}(2) \longrightarrow \mathbb{R}_{>0}, \text{ as follows.}$$

Given $w \in \mathrm{Free}(2)$, trace a path in $\mathbb{R}^2$ by reading w from left to right

$$\begin{cases} a^k : & \text{move } k \text{ steps along } x\text{-axis} \\ & \left(\begin{array}{ll} k \geq 0 & - \ \cancel{\text{to}} \text{ right} \\ k < 0 & - \ |k| \text{ left} \end{array} \right) \\ b^k : & \text{move } k \text{ steps along } y\text{-axis} \end{cases}$$

e.g. $w = a^2 b^{-3} a^{-1} b^2 \rightsquigarrow \gamma_w$: Following path in $\mathbb{R}^2$:

$$d(w) = \max_{p \in \gamma_w} \left(\begin{array}{l} \text{distance b/w} \\ (0,0) \text{ \& a} \\ \text{point } p \in \gamma_w \end{array} \right)$$



(for this path $d(w) =$ distance $\left((0,0) \longrightarrow (2,-3)\right)$ $= \sqrt{13}$)

Remarks:

- $\forall\ w \in H$, endpoint of w is $(0,0)$
- If $w_1, w_2 \in H$

  $d(w_1 w_2) \leq \max\{d(w_1), d(w_2)\}$

If we assume H is finitely-generated, say by $w_1, \ldots, w_\ell \in H$, there will exist a number

$$R = \max \{ d(w_1), \ldots, d(w_\ell) \}$$

s.t. $\quad d(h) \leq R \quad \forall \, h \in H$

But $\quad d(a^n b a^{-n} b^{-1}) = \text{distance}((0,0) - (n,1))$

$$= \sqrt{n^2+1} \quad \text{can be made}$$

larger than R by taking $n > R$.

contradiction! $\square$

(3.3) Presentation of Symmetric group S_n.

Write $\sigma_{ij} = (i\ j) \in S_n \qquad 1 \leq i < j \leq n$

$$s_i = \sigma_{i,i+1} \qquad (1 \leq i \leq n-1)$$

Lemma. S_n is generated by $\{ s_i : 1 \leq i \leq n-1 \}$

(meaning: every permutation $\pi \in S_n$ can be written (NOT uniquely) as a product of s_i's.)

Proof. Step 1. Every permutation can be written as a product of (disjoint) cycles:

[Details: let $X = \{1, 2, 3, \ldots, n\}$ and $\pi \in S_n$.

~~Write~~ Now $X = \bigsqcup_{j=1}^{r} X_j$ where the subsets $X_1, \ldots, X_r$ have the property that

$$a, b \in X_j \iff b = \pi^m(a) \text{ for some } m \geq 0.$$

(e.g. $\pi = (123) \in S_5 \leadsto \{1,2,3,4,5\} = \{1,2,3\} \sqcup \{4\} \sqcup \{5\}$)

if $|X_j| = n_j$, and $x_j \in X_j$ is chosen, then

$$\pi = \underbrace{\left(x_1, \pi(x_1), \ldots, \pi^{n_1-1}(x_1)\right)}_{1^{st}\text{ cycle}} \underbrace{\left(x_2, \pi(x_2), \ldots \pi^{n_2-1}(x_2)\right)}_{2^{nd}\text{ cycle}}$$

$$\cdots \underbrace{\left(x_r, \pi(x_r), \ldots, \pi^{n_r-1}(x_r)\right)}_{r^{th}\text{ cycle}}$$]

Step 2. Every cycle can be written in terms of $\{\sigma_{ij} : 1 \leq i < j \leq n\}$

$$(x_1 \; x_2 \; \ldots \; x_\ell) = (x_1 \, x_2)(x_2 \, x_3) \cdots (x_{\ell-1} \, x_\ell)$$

<u>Step 3</u>. Every σ_{ij} can be written in terms of $\{ s_t : 1 \leq t \leq n-1 \}$

e.g. $$(1 \; n) = (n-1 \; n)(1 \; n-1)(n-1 \; n)$$

[check]

$$= (n-1 \; n)(n-2 \; n-1)(1 \; n-2)(n-2 \; n-1)(n-1 \; n)$$

$$= \cdots = s_{n-1} s_{n-2} \cdots s_2 s_1 s_2 \cdots s_{n-1} s_n$$

□

(3.4) Some relations among $s_1, s_2, \ldots, s_{n-1}$.

$s_i^2 = e$ (obviously)

$s_i s_j = s_j s_i$ if $\{i, i+1\} \cap \{j, j+1\} = \phi$ (disjoint cycles commute).

$(s_i s_{i+1})^3 = e$

$(i \; i+1 \; i+2)$

Define a group by generators and relations.

$$\mathcal{P}_n = \left\langle a_1, \ldots, a_{n-1} \;\middle|\; \begin{array}{ll} a_i^2 = e & (\forall i = 1, \ldots, n-1) \\ a_i \cdot a_j = a_j \cdot a_i & (|i-j| > 1) \\ a_i a_{i+1} a_i = a_{i+1} a_i \cdot a_{i+1} & \\ \quad (1 \leq i \leq n-2) & \end{array} \right\rangle$$

Next week. we will show that $\mathcal{P}_n \cong S_n$.