Lecture 5

(5.0) Compute orders of elements of S_6.

- Arrange them by cycle-type – we know order of elements written as disjoint union of cycles.

Cycle type	Order	# of elements
$1+1+1+1+1+1$	1	1 $\{e\}$ only
$2+1+1+1+1$	2	$\binom{6}{2} = 15$
$2+2+1+1$	2	$\dfrac{\binom{6}{2}\cdot\binom{4}{2}}{2} = 45$
$2+2+2$	2	$\dfrac{\binom{6}{2}\cdot\binom{4}{2}\cdot\binom{2}{2}}{6} = 15$
$3+1+1+1$	3	$\binom{6}{3}\times 2 = 40$
$3+2+1$	6	$\binom{6}{3}\times 2\times 3 = 120$
$3+3$	3	$\binom{6}{3}\times\dfrac{2\times 2}{2} = 40$
$4+1+1$	4	$\binom{6}{4}\times 3\times 2 = 90$
$4+2$	4	same $= 90$
$5+1$	$6\times(4\times3\times2)$	144
6	$5\times4\times3\times2$	120

Total $6! = 720$ ✓

(5.1) Let G be a group and $H \leq G$ a subgroup.

a)

Definition.- H is said to be <u>normal</u> subgroup, denoted by $H \trianglelefteq G$, if

$\forall\, x \in G$, $h \in H$, we have: $x h x^{-1} \in H$.

Equivalently, H is a normal subgroup if for every $x \in G$

$$xH = Hx \quad (\text{as subsets of } G).$$

(left cosets are the same as right cosets).

Main point: it is only when $H \trianglelefteq G$, that G/H has a natural group structure inherited from G. We will prove it below.

→ G/H is called quotient group.

Remark.- If G is abelian, then every subgroup is normal, because, for every $x \in G$ and $h \in H$

$$x h x^{-1} \underset{\text{by}}{=} x x^{-1} h = h \in H$$

(5.2) Group operation on G/H.

Recall – an element of the set G/H is a <u>subset</u> of G, which is of the form $g \cdot H = \{g \cdot h \mid h \in H\}$ for some (<u>NOT</u> uniquely determined) $g \in G$.

How will we "multiply" two such subsets and get another one?

Guess: — $(g_1 H) * (g_2 H) = (g_1 g_2) H$

Since it involved <u>choosing</u> g_1, g_2; it may very well be NOT A DEFINITION! Meaning:

If $g_1 \underset{L}{\sim} g_1'$, $g_2 \underset{L}{\sim} g_2'$, is it true that $g_1 g_2 \underset{L}{\sim} g_1' g_2'$?

Answer: Not always. For example $G = S_4 \geq H = S_3 = \{\sigma \in S_4 \mid \sigma(4) = 4\}$

$$G/H = \{e, (14)\cdot H, (24)\cdot H, (34)\cdot H\}$$

~~$(14) \underset{L}{\sim} (14)$~~

~~$(24) \underset{L}{\sim} (14)\ (124)$~~

~~$(14).(24) = (142)$~~

~~$(14).(124) = (12)$~~ ~~not in the same equiv. class.~~

e.g. $(14) \sim (124)$ $(= (14)(12))$
$(24) \sim (234)$ $(= (24)(23))$

but $(14).(24) = (142)$
$(124).(234) = (123)$

not in the same eq. class

If we assume $H \trianglelefteq G$ is normal!

$g_1 \sim_L g_1'$ (i.e. $g_1^{-1} g_1' \in H$)

$g_2 \sim_L g_2'$ (i.e. $g_2^{-1} g_2' \in H$), then $(g_1 g_2)^{-1} (g_1' g_2')$

$$= g_2^{-1} \underbrace{(g_1^{-1} g_1')}_{\text{an element of } H \text{ say } h_1} g_2' = g_2^{-1} h_1 g_2' = \underbrace{\boxed{g_2^{-1} h_1 g_2}}_{\in H} \overbrace{g_2^{-1} g_2'}^{\in H}$$

because H is normal

assumption

$\Rightarrow (g_1 g_2)^{-1} (g_1' g_2') \in H$, i.e. $g_1 g_2 \sim_L g_1' g_2'$.

So, $\boxed{(g_1 H) * (g_2 H) = (g_1 g_2) H}$ is a valid

function $G/H \times G/H \longrightarrow G/H$. multiplication of G

Associativity $(g_1 H * g_2 H) * g_3 H = \cdots = ((g_1 * g_2) * g_2) H$

$g_1 H * (g_2 H * g_3 H) = \cdots = (g_1 * (g_2 * g_3)) H$

are equal by associativity of multiplication in G

Unit $= e \cdot H$

Inverse $(gH)^{-1} = g^{-1} H$.

(5.3) Examples. $G = S_4 \geq H = \{e, \underbrace{(12).(34),\ (13).(24),\ (14).(23)}_{\substack{\shortparallel \\ \text{all elements of cycle} \\ \text{type } 2+2}}\}$

Ex. H is a subgroup

(e.g. $((12)(34)).((13).(24)) = (14)(23)$)

H is normal because $\sigma \pi \sigma^{-1}$ has the same cycle type as π and $H = \{e\} \cup$ set of all $2+2$ cycle type permutations.

[Proof $\sigma\{(x_1\ x_2\ \dots\ x_\ell)\}\sigma^{-1}$

$= (\sigma(x_1)\ \sigma(x_2)\ \dots\ \sigma(x_\ell))$

still an ℓ-cycle.

And $\sigma\{\pi_1 \cdot \pi_2 \dots \pi_r\}\sigma^{-1}$

$= \sigma \pi_1 \sigma^{-1} . \sigma \pi_2 \sigma^{-1} \dots \sigma \pi_r \sigma^{-1}$ $\square$]

Remark.- The same set $H \subset S_5$ will still be a subgroup, but not ~~necess~~ normal, because

$S_5 \ni \sigma = (15)$ ⇝ $(15)\{(12)(34)\}(15)$

$H \ni \pi = (12)(34)$ $= (25)(34) \notin H$.

(5.4) $G = D_{2n} = \langle s, r \mid s^2 = e = r^n,\ srs = r^{-1} \rangle$

$\quad\ \ \text{IV}$

$\quad\ \ H = \{ e, r, r^2, \ldots, r^{n-1} \}$

Claim: H is normal.

Pf. A typical element of G (not in H) $x = s \cdot r^k$ for some k

A typical element of H $h = r^l$ " " l $(0 \le k, l \le n-1)$

$$\Rightarrow \quad x h x^{-1} = s \cancel{r^k} r^l \cancel{r^{-k}} s^{-1}$$

$$= s r^l s = r^{-l} = r^{n-l} \in H \checkmark$$

$$\Big(= \underbrace{(srs)(srs)\ldots(srs)}_{l \text{ times}} \Big)$$

$\square$

(5.5) $G = S_3 \geq N = \{ e, (123), (132) \}$

is a normal subgroup.

(5.6) $G = GL_2(\mathbb{R})$ = 2×2 matrices with $\det \neq 0$.

$\quad\ \ \text{IU}$

$\quad\ \ N = \{ X \in GL_2(\mathbb{R}) : \det(X) = 1 \}$

Since $\det(AB) = \det(A) \cdot \det(B)$ ← N is a subgroup

$\det(TAT^{-1}) = \det(A)$ ← N is normal.

Notation: $SL_2(\mathbb{R})$ (special linear group).